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A CHARACTERIZATION OF THE SPACES $\mathcal{S}_{1/k+1}^{k/k+1}$ BY MEANS OF HOLOMORPHIC SEMIGROUPS*

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Abstract. The Gel'fand–Shilov spaces $\mathcal{S}_{\alpha}^{\beta}$, $\alpha = 1/(k+1)$, $\beta = k/(k+1)$, are special cases of a general type of test function spaces introduced by de Graaf. We give a self-adjoint operator so that the test functions in those $\mathcal{S}_{\alpha}^{\beta}$ spaces can be expanded in terms of the eigenfunctions of that self-adjoint operator.

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1. Introduction. De Bruijn's theory of generalized functions based on a specific one-parameter semigroup of smoothing operators [1] was generalized considerably by de Graaf [4]. In brief this extended theory can be described as follows: In a Hilbert space $\mathcal{X}$ consider the evolution equation

$$(1.1) \quad \frac{du}{dt} = -\mathfrak{U}u$$

where $\mathfrak{U}$ is a positive, self-adjoint operator, which is unbounded in order that the semigroup $(e^{-t\mathfrak{U}})_{t \geq 0}$ is smoothing. A solution u of (1.1) is called a trajectory if u satisfies

$$(1.2i) \quad \forall t > 0 \quad \forall \tau > 0: \quad e^{-\tau\mathfrak{U}}u(t) = u(t + \tau),$$

$$(1.2ii) \quad \forall t > 0: \quad u(t) \in \mathcal{X}.$$

The limit $\lim_{t \downarrow 0} u(t)$ does not necessarily exist in $\mathcal{X}$!

The complex vector space of all trajectories is denoted by $\mathcal{T}_{\mathcal{X}, \mathfrak{U}}$. The elements of $\mathcal{T}_{\mathcal{X}, \mathfrak{U}}$ are called generalized functions.

The test function space $\mathcal{S}_{\mathcal{X}, \mathfrak{U}}$ is the dense linear subspace of $\mathcal{X}$ consisting of smooth elements of the form $e^{-t\mathfrak{U}}h$, where $h \in \mathcal{X}$ and $t > 0$; we have $\mathcal{S}_{\mathcal{X}, \mathfrak{U}} = \bigcup_{t > 0} e^{-t\mathfrak{U}}(\mathcal{X})$. The densely defined inverse of $e^{-t\mathfrak{U}}$ is denoted by $e^{t\mathfrak{U}}$. For each $\varphi \in \mathcal{S}_{\mathcal{X}, \mathfrak{U}}$ there exists $\tau > 0$ such that $e^{\tau\mathfrak{U}}\varphi$ makes sense. The pairing between $\mathcal{S}_{\mathcal{X}, \mathfrak{U}}$ and $\mathcal{T}_{\mathcal{X}, \mathfrak{U}}$ is defined by

$$(1.3) \quad \langle \varphi, F \rangle = (e^{\tau\mathfrak{U}}\varphi, F(\tau)), \quad \varphi \in \mathcal{S}_{\mathcal{X}, \mathfrak{U}}, \quad F \in \mathcal{T}_{\mathcal{X}, \mathfrak{U}}.$$

Here $(\cdot, \cdot)$ denotes the inner product in $\mathcal{X}$. Definition (1.3) makes sense for $\tau > 0$ sufficiently small, and due to the trajectory property (1.2i) it does not depend on the specific choice of τ . For further results concerning this theory we refer to [4].

The aim of the present paper is to show that for certain Gel'fand–Shilov spaces $\mathcal{S}_{\alpha}^{\beta}$ [2] there exists an operator $\mathfrak{U}$ such that $\mathcal{S}_{\alpha}^{\beta} = \mathcal{S}_{\mathcal{X}, \mathfrak{U}}$. This leads to the result that the elements of the dual of $\mathcal{S}_{\alpha}^{\beta}$ can be interpreted as trajectories. Furthermore, we find that a function in the studied $\mathcal{S}_{\alpha}^{\beta}$ -spaces can be developed in a series of certain orthonormal functions.

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2. Eigenfunction expansions of test functions in $\mathfrak{S}_\alpha^\beta$. Let us consider the following eigenvalue problem in $\mathcal{L}_2(\mathbb{R})$:

$$(2.1) \quad \frac{d^2}{dx^2}y + (\lambda - x^{2k})y = 0,$$

where λ is a real number and k a positive integer. It is well-known that the operator $-d^2/dx^2 + x^{2k}$ has a point spectrum and the set of eigenvalues (λ_n) is real, positive and unbounded. In the sequel we shall regard it as ordered with $\lambda_{n+1} \geq \lambda_n$, $n = 0, 1, \dots$. The corresponding normalized eigenfunctions $\{\psi_n\}$ form a complete orthonormal basis in $\mathcal{L}_2(\mathbb{R})$. So by the Riesz–Fischer theorem every $f \in \mathcal{L}_2(\mathbb{R})$ can be represented by

$$(2.2) \quad f = \sum_{n=0}^{\infty} a_n \psi_n,$$

where $a_n = (f, \psi_n)$ is an $\mathcal{L}_2$ -sequence.

First of all we gather some of the estimates for the eigenvalues λ_n and the eigenfunctions ψ_n of the problem (2.1), and then characterize $\{\psi_n\}$ as elements of certain $\mathfrak{S}_\alpha^\beta$ -spaces. We take $\psi_n(x) > 0$ for large positive values of x , cf. Titchmarsh [5, Chap. VIII].

From Titchmarsh [5, p. 144] we have

$$(2.3) \quad \lambda_n = O(n^{2k/(k+1)}), \quad n \rightarrow \infty.$$

According to Titchmarsh we have the following estimates for the normalized eigenfunctions

$$(2.4) \quad |\psi_n(x)| \leq \frac{2}{3} \lambda_n^{1+3/4k} \quad \text{for all } x \in \mathbb{R}, n \in \mathbb{N} \quad [5, \text{p. 168}],$$

$$(2.5) \quad |\psi_n(x)| \leq \psi_n(x_0) \exp \left\{ - \int_{x_0}^x (u^{2k} - \lambda_n)^{1/2} du \right\} \quad \text{for } x \geq x_0 \geq \lambda_n^{k/2} \quad [5, \text{p. 165}].$$

We take $x_0 = (\frac{4}{3}\lambda_n)^{1/2k}$. From a straightforward calculation it follows that

$$|\psi_n(x)| \leq \frac{2}{3} \lambda_n^{1+3/4k} \exp \left\{ - \frac{1}{4} \frac{1}{k+1} |x|^{k+1} \right\}$$

for $|x| \geq 2\lambda_n^{1/2k}$. For any number a , $0 < a < 1/4(k+1)$, we have

$$(2.6) \quad |\psi_n(x)| \leq K_n \exp(-a|x|^{k+1}), \quad x \in \mathbb{R},$$

where

$$K_n = \frac{2}{3} \lambda_n^{1+3/4k} \exp(2^{k+1} a \lambda_n^{(k+1)/2k}).$$

The eigenfunction $\psi_n(x)$ can be extended to an entire function $\psi_n(z)$. We want to estimate $\psi_n(z)$ in the complex plane. First we produce an estimate for $|\psi_n'(0)|$. Let $\xi > 0$ denote a point at which ψ_n^2 reaches its absolute maximum. We have $0 \leq \xi \leq \lambda_n^{1/2k}$. Integrate the equality

$$-\frac{d}{dx}(\psi_n')^2 = (\lambda_n - x^{2k}) \frac{d}{dx}(\psi_n^2)$$

from 0 to ξ . A crude estimate yields

$$|\psi'_n(0)| \leq \frac{2}{3} \sqrt{1+2k} \lambda_n^{1+3/2+3/4k}.$$

Next, following the technique of Titchmarsh [5, p. 172] it can be shown that

$$\psi_n(z) = y^{(0)}(z) + \sum_{m=1}^{\infty} \{y^{(m)}(z) - y^{(m-1)}(z)\}, \quad z \in \mathbb{C}.$$

Here $y^{(0)}(z) = \psi_n(0) + z\psi'_n(0)$ and $y^{(m)}(z)$, $m \geq 1$, can be obtained from

$$y^{(m)}(z) = y^{(0)}(z) + \int_0^z (s^{2k} - \lambda_n) y^{(m-1)}(s) (w-s) ds.$$

With

$$|y^{(m)}(z) - y^{(m-1)}(z)| \leq |y^{(0)}(z)| \left\{ |z|^{2k} + \lambda_n \right\}^m \frac{|z|^{2m}}{(2m)!}$$

we get the estimate

$$|\psi_n(z)| \leq K_n(|z|) \exp(|z|^{k+1} + \lambda_n^{1/2}|z|).$$

Here

$$K_n(|z|) = \frac{4}{3} \lambda_n^{1+3/4k} (1 + (1+2k)^{1/2} \lambda_n^{1/2}|z|) \geq |y^{(0)}(z)|.$$

Now let $d > 0$. Then

$$\exp(\lambda_n^{1/2}|z|) \leq \exp(d^{-k}|z|^{k+1})$$

whenever $|z| \geq d\lambda_n^{1/2k}$ and

$$\exp(\lambda_n^{1/2}|z|) \leq \exp(d\lambda_n^{(k+1)/2k})$$

whenever $|z| \leq d\lambda_n^{k/2}$. Thus we have

$$(2.7) \quad |\psi_n(z)| \leq K_n(|z|) \exp(d\lambda_n^{(k+1)/2k}) \exp(1+d^{-k})|z|^{k+1}.$$

THEOREM 1. *The eigenfunctions ψ_n of the eigenvalue problem (2.1) are elements of the space $\mathcal{S}_\alpha^\beta$, where $\alpha = 1/(k+1)$ and $\beta = k/(k+1)$.*

Proof. Since ψ_n is an entire function and since it satisfies (2.6) and (2.7), in view of the criterion of Gel'fand and Shilov [2, p. 220], the result follows. $\square$

THEOREM 2. *Let $f \in \mathcal{L}_2(\mathbb{R})$,*

$$f = \sum_{n=0}^{\infty} a_n \psi_n,$$

and suppose there is $\tau > 0$ such that

$$a_n = O(\exp(-\tau \lambda_n^{(k+1)/2k})).$$

Then $f \in \mathcal{S}_{1/(k+1)}^{k/(k+1)}$.

Proof. In (2.6) we can take $a > 0$ so small that $\tau > a2^{k+1}$. Then for some $C > 0$ and all $x \in \mathbb{R}$

$$\begin{aligned} |f(x)| &\leq \sum_{n=0}^{\infty} |a_n| |\psi_n(x)| \\ &\leq C \sum_{n=0}^{\infty} K_n \exp\{-(\tau - a2^{k+1})\lambda_n^{(k+1)/2k}\} \exp(-a|x|^{k+1}). \end{aligned}$$

So $|f(x)| \leq C' \exp(-a|x|^{k+1})$ for some $C' > 0$. Further we can take $d > 0$ and $d < \tau$, so that with the aid of (2.7)

$$\begin{aligned} |f(z)| &\leq \sum_{n=0}^{\infty} |a_n| |\psi_n(z)| \\ &\leq \exp((1 + d^{-k})|z|^{k+1}) \sum_{n=0}^{\infty} K_n(|z|) \exp(-(\tau - d)\lambda_n^{(k+1)/2k}) \\ &\leq C'' \exp((1 + d^{-k})|z|^{k+1}) \end{aligned}$$

for some $C'' > 0$. By the criterion of Gel'fand and Shilov as used in the proof of Theorem 1, $f \in \mathcal{S}_{1/k+1}^{k/k+1}$. $\square$

Let $\mathfrak{A}_k$ be the self-adjoint operator in $\mathcal{L}_2(\mathbb{R})$ defined by

$$(2.8) \quad \mathfrak{A}_k = -\frac{d^2}{dx^2} + x^{2k}.$$

Then as a corollary of Theorem 2 we have

COROLLARY 1. *The test function space $\mathcal{S}_{\mathcal{L}_2(\mathbb{R}), \mathfrak{B}_k}$ is included in $\mathcal{S}_{1/k+1}^{k/k+1}$. Here $\mathfrak{B}_k = (\mathfrak{A}_k)^{(k+1)/2k}$.*

Proof. The functions ψ_n are the eigenfunctions of the positive self-adjoint operator $\mathfrak{B}_k$ with eigenvalues $\lambda_n^{(k+1)/2k}$. Let $f \in \mathcal{S}_{\mathcal{L}_2(\mathbb{R}), \mathfrak{B}_k}$. Then there exists $h \in \mathcal{L}_2(\mathbb{R})$ and $\tau > 0$ such that

$$f = e^{-\tau \mathfrak{B}_k} h.$$

This provides $(f, \psi_n) = \exp(-\tau \lambda_n^{(k+1)/2k}) (h, \psi_n)$. So the coefficients (f, ψ_n) are of the order $\exp(-\tau \lambda_n^{(k+1)/2k})$. By Theorem 2 we have $f \in \mathcal{S}_{1/k+1}^{k/k+1}$. $\square$

We want to prove the converse of Corollary 1:

THEOREM 3.

$$\mathcal{S}_{\mathcal{L}_2(\mathbb{R}), \mathfrak{B}_k} = \mathcal{S}_{1/k+1}^{k/k+1}.$$

In the proof of this theorem we need some lemmas.

LEMMA 1. *Let i_r, j_r be nonnegative integers for $r = 1, 2, \dots, n$. Then*

$$\mathbf{D}^{i_1} x^{j_1} \mathbf{D}^{i_2} \dots \mathbf{D}^{i_n} x^{j_n} = \sum_{l \in \mathbb{N}^n} c_{ij}(l) x^{|j-l|} \mathbf{D}^{|i-l|},$$

where $\mathbf{D}$ is the differential operator d/dx and where the coefficients $c_{ij}(l)$ satisfy

$$|c_{ij}(l)| \leq \frac{1}{l!} \frac{j!}{(j-l)!} \frac{|i|!}{|i-l|!}$$

($c_{ij}(l) = 0$ if $l > \min(i, j)$).

We use multi-indices, and $|i| = i_1 + i_2 + \dots + i_n$, $i! = i_1! i_2! \dots i_n!$, etc.

Proof. See Goodman [3, p. 67]. $\square$

LEMMA 2. Let f be an infinitely differentiable function which satisfies the following inequalities for fixed $A, B, C > 0$ and $\alpha, \beta > 0$, $\alpha + \beta \geq 1$:

$$(2.9) \quad |(x^k \mathbf{D}^l f)(x)| \leq C A^k B^l k^{\alpha k} l^{\beta l}, \quad k, l = 0, 1, 2, \dots$$

Then for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}^n$

$$|(\mathbf{D}^{i_1} x^{j_1} \dots \mathbf{D}^{i_n} x^{j_n} f)(x)| \leq C_1 A_1^{|j|} B_1^{|i|} |j|^{\alpha|j|} |i|^{\beta|i|}$$

where $C_1 = C$, $A_1 = 2^{\sigma\beta+1} e^{\sigma\alpha} A$, $B_1 = 2^{\sigma\alpha} e^{\sigma\beta} B$ and $\sigma = (\alpha + \beta)^{-1}$.

Proof. Let $n \in \mathbb{N}$ and $i, j \in \mathbb{N}^n$. Then by Lemma 1

$$|(\mathbf{D}^{i_1} x^{j_1} \dots \mathbf{D}^{i_n} x^{j_n} f)(x)| \leq \sum_l |c_{ij}(l)| |(x^{|j-l|} \mathbf{D}^{|i-l|} f)(x)|.$$

With the assumption (2.9) we estimate this series as follows:

$$\begin{aligned} & \sum_{l \leq \min(i, j)} |c_{ij}(l)| |(x^{|j-l|} \mathbf{D}^{|i-l|} f)(x)| \\ & \leq C \sum_l \frac{1}{l!} \frac{j!}{(j-l)!} \frac{|i|!}{|i-l|!} A^{|j-l|} B^{|i-l|} |j-l|^{\alpha|j-l|} |i-l|^{\beta|i-l|} \\ & \leq C A^{|j|} B^{|i|} \sum_l \frac{1}{l!} \frac{j!}{(j-l)!} \frac{|i|!}{|i-l|!} |j-l|^{\alpha|j-l|} |i-l|^{\beta|i-l|}. \end{aligned}$$

The latter series can be treated as follows

$$\begin{aligned} & \sum_{l \leq \min(i, j)} \frac{1}{l!} \frac{j!}{(j-l)!} \frac{|i|!}{|i-l|!} |j-l|^{\alpha|j-l|} |i-l|^{\beta|i-l|} \\ & \leq \sup_{|l| \leq |i|} \frac{|i|! |i-l|^{\beta|i-l|}}{|i-l|! (|l|!)^{\sigma\alpha}} \sup_{|l| \leq |j|} \left(\frac{|j|! |j-l|^{\alpha|j-l|}}{|j-l|! (|l|!)^{\sigma\beta}} \right) \sum_{l \leq j} \binom{j}{l} \left(\frac{|j|}{|l|} \right)^{-1}. \end{aligned}$$

We have

$$\sum_{l \leq j} \binom{j}{l} \left(\frac{|j|}{|l|} \right)^{-1} \leq \sum_{l \leq j} \binom{j}{l} = 2^{|j|}.$$

With the aid of the inequality $n! < n^n < n! e^n$:

$$\begin{aligned} \left(\frac{|i|! |i-l|^{\beta|i-l|}}{|i-l|! (|l|!)^{\sigma\alpha}} \right) & \leq \left(\frac{|i|}{|l|} \right)^{\sigma\alpha} \frac{(|i|!)^{\sigma\beta}}{(|i-l|!)^{\sigma\beta}} |i-l|^{\beta|i-l|} \\ & \leq 2^{\sigma\alpha|i|} e^{\sigma\beta|i|} (|i|!)^{\sigma\beta} (|i-l|)^{(1-\sigma)\beta|i-l|} \leq 2^{\sigma\alpha|i|} e^{\sigma\beta|i|} |i|^{\beta|i|} \end{aligned}$$

and similarly

$$\left(\frac{|j|! |j-l|^{\alpha|j-l|}}{|j-l|! (|l|!)^{\sigma\beta}} \right) \leq 2^{\sigma\beta|j|} e^{\sigma\alpha|j|} |j|^{\alpha|j|}.$$

Combining these results, we derive

$$|(\mathbf{D}^{i_1} x^{j_1} \cdots \mathbf{D}^{i_n} x^{j_n} f)(x)| \leq C A_1^{|j|} B_1^{|j|} |j|^{\alpha|j|} |i|^{\beta|i|},$$

where $A_1 = 2^{\sigma\beta+1} e^{\sigma\alpha} A$, $B_1 = 2^{\sigma\alpha} e^{\sigma\beta} B$. $\square$

LEMMA 3. For $f \in \mathfrak{S}_{1/k+1}^{k/k+1}$ we have

$$|(\mathbf{D}^2 - x^{2k})^p f(x)| \leq K N^p p^{2pk/(k+1)}, \quad p = 0, 1, 2, \dots,$$

where K and N are fixed positive constants depending on f .

Proof. Let $\alpha = 1/(k+1)$, $\beta = k/(k+1)$. Let $f \in \mathfrak{S}_{\alpha}^{\beta}$. Then there are positive constants A, B, C such that for all $x \in \mathbb{R}$

$$|(x^l \mathbf{D}^q f)(x)| \leq C A^l B^q l^{\alpha l} q^{\beta q},$$

with $l, q = 0, 1, 2, \dots$.

Now let $p \in \mathbb{N}$. Then

$$(\mathbf{D}^2 - x^{2k})^p = \sum_{s=0}^p V_s(\mathbf{D}^2, x^{2k}),$$

where $V_s(\mathbf{D}^2, x^{2k})$ consists of a sum of $\binom{p}{s}$ combinations of the form

$$(\mathbf{D}^2)^{i_1} (x^{2k})^{j_1} \cdots (\mathbf{D}^2)^{i_n} (x^{2k})^{j_n}$$

where $i_1 + \cdots + i_n = s$ and $j_1 + \cdots + j_n = p - s$. With the aid of Lemma 2 we have

$$|V_s(\mathbf{D}^2, x^{2k}) f(x)| \leq \binom{p}{s} C A_1^{2k(p-s)} B_1^{2s} (2k(p-s))^{2\alpha k(p-s)} (2s)^{2\beta s},$$

with $A_1 = 2^{\beta+1} e^{\alpha} A$ and $B_1 = 2^{\alpha} e^{\beta} B$. So

$$\begin{aligned} |(\mathbf{D}^2 - x^{2k})^p f(x)| &\leq C \sum_{s=0}^p \binom{p}{s} A_2^{p-s} B_2^s (p-s)^{2\alpha k(p-s)} s^{2\beta s} \\ &\leq C \sum_{s=0}^p \binom{p}{s} A_2^{p-s} B_2^s (p^{2\alpha k})^{p-s} (p^{2\beta})^s \\ &= C (A_2 p^{2\alpha k} + B_2 p^{2\beta})^p. \end{aligned}$$

Substituting the values of α and β it follows that

$$|(\mathbf{D}^2 - x^{2k})^p f(x)| \leq C (A_2 + B_2)^p p^{2pk/(k+1)}$$

where $A_2 = ((2k)^{\alpha} A_1)^{2k}$ and $B_2 = 2^{\beta} B_1$. $\square$

Proof of Theorem 3. Because of Corollary 1 we only have to prove the inclusion

$$\mathfrak{S}_{1/k+1}^{k/k+1} \subset \mathfrak{S}_{\mathcal{L}_2(\mathbb{R}), \mathfrak{B}_k}.$$

So let $f \in \mathfrak{S}_{1/k+1}^{k/k+1}$. Put $a_n = (f, \psi_n)$, $n \in \mathbb{N}$. Then for each $p \in \mathbb{N}$ fixed

$$a_n = (f, \psi_n) = \lambda_n^{-p} ((-\mathbf{D}^2 + x^{2k})^p f, \psi_n).$$

With the aid of Lemma 3 we get positive constants K_f and N_f such that

$$\|(-\mathbf{D}^2 + x^{2k})^p f\|_{\infty} \leq K_f N_f^p p^{2pk/(k+1)}.$$

And

$$|a_n| \leq \lambda_n^{-p} \|(-\mathbf{D}^2 + x^{2k})^p f\|_\infty \|\psi_n\|_1, \quad n = 0, 1, 2, \dots$$

By (2.4) and (2.5)

$$\begin{aligned} \|\psi_n\|_1 &= \int_{-\infty}^{\infty} |\psi_n(x)| dx = \left(\int_{|x| \leq 2\lambda_n^{k/2}} + \int_{|x| > 2\lambda_n^{k/2}} \right) |\psi_n(x)| dx \\ &\leq \frac{8}{3} \lambda_n^{1+5/4k} + c_k \lambda_n^{1+3/4k} \end{aligned}$$

where c_k only depends on k . Therefore

$$|a_n| \leq c'_k \lambda_n^{1+5/4k} \lambda_n^{-p} K_f N_f^p p^{2pk/(k+1)}.$$

Finally taking the infimum of the right-hand side with respect to p we arrive at

$$|a_n| \leq c'_k K_f \lambda_n^{1+5/4k} \exp - \{2\beta e^{-1} N_f^{-1/2\beta}\} \lambda_n^{1/2\beta}$$

with $\beta = k/(k+1)$. From this the assertion follows. $\square$

By taking Fourier transforms in Theorem 3 we derive easily

THEOREM 4.

$$\mathfrak{S}_{k/k+1}^{1/k+1} = \mathfrak{S}_{\mathbb{E}_2(\mathbf{R}), \tilde{\mathfrak{B}}_k},$$

where $\tilde{\mathfrak{B}}_k = ((-d^2/dx^2)^k + x^2)^{(k+1)/2k}$.

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