A characterization of the spaces $S^{k/k+1}_{1/k+1}$ by means of holomorphic semigroups

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A CHARACTERIZATION OF THE SPACES $S^{\alpha/(k+1)}_{k}$
BY MEANS OF HOLOMORPHIC SEMIGROUPS*

S. J. L. VAN EIJNDHOVEN,† J. DE GRAAF † AND R. S. PATHAK †

Abstract. The Gel'fand-Shilov spaces $S^{\alpha/(k+1)}_{k}$, $\alpha = 1/(k+1)$, $\beta = k/(k+1)$, are special cases of a general type of test function spaces introduced by de Graaf. We give a self-adjoint operator so that the test functions in those $S_{\alpha}$ spaces can be expanded in terms of the eigenfunctions of that self-adjoint operator.

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1. Introduction. De Bruijn's theory of generalized functions based on a specific one-parameter semigroup of smoothing operators [1] was generalized considerably by de Graaf [4]. In brief this extended theory can be described as follows: In a Hilbert space $\mathcal{H}$ consider the evolution equation

$$\frac{du}{dt} = -\mathbb{A} u$$

where $\mathbb{A}$ is a positive, self-adjoint operator, which is unbounded in order that the semigroup $(e^{-t\mathbb{A}})_{t \geq 0}$ is smoothing. A solution $u$ of (1.1) is called a trajectory if $u$ satisfies

$$\forall t > 0 \, \forall \tau > 0: \quad e^{-t \mathbb{A}} u(t) = u(t + \tau),$$

$$\forall t > 0: \quad u(t) \in \mathcal{H}.$$  

The limit $\lim_{t \to 0} u(t)$ does not necessarily exist in $\mathcal{H}$!

The complex vector space of all trajectories is denoted by $\mathcal{S}_{\mathcal{H}, \mathbb{A}}$. The elements of $\mathcal{S}_{\mathcal{H}, \mathbb{A}}$ are called generalized functions.

The test function space $\mathcal{S}_{\mathcal{H}, \mathbb{A}}$ is the dense linear subspace of $\mathcal{H}$ consisting of smooth elements of the form $e^{-t \mathbb{A}} h$, where $h \in \mathcal{H}$ and $t > 0$; we have $\mathcal{S}_{\mathcal{H}, \mathbb{A}} = \bigcup_{t > 0} e^{-t \mathbb{A}}(\mathcal{H})$. The densely defined inverse of $e^{-t \mathbb{A}}$ is denoted by $e^{t \mathbb{A}}$. For each $\varphi \in \mathcal{S}_{\mathcal{H}, \mathbb{A}}$ there exists $t > 0$ such that $e^{t \mathbb{A}} \varphi$ makes sense. The pairing between $\mathcal{S}_{\mathcal{H}, \mathbb{A}}$ and $\mathcal{S}_{\mathcal{H}, \mathbb{A}}$ is defined by

$$\langle \varphi, F \rangle := (e^{t \mathbb{A}} \varphi, F(t)), \quad \varphi \in \mathcal{S}_{\mathcal{H}, \mathbb{A}}, \quad F \in \mathcal{S}_{\mathcal{H}, \mathbb{A}}.$$  

Here $(\cdot, \cdot)$ denotes the inner product in $\mathcal{H}$. Definition (1.3) makes sense for $\tau > 0$ sufficiently small, and due to the trajectory property (1.2i) it does not depend on the specific choice of $\tau$. For further results concerning this theory we refer to [4].

The aim of the present paper is to show that for certain Gel'fand–Shilov spaces $S^{\alpha}_{\beta}$ [2] there exists an operator $\mathbb{A}$ such that $S^{\alpha}_{\beta} = \mathcal{S}_{\mathcal{H}, \mathbb{A}}$. This leads to the result that the elements of the dual of $S^{\alpha}_{\beta}$ can be interpreted as trajectories. Furthermore, we find that a function in the studied $S^{\alpha}_{\beta}$-spaces can be developed in a series of certain orthonormal functions.

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†Department of Mathematics, Technological University, Eindhoven, the Netherlands. One of the authors (S.JLVE) was supported by a grant from the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).
2. Eigenfunction expansions of test functions in $S^\beta_n$. Let us consider the following eigenvalue problem in $L^2(\mathbb{R})$:

$$\frac{d^2}{dx^2} y + (\lambda - x^{2k}) y = 0,$$

where $\lambda$ is a real number and $k$ a positive integer. It is well-known that the operator $-d^2/dx^2 + x^{2k}$ has a point spectrum and the set of eigenvalues $(\lambda_n)$ is real, positive and unbounded. In the sequel we shall regard it as ordered with $\lambda_{n+1} \geq \lambda_n$, $n = 0, 1, \ldots$. The corresponding normalized eigenfunctions $\{\psi_n\}$ form a complete orthonormal basis in $L^2(\mathbb{R})$. So by the Riesz–Fischer theorem every $f \in L^2(\mathbb{R})$ can be represented by

$$f = \sum_{n=0}^{\infty} a_n \psi_n,$$

where $a_n = (f, \psi_n)$ is an $L^2$-sequence.

First of all we gather some of the estimates for the eigenvalues $\lambda_n$ and the eigenfunctions $\psi_n$ of the problem (2.1), and then characterize $\{\psi_n\}$ as elements of certain $S^\beta_n$-spaces. We take $\psi_n(x) > 0$ for large positive values of $x$, cf. Titchmarsh [5, Chap. VIII].

From Titchmarsh [5, p. 144] we have

$$|\psi_n(x)| \leq \frac{2}{3} \frac{\lambda^{1/4+3/4k}}{n}, \quad n \in \mathbb{N} \quad [5, \text{p. 168}],$$

$$|\psi_n(x)| \leq \psi_n(x_0) \exp \left( - \int_{x_0}^{x} (\mu^{2k} - \lambda_n)^{1/2} \, d\mu \right) \quad \text{for } x \geq x_0 \geq \lambda_n^{1/2} \quad [5, \text{p. 165}].$$

We take $x_0 = (\frac{2}{3} \lambda_n)^{1/2k}$. From a straightforward calculation it follows that

$$|\psi_n(x)| \leq \frac{2}{3} \lambda_n^{1/4+3/4k} \exp \left( - \frac{1}{4} \frac{1}{k+1} |x|^{k+1} \right)$$

for $|x| \geq 2 \lambda_n^{1/2k}$. For any number $a$, $0 < a < 1/4(k+1)$, we have

$$|\psi_n(x)| \leq K_n \exp(-a|x|^{k+1}), \quad x \in \mathbb{R},$$

where

$$K_n = \frac{2}{3} \lambda_n^{1/4+3/4k} \exp(2^{k+1}a \lambda_n^{(k+1)/2k}).$$

The eigenfunction $\psi_n(x)$ can be extended to an entire function $\psi_n(z)$. We want to estimate $\psi_n(z)$ in the complex plane. First we produce an estimate for $|\psi_n'(0)|$. Let $\xi > 0$ denote a point at which $\psi_n^2$ reaches its absolute maximum. We have $0 \leq \xi \leq \lambda_n^{1/2k}$. Integrate the equality

$$- \frac{d}{dx} (\psi_n')^2 = (\lambda_n - x^{2k}) \frac{d}{dx} (\psi_n^2)$$
from 0 to $\xi$. A crude estimate yields
\[ |\psi'(0)| \leq \frac{2}{3} \sqrt{1 + 2k} \lambda_n^{1/2 + 3/4k}. \]

Next, following the technique of Titchmarsh [5, p. 172] it can be shown that
\[ \psi_n(z) = y^{(0)}(z) + \sum_{m=1}^{\infty} \{ y^{(m)}(z) - y^{(m-1)}(z) \}, \quad z \in \mathbb{C}. \]

Here $y^{(0)}(z) = \psi_n(0) + z \psi'_n(0)$ and $y^{(m)}(z)$, $m \geq 1$, can be obtained from
\[ y^{(m)}(z) = y^{(0)}(z) + \int_0^z (s^{2k} - \lambda_n) y^{(m-1)}(s)(w-s) \, ds. \]

With
\[ |y^{(m)}(z) - y^{(m-1)}(z)| \leq |y^{(0)}(z)| \left( |z|^{2k} + \lambda_n \right)^m \frac{|z|^{2m}}{(2m)!}, \]
we get the estimate
\[ |\psi_n(z)| \leq K_n(|z|) \exp\left(|z|^{k+1} + \lambda_n^{1/2}|z|\right). \]

Here
\[ K_n(|z|) = \frac{4}{3} \lambda_n^{1/2 + 3/4k} \left(1 + (1 + 2k)^{1/2} \lambda_n^{1/2} |z|\right) \geq |y^{(0)}(z)|. \]

Now let $d > 0$. Then
\[ \exp(\lambda_n^{1/2}|z|) \leq \exp(d^{-k}|z|^{k+1}) \]
whenever $|z| \geq d \lambda_n^{1/2k}$ and
\[ \exp(\lambda_n^{1/2}|z|) \leq \exp(d \lambda_n^{(k+1)/2k}) \]
whenever $|z| \leq d \lambda_n^{k/2}$. Thus we have
\[ |\psi_n(z)| \leq K_n(|z|) \exp\left(d \lambda_n^{(k+1)/2k} \right) \exp(1 + d^{-k}) |z|^{k+1}. \]

**Theorem 1.** The eigenfunctions $\psi_n$ of the eigenvalue problem (2.1) are elements of the space $S_{\alpha}^{\beta}$, where $\alpha = 1/(k+1)$ and $\beta = k/(k+1)$.

**Proof.** Since $\psi_n$ is an entire function and since it satisfies (2.6) and (2.7), in view of the criterion of Gel'fand and Shilov [2, p. 220], the result follows. \qed

**Theorem 2.** Let $f \in L_2(\mathbb{R})$,
\[ f = \sum_{n=0}^{\infty} a_n \psi_n, \]
and suppose there is $\tau > 0$ such that
\[ a_n = O\left(\exp\left(-\tau \lambda_n^{(k+1)/2k}\right)\right). \]

Then $f \in S_{1/k+1}^{1/k+1}$. 

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Proof. In (2.6) we can take $a>0$ so small that $\tau > a2^{k+1}$. Then for some $C>0$ and all $x \in \mathbb{R}$

$$|f(x)| \leq \sum_{n=0}^{\infty} |a_n| |\psi_n(x)|$$

$$\leq C \sum_{n=0}^{\infty} K_n \exp\left\{ - (\tau - a2^{k+1}) \lambda_n^{(k+1)/2k} \right\} \exp\left\{ -a|x|^{k+1} \right\}.$$ 

So $|f(x)| \leq C \exp\left( -a|x|^{k+1} \right)$ for some $C>0$. Further we can take $d>0$ and $d<\tau$, so that with the aid of (2.7)

$$|f(z)| \leq \sum_{n=0}^{\infty} |a_n| |\psi_n(z)|$$

$$\leq \exp\left( (1+d^{-k})|z|^{k+1} \right) \sum_{n=0}^{\infty} K_n (|z|) \exp\left( - (\tau - d) \lambda_n^{(k+1)/2k} \right)$$

$$\leq C'' \exp\left( (1+d^{-k})|z|^{k+1} \right)$$

for some $C''>0$. By the criterion of Gel'fand and Shilov as used in the proof of Theorem 1, $f \in \mathcal{S}_{1/k+1}$. □

Let $\mathcal{A}_k$ be the self-adjoint operator in $\mathcal{L}_2(\mathbb{R})$ defined by

$$(2.8) \quad \mathcal{A}_k = -\frac{d^2}{dx^2} + x^{2k}.$$ 

Then as a corollary of Theorem 2 we have

**Corollary 1.** The test function space $\mathcal{S}_{1/k+1}$ is included in $\mathcal{S}_{k/k+1}^{1/k+1}$. Here $\mathcal{A}_k = (\mathcal{A}_k)^{(k+1)/2k}$.

**Proof.** The functions $\psi_n$ are the eigenfunctions of the positive self-adjoint operator $\mathcal{A}_k$ with eigenvalues $\lambda_n^{(k+1)/2k}$. Let $f \in \mathcal{S}_{1/k+1}$. Then there exists $h \in \mathcal{L}_2(\mathbb{R})$ and $\tau>0$ such that

$$f = e^{-\tau \mathcal{A}_k} h.$$ 

This provides $(f, \psi_n) = \exp(-\tau \lambda_n^{(k+1)/2k}) (h, \psi_n)$. So the coefficients $(f, \psi_n)$ are of the order $\exp(-\tau \lambda_n^{(k+1)/2k})$. By Theorem 2 we have $f \in \mathcal{S}_{k/k+1}^{1/k+1}$. □

We want to prove the converse of Corollary 1:

**Theorem 3.**

$$\mathcal{S}_{1/k+1} \subseteq \mathcal{S}_{k/k+1}.$$ 

In the proof of this theorem we need some lemmas.

**Lemma 1.** Let $i, j, k$ be nonnegative integers for $r=1, 2, \cdots, n$. Then

$$\mathbf{D}^{i_1} x^{j_1} \mathbf{D}^{i_2} x^{j_2} \cdots \mathbf{D}^{i_r} x^{j_r} = \sum_{l \in \mathbb{N}^n} c_{ij}(l) x^{l-j} (\mathbf{D}^{i-j}),$$

where $\mathbf{D}$ is the differential operator $d/dx$ and where the coefficients $c_{ij}(l)$ satisfy

$$|c_{ij}(l)| \leq \frac{1}{l!} \frac{j!}{(j-l)!} \frac{|l|!}{|i-l|!}$$

($c_{ij}(l) = 0$ if $l > \min(i, j)$).
We use multi-indices, and $|i| = i_1 + i_2 + \cdots + i_n$, $i! = i_1! i_2! \cdots i_n!$, etc.

**Proof.** See Goodman [3, p. 67]. \hfill \Box

**Lemma 2.** Let $f$ be an infinitely differentiable function which satisfies the following inequalities for fixed $A, B, C > 0$ and $\alpha, \beta > 0$, $\alpha + \beta \geq 1$:

\[(x^k D^l f)(x) \leq C A^k B^l \alpha^k \beta^l, \quad k, l = 0, 1, 2, \ldots.
\]

Then for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}^n$

\[
|\left( D^{i_1} x^{j_1} \cdots D^{i_n} x^{j_n} f \right)(x) | \leq C_1 A_1^j B_1^j |i|^\alpha |j|^\beta
\]

where $C_1 = C, A_1 = 2^{\alpha + \beta + 1} \alpha \alpha A, B_1 = 2^{\alpha + \beta} \beta B$ and $\sigma = (\alpha + \beta)^{-1}$.

**Proof.** Let $n \in \mathbb{N}$ and $i, j \in \mathbb{N}^n$. Then by Lemma 1

\[
|\left( D^{i_1} x^{j_1} \cdots D^{i_n} x^{j_n} f \right)(x) | \leq \sum_i |c_{ij}(l)| \left| (x^{j-l} D^{i-l} f)(x) \right|
\]

With the assumption (2.9) we estimate this series as follows:

\[
\sum_{i \leq \min(i, j)} |c_{ij}(l)| \left| (x^{j-l} D^{i-l} f)(x) \right|
\leq C \sum_i \frac{1}{i!} \frac{1}{(j-l)!} |i|! \frac{|i|!}{|j|!} A_1^{j-l} B_1^{j-l} |j-l|^\alpha |l|^\beta
\]

\[
\leq C A_1^{j-l} B_1^{j-l} \sum_i \frac{1}{i!} \frac{1}{(j-l)!} |i|! \frac{|i|!}{|j|!} |j-l|^\alpha |l|^\beta.
\]

The latter series can be treated as follows:

\[
\sum_{i \leq \min(i, j)} \frac{1}{i!} \frac{1}{(j-l)!} \frac{|i|!}{|j|!} |j-l|^\alpha |i-l|^\beta
\]

\[
\leq \sup_{|i| = |i|} \frac{|i|!}{|l|!} |l|^\beta |l-\alpha \alpha\sup_{|i| = |i|} \frac{|i|!}{|j-\alpha\alpha|} \frac{|i|!}{(|i|)!} |l|^\beta
\]

\[
\sum_{i \leq j} \frac{j!}{l!} \frac{|j|!}{|l|!} \left( \frac{|j|!}{|l|!} \right)^{-1}.
\]

We have

\[
\sum_{i \leq j} \frac{j!}{l!} \frac{|j|!}{|l|!} \left( \frac{|j|!}{|l|!} \right)^{-1} \leq \sum_{i \leq j} \frac{j!}{l!} = 2^{l}.\]

With the aid of the inequality $n! < n^n < n!e^n$:

\[
\left( \frac{|i|!}{|l|!} |l-\alpha \alpha\right) \leq \left( \frac{|i|!}{|l|!} \right)^\alpha (|l|)^\alpha (|l-\alpha \alpha\right) \leq 2^{\alpha \alpha |l|} \leq 2^{\alpha \alpha |l|} \leq 2^{\alpha |l|} \leq 2^{|l|}.
\]

and similarly

\[
\left( \frac{|j!|}{|l|!} |l-\alpha \alpha\right) \leq 2^{\alpha |l|}.
\]
Combining these results, we derive

$$\left| \left( D^i x^j \cdots D^i x^j f \right)(x) \right| \leq CA_i^i B_i^i \| j \|^i \| i \|^i,$$

where $A_i = 2^{q+1} e^q A$, $B_i = 2^{q+1} e^q B$. □

**Lemma 3.** For $f \in S_{k/(k+1)}$ we have

$$\left| \left( D^2 - x^{2k} \right)^p f(x) \right| \leq KN^p p^{2pk/(k+1)}, \quad p = 0, 1, 2, \cdots,$$

where $K$ and $N$ are fixed positive constants depending on $f$.

**Proof.** Let $\alpha = 1/(k+1), \beta = k/(k+1)$. Let $f \in S_{\alpha}$. Then there are positive constants $A, B, C$ such that for all $x \in \mathbb{R}$

$$\left| (x^i D^q f)(x) \right| \leq CA^{q} B^{q} \| j \|^q \| i \|^q,$$

with $l, q = 0, 1, 2, \cdots$.

Now let $p \in \mathbb{N}$. Then

$$\left( D^2 - x^{2k} \right)^p = \sum_{s=0}^{p} V_s(D^2, x^{2k}),$$

where $V_s(D^2, x^{2k})$ consists of a sum of $(s)$ combinations of the form

$$\left( D^2 \right)^i (x^{2k})^i \cdots \left( D^2 \right)^n (x^{2k})^n$$

where $i_1 + \cdots + i_n = s$ and $j_1 + \cdots + j_n = p-s$. With the aid of Lemma 2 we have

$$\left| V_s(D^2, x^{2k}) f(x) \right| \leq \left( \begin{array}{c} p \\ s \end{array} \right) C A_i^{2k(p-s)} B_i^{2s} (2k(p-s))^{2ak(p-s)} (2s)^{2\beta s},$$

with $A_i = 2^{q+1} e^q A$ and $B_i = 2^{q+1} e^q B$. So

$$\left| \left( D^2 - x^{2k} \right)^p f(x) \right| \leq C \sum_{s=0}^{p} \left( \begin{array}{c} p \\ s \end{array} \right) A_i^{2k(p-s)} B_i^{2s} (p-s)^{2ak(p-s)} s^{2\beta s}$$

$$\leq C \sum_{s=0}^{p} \left( \begin{array}{c} p \\ s \end{array} \right) A_i^{2k(p-s)} B_i^{2s} (p^{2ak})^{p-s} (p^{2\beta})^s$$

$$= C \left( A_i^{2k(p-s)} B_i^{2s} (p^{2ak})^p \right).$$

Substituting the values of $\alpha$ and $\beta$ it follows that

$$\left| \left( D^2 - x^{2k} \right)^p f(x) \right| \leq C \left( A_2 + B_2 \right)^p p^{2pk/(k+1)}$$

where $A_2 = (2k)^{q+1} A_i^{2k}$ and $B_2 = 2^{q+1} B_i$. □

**Proof of Theorem 3.** Because of Corollary 1 we only have to prove the inclusion

$$S_{k/(k+1)} \subseteq S_{\gamma}(\mathbb{R}) \subseteq \mathbb{B}.$$

So let $f \in S_{k/(k+1)}$. Put $a_n = (f, \psi_n)$, $n \in \mathbb{N}$. Then for each $p \in \mathbb{N}$ fixed

$$a_n = (f, \psi_n) = \lambda_n^{-p} \left( -D^2 + x^{2k} \right)^p f, \psi_n).$$

With the aid of Lemma 3 we get positive constants $K_f$ and $N_f$ such that

$$\left\| \left( -D^2 + x^{2k} \right)^p f \right\|_\infty \leq K_f N_f p^{2pk/(k+1)}.$$
And
\[ |a_n| \leq \lambda_n^{-p} \left\| \left( -D^2 + x^{2k} \right)^p \right\|_\infty \| \psi_n \|_1, \quad n = 0, 1, 2, \ldots. \]

By (2.4) and (2.5)
\[ \| \psi_n \|_1 = \int_{-\infty}^{\infty} |\psi_n(x)| \, dx = \left( \int_{|x| \leq 2\lambda_n^{k/2}} + \int_{|x| > 2\lambda_n^{k/2}} \right) |\psi_n(x)| \, dx \]
\[ \leq \frac{8}{3} \lambda_n^{1+5/4k} + c_k \lambda_n^{1+3/4k} \]
where \( c_k \) only depends on \( k \). Therefore
\[ |a_n| \leq c_k' \lambda_n^{1+5/4k} \lambda_n^{-p} K_F N_f p^{2 p k / (k+1)}. \]

Finally taking the infimum of the right-hand side with respect to \( p \) we arrive at
\[ |a_n| \leq c_k' K_F \lambda_n^{1+5/4k} \exp \left\{ 2 \beta e^{-1} N_f^{-1/2} \right\} \lambda_n^{1/2} \]
with \( \beta = k / (k + 1) \). From this the assertion follows. \( \square \)

By taking Fourier transforms in Theorem 3 we derive easily

**Theorem 4.**
\[ S_{k+1}^{1/k+1} \equiv S_{\hat{\xi}_k}(R), \hat{\xi}_k, \]

where \( \hat{\xi}_k = \left\{ (\frac{-d^2}{dx^2})^k + x^2 \right\}^{(k+1)/2k} \).

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