Interpolating between the compound-geometric and compound-Poisson distributions

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Interpolating between the compound-geometric and
compound-Poisson distributions

by

K. van Harn and F.W. Steutel

Eindhoven, July 1975

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0. Summary

The starting point of this paper is the characterization of the compound-Poisson, i.e. the infinitely divisible lattice distributions (class C_1) and the subset of compound-geometric lattice distributions (class C_0) by the non-negativity of recursively defined quantities (section 1). In section 2 we generalize these recursion relations by introducing sequences $c_n(\alpha)$ for $\alpha \in [0,1]$ and thus we obtain classes of distributions C_α , which we wish to be increasing with α . This can be achieved by an appropriate choice of $c_n(\alpha)$ (section 3). For the classes C_α we obtain other properties, which generalize known properties of the class C_1 . Another subdivision of this class is given in section 4, where we consider the compound-negative-binomial lattice distributions.

1. Preliminaries

We only consider lattice distributions $(p_n)_{n \geq 0}$, i.e. distributions on the nonnegative integers, with $p_0 > 0$. The corresponding probability generating function (pgf)

$$P(z) := \sum_{n=0}^{\infty} p_n z^n, \quad |z| \leq 1$$

is then unequal to zero on $|z| < \rho$ for some $\rho > 0$. If $(p_n)_{n \geq 0}$ (or P) is infinitely divisible (inf div), then P is unequal to zero on the closed unit disk $|z| \leq 1$ (cf. [4]).

For an inf div pgf P we have the following representation (see [1]).

Theorem 1.1. A pgf P is inf div if and only if there is a $\lambda > 0$ and a pgf Q_1 , such that

$$(1.1) \quad P(z) = \exp[\lambda(Q_1(z) - 1)] .$$

The representation is unique, if we take $Q_1(0) = 0$.

Thus, the class of inf div distributions on the nonnegative integers coincides with the class of compound-Poisson distributions on the nonnegative integers. We recall the definition of such distributions.

Definition 1.2. A compound (lattice) distribution is a distribution with pgf $P(z) = G(Q(z))$, where G and Q are pgf's. The corresponding random variable $\underline{x}$ can then be written as

$$\underline{x} = \underline{x}_1 + \underline{x}_2 + \dots + \underline{x}_{\underline{n}},$$

where $\underline{n}, \underline{x}_1, \underline{x}_2, \dots$ are independent, while $\underline{n}$ has pgf G and each of the $\underline{x}_i$ has pgf Q .

Example 1.3.

i) $\underline{n}$ has a geometric (p) distribution: $G(z) = \frac{1-p}{1-pz}$, so a *compound-geometric* distribution $(p_n)_{n \geq 0}$ has pgf

$$(1.2) \quad P(z) = \frac{1-p}{1-pQ_0(z)}.$$

ii) $\underline{n}$ has a Poisson (λ) distribution: $G(z) = \exp[\lambda(z-1)]$, so a *compound-Poisson* distribution $(p_n)_{n \geq 0}$ has pgf

$$(1.3) \quad P(z) = \exp[\lambda(Q_1(z) - 1)].$$

Definition 1.4. C_0 is the set of all compound-geometric distributions. C_1 is the set of all compound-Poisson distributions, which, according to theorem 1.1, is equal to the set of all inf div distributions. If P is a pgf corresponding to a distribution in a class C , we shall also say, that $P \in C$.

Remark. To some of the generating functions, corresponding to distributions in C_0 or C_1 , we add an index 0 or 1, to be able to fit them in the more general notation of the following section. Further we denote the coefficient of z^n of a generating function $R_\alpha(z)$ by $r_n(\alpha)$, or r_n , if no confusion is possible, while the sequence $(r_n(\alpha))_{n \geq 0}$ is simply denoted by $r_n(\alpha)$.

We now formulate some more or less known theorems, which will be the starting point for our investigations in the next sections. For the sake of completeness we write down the proofs in detail and in a few lemma's we give some properties of the appearing quantities. We start with the definition of absolute monotonicity, and then consider the distributions $(p_n)_{n \geq 0}$ in C_1 .

Definition 1.5. A function R on the complex numbers is called *absolutely monotone* (abs mon), if there are $r_n \geq 0$ ($n = 0, 1, 2, \dots$) and $\rho > 0$, such that

$$R(z) = \sum_{n=0}^{\infty} r_n z^n, \quad \text{for } |z| < \rho.$$

Theorem 1.6. A pgf P is inf div ($P \in C_1$) if and only if

$$R_1(z) := \frac{P'(z)}{P(z)}$$

is absolutely monotone.

Proof. Let $P \in C_1$. Then there are $\lambda > 0$ and pgf Q_1 , such that

$$\log P(z) = \lambda(Q_1(z) - 1).$$

Hence:

$$R_1(z) = P'(z)/P(z) = [\log P(z)]' = \lambda Q_1'(z),$$

which is a abs mon function ($|z| < 1$).

Conversely, let $P'(z)/P(z) = R_1(z)$ be abs mon, i.e.

$$P'(z)/P(z) = R_1(z) = \sum_{n=0}^{\infty} r_n(1) z^n, \quad \text{for } |z| < \rho,$$

with $r_n(1) \geq 0$ for all n and $\rho > 0$. Integrating from 0 to z ($|z| < \rho$), we obtain

$$\log P(z) = \log p_0 + \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} z^{n+1}, \quad |z| < \rho,$$

or, if we define $\lambda := -\log p_0$ and $Q_1(z) := \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} z^{n+1}$:

$$P(z) = \exp[\lambda(Q_1(z) - 1)], \quad |z| < \rho,$$

with $\lambda > 0$ and Q_1 abs mon.

Since $P'(z) = P(z) \cdot R_1(z)$, $|z| < \rho$, we have the following relations

$$(1.4) \quad (n+1)p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(1), \quad n = 0, 1, 2, \dots,$$

where $r_n(1) \geq 0$ for all n . It follows (see lemma 1.7), that

$$\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} < \infty .$$

Now $Q_1(z)$ is convergent for $|z| \leq 1$, so that $\exp[\lambda(Q_1(z) - 1)]$ is an analytic continuation of $P(z)$ to the whole disk $|z| \leq 1$. As P itself is analytic on $|z| \leq 1$, we have

$$P(z) = \exp[\lambda(Q_1(z) - 1)], \quad \text{for } |z| \leq 1 .$$

From $P(1) = 1$ we get now $Q_1(1) = 1$, so Q_1 is a pgf and P is a compound-Poisson pgf, i.e. $P \in C_1$.

From $Q_1(1) = 1$ we then obtain:

$$\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} = 1, \text{ or: } \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} = \lambda = -\log p_0 . \quad \square$$

Remark. For the quantities R_1 , λ and Q_1 from the preceding proof we have the relations

$$(1.5) \quad R_1(z) = \lambda Q_1'(z), \quad \lambda = -\log p_0 = \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1}, \quad Q_1(z) = \frac{1}{\lambda} \int_0^z R_1(u) du .$$

Some properties of the recursions (1.4) are now summarized in the following lemma.

Lemma 1.7. The sequence $r_n(1)$, defined by (1.4), and its generating function R_1 have the following properties

i) R_1 has a radius of convergence $\rho > 0$, and

$$R_1(z) = \frac{P'(z)}{P(z)} \quad \text{for } |z| < \rho .$$

ii) If $r_n(1) \geq 0$ for all n , then

$$(1.6) \quad \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} < \infty ,$$

so that $R_1(z)$ is convergent for $|z| < 1$.

Conversely, if a sequence $r_n(1)$ with $r_n(1) \geq 0$ for all n and satisfying (1.6) is given, then (1.4) defines a unique probability distribution $(p_n)_{n \geq 0}$.

Proof. From (1.4) we obtain

$$p_0 r_n = (n+1)p_{n+1} - \sum_{k=1}^n p_k r_{n-k} ,$$

so that

$$p_0 |r_n| \leq (n+1)p_{n+1} + \sum_{k=1}^n p_k |r_{n-k}| ,$$

or

$$2p_0 |r_n| \leq (n+1)p_{n+1} + \sum_{k=0}^n p_k |r_{n-k}| .$$

Hence, if we define

$$|R_N|(x) := \sum_{n=0}^N |r_n| x^n \quad (x > 0, N \in \mathbb{N}) ;$$

$$\begin{aligned} 2p_0 |R_N|(x) &\leq \sum_{n=0}^N (n+1)p_{n+1} x^n + \sum_{n=0}^N \sum_{k=0}^n p_k |r_{n-k}| x^n \leq \\ &\leq \sum_{n=0}^{\infty} (n+1)p_{n+1} x^n + \sum_{k=0}^N p_k \sum_{n=k}^N |r_{n-k}| x^n \leq \\ &\leq P'(x) + \sum_{k=0}^N p_k \sum_{n=0}^N |r_n| x^{k+n} \leq P'(x) + P(x) |R_N|(x) . \end{aligned}$$

Now, if we choose a $x_0 > 0$, such that $P(x_0) < 2p_0$, then we have

$$|R_N|(x_0) \leq \frac{P'(x_0)}{2p_0 - P(x_0)} < \infty \quad \text{for all } N ,$$

from which we can conclude that $\sum_{n=0}^{\infty} |r_n| x_0^n < \infty$, and $R_1(z) = \sum_{n=0}^{\infty} r_n z^n$ has a radius of convergence $\rho \geq x_0$. Taking generating functions, from (1.4) we immediately have

$$P'(z) = P(z)R_1(z) .$$

Let $r_n \geq 0$ for all n . Then we can write

$$\begin{aligned} 1 - p_0 &= \sum_{n=0}^{\infty} p_{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n p_k r_{n-k} = \sum_{k=0}^{\infty} p_k \sum_{n=k}^{\infty} \frac{r_{n-k}}{n+1} = \\ &= \sum_{k=0}^{\infty} p_k \sum_{n=0}^{\infty} \frac{r_n}{n+1+k} \geq p_0 \sum_{n=0}^{\infty} \frac{r_n}{n+1}, \end{aligned}$$

so that

$$\sum_{n=0}^{\infty} \frac{r_n}{n+1} \leq \frac{1 - p_0}{p_0} < \infty,$$

and $\sum_{n=0}^{\infty} \frac{r_n}{n+1} z^{n+1}$ convergent for $|z| \leq 1$. As $R_1(z) = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{r_n}{n+1} z^{n+1}$, it follows that $R_1(z)$ is convergent for $|z| < 1$. Conversely, let $r_n \geq 0$ for all n and $a := \sum_{n=0}^{\infty} \frac{r_n}{n+1} < \infty$. Choose $p_0 \in (0, 1]$ and define p_n , $n \geq 1$ by

$$(n+1)p_{n+1} = \sum_{k=0}^n p_k r_{n-k}, \quad n = 0, 1, 2, \dots$$

It follows that $p_n \geq 0$ for all n . Now define $q_n := p_n/p_0$, $n \geq 0$, then also

$$(n+1)q_{n+1} = \sum_{k=0}^n q_k r_{n-k}, \quad n = 0, 1, 2, \dots,$$

with $q_0 = 1$. If μ is the counting measure on $\{0, 1, 2, \dots\}$ and $f_n(k) := \frac{r_k}{k+1+n}$, $k \geq 0$, $n \geq 0$, then $f_n(k) \leq f_0(k)$ for all n and $\int f_0 d\mu = a < \infty$, so that by the dominated-convergence theorem

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \frac{r_k}{k+1+n} = \lim_{n \rightarrow \infty} \int f_n(k) d\mu(k) = \int \lim_{n \rightarrow \infty} f_n(k) d\mu(k) = 0.$$

So we can choose n_0 and $a_0 < 1$, such that

$$\forall n \geq n_0 \quad \sum_{k=0}^{\infty} \frac{r_k}{k+1+n} \leq a_0.$$

We now estimate $\sum_{n=0}^{\infty} q_n$ as follows. For $N > n_0$ we have

$$\begin{aligned}
 \sum_{n=1}^{N+1} q_n &= \sum_{n=0}^N q_{n+1} = \sum_{n=0}^N \frac{1}{n+1} \sum_{k=0}^n q_k r_{n-k} = \sum_{k=0}^N q_k \sum_{n=k}^N \frac{r_{n-k}}{n+1} \leq \\
 &\leq \sum_{n=0}^N q_n \sum_{k=0}^{\infty} \frac{r_k}{k+1+n} \leq a \sum_{n=0}^{n_0-1} q_n + a_0 \sum_{n=n_0}^N q_n \leq \\
 &\leq a \sum_{n=0}^{n_0-1} q_n + a_0 \sum_{n=1}^{N+1} q_n .
 \end{aligned}$$

As $a_0 < 1$, we have for $N > n_0$

$$\sum_{n=1}^{N+1} q_n \leq \frac{a}{1-a_0} \sum_{n=0}^{n_0-1} q_n, \text{ and so } \sum_{n=0}^{\infty} q_n =: c < \infty .$$

Since $q_n = p_n/p_0$, it follows that $c \geq 1$ and $\sum_{n=0}^{\infty} p_n = p_0 c$, so that $(p_n)_{n \geq 0}$ becomes a probability distribution, if we take $p_0 = 1/c$. $\square$

Remark. The $r_n(1)$ may be expressed explicitly in the p_n : from the first n equations of (1.4) by Cramer's rule we obtain

$$(1.7) \quad r_{n-1}(1) = p_0^{-n} \det \begin{vmatrix} p_0 & & & & p_1 \\ p_1 & p_0 & & & 2p_2 \\ p_2 & p_1 & p_0 & & 3p_3 \\ \vdots & & \ddots & \ddots & \vdots \\ p_{n-1} & \cdot & \cdot & \cdot & p_1 & np_n \end{vmatrix} .$$

From theorem 1.6 and lemma 1.7 one easily derives the following characterization of inf div lattice distributions, first given by Katti [3].

Theorem 1.8. A distribution $(p_n)_{n \geq 0}$ is inf div if and only if the quantities $r_n(1)$, $n = 0, 1, \dots$ defined by

$$(n+1)p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(1)$$

are nonnegative.

Proof. Let $(p_n)_{n \geq 0}$ be inf div. According to lemma 1.7i) the generating function R_1 of the $r_n(1)$ exists and is equal to $P'(z)/P(z)$ for $|z| < \rho$. As this function is abs mon, by theorem 1.6 we have: $r_n(1) \geq 0$ for all n . Conversely, let $r_n(1) \geq 0$ for all n . Then (lemma 1.7ii)) the generating function R_1 of the $r_n(1)$ exists and is equal to $P'(z)/P(z)$ for $|z| < 1$, so that $P'(z)/P(z)$ is abs mon. The inf div of P follows now from theorem 1.6. $\square$

We now turn to the distributions in C_0 , and derive some analogous results.

Theorem 1.9. A pgf P is compound-geometric ($P \in C_0$) if and only if

$$R_0(z) := \frac{P(z) - p_0}{zP(z)}$$

is absolutely monotone.

Proof. Let $P \in C_0$. Then there are $p \in (0,1)$ and pgf Q_0 , such that

$$P(z) = \frac{1-p}{1-pQ_0(z)}.$$

Hence:

$$\begin{aligned} R_0(z) &= \frac{1}{z} \{1 - p_0/P(z)\} = \frac{1}{z} \{1 - \frac{p_0}{1-p} (1 - pQ_0(z))\} = \\ &= \frac{1}{z} \{1 - \frac{1}{1-pq_0} [1 - pq_0 - p(Q_0(z) - q_0)]\} = \frac{p}{1-pq_0} \frac{Q_0(z) - q_0}{z}, \end{aligned}$$

which is an abs mon function ($|z| \leq 1$).

Conversely, let R_0 be abs mon, i.e.

$$R_0(z) = \sum_{n=0}^{\infty} r_n(0)z^n, \quad \text{for } |z| < \rho,$$

with $r_n(0) \geq 0$ for all n and $\rho > 0$. As $P(z) - p_0 = zP(z)R_0(z)$, we have

$$P(z) = \frac{p_0}{1 - zR_0(z)}, \quad |z| < \rho,$$

or, if we define $p := 1 - p_0$ and $Q_0(z) := \frac{1}{p} zR_0(z)$:

$$P(z) = \frac{1-p}{1 - pQ_0(z)}, \quad |z| < \rho,$$

with $p \in (0,1)$ and Q_0 abs mon.

Since

$$\frac{P(z) - p_0}{z} = P(z) \cdot R_0(z), \quad |z| < \rho,$$

we have the following relations

$$(1.8) \quad p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(0), \quad n = 0, 1, 2, \dots,$$

where $r_n(0) \geq 0$ for all n , from which we can derive (see lemma 1.10), that

$$\sum_{n=0}^{\infty} r_n(0) < 1.$$

Now $R_0(z)$, and hence $Q_0(z)$, is convergent for $|z| \leq 1$, and, as

$$|pQ_0(z)| = |zR_0(z)| \leq R_0(1) = \sum_{n=0}^{\infty} r_n(0) < 1, \quad |z| \leq 1,$$

$\frac{1-p}{1-pQ_0(z)}$ is an analytic continuation of $P(z)$ to the closed unit disk $|z| \leq 1$. As P itself is analytic for $|z| \leq 1$, we have

$$P(z) = \frac{1-p}{1-pQ_0(z)}, \quad \text{for } |z| \leq 1.$$

From $P(1) = 1$ we now get $Q_0(1) = 1$, so Q_0 is a pgf and P is a compound-geometric pgf: $P \in C_0$.

From $Q_0(1) = 1$ we then obtain: $\frac{1}{p} R_0(1) = 1$, or: $\sum_{n=0}^{\infty} r_n(0) = p = 1 - p_0$. $\square$

Remark. The following relations now hold for the quantities R_0 , p and Q_0 from the preceding proof

$$(1.9) \quad R_0(z) = \frac{p}{1-pq_0} \frac{Q_0(z) - q_0}{z}, \quad \text{or if we take } q_0 = 0,$$

$$R_0(z) = \frac{1}{z} pQ_0(z), \quad p = 1 - p_0 = \sum_{n=0}^{\infty} r_n(0) = R_0(1), \quad Q_0(z) = \frac{1}{p} zR_0(z).$$

We have the following analogue of lemma 1.7 for the recursions (1.8).

Lemma 1.10. The sequence $r_n(0)$, defined by (1.8), and its generating function R_0 have the following properties

i) R_0 has a radius of convergence $\rho > 0$, and

$$(1.10) \quad R_0(z) = \frac{P(z) - p_0}{zP(z)}, \quad \text{for } |z| < \rho.$$

ii) If $r_n(0) \geq 0$ for all n , then

$$(1.11) \quad \sum_{n=0}^{\infty} r_n(0) < 1,$$

so that $R_0(z)$ is convergent for $|z| \leq 1$.

Conversely, if a sequence $r_n(0)$ with $r_n(0) \geq 0$ for all n and satisfying (1.11) is given, then (1.8) defines a unique probability distribution $(p_n)_{n \geq 0}$.

Proof. Exactly as in the proof of lemma 1.7 we can show, that R_0 has a radius of convergence $\rho > 0$ (replace the factor $n+1$ by 1). Then, taking generating functions, from (1.8) we immediately obtain

$$\frac{1}{z}[P(z) - p_0] = P(z)R_0(z).$$

Let $r_n(0) \geq 0$ for all n . Then we can write

$$1 - p_0 = \sum_{n=0}^{\infty} p_{n+1} = \sum_{n=0}^{\infty} \sum_{k=0}^n p_k r_{n-k}(0) = \sum_{k=0}^{\infty} p_k \sum_{n=0}^{\infty} r_n(0) = \sum_{n=0}^{\infty} r_n(0),$$

so that

$$\sum_{n=0}^{\infty} r_n(0) = 1 - p_0 < 1,$$

and $R_0(z) = \sum_{n=0}^{\infty} r_n(0)z^n$ is convergent for $|z| \leq 1$.

Conversely, let $r_n \geq 0$ for all n and $a := \sum_{n=0}^{\infty} r_n < 1$. Choose $p_0 \in (0, 1]$, define p_n , $n \geq 1$ by (1.8) (then $p_n \geq 0$ for all n) and define $q_n := p_n/p_0$, $n \geq 0$, then also

$$q_{n+1} = \sum_{k=0}^n q_k r_{n-k}, \quad n = 0, 1, 2, \dots, \text{ with } q_0 = 1.$$

We can write now

$$\sum_{n=0}^{\infty} q_{n+1} = \sum_{n=0}^{\infty} \sum_{k=0}^n q_k r_{n-k} = \sum_{k=0}^{\infty} q_k \sum_{n=k}^{\infty} r_{n-k} = a(1 + \sum_{n=0}^{\infty} q_{n+1}) .$$

Hence, as $a < 1$,

$$\sum_{n=0}^{\infty} q_{n+1} = \frac{a}{1-a} , \text{ or } \sum_{n=0}^{\infty} q_n = \frac{1}{1-a} =: c < \infty .$$

Since $q_n = p_n/p_0$, it follows that $c \geq 1$ and $\sum_{n=0}^{\infty} p_n = p_0 c$, so $(p_n)_{n \geq 0}$ becomes a probability distribution, if we take $p_0 = \frac{1}{c}$. $\square$

Remark. The determinant representation (cf. (1.7)) for the $r_n(0)$ now becomes

$$(1.12) \quad r_{n-1}(0) = p_0^{-n} \det \begin{vmatrix} p_0 & & & & p_1 \\ p_1 & p_0 & & & p_2 \\ p_2 & p_1 & p_0 & & p_3 \\ \vdots & & \ddots & \ddots & \vdots \\ p_{n-1} & \dots & \dots & p_1 & p_n \end{vmatrix} .$$

As an analogue to theorem 1.8 we now have from theorem 1.9 and lemma 1.10 (see also [4]):

Theorem 1.11. A distribution $(p_n)_{n \geq 0}$ is compound-geometric if and only if the quantities $r_n(0)$, $n = 0, 1, \dots$ defined by

$$p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(0)$$

are nonnegative.

Two other classes of lattice distributions, which are known to be inf div, are the log-convex and the completely monotone distributions. For the latter we give a useful characterization in theorem 1.14 (see [4]).

Definition 1.12. A distribution $(p_n)_{n \geq 0}$ is called *log-convex* if

$$p_{n+1} p_{n-1} \geq p_n^2, \quad n \geq 1 .$$

$\mathcal{B}$ is the set of all log-convex lattice distributions.

Definition 1.13. A distribution $(p_n)_{n \geq 0}$ is called *completely monotone* if

$$(-1)^k \Delta^k p_n \geq 0, \quad n \geq 0 \text{ and } k \geq 0.$$

A is the set of all completely monotone lattice distributions.

Theorem 1.14. $(p_n)_{n \geq 0}$ is completely monotone if and only if there is a finite measure μ on $[0,1]$ such that

$$p_n = \int_0^1 p^n d\mu(p),$$

or, equivalently, $(p_n)_{n \geq 0}$ is a mixture of geometric distributions:

$$p_n = \int_0^1 (1-p)p^n dF(p),$$

with F is a distribution function on $[0,1]$.

We conclude this section with a theorem concerning the relation between the classes A , B , C_0 and C_1 of (inf div) distributions (see [2], [4] and [5]).

Theorem 1.15. $A \subset B \subset C_0 \subset C_1$, or: the following implications hold:

$$\begin{aligned} & (p_n)_{n \geq 0} \text{ completely monotone} \Rightarrow \\ & \Rightarrow (p_n)_{n \geq 0} \text{ log-convex} \Rightarrow \\ & \Rightarrow (p_n)_{n \geq 0} \text{ compound-geometric} \Rightarrow \\ & \Rightarrow (p_n)_{n \geq 0} \text{ compound-Poisson, or: } (p_n)_{n \geq 0} \text{ inf div.} \end{aligned}$$

Proof.

$A \subset B$: Let $(p_n)_{n \geq 0}$ be completely monotone. Then (theorem 1.14)

$$p_n = \int_0^1 (1-p)p^n dF(p), \quad n \geq 0,$$

with F a distribution function on $[0,1]$. If $\underline{p}$ is a random variable with distribution function F , $\underline{x} := (1-\underline{p})^{\frac{1}{2}} \underline{p}^{\frac{1}{2}(n+1)}$ and $\underline{y} := (1-\underline{p})^{\frac{1}{2}} \underline{p}^{\frac{1}{2}(n-1)}$, then,

using Schwarz's inequality, we have

$$p_{n+1}p_{n-1} = E\bar{x}^2 \cdot E\bar{y}^2 \geq [E\bar{x}\bar{y}]^2 = [E(1-p)\bar{p}^n]^2 = p_n^2,$$

so that $(p_n)_{n \geq 0}$ is log-convex.

$B \subset C_0$: Let $(p_n)_{n \geq 0}$ be log-convex. If $p_1 = 0$, then it follows by induction from $p_n^2 \leq p_{n+1}p_{n-1}$, that $p_n = 0$ for all $n \geq 1$, which is a trivial case. So $p_1 > 0$, but then it follows by induction from $p_n \geq p_{n-1}^2/p_{n-2}$, that $p_n > 0$ for all n . Now we can write

$$\frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}} \geq \dots \geq \frac{p_k}{p_{k-1}} \geq \dots \geq \frac{p_1}{p_0}, \quad k = 1, \dots, n,$$

and so

$$(1.13) \quad p_{n+1}p_{k-1} \geq p_n p_k, \quad k = 1, \dots, n.$$

We want to use theorem 1.11 and prove the nonnegativity of the $r_n(0)$ by induction from (1.13): $r_0(0) = p_1/p_0 > 0$. Let $r_k(0) \geq 0$ for $k \leq n-1$, then it follows

$$\begin{aligned} p_n p_0 r_n(0) &= p_n \{ p_{n+1} - \sum_{k=1}^n p_k r_{n-k}(0) \} = \\ &= p_n p_{n+1} - \sum_{k=1}^n p_n p_k r_{n-k}(0) \geq p_n p_{n+1} - \sum_{k=1}^n p_{n+1} p_{k-1} r_{n-k}(0) = \\ &= p_{n+1} \{ p_n - \sum_{k=0}^{n-1} p_k r_{n-1-k}(0) \} = 0, \end{aligned}$$

so $r_n(0) \geq 0$, and (theorem 1.11): $(p_n)_{n \geq 0} \in C_0$.

$C_0 \subset C_1$: We can simply prove this by showing that a compound-geometric distribution is also compound-Poisson.

If $P(z) = \frac{1-p}{1-pQ_0(z)} \in C_0$, then

$$(1.14) \quad \lambda := -\log(1-p) > 0, \quad Q_1(z) := \frac{\log(1-pQ_0(z))}{\log(1-p)} = \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n} (pQ_0(z))^n$$

is abs mon, with $Q_1(1) = 1$, while

$$\log P(z) = \log \frac{1-p}{1-pQ_0(z)} = \lambda(Q_1(z) - 1), \quad \text{so } P \in C_1.$$

An other method uses the theorems 1.6 and 1.9: If $P \in C_0$, then

$$P(z) = \frac{p_0}{1 - zR_0(z)},$$

with R_0 abs mon, from which we derive

$$\begin{aligned} R_1(z) &= P'(z)/P(z) = [\log P(z)]' = [-\log(1 - zR_0(z))]' = \\ &= \frac{[zR_0(z)]'}{1 - zR_0(z)} = \frac{1}{p_0} P(z)[zR_0(z)]', \end{aligned}$$

which is an abs mon function, so: $P \in C_1$. □

Remark. The relations between R_1 and R_0 , derived in the preceding proof,

$$(1.15) \quad R_1(z) = \frac{[zR_0(z)]'}{1 - zR_0(z)},$$

and

$$(1.16) \quad R_1(z) = \frac{P(z)}{p_0} [zR_0(z)]',$$

give by equating the coefficients of z^n the following relations between the sequences $r_n(1)$ and $r_n(0)$ ($[zR_0(z)]' = \sum_{n=0}^{\infty} (n+1)r_n(0)z^n$)

$$(1.17) \quad r_n(1) = (n+1)r_n(0) + \sum_{k=0}^{n-1} r_k(1)r_{n-1-k}(0),$$

and

$$(1.18) \quad r_n(1) = \frac{1}{p_0} \sum_{k=0}^n (k+1)r_k(0)p_{n-k}.$$

From both relations we immediately obtain: If $r_n(0) \geq 0$ for all n , then

$$(1.19) \quad r_n(1) \geq (n+1)r_n(0), \quad n = 0, 1, 2, \dots$$

In view of the theorems 1.8 and 1.11 we can formulate the following

Corollary 1.16. If, for a given distribution $(p_n)_{n \geq 0}$, the $r_n(0)$, $n \geq 0$, defined by

$$p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(0), \quad n = 0, 1, 2, \dots$$

are all nonnegative, then the $r_n(1)$, $n \geq 0$, defined by

$$(n+1)p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(1), \quad n = 0, 1, 2, \dots$$

are all nonnegative, too.

2. Interpolating between 1 and $n+1$

In the preceding section we have given characterizations of the distributions $(p_n)_{n \geq 0}$ in C_0 and C_1 by the nonnegativity of the recursively defined quantities $r_n(0)$ and $r_n(1)$ ($n \geq 0$) (see theorem 1.8 and 1.11). This suggests the possibility of dividing the class of distributions $C_1 \setminus C_0$ (cf. corollary 1.16) in a number of increasing subclasses of distributions, characterized in a similar way.

Let us introduce therefore sequences $(c_n(\alpha))_{n \geq 0}$, defined for $\alpha \in [0, 1]$, with the following properties

$$(2.1) \quad \begin{aligned} c_n(0) &= 1 \text{ and } c_n(1) = n + 1 \text{ for all } n \geq 0, \\ c_n(\alpha) &\text{ is nondecreasing in } n \text{ for each } \alpha, \text{ and in } \alpha \text{ for each } n. \end{aligned}$$

For such a sequence $c_n(\alpha)$, and a lattice distribution $(p_n)_{n \geq 0}$, define the sequence $(r_n(\alpha))_{n \geq 0}$ by

$$(2.2) \quad c_n(\alpha)p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(\alpha), \quad n = 0, 1, 2, \dots,$$

with generating function

$$R_\alpha(z) := \sum_{n=0}^{\infty} r_n(\alpha) z^n.$$

For $\alpha \in [0, 1]$ define the following class of distributions

$$C_\alpha := \{(p_n)_{n \geq 0} \mid \forall_{n \geq 0} r_n(\alpha) \geq 0\},$$

or, in terms of generating functions,

$$C_\alpha := \{\text{pgf } P \mid R_\alpha \text{ is abs mon}\}.$$

Of course, for $\alpha = 0$ we get the recursion (1.8) with the $r_n(0)$, R_0 and C_0 from section 1, and for $\alpha = 1$ the recursion (1.4) with the $r_n(1)$, R_1 and C_1 from that section.

Examples of $c_n(\alpha)$ we can obtain by an obvious interpolation between $c_n(0) = 1$ and $c_n(1) = n+1$, or, if $C_\alpha(z)$ is the generating function of the sequence $c_n(\alpha)$, between $C_0(z) = (1-z)^{-1}$ and $C_1(z) = (1-z)^{-2}$.

Example 2.1. (the properties (2.1) are easily seen to hold)

$$\text{i)} \quad C_\alpha(z) = (1-z)^{-1-\alpha} = \sum_{n=0}^{\infty} \binom{-1-\alpha}{n} (-z)^n = \sum_{n=0}^{\infty} \binom{\alpha+n}{n} z^n, \text{ so } c_n(\alpha) = \binom{\alpha+n}{n}.$$

$$\text{ii)} \quad C_\alpha(z) = (1-z)^{-1} (1-\alpha z)^{-1} = \sum_{m=0}^{\infty} z^m \sum_{k=0}^{\infty} \alpha^k z^k = \sum_{n=0}^{\infty} \sum_{k=0}^n \alpha^k z^n, \text{ so}$$

$$c_n(\alpha) = 1 + \alpha + \alpha^2 + \dots + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha}.$$

$$\text{iii)} \quad C_\alpha(z) = (1-z)^{-1} (1-\alpha z)^{-\alpha} = \sum_{m=0}^{\infty} z^m \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-\alpha z)^k = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{-\alpha}{k} (-\alpha)^k z^n, \text{ so}$$

$$c_n(\alpha) = \sum_{k=0}^n \binom{-\alpha}{k} (-\alpha)^k = \sum_{k=0}^n \binom{\alpha+k-1}{k} \alpha^k.$$

$$\begin{aligned} \text{iv)} \quad C_\alpha(z) &= (1-z)^{-2} [1 - (1-\alpha)z] = (1-z)^{-1} + \alpha z (1-z)^{-2} = \\ &= \sum_{n=0}^{\infty} z^n + \alpha z \sum_{n=0}^{\infty} (n+1) z^n, \text{ so } c_n(\alpha) = 1 + \alpha n. \end{aligned}$$

$$\text{v)} \quad c_n(\alpha) = (n+1)^\alpha.$$

$$\text{vi)} \quad c_n(\alpha) = (1 + \alpha n)^\alpha.$$

$$\text{vii)} \quad c_n(\alpha) = \sum_{k=0}^n a_{k,n} \alpha^k,$$

with $a_{k,n} \geq 0$, $a_{0,n} = 1$ and $\sum_{k=1}^n a_{k,n} = n$.

In the following lemma we summarize the main properties of the sequence $r_n(\alpha)$ and its generating function R_α (cf. lemma 1.7 and 1.10).

Lemma 2.2. For every $\alpha \in [0,1]$ and a fixed sequence $c_n(\alpha)$ we have

i) R_α has a radius of convergence $\rho > 0$, and

$$(2.3) \quad R_\alpha(z) = P(z)^{-1} \sum_{n=0}^{\infty} c_n(\alpha) p_{n+1} z^n, \quad \text{for } |z| < \rho.$$

ii) If $(p_n)_{n \geq 0} \in C_\alpha$, then

$$(2.4) \quad \sum_{n=0}^{\infty} \frac{r_n(\alpha)}{c_n(\alpha)} < \infty,$$

so that $R_\alpha(z)$ is convergent for $|z| < 1$. If, furthermore, $\lim_{n \rightarrow \infty} c_n(\alpha) =: c(\alpha) < \infty$, then

$$(2.5) \quad \sum_{n=0}^{\infty} r_n(\alpha) < c(\alpha),$$

so that $R_\alpha(z)$ is convergent for $|z| \leq 1$.

iii) Conversely, every sequence $r_n(\alpha)$ with $r_n(\alpha) \geq 0$ for all n , and satisfying (2.4), or, if $\lim_{n \rightarrow \infty} c_n(\alpha) =: c(\alpha) < \infty$, (2.5), by (2.2) defines a probability distribution $(p_n)_{n \geq 0} \in C_\alpha$.

Proof. We can prove i) and the first part of ii) in the same way as in lemma 1.7 (with the factor $n+1$ replaced by $c_n(\alpha)$). For the second part of ii) we can write, since $c_n(\alpha)$ is nondecreasing in n ,

$$1 - p_0 = \sum_{n=0}^{\infty} p_{n+1} = \sum_{n=0}^{\infty} \frac{1}{c_n(\alpha)} \sum_{k=0}^n p_k r_{n-k}(\alpha) \geq \frac{1}{c(\alpha)} \sum_{k=0}^{\infty} p_k \sum_{n=0}^{\infty} r_n(\alpha),$$

so

$$\sum_{n=0}^{\infty} r_n(\alpha) \leq (1 - p_0) c(\alpha) < c(\alpha).$$

Following again the proof of lemma 1.7, we see (using again the monotonicity of $c_n(\alpha)$), that for iii) it is sufficient to prove

$$(2.6) \quad \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \frac{r_k(\alpha)}{c_{k+n}(\alpha)} < 1.$$

If $\lim_{n \rightarrow \infty} c_n(\alpha) = \infty$, then with Lebesgue's convergence theorem it follows that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \frac{r_k(\alpha)}{c_{k+n}(\alpha)} = \sum_{k=0}^{\infty} \lim_{n \rightarrow \infty} \frac{r_k(\alpha)}{c_{k+n}(\alpha)} = 0 ,$$

so that (2.6) holds. If $\lim_{n \rightarrow \infty} c_n(\alpha) = c(\alpha) < \infty$, then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \frac{r_k(\alpha)}{c_{k+n}(\alpha)} = \sum_{k=0}^{\infty} \frac{r_k(\alpha)}{c(\alpha)} ,$$

and this is less than 1, if (2.5) holds. □

One would like to have, for all α and $\beta \in [0,1]$

$$(2.7) \quad \alpha \leq \beta \Rightarrow C_\alpha \subset C_\beta .$$

We now investigate for sequences $c_n(\alpha)$, satisfying (2.1), to what extent this implication holds.

Lemma 2.3. For all $n \geq 0$ and all $\alpha, \beta \in [0,1]$

$$(2.8) \quad \sum_{k=0}^n c_k(\alpha) p_{k+1} r_{n-k}(\beta) = \sum_{k=0}^n c_k(\beta) p_{k+1} r_{n-k}(\alpha) .$$

Proof. Define

$$A_\alpha(z) := \sum_{n=0}^{\infty} c_n(\alpha) p_{n+1} z^n \quad (\text{convergent for } |z| < 1) ,$$

then it follows from lemma 2.2i): $A_\alpha(z) = P(z)R_\alpha(z)$ and $A_\beta(z) = P(z)R_\beta(z)$, from which

$$A_\alpha(z)R_\beta(z) = A_\beta(z)R_\alpha(z) .$$

Equating the coefficients of z^n , we obtain (2.8). □

Theorem 2.4. For all $\alpha \in [0,1]$

$$C_0 \subset C_\alpha .$$

This is an immediate consequence of the following:

If $r_n(0) \geq 0$ for all n , then

i) For all $\alpha \in [0,1]$ and all n

$$(2.9) \quad r_n(\alpha) \geq c_n(\alpha) r_n(0) .$$

ii) If

$$(2.10) \quad \frac{c_{n+1}(\alpha)}{c_n(\alpha)} \text{ is nondecreasing in } \alpha \text{ for all } n ,$$

then

$$\frac{r_n(\alpha)}{c_n(\alpha)} \text{ is nondecreasing in } \alpha \text{ for all } n ,$$

and

$$r_n(\alpha) \text{ is nondecreasing in } \alpha \text{ for all } n .$$

Proof. Let $r_n(0) \geq 0$ for all n (or: $(p_n)_{n \geq 0} \in C_0$). According to (2.8) (with $\beta = 0$, and hence $c_k(\beta) = 1$) we can write

$$\begin{aligned} p_0 r_n(\alpha) &= c_n(\alpha) p_{n+1} - \sum_{k=1}^n p_k r_{n-k}(\alpha) = c_n(\alpha) p_{n+1} - \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(\alpha) = \\ &= c_n(\alpha) p_{n+1} - \sum_{k=0}^{n-1} c_k(\alpha) p_{k+1} r_{n-1-k}(0) , \end{aligned}$$

so that

$$\begin{aligned} p_0 [r_n(\alpha) - c_n(\alpha) r_n(0)] &= \\ &= c_n(\alpha) \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(0) - \sum_{k=0}^{n-1} c_k(\alpha) p_{k+1} r_{n-1-k}(0) = \\ &= \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(0) [c_n(\alpha) - c_k(\alpha)] , \end{aligned}$$

which is nonnegative, as $r_n(0) \geq 0$ for all n and $c_n(\alpha)$ is nondecreasing in n .

We conclude

$$r_n(\alpha) \geq c_n(\alpha) r_n(0), \quad n = 0, 1, 2, \dots ,$$

so that all $r_n(\alpha)$ are nonnegative, and $(p_n)_{n \geq 0} \in C_\alpha$.

Furthermore, let $\frac{c_{n+1}(\alpha)}{c_n(\alpha)}$ be nondecreasing in α for all n . Then we have for $\beta > \alpha$ and $k \leq n$

$$\frac{c_n(\beta)}{c_k(\beta)} = \prod_{\ell=k}^{n-1} \frac{c_{\ell+1}(\beta)}{c_\ell(\beta)} \geq \prod_{\ell=k}^{n-1} \frac{c_{\ell+1}(\alpha)}{c_\ell(\alpha)} = \frac{c_n(\alpha)}{c_k(\alpha)} ,$$

or

$$(2.11) \quad c_n(\beta)c_k(\alpha) \geq c_n(\alpha)c_k(\beta) \quad \text{for } \alpha < \beta, k = 0, 1, \dots, n.$$

From (2.2) and (2.8) we obtain for $\alpha < \beta$

$$\begin{aligned} p_0[c_n(\alpha)r_n(\beta) - c_n(\beta)r_n(\alpha)] &= \\ &= c_n(\beta) \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(\alpha) - c_n(\alpha) \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(\beta) = \\ &= \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(0) [c_n(\beta)c_k(\alpha) - c_n(\alpha)c_k(\beta)], \end{aligned}$$

which is nonnegative on account of (2.11). It follows that

$$c_n(\alpha)r_n(\beta) \geq c_n(\beta)r_n(\alpha), \quad \text{for } \beta > \alpha \text{ and all } n,$$

which implies that $\frac{r_n(\alpha)}{c_n(\alpha)}$ is nondecreasing in α for all n . Consequently

$$r_n(\beta) \geq \frac{c_n(\beta)}{c_n(\alpha)} r_n(\alpha) \geq r_n(\alpha), \quad \text{for } \beta > \alpha,$$

because $c_n(\alpha)$ is nondecreasing in α . So we finally have, that $r_n(\alpha)$ is nondecreasing in α for all n . $\square$

Remark. One easily verifies, that condition (2.10) is satisfied in the following cases

$$c_n(\alpha) = \binom{\alpha+n}{n}, \frac{1 - \alpha^{n+1}}{1 - \alpha}, 1 + \alpha n \text{ and } (n+1)^\alpha.$$

We now consider distributions $(p_n)_{n \geq 0}$ in C_α with $\alpha > 0$. It turns out that for many choices of $c_n(\alpha)$ we do not have the desired property (2.7). We shall give some simple, necessary conditions for (2.7), from which we obtain counterexamples for almost all our choices of the sequence $c_n(\alpha)$. To this end in the following lemma we introduce a special distribution.

Lemma 2.5. For $\alpha_0 \in (0, 1)$, the sequence $r_n(\alpha_0)$ with $r_0(\alpha_0) = 1$ and $r_n(\alpha_0) = 0$ for all $n \geq 1$ by (2.2) defines a probability distribution $(\tilde{p}_n)_{n \geq 0} \in C_{\alpha_0}$, for which

$$(2.12) \quad \tilde{p}_{n+1} = \tilde{p}_0 \prod_{k=0}^n c_k(\alpha_0)^{-1}, \quad n = 0, 1, 2, \dots$$

Proof. The sequence $r_n(\alpha_0)$ satisfies the conditions of lemma 2.2iii) and hence defines a distribution $(\tilde{p}_n)_{n \geq 0} \in C_{\alpha_0}$ by the recursion, from which we obtain: $c_n(\alpha_0)\tilde{p}_{n+1} = \tilde{p}_n$, and from this (2.12) immediately follows. $\square$

Lemma 2.6.

i) If there is a $\alpha_0 \in (0,1)$ such that

$$(2.13) \quad \frac{1}{3} c_1(\alpha_0)c_2(\alpha_0) + 1 < c_2(\alpha_0) ,$$

then for $r_2(\alpha)$ corresponding to $(\tilde{p}_n)_{n \geq 0}$ we have: $r_2(1) < 0$.

ii) If there is a $\alpha_0 \in (0,1)$ such that

$$(2.14) \quad c_2'(\alpha_0) < c_1'(\alpha_0)c_2(\alpha_0) ,$$

then for $r_2(\alpha)$ corresponding to $(\tilde{p}_n)_{n \geq 0}$ we have: $r_2(\alpha_0) = 0$ and $r_2'(\alpha_0) < 0$ (here c_1 and c_2 are supposed to be differentiable).

So, in both cases we do not have (2.7).

Proof. For the $r_n(\alpha)$ corresponding to $(\tilde{p}_n)_{n \geq 0}$ we have on account of (2.12)

$$\begin{aligned} r_n(\alpha) &= \frac{1}{\tilde{p}_0} \{c_n(\alpha)\tilde{p}_{n+1} - \sum_{k=1}^n \tilde{p}_k r_{n-k}(\alpha)\} = \\ &= c_n(\alpha) \prod_{k=0}^n c_k(\alpha_0)^{-1} - \sum_{k=0}^{n-1} \prod_{\ell=0}^k c_\ell(\alpha_0)^{-1} r_{n-1-k}(\alpha) , \end{aligned}$$

from which we successively obtain

$$r_0(\alpha) = r_0(\alpha_0) = 1 ,$$

$$r_1(\alpha) = \frac{1}{c_1(\alpha_0)} c_1(\alpha) - 1 ,$$

$$\begin{aligned} r_2(\alpha) &= \frac{1}{c_1(\alpha_0)c_2(\alpha_0)} c_2(\alpha) - \left\{ \frac{1}{c_1(\alpha_0)} c_1(\alpha) - 1 \right\} - \frac{1}{c_1(\alpha_0)} = \\ &= \frac{1}{c_1(\alpha_0)} \left\{ \frac{1}{c_2(\alpha_0)} c_2(\alpha) - c_1(\alpha) + c_1(\alpha_0) - 1 \right\} . \end{aligned}$$

It follows, that, if (2.13) holds, then

$$r_2(1) = \frac{1}{c_1(\alpha_0)} \left\{ \frac{3}{c_2(\alpha_0)} - 3 + c_1(\alpha_0) \right\} < 0 ,$$

and consequently, $(\tilde{p}_n)_{n \geq 0}$ in C_{α_0} , but not in C_1 .

If (2.14) holds, then

$$r_2'(\alpha_0) = \frac{1}{c_1(\alpha_0)} \left\{ \frac{1}{c_2(\alpha_0)} c_2'(\alpha_0) - c_1'(\alpha_0) \right\} < 0 ,$$

whence $(\tilde{p}_n)_{n \geq 0}$ in C_{α_0} , but not in C_α for $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$, with $\varepsilon > 0$ sufficiently small. $\square$

In the following lemma, for the different choices of $c_n(\alpha)$ (cf. lemma 2.1) we list values of α_0 , for which (2.13) and (2.14) hold.

Lemma 2.7.

- i) $c_n(\alpha) = \binom{\alpha+n}{n}$: satisfies (2.13) for $\alpha_0 > \sqrt{3} - 1$ ($\simeq 0.732$), and (2.14) for $\alpha_0 > \frac{1}{2}(\sqrt{5} - 1)$ ($\simeq 0.618$).
- ii) $c_n(\alpha) = \frac{1 - \alpha^{n+1}}{1 - \alpha}$: satisfies neither (2.13) nor (2.14).
- iii) $c_n(\alpha) = \sum_{k=0}^n \binom{\alpha+k-1}{k} \alpha^k$: satisfies (2.13) and (2.14) for $\alpha_0 > 1 - \varepsilon$, with ε positive and sufficiently small.
- iv) $c_n(\alpha) = 1 + \alpha n$: satisfies (2.13) and (2.14) for $\alpha_0 > \frac{1}{2}$.
- v) $c_n(\alpha) = (n+1)^\alpha$: satisfies (2.13) for $\alpha_0 > 1 - \varepsilon$ (ε positive and sufficiently small) and (2.14) for $\alpha_0 > {}^2\log^2 \log 3$.
- vi) $c_n(\alpha) = (1 + \alpha n)^\alpha$: satisfies (2.14) for $\alpha_0 > 1 - \varepsilon$ (ε positive and sufficiently small).

We further have (cf. example 2.1ii) and vii)):

Lemma 2.8. If c_1 is linear and c_2 quadratic in α , then in order to exclude (2.13) and (2.14) for all $\alpha_0 \in (0,1)$ it is necessary and sufficient that for all $\alpha \in (0,1)$

$$(2.15) \quad c_1(\alpha) = 1 + \alpha, \quad c_2(\alpha) = 1 + \alpha + \alpha^2 .$$

Proof. As $c_n(0) = 1$ and $c_n(1) = n + 1$, c_1 and c_2 already have the following form

$$c_1(\alpha) = 1 + \alpha, \quad c_2(\alpha) = 1 + (2 - b)\alpha + b\alpha^2,$$

with $b \neq 0$. Further, from the requirement $c_1(\alpha) \leq c_2(\alpha)$ for all $\alpha \in (0, 1)$ we obtain: $b \leq \frac{1}{1 - \alpha}$ for all $\alpha \in (0, 1)$, so: $b \leq 1$.

Now suppose, that (2.13) does not hold for all $\alpha_0 \in (0, 1)$. Then it follows that $b \geq \frac{2\alpha - 1}{\alpha(2 - \alpha)}$ for all $\alpha \in (0, 1)$, which implies: $b \geq 1$. So we obtain (2.15) as the only possibility for c_1 and c_2 , which gives no counter-examples according to lemma 2.7ii). $\square$

It is a pity, that $c_n(\alpha) = (n + 1)^\alpha$ and $c_n(\alpha) = \binom{\alpha+n}{n} (\sim \frac{n^\alpha}{\Gamma(\alpha + 1)}, n \rightarrow \infty)$ do not have the required properties, because they seem to give a "better" subdivision of $C_1 \setminus C_0$, than e.g.

$$c_n(\alpha) = 1 + \alpha + \dots + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha} (\sim \frac{1}{1 - \alpha}, n \rightarrow \infty).$$

However, since the latter yields no counter-examples and is the most obvious in view of lemma 2.8, we shall consider it in more detail in the next section. If we drop the monotonicity of $c_0(\alpha)$, then we can consider sequences $c_n(\alpha)$ like $(n + \alpha)^\alpha$, and $\frac{1}{\Gamma(1 - \alpha)} \binom{n+1-\alpha}{n+1}^{-1}$. The latter we obtain by means of "fractional differentiation":

$$\begin{aligned} C_\alpha(z) &:= z^{\alpha-1} \left(\frac{d}{dz}\right)^\alpha [z(1 - z)^{-1}] = z^{\alpha-1} \left(\frac{d}{dz}\right)^\alpha \left[\sum_{n=1}^{\infty} z^n \right] = \\ &= z^{\alpha-1} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n - \alpha + 1)} z^{n-\alpha} = \sum_{n=0}^{\infty} \frac{(n+1)!}{\Gamma(n+2-\alpha)} z^n, \end{aligned}$$

from which

$$c_n(\alpha) = \frac{(n+1)!}{\Gamma(n+2-\alpha)} = \frac{1}{\Gamma(1-\alpha)} \binom{n+1-\alpha}{n+1}^{-1}.$$

We shall not pursue this in this report.

We conclude this section with a property of C_α , which holds for every choice of $c_n(\alpha)$ and is already known for $\alpha = 0$ and 1.

Theorem 2.9. If $(p_n)_{n \geq 0} \in C_\alpha$, then also $(q_n^{(\gamma)})_{n \geq 0} \in C_\alpha$ for all $\gamma \in [0,1]$, with

$$q_n^{(\gamma)} := \frac{\gamma^n}{P(\gamma)} p_n, \quad n = 0, 1, 2, \dots$$

Proof. Let $(p_n)_{n \geq 0} \in C_\alpha$, so $r_n(\alpha) \geq 0$ for all n . Define the sequence $r_n^{(\gamma)}(\alpha)$ by

$$c_n(\alpha) q_{n+1}^{(\gamma)} = \sum_{k=0}^n q_k^{(\gamma)} r_{n-k}^{(\gamma)}(\alpha), \quad n = 0, 1, 2, \dots$$

On the other side we can write

$$\begin{aligned} c_n(\alpha) q_{n+1}^{(\gamma)} &= \frac{\gamma^{n+1}}{P(\gamma)} c_n(\alpha) p_{n+1} = \frac{\gamma^{n+1}}{P(\gamma)} \sum_{k=0}^n p_k r_{n-k}(\alpha) = \\ &= \sum_{k=0}^n q_k^{(\gamma)} \gamma^{n-k+1} r_{n-k}(\alpha), \end{aligned}$$

so that

$$r_n^{(\gamma)}(\alpha) = \gamma^{n+1} r_n(\alpha) \geq 0 \quad \text{for all } n.$$

It follows by definition, that $(q_n^{(\gamma)})_{n \geq 0} \in C_\alpha$. □

3. Case $c_n(\alpha) = 1 + \alpha + \alpha^2 + \dots + \alpha^n$

In this section we investigate the recursion relations (2.2) with

$$c_n(\alpha) = 1 + \alpha + \alpha^2 + \dots + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha}, \quad n = 0, 1, 2, \dots$$

So, the quantities $r_n(\alpha)$, $R_\alpha(z)$ and the class C_α , defined in section 2, now correspond to this choice of $c_n(\alpha)$.

First, we formulate lemma 2.2 more precisely.

Lemma 3.1. For all $\alpha \in [0,1)$ we have

i) R_α has a radius of convergence $\rho > 0$, and

$$(3.1) \quad R_\alpha(z) = \frac{1}{1-\alpha} \frac{1}{z} \left\{ 1 - \frac{P(\alpha z)}{P(z)} \right\}, \quad \text{for } |z| < \rho.$$

ii) If $(p_n)_{n \geq 0} \in C_\alpha$, then

$$(3.2) \quad \sum_{n=0}^{\infty} r_n(\alpha) < \frac{1}{1-\alpha},$$

so that $R_\alpha(z)$ is convergent for $|z| \leq 1$.

iii) Conversely, every sequence $r_n(\alpha)$ with $r_n(\alpha) \geq 0$ for all n , and satisfying (3.2), by (2.2) defines a unique probability distribution $(p_n)_{n \geq 0} \in C_\alpha$.

Proof. For $|z| < 1$ we can write

$$\begin{aligned} \sum_{n=0}^{\infty} c_n(\alpha) p_{n+1} z^n &= \frac{1}{1-\alpha} \left\{ \sum_{n=0}^{\infty} p_{n+1} z^n - \sum_{n=0}^{\infty} \alpha^{n+1} p_{n+1} z^n \right\} = \\ &= \frac{1}{1-\alpha} \left\{ \frac{P(z) - p_0}{z} - \frac{P(\alpha z) - p_0}{z} \right\} = \\ &= \frac{1}{1-\alpha} \frac{1}{z} \{P(z) - P(\alpha z)\}. \end{aligned}$$

Now (3.1) immediately follows from (2.3). As $\lim_{n \rightarrow \infty} c_n(\alpha) = \frac{1}{1-\alpha}$, the rest is a reformulation of lemma 2.2. $\square$

Remark. In lemma 3.1 we only consider $\alpha < 1$. The case $\alpha = 1$, which is essentially different, has been treated in lemma 1.7. It is a limiting case in the following sense:

$$\begin{aligned} \lim_{\alpha \uparrow 1} R_\alpha(z) &= \lim_{\alpha \uparrow 1} \frac{1}{1-\alpha} \frac{1}{z} \left\{ 1 - \frac{P(\alpha z)}{P(z)} \right\} = \frac{1}{P(z)} \lim_{\alpha \uparrow 1} \frac{P(z) - P(\alpha z)}{z - \alpha z} = \\ &= \frac{P'(z)}{P(z)} = R_1(z). \end{aligned}$$

Since (3.1) gives an explicit expression of R_α in P , we can also prove theorem 2.4 using generating functions.

Lemma 3.2. For all $\alpha \in [0, 1)$ we have

$$(3.3) \quad R_\alpha(z) = \frac{1}{1-\alpha} \frac{R_0(z) - \alpha R_0(\alpha z)}{1 - \alpha z R_0(\alpha z)}$$

and

$$(3.4) \quad R_\alpha(z) = \frac{P(\alpha z)}{p_0} \frac{R_0(z) - \alpha R_0(\alpha z)}{1 - \alpha},$$

or, in terms of the coefficients,

$$(3.5) \quad r_n(\alpha) = c_n(\alpha)r_n(0) + \sum_{k=0}^{n-1} \alpha^{k+1} r_k(0) r_{n-1-k}(\alpha)$$

and

$$(3.6) \quad r_n(\alpha) = \frac{1}{p_0} \sum_{k=0}^n c_k(\alpha) r_k(0) \alpha^{n-k} p_{n-k}.$$

Proof. As $R_0(z) = \frac{1}{z} \{1 - \frac{p_0}{P(z)}\}$, we have $P(z) = \frac{p_0}{1 - zR_0(z)}$. Substituting this in (3.1) we obtain (3.3):

$$R_\alpha(z) = \frac{1}{1-\alpha} \frac{1}{z} \left\{1 - \frac{1 - zR_0(z)}{1 - \alpha zR_0(\alpha z)}\right\} = \frac{1}{1-\alpha} \frac{R_0(z) - \alpha R_0(\alpha z)}{1 - \alpha zR_0(\alpha z)},$$

from which, using (3.1) once more, (3.4) follows. The relations (3.5) and (3.6) now follow from (3.3) and (3.4) by equating the coefficients of z^n . $\square$

From this lemma we immediately obtain the abs mon of R_α from that of R_0 , and so: $C_0 \subset C_\alpha$, while the inequality (2.9) follows from (3.5) or (3.6). Furthermore, letting $\alpha \uparrow 1$ in lemma 3.2, we get the relations (1.15), ..., (1.18).

The following lemma expresses P in terms of R_α .

Lemma 3.3. For all $\alpha \in [0,1)$ we have (with the ρ from lemma 3.1)

$$(3.7) \quad P(z) = p_0 \prod_{k=0}^{\infty} [1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)]^{-1}, \quad \text{for } |z| < \rho.$$

If $P \in C_\alpha$, then (3.7) holds for $|z| \leq 1$ and we can write

$$(3.8) \quad P(z) = \prod_{k=0}^{\infty} \frac{1 - (1-\alpha)\alpha^k R_\alpha(\alpha^k)}{1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)}, \quad \text{for } |z| \leq 1.$$

Proof. According to (3.1) for $|z| < \rho$ we can write

$$P(z) = [1 - (1-\alpha)zR_\alpha(z)]^{-1} P(\alpha z),$$

from which we obtain for every $n \in \mathbb{N}$

$$P(z) = \prod_{k=0}^{n-1} [1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)]^{-1} \cdot P(\alpha^n z),$$

or

$$\prod_{k=0}^{n-1} [1 - (1 - \alpha)\alpha^k z R_\alpha(\alpha^k z)] = \frac{P(\alpha^n z)}{P(z)}.$$

This tends to $\frac{P_0}{P(z)}$, for $n \rightarrow \infty$, from which (3.7) follows. If $P \in C_\alpha$, then on account of lemma 3.1 the derivation above is valid for $|z| \leq 1$. Taking $z = 1$ in (3.7) we get

$$P_0 = \prod_{k=0}^{\infty} [1 - (1 - \alpha)\alpha^k R_\alpha(\alpha^k)] ,$$

and (3.8) follows from (3.7). □

Lemma 3.3 gives us a characterization of C_α . For $\alpha = 0$ and 1 we already have one: C_0 is the set of compound-geometric distributions and C_1 the set of compound-Poisson, or inf div distributions. C_α now appears here as the rather special set of infinite products of compound-geometric pgf's, given by (3.8). Condensing the notation a little, we have:

Lemma 3.4. For $\alpha \in (0,1)$: $P \in C_\alpha$ if and only if there is $p \in (0,1)$ and pgf Q with $Q(0) = 0$, such that

$$(3.9) \quad P(z) = \prod_{k=0}^{\infty} \frac{1 - pQ(\alpha^k)}{1 - pQ(\alpha^{k+1})}.$$

Proof. Let $P \in C_\alpha$. Then P has the representation (3.8), which becomes (3.9), if we define $Q(z) := \frac{z R_\alpha(z)}{R_\alpha(1)}$ and $p := (1 - \alpha) R_\alpha(1)$. As R_α is abs mon, Q is a pgf, while $p \in (0,1)$ on account of (3.2).

Conversely, if P has the representation (3.9), then we have for the corresponding R_α

$$\begin{aligned} R_\alpha(z) &= \frac{1}{1 - \alpha} \frac{1}{z} \left\{ 1 - \frac{P(\alpha z)}{P(z)} \right\} = \frac{1}{1 - \alpha} \frac{1}{z} \left\{ 1 - \prod_{k=0}^{\infty} \frac{1 - pQ(\alpha^k z)}{1 - pQ(\alpha^{k+1} z)} \right\} = \\ &= \frac{1}{1 - \alpha} \frac{1}{z} \{ 1 - (1 - pQ(z)) \} = \frac{p}{1 - \alpha} \frac{Q(z)}{z}, \end{aligned}$$

which, as $Q(0) = 0$, is an abs mon function. So: $P \in C_\alpha$. □

Theorem 3.5. For all $\alpha \in [0,1]$ we have: $C_\alpha \subset C_1$. Or: every pgf P with an absolutely monotone R_α is inf div.

Proof. Let $P \in C_\alpha$ with $\alpha \in (0,1)$. Then P has the representation (3.9), or

$$(3.10) \quad P(z) = \prod_{k=0}^{\infty} \frac{1 - \pi_k}{1 - \pi_k S_k(z)},$$

with $\pi_k := pQ(\alpha^k) \in (0,1)$ and $S_k(z) := Q(\alpha^k z)/Q(\alpha^k)$ a pgf. So, P is an infinite product of compound-geometric pgf's, which is inf div. $\square$

Remark. We can also prove theorem 3.5 using the following relation between R_1 and R_α : from (3.7) we obtain

$$\begin{aligned} R_1(z) &= \frac{P'(z)}{P(z)} = \frac{d}{dz} \log P(z) = - \frac{d}{dz} \log \prod_{k=0}^{\infty} [1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)] = \\ &= - \sum_{k=0}^{\infty} \frac{d}{dz} \log [1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)] = \sum_{k=0}^{\infty} \frac{(1-\alpha)\alpha^k [z R_\alpha(\alpha^k z)]'}{1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)}, \end{aligned}$$

so

$$(3.11) \quad R_1(z) = (1-\alpha) \sum_{k=0}^{\infty} \alpha^k \frac{[z R_\alpha(\alpha^k z)]'}{1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)}.$$

Now, if R_α is abs mon, then it follows from (3.11), that R_1 is abs mon, too.

Before we investigate, whether the implication

$$(3.12) \quad \alpha < \beta \Rightarrow C_\alpha \subset C_\beta$$

holds generally, we prove, that the special distribution $(\tilde{p}_n)_{n \geq 0} \in C_{\alpha_0}$ (see lemma 2.5), from which we obtained counter-examples for many choices of $c_n(\alpha)$, belongs to every C_β with $\beta \geq \alpha_0$.

Lemma 3.6. For the $r_n(\alpha)$ corresponding to the distribution $(\tilde{p}_n)_{n \geq 0} \in C_{\alpha_0}$ from lemma 2.5 we have

$$(3.13) \quad r_n(\alpha) = \prod_{k=1}^n \frac{\alpha - \alpha_0^k}{c_k(\alpha_0)}, \quad n = 0, 1, \dots, \text{ and } \alpha \in [0, 1].$$

From this it follows that

$$(3.14) \quad (\tilde{p}_n)_{n \geq 0} \in C_\alpha \Leftrightarrow \alpha \in [\alpha_0, 1]$$

and

$$(3.15) \quad c_n(\alpha) r_n(\beta) \geq c_n(\beta) r_n(\alpha), \quad \text{for } \beta \geq \alpha \geq \alpha_0.$$

Proof. On account of (3.7) for $\tilde{P}(z) := \sum_{n=0}^{\infty} \tilde{p}_n z^n$ we have, as $R_{\alpha_0}(z) \equiv 1$

$$\tilde{P}(z) = \tilde{p}_0 \prod_{k=0}^{\infty} [1 - (1 - \alpha_0) \alpha_0^k z]^{-1},$$

from which for $\alpha \in [0, 1)$ we obtain

$$R_{\alpha}(z) = \frac{1}{1 - \alpha} \frac{1}{z} \left\{ 1 - \frac{\tilde{P}(\alpha z)}{\tilde{P}(z)} \right\} = \frac{1}{1 - \alpha} \frac{1}{z} \left\{ 1 - \prod_{k=0}^{\infty} \frac{1 - (1 - \alpha_0) \alpha_0^k z}{1 - (1 - \alpha_0) \alpha_0^k \alpha z} \right\}.$$

Hence, for $m \geq 1$

$$R_{\alpha_0^m}(z) = \frac{1}{1 - \alpha_0^m} \frac{1}{z} \left\{ 1 - \prod_{k=0}^{m-1} [1 - (1 - \alpha_0) \alpha_0^k z] \right\},$$

which is a polynomial in z of degree $(m-1)$, so that $r_n(\alpha_0^m) = 0$ for $n \geq m$, or

$$(3.16) \quad \forall_{n \geq 1} \quad \forall_{m \leq n} \quad r_n(\alpha_0^m) = 0.$$

However, with (2.12),

$$r_n(\alpha) = \tilde{p}_0^{-1} c_n(\alpha) \tilde{p}_{n+1} - \tilde{p}_0^{-1} \sum_{k=0}^{n-1} \tilde{p}_{k+1} r_{n-1-k}(\alpha) = \prod_{k=1}^n c_k(\alpha_0)^{-1} \cdot \alpha^n + \dots,$$

which is a polynomial in α of degree n . Hence, (3.13) immediately follows from (3.16). From (3.13) we easily obtain (3.14) and prove (3.15) by proving the equivalent inequalities

$$(3.17) \quad c_n(\alpha) \prod_{k=1}^n (\beta - \alpha_0^k) \geq c_n(\beta) \prod_{k=1}^n (\alpha - \alpha_0^k), \quad n = 1, 2, \dots, \beta \geq \alpha \geq \alpha_0,$$

using mathematical induction.

For $n = 1$:

$$c_1(\alpha)(\beta - \alpha_0) = (1 + \alpha)(\beta - \alpha_0) \geq (1 + \beta)(\alpha - \alpha_0) = c_1(\beta)(\alpha - \alpha_0).$$

Suppose that (3.17) holds for a fixed n , then

$$\begin{aligned}
 c_{n+1}(\alpha) \prod_{k=1}^{n+1} (\beta - \alpha_0^k) &= c_n(\alpha) \prod_{k=1}^{n+1} (\beta - \alpha_0^k) + \prod_{k=1}^{n+1} \alpha(\beta - \alpha_0^k) \geq \\
 &\geq (\beta - \alpha_0^{n+1}) c_n(\beta) \prod_{k=1}^n (\alpha - \alpha_0^k) + \prod_{k=1}^{n+1} \beta(\alpha - \alpha_0^k) \geq \\
 &\geq c_n(\beta) \prod_{k=1}^{n+1} (\alpha - \alpha_0^k) + \beta^{n+1} \prod_{k=1}^{n+1} (\alpha - \alpha_0^k) = c_{n+1}(\beta) \prod_{k=1}^{n+1} (\alpha - \alpha_0^k) . \quad \square
 \end{aligned}$$

The result of the preceding lemma gives us hope, that (3.12) does indeed hold. Therefore, we look for a convenient expression for R_β in terms of R_α .

Lemma 3.7. For $\alpha, \beta \in [0, 1)$ the following relations between R_α and R_β hold

$$(3.18) \quad \frac{1 - (1 - \beta)zR_\beta(z)}{1 - (1 - \beta)\alpha zR_\beta(\alpha z)} = \frac{1 - (1 - \alpha)zR_\alpha(z)}{1 - (1 - \alpha)\beta zR_\alpha(\beta z)}$$

and

$$(3.19) \quad R_\beta(z) = \frac{1}{1 - \beta} \frac{1}{z} \left\{ 1 - \prod_{k=0}^{\infty} \frac{1 - (1 - \alpha)\alpha^k zR_\alpha(\alpha^k z)}{1 - (1 - \alpha)\alpha^k \beta zR_\alpha(\alpha^k \beta z)} \right\} .$$

Proof. According to (3.1) we have: $1 - (1 - \alpha)zR_\alpha(z) = \frac{P(\alpha z)}{P(z)}$, so

$$\begin{aligned}
 [1 - (1 - \alpha)zR_\alpha(z)][1 - (1 - \beta)\alpha zR_\beta(\alpha z)] &= \frac{P(\alpha z)}{P(z)} \cdot \frac{P(\beta \alpha z)}{P(\alpha z)} = \\
 &= \frac{P(\beta z)}{P(z)} \cdot \frac{P(\alpha \beta z)}{P(\beta z)} = [1 - (1 - \beta)zR_\beta(z)][1 - (1 - \alpha)\beta zR_\alpha(\beta z)] ,
 \end{aligned}$$

which gives (3.18). By iteration of (3.18) we obtain for every $n \geq 1$

$$1 - (1 - \beta)zR_\beta(z) = \prod_{k=0}^{n-1} \frac{1 - (1 - \alpha)\alpha^k zR_\alpha(\alpha^k z)}{1 - (1 - \alpha)\alpha^k \beta zR_\alpha(\alpha^k \beta z)} \cdot [1 - (1 - \beta)\alpha^n zR_\beta(\alpha^n z)] .$$

Since the last factor tends to 1, if $n \rightarrow \infty$, (3.19) follows. $\square$

Though (3.19) expresses R_β explicitly in R_α , it does not provide an easy proof of the abs mon of R_β , if we know that R_α is abs mon and if $\beta \geq \alpha$. Therefore, we return to relation (3.18), which we rewrite in the following lemma.

Lemma 3.8. For $\alpha, \beta \in [0, 1)$ we have

$$(3.20) \quad \frac{1}{1-\alpha} [R_\beta(z) - \alpha R_\beta(\alpha z)] = \frac{1}{1-\beta} [R_\alpha(z) - \beta R_\alpha(\beta z)] + \\ + z\{\beta R_\alpha(\beta z)R_\beta(z) - \alpha R_\beta(\alpha z)R_\alpha(z)\} ,$$

or, in terms of the coefficients,

$$(3.21) \quad c_{n+1}(\alpha)r_{n+1}(\beta) = c_{n+1}(\beta)r_{n+1}(\alpha) + \sigma_n(\alpha, \beta) ,$$

with

$$(3.22) \quad \sigma_n(\alpha, \beta) := \sum_{k=0}^n (\beta^{k+1} - \alpha^{n-k+1}) r_k(\alpha) r_{n-k}(\beta) .$$

Proof. Writing out (3.18) we obtain (3.20), from which, by equating the coefficients of z^{n+1} , (3.21) follows. $\square$

Lemma 3.9. If $P \in C_\alpha$, then for all $n \geq 1$

$$(3.23) \quad c_n(\alpha)r_n(\beta) \geq c_n(\beta)r_n(\alpha), \quad \text{for } \beta \in [\alpha^{1/n}, 1] .$$

Proof. Let $P \in C_\alpha$. According to (3.21), (3.23) is equivalent to

$$(3.24) \quad \sigma_{n-1}(\alpha, \beta) \geq 0 \quad \text{for } \beta \in [\alpha^{1/n}, 1] .$$

For $n = 1$: $\sigma_0(\alpha, \beta) = (\beta - \alpha)r_0(\alpha)r_0(\beta) \geq 0$ for $\beta \geq \alpha$. Suppose $\beta \in [\alpha^{1/n+1}, 1]$ and $\sigma_k(\alpha, \beta) \geq 0$ for $k = 1, 2, \dots, n-1$. Then it follows from (3.21), that $r_k(\beta) \geq 0$ for $k = 1, 2, \dots, n$, so that with (3.22)

$$\sigma_n(\alpha, \beta) \geq (\beta^{n+1} - \alpha) \sum_{k=0}^n r_k(\alpha) r_{n-k}(\beta) \geq 0 .$$

Thus, (3.24) has been proved by induction. $\square$

Remark. Using (3.21) and (3.22), we can prove something more than (3.23), namely

$$c_n(\alpha)r_n(\beta) \geq c_n(\beta)r_n(\alpha), \quad \text{for } \beta \geq \alpha, \text{ if } n = 0, 1, \dots, 5, \\ \text{for } \beta \geq \alpha^{3/n-2}, \text{ if } n \geq 6 .$$

However, it is difficult to proceed by induction in this way to prove (3.12).

In the following lemma we formulate some relations, from which another partial result follows.

Lemma 3.10.

i) If $\alpha = \beta\gamma$ with $\beta, \gamma \in [0, 1]$, then

$$(3.25) \quad 1 - (1 - \alpha)zR_\alpha(z) = [1 - (1 - \beta)zR_\beta(z)][1 - (1 - \gamma)\beta zR_\gamma(\beta z)] .$$

ii) For all $\beta \in [0, 1]$

$$(3.26) \quad (1 + \beta)R_{\beta^2}(z) = R_\beta(z) + \beta R_\beta(\beta z) - \beta(1 - \beta)zR_\beta(z)R_\beta(\beta z) .$$

iii) For all $\beta \in [0, 1]$ and all $n \geq 0$

$$(3.27) \quad \sigma_n(\beta^2, \beta) = \frac{\beta}{1 + \beta} (1 - \beta^{n+2}) \sum_{k=0}^n \beta^k r_k(\beta) r_{n-k}(\beta) .$$

iv) If $P \in C_{\beta^2}$ with $\beta \in [0, 1]$, then

$$(3.28) \quad c_n(\beta^2)r_n(\beta) \geq c_n(\beta)r_n(\beta^2), \quad n = 0, 1, 2, \dots .$$

v) For all $\beta \in [0, 1]$ $C_{\beta^2} \subset C_\beta$.

Proof. If $\alpha = \beta\gamma$, then, on account of (3.19), we can write

$$\begin{aligned} 1 - (1 - \beta)zR_\beta(z) &= \prod_{k=0}^{\infty} \frac{1 - (1 - \alpha)\alpha^k zR_\alpha(\alpha^k z)}{1 - (1 - \alpha)\alpha^k \beta zR_\alpha(\alpha^k \beta z)} = \\ &= [1 - (1 - \alpha)zR_\alpha(z)] \prod_{k=0}^{\infty} \frac{1 - (1 - \alpha)\alpha^{k+1} zR_\alpha(\alpha^{k+1} z)}{1 - (1 - \alpha)\alpha^k \beta zR_\alpha(\alpha^k \beta z)} = \\ &= [1 - (1 - \alpha)zR_\alpha(z)] \left\{ \prod_{k=0}^{\infty} \frac{1 - (1 - \alpha)\alpha^k \beta zR_\alpha(\alpha^k \beta z)}{1 - (1 - \alpha)\alpha^k \gamma \beta zR_\alpha(\alpha^k \gamma \beta z)} \right\}^{-1} = \\ &= [1 - (1 - \alpha)zR_\alpha(z)][1 - (1 - \gamma)\beta zR_\gamma(\beta z)]^{-1} , \end{aligned}$$

from which (3.25) follows. Taking $\gamma = \beta$, and multiplying out, we obtain (3.26), from which, by equating the coefficients of z^{n+1} , it follows that

$$(1 + \beta)r_{n+1}(\beta^2) = (1 + \beta^{n+2})r_{n+1}(\beta) - \beta(1 - \beta) \sum_{k=0}^n \beta^k r_k(\beta) r_{n-k}(\beta) .$$

Hence, with (3.21),

$$\begin{aligned}\sigma_n(\beta^2, \beta) &= c_{n+1}(\beta^2) r_{n+1}(\beta) - c_{n+1}(\beta) r_{n+1}(\beta^2) = \\ &= \frac{1 - \beta^{n+2}}{1 - \beta^2} \{ (1 + \beta) r_{n+1}(\beta^2) + \beta(1 - \beta) \sum_{k=0}^n \beta^k r_k(\beta) r_{n-k}(\beta) \} - \\ &- \frac{1 - \beta^{n+2}}{1 - \beta} r_{n+1}(\beta^2) = \frac{\beta}{1 + \beta} (1 - \beta^{n+2}) \sum_{k=0}^n \beta^k r_k(\beta) r_{n-k}(\beta) ,\end{aligned}$$

which is (3.27). If furthermore, $P \in C_{\beta^2}$, then all $r_n(\beta^2)$ are nonnegative and from (3.21) and (3.27) by induction we obtain

$$\sigma_n(\beta^2, \beta) \geq 0, \quad \text{for } n = 0, 1, 2, \dots,$$

which gives (3.28). From this we see, that all $r_n(\beta)$ are nonnegative, too, i.e. $P \in C_\beta$. $\square$

Corollary 3.11. For all $\alpha \in [0, 1]$ we have

$$C_\alpha \subset C_{\alpha^{\frac{1}{2}}} \subset C_{\alpha^{\frac{1}{4}}} \subset \dots \subset C_{\alpha^{\frac{1}{2^m}}} \subset \dots \subset C_1 .$$

We have now obtained in a rather laborious way a few results (theorem 3.5, lemma 3.9 and corollary 3.11), which indicate, that (3.12) is true. Indeed, we can prove (3.12) by a method, inspired by still another proof of " $C_\alpha \subset C_1$ ", which follows from the next theorem and lemma.

Theorem 3.12.

- i) If $P \in C_1$, then for all $\alpha \in [0, 1]$ $\frac{P(z)}{P(\alpha z)}$ is abs mon and $\frac{P(\alpha)P(z)}{P(\alpha z)} \in C_1$.
- ii) For all $\alpha \in [0, 1]$ we have the following characterization of C_α :

$$(3.29) \quad P \in C_\alpha \Leftrightarrow \frac{P(\alpha)P(z)}{P(\alpha z)} \in C_0 .$$

Proof. If $P \in C_1$, then P has the representation (1.1), from which it follows that

$$(3.30) \quad \frac{P(\alpha)P(z)}{P(\alpha z)} = \exp[\lambda(Q(z) - Q(\alpha z) + Q(\alpha) - 1)] .$$

Again, this is of the form (1.1), so $\frac{P(\alpha)P(z)}{P(\alpha z)} \in C_1$.

Now let $P \in C_\alpha$. First we see from

$$(3.31) \quad \frac{P(\alpha)P(z)}{P(\alpha z)} = \frac{P(\alpha)}{1 - (1 - \alpha)zR_\alpha(z)},$$

that $\frac{P(\alpha)P(z)}{P(\alpha z)}$ is a pgf. If we define $p := 1 - P(\alpha)$ and $Q(z) := \frac{1 - \alpha}{1 - P(\alpha)} zR_\alpha(z)$, then it follows that $p \in [0, 1)$ and Q is a pgf (as $R_\alpha(1) = \frac{1 - P(\alpha)}{1 - \alpha}$). Hence

$$\frac{P(\alpha)P(z)}{P(\alpha z)} = \frac{1 - p}{1 - pQ(z)} \in C_0.$$

However, this and its converse, i.e. (3.29), we can immediately prove using the following relation between the R_0 corresponding to the pgf $\frac{P(\alpha)P(z)}{P(\alpha z)}$ (denoted by $R_0^{(\alpha)}$) and R_α :

$$(3.32) \quad R_0^{(\alpha)}(z) = \frac{1}{z} \left\{ 1 - \frac{P(\alpha)}{P(\alpha)P(z)/P(\alpha z)} \right\} = \frac{1}{z} \left\{ 1 - \frac{P(\alpha z)}{P(z)} \right\} = (1 - \alpha)R_\alpha(z).$$

Thus, $R_0^{(\alpha)}$ is abs mon if and only if R_α is abs mon, which is, by definition, equivalent to (3.29). $\square$

Lemma 3.13. The following relation between R_1 and R_α holds

$$(3.33) \quad \frac{1}{1 - \alpha} \{R_1(z) - \alpha R_1(\alpha z)\} = \frac{P(z)}{P(\alpha z)} [zR_\alpha(z)]',$$

or

$$(3.34) \quad \frac{1}{1 - \alpha} \{R_1(z) - \alpha R_1(\alpha z)\} = \frac{[zR_\alpha(z)]'}{1 - (1 - \alpha)zR_\alpha(z)}.$$

In terms of the coefficients

$$(3.35) \quad c_n(\alpha)r_n(1) = (n+1)r_n(\alpha) + (1 - \alpha) \sum_{k=0}^{n-1} c_k(\alpha)r_k(1)r_{n-1-k}(\alpha).$$

Proof. On account of (3.1) we can write

$$\begin{aligned} (1 - \alpha)[zR_\alpha(z)]' &= -\left[\frac{P(\alpha z)}{P(z)}\right]' = \frac{P(\alpha z)P'(z)}{P(z)^2} - \alpha \frac{P'(\alpha z)}{P(z)} = \\ &= \frac{P(\alpha z)}{P(z)} \{R_1(z) - \alpha R_1(\alpha z)\}, \end{aligned}$$

from which (3.33) follows. Using (3.1) once more we obtain (3.34), from which, equating the coefficients of z^n , we get (3.35) by using

$$(3.36) \quad \frac{1}{1-\alpha} \{R_1(z) - \alpha R_1(\alpha z)\} = \sum_{n=0}^{\infty} c_n(\alpha) r_n(1) z^n ,$$

and

$$(3.36)' \quad [zR_{\alpha}(z)]' = \sum_{n=0}^{\infty} (n+1) r_n(\alpha) z^n .$$

□

Corollary 3.14. For all $\alpha \in [0,1]$ we have $C_{\alpha} \subset C_1$. Furthermore, if $P \in C_{\alpha}$, then

$$(3.37) \quad c_n(\alpha) r_n(1) \geq (n+1) r_n(\alpha), \quad n = 0, 1, 2, \dots .$$

Proof. Let $P \in C_{\alpha}$. Then R_{α} is abs mon, so that the abs mon of R_1 follows from (3.33) and (3.29), or immediately from (3.34). By induction we obtain the inequality (3.37) from (3.35). □

We now try to find a relation between R_{α} and R_{β} similar to the relation (3.33) between R_{α} and R_1 . On the one hand we can write for $(1-\alpha)[zR_{\alpha}(z)]'$ the limit

$$(1-\alpha) \lim_{\beta \uparrow 1} \frac{zR_{\alpha}(z) - \beta zR_{\alpha}(\beta z)}{z - \beta z} = \lim_{\beta \uparrow 1} \frac{1-\alpha}{1-\beta} \{R_{\alpha}(z) - \beta R_{\alpha}(\beta z)\} ,$$

and on the other hand, according to (3.33),

$$\frac{P(\alpha z)}{P(z)} \{R_1(z) - \alpha R_1(\alpha z)\} = \lim_{\beta \uparrow 1} \frac{P(\alpha z)}{P(\beta z)} \{R_{\beta}(z) - \alpha R_{\beta}(\alpha z)\} .$$

Equating the two expressions, we may hope, that the following relation holds

$$\frac{1-\alpha}{1-\beta} \{R_{\alpha}(z) - \beta R_{\alpha}(\beta z)\} = \frac{P(\alpha z)}{P(\beta z)} \{R_{\beta}(z) - \alpha R_{\beta}(\alpha z)\} ,$$

or, symmetrizing in α and β ,

$$(3.38) \quad \frac{1-\beta}{P(\beta z)} \{R_{\beta}(z) - \alpha R_{\beta}(\alpha z)\} = \frac{1-\alpha}{P(\alpha z)} \{R_{\alpha}(z) - \beta R_{\alpha}(\beta z)\} .$$

The truth of (3.38) is expressed in the following lemma.

Lemma 3.15. The following relation between R_{α} and R_{β} holds

$$(3.39) \quad \frac{1}{1-\alpha} \{R_{\beta}(z) - \alpha R_{\beta}(\alpha z)\} = \frac{P(\beta z)}{P(\alpha z)} \frac{1}{1-\beta} \{R_{\alpha}(z) - \beta R_{\alpha}(\beta z)\} .$$

Proof. According to (3.1) the left-hand side of (3.38) can be written as

$$\begin{aligned} \frac{1-\beta}{P(\beta z)} \{R_{\beta}(z) - \alpha R_{\beta}(\alpha z)\} &= \frac{1}{zP(\beta z)} \left\{ \left(1 - \frac{P(\beta z)}{P(z)}\right) - \left(1 - \frac{P(\beta \alpha z)}{P(\alpha z)}\right) \right\} = \\ &= \frac{1}{z} \left\{ \frac{P(\alpha \beta z)}{P(\alpha z)P(\beta z)} - \frac{1}{P(z)} \right\}, \end{aligned}$$

which is symmetric in α and β , and hence equal to the right-hand side of (3.38). Rewriting (3.38) gives (3.39). $\square$

Now, using relation (3.39), we can easily prove the desired property (3.12).

Theorem 3.16. For all $\alpha, \beta \in [0,1]$ the following implication holds

$$\alpha \leq \beta \Rightarrow C_{\alpha} \subset C_{\beta}.$$

Proof. The case $\alpha < 1, \beta = 1$ has been treated in corollary 3.14. Now let $\alpha < \beta < 1$ and $P \in C_{\alpha}$. Then also $P \in C_1$, and on account of lemma 3.12i) it follows that $\frac{P(z)}{P(\frac{\alpha}{\beta} z)}$, and therefore $\frac{P(\beta z)}{P(\alpha z)}$, is abs mon. Considering the relation (3.39) and noting that

$$(3.40) \quad \frac{1}{1-\beta} \{R_{\alpha}(z) - \beta R_{\alpha}(\beta z)\} = \sum_{n=0}^{\infty} c_n(\beta) r_n(\alpha) z^n,$$

we obtain the abs mon of R_{β} from that of R_{α} and $\frac{P(\beta z)}{P(\alpha z)}$. Hence $P \in C_{\beta}$. $\square$

Using lemma 3.15 we prove inequalities, stronger than $r_n(\beta) \geq 0$ ($n \geq 0$), and of which (1.19), (2.9), (3.15), (3.23), (3.28) and (3.37) are special cases.

Corollary 3.17. If $P \in C_{\alpha}$, then we have for all $\beta \in [\alpha, 1]$

$$(3.41) \quad c_n(\alpha) r_n(\beta) \geq c_n(\beta) r_n(\alpha), \quad n = 0, 1, 2, \dots,$$

and so: every r_n is nondecreasing on $[\alpha, 1]$.

Proof. As we saw in the proof of theorem 3.16, if $P \in C_{\alpha}$ and $\alpha < \beta$, we can write

$$\frac{P(\beta z)}{P(\alpha z)} = \sum_{n=0}^{\infty} s_n(\alpha, \beta) z^n,$$

with $s_n(\alpha, \beta) \geq 0$ for all n and $s_0(\alpha, \beta) = 1$. But then, by equating the coeffi-

icients of z^n , it follows from (3.39) and (3.40), that

$$c_n(\alpha)r_n(\beta) = \sum_{k=0}^n c_k(\beta)r_k(\alpha)s_{n-k}(\alpha,\beta) ,$$

from which (using $s_0 = 1$ and $s_n \geq 0$) we obtain (3.41). The last assertion of this corollary is obtained from (3.41) by the fact, that $c_n(\alpha)$ is nondecreasing in α . $\square$

From theorem 3.16 we obtain an extension of corollary 1.16, as follows.

Corollary 3.18. If the $r_n(\alpha)$, $n \geq 0$, defined by

$$\frac{1 - \alpha^{n+1}}{1 - \alpha} p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(\alpha), \quad n = 0, 1, 2, \dots$$

with $\alpha \in [0, 1]$, are all nonnegative, then the $r_n(\beta)$, $n \geq 0$, defined by

$$\frac{1 - \beta^{n+1}}{1 - \beta} p_{n+1} = \sum_{k=0}^n p_k r_{n-k}(\beta), \quad n = 0, 1, 2, \dots$$

with $\beta \in [\alpha, 1]$, are all nonnegative, too. We then, in fact, have

$$\frac{1 - \alpha^{n+1}}{1 - \alpha} r_n(\beta) \geq \frac{1 - \beta^{n+1}}{1 - \beta} r_n(\alpha), \quad n = 0, 1, 2, \dots .$$

Theorem 3.16 says, that for $\alpha \leq \beta$ the abs mon of R_β follows from that of R_α . As on account of lemma 3.1iii) an abs mon R_α may be any abs mon function f with $f(1) < \frac{1}{1 - \alpha}$, with lemma 3.7 we can formulate the following assertion about abs mon functions.

Corollary 3.19. If φ is an abs mon function with $\varphi(0) = 0$ and $\varphi(1) < 1$, then the function ψ , for $0 \leq \alpha \leq \beta \leq 1$ defined by

$$\frac{1 - \psi(z)}{1 - \psi(\alpha z)} = \frac{1 - \varphi(z)}{1 - \varphi(\beta z)} ,$$

or by

$$\psi(z) = 1 - \prod_{k=0}^{\infty} \frac{1 - \varphi(\alpha^k z)}{1 - \varphi(\alpha^k \beta z)} ,$$

is abs mon, too.

In theorem 2.9 we proved the following implication (now formulated in terms of generating functions)

$$(3.42) \quad P \in C_{\alpha} \Rightarrow \frac{P(\gamma z)}{P(\gamma)} \in C_{\alpha}, \quad \text{for } \gamma \in [0,1] .$$

If we denote the generating function of the $r_n(\alpha)$ corresponding to the pgf $P^{(\gamma)}(z) := \frac{P(\gamma z)}{P(\gamma)}$ by $R_{\alpha}^{(\gamma)}$ (in the sequel we shall use such a notation without further explication), then with (3.1) we can write

$$(1-\alpha)zR_{\alpha}^{(\gamma)}(z) = 1 - \frac{P^{(\gamma)}(\alpha z)}{P^{(\gamma)}(z)} = 1 - \frac{P(\gamma \alpha z)}{P(\gamma z)} = (1-\alpha)\gamma z R_{\alpha}(\gamma z) .$$

Hence

$$(3.43) \quad R_{\alpha}^{(\gamma)}(z) = \gamma R_{\alpha}(\gamma z) ,$$

from which (3.42) immediately follows.

By means of similar methods we can obtain other properties of C_{α} . We formulate them in the following theorems.

Theorem 3.20. A pgf P , which is the limit of a sequence pgf's $P_n \in C_{\alpha}$, belongs to C_{α} (i.e. C_{α} is closed under weak convergence).

Proof. Let $P_n \in C_{\alpha}$ for $n \geq 1$, so $R_{\alpha}^{(n)}(z) = \frac{1}{1-\alpha} \frac{1}{z} \{1 - \frac{P_n(\alpha z)}{P_n(z)}\}$ is abs mon. But then we have, if $P(z) = \lim_{n \rightarrow \infty} P_n(z)$,

$$\begin{aligned} R_{\alpha}(z) &= \frac{1}{1-\alpha} \frac{1}{z} \{1 - \frac{P(\alpha z)}{P(z)}\} = \frac{1}{1-\alpha} \frac{1}{z} \{1 - \lim_{n \rightarrow \infty} \frac{P_n(\alpha z)}{P_n(z)}\} = \\ &= \lim_{n \rightarrow \infty} R_{\alpha}^{(n)}(z) , \end{aligned}$$

which is abs mon, too, so: $P \in C_{\alpha}$. □

Theorem 3.21. For all $\alpha \in [0,1]$ we have:

$$(3.44) \quad P \in C_{\alpha} \Rightarrow \frac{P(\gamma)P(z)}{P(\gamma z)} \in C_{\alpha}, \quad \text{for } \gamma \in [0,\alpha) \\ \in C_0, \quad \text{for } \gamma \in [\alpha,1] .$$

Proof. Define $P^{(\gamma)}(z) := \frac{P(\gamma)P(z)}{P(\gamma z)}$, for $\gamma \in [0,1]$. If $P \in C_1$, then on account of theorem 3.12i) we know that $P^{(\gamma)} \in C_1$, too, for all γ . Now let $\alpha \in [0,1)$ and $P \in C_\alpha$. Then we can write

$$\begin{aligned} (1-\alpha)zR_\alpha^{(\gamma)}(z) &= 1 - \frac{P^{(\gamma)}(\alpha z)}{P^{(\gamma)}(z)} = 1 - \frac{P(\alpha z)}{P(\gamma \alpha z)} \frac{P(\gamma z)}{P(z)} = \\ &= \frac{P(\gamma z)}{P(\alpha \gamma z)} \left\{ \frac{P(\alpha \gamma z)}{P(\gamma z)} - \frac{P(\alpha z)}{P(z)} \right\} = \frac{P(\gamma z)}{P(\alpha \gamma z)} (1-\alpha)z\{R_\alpha(z) - \gamma R_\alpha(\gamma z)\}, \end{aligned}$$

so that

$$(3.45) \quad R_\alpha^{(\gamma)}(z) = \frac{P(\gamma z)}{P(\alpha \gamma z)} \{R_\alpha(z) - \gamma R_\alpha(\gamma z)\}.$$

Using (3.39) we can rewrite this

$$(3.46) \quad R_\alpha^{(\gamma)}(z) = \frac{1-\gamma}{1-\alpha} \frac{P(\alpha z)}{P(\gamma \alpha z)} \{R_\gamma(z) - \alpha R_\gamma(\alpha z)\}.$$

As according to theorem 3.12i), $\frac{P(\gamma z)}{P(\alpha \gamma z)}$ is abs mon, using (3.45) we obtain for all $\gamma \in [0,1]$ the abs mon of $R_\alpha^{(\gamma)}$ from that of R_α . Hence $P^{(\gamma)} \in C_\alpha$ for all $\gamma \in [0,1]$. However, $P \in C_\gamma$ for $\gamma \geq \alpha$, so on account of (3.29) we can conclude that $P^{(\gamma)} \in C_0$ for all $\gamma \in [\alpha,1]$. $\square$

Theorem 3.22. For all $\alpha \in [0,1]$ we have:

$$(3.47) \quad P \in C_\alpha \Rightarrow \prod_{k=0}^{n-1} \frac{P(\gamma^k z)}{P(\gamma^k)} \in C_\gamma, \quad \text{for } n \geq 1, \gamma \in [\alpha^{1/n}, 1].$$

Proof. Take a fixed $n \geq 1$ and define

$$P^{(\gamma)}(z) := \prod_{k=0}^{n-1} \frac{P(\gamma^k z)}{P(\gamma^k)}, \quad \text{for } \gamma \in [0,1].$$

If $P \in C_1$, then $P^{(\gamma)}$ as product of pgf's in C_1 belongs to C_1 , too, for all γ . Now let $\alpha \in [0,1)$ and $P \in C_\alpha$. For $\beta \in [0,1)$ we can write

$$(1-\beta)zR_\beta^{(\gamma)}(z) = 1 - \frac{P^{(\gamma)}(\beta z)}{P^{(\gamma)}(z)} = 1 - \prod_{k=0}^{n-1} \frac{P(\gamma^k \beta z)}{P(\gamma^k z)},$$

or, with $\beta = \gamma$,

$$(1-\gamma)zR_\gamma^{(\gamma)}(z) = 1 - \frac{P(\gamma^n z)}{P(z)} = (1-\gamma^n)zR_\gamma^n(z).$$

Hence

$$(3.48) \quad R_Y^{(\gamma)}(z) = c_{n-1}(\gamma) R_{Y^n}(z) .$$

As R_{Y^n} is abs mon for $\gamma \geq \alpha^{1/n}$, it follows from (3.48), that $R_Y^{(\gamma)}$ is abs mon for $\gamma \geq \alpha^{1/n}$. Hence $P^{(\gamma)} \in C_Y$ for $\gamma \geq \alpha^{1/n}$. $\square$

Remark. In view of theorem 3.16 it is evident, that theorem 3.22 is equivalent to the assertion that for all $\gamma \in [0,1]$:

$$(3.49) \quad P \in C_{Y^n} \Rightarrow \prod_{k=0}^{n-1} \frac{P(\gamma^k z)}{P(\gamma^k)} \in C_Y .$$

We can easily prove (3.49) using lemma 3.4: If $P \in C_{Y^n}$, then there are $p \in (0,1)$ and pgf Q with $Q(0) = 0$, such that

$$P(z) = \prod_{\ell=0}^{\infty} \frac{1 - pQ(\gamma^{n\ell} z)}{1 - pQ(\gamma^{n\ell} z)} .$$

But then

$$\prod_{k=0}^{n-1} \frac{P(\gamma^k z)}{P(\gamma^k)} = \prod_{\ell=0}^{\infty} \prod_{k=0}^{n-1} \frac{1 - pQ(\gamma^{n\ell+k} z)}{1 - pQ(\gamma^{n\ell+k} z)} = \prod_{\ell=0}^{\infty} \frac{1 - pQ(\gamma^{\ell} z)}{1 - pQ(\gamma^{\ell} z)} \in C_Y .$$

Still another proof of (3.49) we obtain from the characterization (3.29): If $P \in C_{Y^n}$, then $\frac{P(\gamma^n)P(z)}{P(\gamma^n z)} \in C_0$. But for $P^{(\gamma)}(z) := \prod_{k=0}^{n-1} \frac{P(\gamma^k z)}{P(\gamma^k)}$ we have

$$\frac{P^{(\gamma)}(\gamma)P^{(\gamma)}(z)}{P^{(\gamma)}(\gamma z)} = \prod_{k=0}^{n-1} \frac{P(\gamma^{k+1})}{P(\gamma^k)} \prod_{k=0}^{n-1} \frac{P(\gamma^k z)}{P(\gamma^{k+1} z)} = P(\gamma^n) \frac{P(z)}{P(\gamma^n z)} \in C_0 ,$$

so $P^{(\gamma)} \in C_Y$.

Corollary 3.23. For all $n \geq 1$ and all $\alpha \in [0,1]$ we have

$$(3.50) \quad \prod_{k=0}^{n-1} \frac{1 - pQ(\alpha^k)}{1 - pQ(\alpha^k z)} \in C_{\alpha} ,$$

with $p \in [0,1)$ and Q a pgf.

Proof. Take $P(z) = \frac{1-p}{1-pQ(z)} \in C_0$ in (3.47). $\square$

Theorem 3.24. For all $\alpha \in [0,1]$ we have:

$$(3.51) \quad P \in C_\alpha \Rightarrow P^\gamma \in C_\alpha, \quad \text{for } \gamma \in [0,1] .$$

Proof. If $P \in C_1$, then, according to theorem 1.1, $P^\gamma \in C_1$ for all $\gamma \geq 0$. Now let $\alpha \in [0,1)$ and $P \in C_\alpha$. Then we can write

$$(1 - \alpha)zR_\alpha^{(\gamma)}(z) = 1 - \frac{P^\gamma(\alpha z)}{P^\gamma(z)} = 1 - \left(\frac{P(\alpha z)}{P(z)}\right)^\gamma ,$$

so that

$$\begin{aligned} \frac{\partial}{\partial z} [(1 - \alpha)zR_\alpha^{(\gamma)}(z)] &= -\gamma \left(\frac{P(\alpha z)}{P(z)}\right)^{\gamma-1} \frac{\partial}{\partial z} \left[\frac{P(\alpha z)}{P(z)}\right] = \\ &= \gamma \left(\frac{P(z)}{P(\alpha z)}\right)^{1-\gamma} \frac{\partial}{\partial z} [(1 - \alpha)zR_\alpha(z)] . \end{aligned}$$

Hence

$$(3.52) \quad \frac{\partial}{\partial z} [zR_\alpha^{(\gamma)}(z)] = \gamma \left(\frac{P(z)}{P(\alpha z)}\right)^{1-\gamma} \frac{\partial}{\partial z} [zR_\alpha(z)] .$$

As on account of theorem 3.12i) $\frac{P(\alpha)P(z)}{P(\alpha z)} \in C_1$, and so $\left(\frac{P(\alpha)P(z)}{P(\alpha z)}\right)^{1-\gamma} \in C_1$ for $\gamma \in [0,1]$, it follows that $\left(\frac{P(z)}{P(\alpha z)}\right)^{1-\gamma}$ is abs mon for $\gamma \in [0,1]$. Now, using (3.52), we obtain the abs mon of $R_\alpha^{(\gamma)}$ from that of R_α , and we conclude: $P^\gamma \in C_\alpha$. $\square$

Remark. If we wish to use characterization (3.29) for the proof of theorem 3.24, then we have to prove (3.51) for $\alpha = 0$. If $P(z) = \frac{1-p}{1-pA(z)} \in C_0$, with $p \in [0,1)$ and A a pgf, then we can write $[P(z)]^\gamma = \frac{1-q}{1-qB(z)}$, with

$$\begin{aligned} q &:= 1 - (1-p)^\gamma , \\ B(z) &:= \frac{1}{q} \{1 - (1-pA(z))^\gamma\} = \frac{1}{q} \left\{1 - \sum_{n=0}^{\infty} \binom{\gamma}{n} (-pA(z))^n\right\} = \\ &= \frac{1}{q} \sum_{n=1}^{\infty} \frac{\gamma}{n} \binom{n-1-\gamma}{n-1} (pA(z))^n , \end{aligned}$$

from which we see, that $q \in [0,1)$ and B is a pgf, for $\gamma \in [0,1]$. Hence $P^\gamma \in C_0$.

Now let $P \in C_\alpha$. Then, according to (3.29), $\frac{P(\alpha)P(z)}{P(\alpha z)} \in C_0$, so that

$$\frac{P(\alpha)^\gamma P(z)^\gamma}{P(\alpha z)^\gamma} = \left(\frac{P(\alpha)P(z)}{P(\alpha z)} \right)^\gamma \in C_0,$$

too, for $\gamma \in [0,1]$. Using (3.29) once more, we conclude that $P^\gamma \in C_\alpha$ for $\gamma \in [0,1]$.

We conclude the properties of the classes C_α with the assertion that $\bigcup_{\alpha < 1} C_\alpha$ is dense in C_1 in the following sense:

Theorem 3.25. If $P \in C_1$, then there is an increasing sequence α_n , with $\lim_{n \rightarrow \infty} \alpha_n = 1$, and there are pgf's $P_n \in C_{\alpha_n}$, such that

$$P(z) = \lim_{n \rightarrow \infty} P_n(z), \quad \text{for } |z| \leq 1.$$

Proof. Let $P \in C_1$. Then there is a $\lambda > 0$ and a pgf Q , such that

$$P(z) = \exp[\lambda(Q(z) - 1)].$$

Take $\alpha_n := 1 - \frac{1}{n}$. If for $n > \lambda$ we define the pgf's P_n by

$$(3.53) \quad P_n(z) = \prod_{k=0}^{n-1} \frac{1 - \lambda/n \cdot Q(\alpha_n^k)}{1 - \lambda/n \cdot Q(\alpha_n^k z)},$$

then, on account of corollary 3.23, we know that $P_n \in C_{\alpha_n}$. As $P(1) = P_n(1) = 1$, it is sufficient to prove the convergence of $P_n(z)$ to $P(z)$ for $z \in [0,1)$. So, take a fixed $z \in [0,1)$ and write

$$P_n(z) = \prod_{k=0}^{n-1} \left\{ 1 + \frac{\lambda[Q(\alpha_n^k z) - Q(\alpha_n^k)]}{n - \lambda Q(\alpha_n^k z)} \right\} = \prod_{k=0}^{n-1} \left\{ 1 + \frac{\lambda[Q(z) - 1]}{n} + f_k(n) \right\},$$

with

$$f_k(n) := \frac{\lambda[Q(\alpha_n^k z) - Q(\alpha_n^k)]}{n - \lambda Q(\alpha_n^k z)} - \frac{\lambda[Q(z) - 1]}{n} =$$

$$\begin{aligned}
 &= \frac{\lambda}{n} \left\{ 1 - Q(z) + [Q(\alpha_n^k z) - Q(\alpha_n^k)] \frac{n}{n - \lambda Q(\alpha_n^k z)} \right\} = \\
 &= \frac{\lambda}{n} \left\{ 1 - Q(z) + Q(\alpha_n^k z) - Q(\alpha_n^k) + \frac{\lambda Q(\alpha_n^k z)}{n - \lambda Q(\alpha_n^k z)} [Q(\alpha_n^k z) - Q(\alpha_n^k)] \right\}.
 \end{aligned}$$

Hence

$$(3.54) \quad |f_k(n)| \leq \frac{\lambda}{n} \left\{ |1 - Q(\alpha_n^k)| + |Q(z) - Q(\alpha_n^k z)| + \frac{\lambda}{n - \lambda} \right\}.$$

For $k < n$ we estimate

$$\begin{aligned}
 |Q(z) - Q(\alpha_n^k z)| &= |Q(z) - Q(z - (1 - \alpha_n^k)z)| = (1 - \alpha_n^k) z Q'(\xi_{n,k}) \leq \\
 &\leq z Q'(z) \cdot (1 - \alpha_n^k),
 \end{aligned}$$

because $(1 - \alpha_n^k)z < \xi_{n,k} < z$ and Q is convex. As for $k < n$

$$1 - \alpha_n^k = 1 - \left(1 - \frac{1}{n^2}\right)^k \leq 1 - \left(1 - k \frac{1}{n^2}\right) \leq \frac{1}{n},$$

we finally have for some $C(z) > 0$ and all $k < n$

$$(3.55) \quad |Q(z) - Q(\alpha_n^k z)| \leq \frac{C(z)}{n}.$$

As $\lim_{z \uparrow 1} Q(z) = Q(1) = 1$ and $\lim_{n \rightarrow \infty} \alpha_n^n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2}\right)^n = 1$, we further have for all $k < n$

$$|1 - Q(\alpha_n^k)| \leq 1 - Q(\alpha_n^n) \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

Using this and (3.55) we conclude from (3.54) that

$$(3.56) \quad \forall_{k < n} |f_k(n)| \leq \varepsilon(n), \text{ with } \varepsilon(n) = o\left(\frac{1}{n}\right) \quad (n \rightarrow \infty).$$

Now it follows that

$$\begin{aligned}
 P_n(z) &\leq \prod_{k=0}^{n-1} \left\{ 1 + \frac{\lambda[Q(z) - 1]}{n} + |f_k(n)| \right\} \leq \\
 &\leq \left\{ 1 + \frac{\lambda[Q(z) - 1]}{n} + \varepsilon(n) \right\}^n \rightarrow \exp[\lambda(Q(z) - 1)], \quad (n \rightarrow \infty),
 \end{aligned}$$

and

$$P_n(z) \geq \prod_{k=0}^{n-1} \left\{ 1 + \frac{\lambda[Q(z) - 1]}{n} - |f_k(n)| \right\} \geq$$

$$\geq \left\{ 1 + \frac{\lambda[Q(z) - 1]}{n} - \varepsilon(n) \right\}^n \rightarrow \exp[\lambda(Q(z) - 1)], \quad (n \rightarrow \infty).$$

Hence

$$\lim_{n \rightarrow \infty} P_n(z) = P(z).$$

□

Finally, we give a few examples of distributions in C_α . First we consider the Poisson-distribution

$$(3.57) \quad p_n = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, \dots, \text{ with pgf } P(z) = \exp[\lambda(z - 1)].$$

To obtain the $\alpha \in [0, 1]$ for which $P \in C_\alpha$, we consider R_α

$$(1 - \alpha)zR_\alpha(z) = 1 - \frac{P(\alpha z)}{P(z)} = 1 - \exp[\lambda z(\alpha - 1)] =$$

$$= 1 - \sum_{n=0}^{\infty} (-1)^n (1 - \alpha)^n \frac{(\lambda z)^n}{n!} = \sum_{n=1}^{\infty} (-1)^{n-1} (1 - \alpha)^n \frac{(\lambda z)^n}{n!}.$$

Hence

$$(3.58) \quad r_n(\alpha) = (-1)^n (1 - \alpha)^n \frac{\lambda^{n+1}}{(n+1)!}, \quad n = 0, 1, 2, \dots,$$

from which it follows that

$$(3.59) \quad \exp[\lambda(z - 1)] \notin C_\alpha \quad \text{for all } \alpha < 1.$$

Using the characterization (3.29), from pgf's in C_0 we can construct pgf's in C_α . For example, if we take the geometric pgf in C_0 , then we may conclude that the pgf P defined by

$$\frac{P(\alpha)P(z)}{P(\alpha z)} = \frac{1 - p}{1 - pz},$$

with $p \in [0, 1)$, is in C_α . It follows that $P(\alpha) = 1 - p$ and $P(z) = \frac{1}{1 - pz} P(\alpha z)$, so

$$P(z) = p_0 \prod_{k=0}^{\infty} \frac{1}{1 - p\alpha^k z} = \prod_{k=0}^{\infty} \frac{1 - p\alpha^k}{1 - p\alpha^k z}.$$

We thus have for all $p \in [0,1)$:

$$(3.60) \quad \prod_{k=0}^{\infty} \frac{1 - p\alpha^k}{1 - p\alpha^k z} \in C_{\alpha}.$$

Note that for $p = 1 - \alpha$ we get the pgf $\tilde{P}$ from lemma 2.5 (cf. lemma 3.6, where the $r_n(\alpha)$ have been given).

Next we consider products of geometric pgf's, which we can write in the following form

$$(3.61) \quad P(z) = \frac{1-p}{1-pz} \cdot \frac{1-p\gamma_1}{1-p\gamma_1 z} \cdots \frac{1-p\gamma_1\gamma_2 \cdots \gamma_{m-1}}{1-p\gamma_1\gamma_2 \cdots \gamma_{m-1} z},$$

with $p \in [0,1)$ and $\gamma_1, \gamma_2, \dots, \gamma_{m-1} \in [0,1]$. If $\gamma_1 = \gamma_2 = \dots = \gamma_{m-1} =: \gamma$, then we have a special case of corollary 3.23:

$$(3.62) \quad \prod_{k=0}^{m-1} \frac{1 - p\gamma^k}{1 - p\gamma^k z} \in C_{\gamma}.$$

Taking $m = 2$ in (3.61) we can write

$$\begin{aligned} P(z) &= \frac{1-p}{1-pz} \cdot \frac{1-p\gamma}{1-p\gamma z} = \frac{(1-p)(1-p\gamma)}{1-\gamma} \left\{ \frac{1}{1-pz} - \gamma \frac{1}{1-p\gamma z} \right\} = \\ &= \frac{(1-p)(1-p\gamma)}{1-\gamma} \sum_{n=0}^{\infty} (1-\gamma^{n+1}) p^n z^n, \end{aligned}$$

so for the corresponding distribution $(p_n)_{n \geq 0}$ we have

$$(3.63) \quad p_n = (1-p)(1-p\gamma)c_n(\gamma)p^n, \quad n = 0, 1, 2, \dots$$

It follows that

$$\begin{aligned} R_{\alpha}(z) &= \frac{1}{1-\alpha} \frac{1}{z} \left\{ 1 - \frac{P(\alpha z)}{P(z)} \right\} = \frac{1}{1-\alpha} \frac{1}{z} \left\{ 1 - \frac{(1-pz)(1-p\gamma z)}{(1-p\alpha z)(1-p\alpha\gamma z)} \right\} = \\ &= \frac{1}{1-\alpha} \frac{1}{z} \left\{ 1 - (1-pz)(1-p\gamma z) \sum_{n=0}^{\infty} c_n(\gamma) p^n (\alpha z)^n \right\} = \\ &= (1+\gamma)p + \frac{1}{1-\alpha} \sum_{n=1}^{\infty} [(1+\gamma)c_n(\gamma)\alpha - c_{n+1}(\gamma)\alpha^2 + \\ &\quad + \gamma c_{n-1}(\gamma)] \alpha^{n-1} p^{n+1} z^n, \end{aligned}$$

so

$$(3.64) \quad \begin{aligned} r_0 &= (1 + \gamma)p, \\ r_n(\alpha) &= [c_{n+1}(\gamma)\alpha - \gamma c_{n-1}(\gamma)]p^{n+1}\alpha^{n-1}, \quad n \geq 1. \end{aligned}$$

Now, for the zero $\alpha_n := \gamma \frac{c_{n-1}(\gamma)}{c_{n+1}(\gamma)}$ of $r_n(\alpha)$, we have that α_n is nondecreasing in n and $\lim_{n \rightarrow \infty} \alpha_n = \gamma$. Hence

$$r_n(\alpha) \geq 0 \text{ for all } n \Leftrightarrow \alpha \geq \gamma,$$

from which, by definition, we obtain:

$$(3.65) \quad \frac{1-p}{1-pz} \cdot \frac{1-p\gamma}{1-p\gamma z} \in C_\alpha \Leftrightarrow \alpha \geq \gamma.$$

Note that for $\gamma = 1$ we get

$$(3.66) \quad \left(\frac{1-p}{1-pz}\right)^2 \notin C_\alpha \text{ for all } \alpha < 1.$$

By similar methods we can even prove that for all $\epsilon > 0$:

$$(3.67) \quad \left(\frac{1-p}{1-pz}\right)^{1+\epsilon} \notin C_\alpha \text{ for all } \alpha < 1,$$

from which we see, that theorem 3.24 does not hold for $\gamma > 1$.

The case $m = 3$ is already much more difficult. As in the case $m = 2$ we can prove that

$$(3.68) \quad p_n = (1-p)(1-p\gamma_1)(1-p\gamma_1\gamma_2)d_n(\gamma_1, \gamma_2)p^n, \quad n = 0, 1, 2, \dots,$$

with

$$d_n(\gamma_1, \gamma_2) := \sum_{k=0}^n \gamma_1^k c_k(\gamma_2) = \frac{1}{1-\gamma_2} \{c_n(\gamma_1) - \gamma_2 c_n(\gamma_1\gamma_2)\}.$$

For R_α we obtain

$$\begin{aligned} R_\alpha(z) &= \frac{1}{1-\alpha} \frac{1}{z} \{1 - (1-pz)(1-p\gamma_1 z)(1-p\gamma_1\gamma_2 z) \sum_{n=0}^{\infty} d_n(p\alpha z)^n\} = \\ &= d_1 p + \frac{1}{1-\alpha} \{[-d_2 \alpha^2 + d_1^2 \alpha - \gamma_1(1+\gamma_2+\gamma_1\gamma_2)]p^2 z + \\ &+ \sum_{n=2}^{\infty} [-d_{n+1} \alpha^3 + d_1 d_n \alpha^2 - \gamma_1(1+\gamma_2+\gamma_1\gamma_2)d_{n-1} \alpha + \\ &+ \gamma_1^2 \gamma_2 d_{n-2}]p^{n+1} \alpha^{n-2} z^n\}, \end{aligned}$$

so

$$\begin{aligned}
 r_0 &= d_1 p = (1 + \gamma_1 + \gamma_1 \gamma_2) p, \\
 (3.69) \quad r_1(\alpha) &= [d_2 \alpha - \gamma_1 (1 + \gamma_2 + \gamma_1 \gamma_2)] p^2, \\
 r_n(\alpha) &= [d_{n+1} \alpha^2 - (d_1 d_n - d_{n+1}) \alpha + \gamma_1^2 \gamma_2 d_{n-2}] p^{n+1} \alpha^{n-2}, \quad n \geq 2.
 \end{aligned}$$

From $\lim_{n \rightarrow \infty} d_n = (1 - \gamma_1)^{-1} (1 - \gamma_1 \gamma_2)^{-1}$, for the two nonvanishing zero's $\alpha_n^{(1)} \leq \alpha_n^{(2)}$ of $r_n(\alpha)$ it follows that

$$(3.70) \quad \lim_{n \rightarrow \infty} \alpha_n^{(1)} = \gamma_1 \gamma_2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha_n^{(2)} = \gamma_1.$$

Furthermore, we can prove that

$$\alpha_n^{(2)} \leq \gamma_1 \quad \text{for all } n \Leftrightarrow \gamma_1 \geq \gamma_2,$$

and

$$\gamma_1 \leq \gamma_2 \Rightarrow \alpha_n^{(2)} \leq \gamma_2 \quad \text{for all } n.$$

So, if $\gamma_1 > \gamma_2$, then $P \in C_{\gamma_1}$, while, on account of (3.70) there is no $\alpha < \gamma_1$ with $P \in C_\alpha$. However, if $\gamma_1 < \gamma_2$, then $P \in C_{\gamma_2}$, but now there may be an $\alpha < \gamma_2$ with $P \in C_\alpha$ (namely $\alpha := \max_n \alpha_n^{(2)}$). Anyhow we have

$$(3.71) \quad \frac{1-p}{1-pz} \cdot \frac{1-p\gamma_1}{1-p\gamma_1 z} \cdot \frac{1-p\gamma_1\gamma_2}{1-p\gamma_1\gamma_2 z} \in C_{\max(\gamma_1, \gamma_2)}.$$

4. Subdivision of C_1 by the compound-negative-binomial distributions

In section 3 we obtained a subdivision of C_1 by considering the nonnegativity of the recursively defined $r_n(\alpha)$, from which it followed that a $P \in C_\alpha$ could be represented by (3.9). In this section we start from a representation for P : we consider the classes $\mathcal{L}_u$ of the compound-negative-binomial lattice distributions, with parameter $u \geq 0$. As the negative-binomial distribution with parameters $p \in [0, 1)$ and $u \geq 0$:

$$(4.1) \quad p_n = \binom{-u}{n} (1-p)^u (-p)^n = \binom{n+u-1}{n} (1-p)^u p^n, \quad n = 0, 1, 2, \dots$$

has pgf P equal to

$$P(z) = \left(\frac{1-p}{1-pz} \right)^u ,$$

we have

$$(4.2) \quad \mathcal{L}_u = \{ \text{pgf } P \mid \exists_{p \in [0,1)} \exists_{\text{pgf } Q} P(z) = \left(\frac{1-p}{1-pQ(z)} \right)^u \} ,$$

or briefly

$$(4.3) \quad \mathcal{L}_u = \{ \text{pgf } P \mid P^{1/u} \in C_0 \} .$$

We immediately see, that $\mathcal{L}_0$ only consists of the distribution concentrated in 0, and that $\mathcal{L}_1 = C_0$, the class of all compound-geometric distributions. Furthermore, if $u \leq 1$ and $P \in \mathcal{L}_u$, then $P^{1/u} \in C_0$, so that, according to theorem 3.24, $P = (P^{1/u})^u \in C_0$. Hence $\mathcal{L}_u \subset C_0$ for $u \leq 1$. However, we can generalize this:

Theorem 4.1. For all u and $v \geq 0$ we have the following implication

$$(4.4) \quad u \leq v \Rightarrow \mathcal{L}_u \subset \mathcal{L}_v .$$

Proof. Let $u \leq v$ and $P \in \mathcal{L}_u$. Then $P^{1/u} \in C_0$, so that, according to theorem 3.24, $P^{1/v} = (P^{1/u})^{u/v} \in C_0$. Hence $P \in \mathcal{L}_v$. $\square$

We now have subclasses $\mathcal{L}_u$ of C_1 , increasing with u . They even are dense in C_1 (cf. theorem 3.25).

Theorem 4.2. If $P \in C_1$, then there are pgf's $P_u \in \mathcal{L}_u$, such that

$$P(z) = \lim_{u \rightarrow \infty} P_u(z), \quad \text{for } |z| \leq 1 .$$

Proof. Let $P \in C_1$. Then there is a $\lambda > 0$ and a pgf Q such that

$$P(z) = \exp[\lambda(Q(z) - 1)] .$$

Define for $u > \lambda$

$$(4.5) \quad P_u(z) := \left(\frac{1 - \lambda/u}{1 - \lambda/u \cdot Q(z)} \right)^u ,$$

then $P_u \in \mathcal{L}_u$ and

$$\lim_{u \rightarrow \infty} P_u(z) = \lim_{u \rightarrow \infty} \left\{ 1 + \frac{\lambda[Q(z) - 1]}{u - \lambda Q(z)} \right\}^u = \exp[\lambda(Q(z) - 1)] . \quad \square$$

In the following two theorems we give characterizations of $\mathcal{L}_u$. First a lemma.

Lemma 4.3. If $P \in C_0$, then there is a $p \in [0,1)$ and a pgf Q , with $Q(0) = 0$, such that

$$P(z) = \frac{1-p}{1-pQ(z)}.$$

Proof. Let $P \in C_0$. Then there is a $p' \in [0,1)$ and a pgf Q_1 such that $P(z) = \frac{1-p'}{1-p'Q_1(z)}$. Now, if we define

$$p := \frac{p'(1-Q_1(0))}{1-p'Q_1(0)} \quad \text{and} \quad Q(z) := \frac{Q_1(z)-Q_1(0)}{1-Q_1(0)},$$

then $p \in [0,1)$ and Q is a pgf with $Q(0) = 0$, while $P(z) = \frac{1-p}{1-pQ(z)}$. $\square$

Theorem 4.4. For every $u > 0$ we have

$$(4.6) \quad \mathcal{L}_u = \{\text{pgf } P \mid 1 - (\frac{p_0}{P(z)})^{1/u} \text{ abs mon}\}.$$

Proof. Let $P \in \mathcal{L}_u$. Then, on account of lemma 4.3, there is a $p \in [0,1)$ and a pgf Q , with $Q(0) = 0$, such that

$$P(z) = \left(\frac{1-p}{1-pQ(z)}\right)^u.$$

As $p_0 = (1-p)^u$, hence $1-p = p_0^{1/u}$, we have

$$pQ(z) = 1 - (1-p)P(z)^{-1/u} = 1 - \left(\frac{p_0}{P(z)}\right)^{1/u},$$

which is abs mon, as Q is a pgf.

Conversely, let $S_u(z) := 1 - (\frac{p_0}{P(z)})^{1/u}$ be abs mon. If we define

$p := S_u(1) = 1 - p_0^{1/u}$ and $Q(z) := \frac{1}{p} S_u(z)$, then

$$P(z)^{1/u} = \frac{p_0^{1/u}}{1-S_u(z)} = \frac{1-p}{1-pQ(z)} \in C_0.$$

Hence $P \in \mathcal{L}_u$. $\square$

Theorem 4.5. For every $u > 0$ we have

$$(4.7) \quad \mathcal{L}_u = \{ \text{pgf } P \mid \frac{P'(z)}{P(z)^{1+1/u}} \text{ abs mon} \} .$$

Proof. As $1 - (\frac{P_0}{P(z)})^{1/u}$ vanishes for $z = 0$ and

$$\frac{\partial}{\partial z} [1 - (\frac{P_0}{P(z)})^{1/u}] = -P_0^{1/u} \frac{\partial}{\partial z} [P(z)^{-1/u}] = \frac{1}{u} P_0^{1/u} P(z)^{-1-1/u} P'(z),$$

(4.7) immediately follows from (4.6). $\square$

Except for the fact that $\mathcal{L}_1 = \mathcal{C}_0$ we can say little about the relation between the $\mathcal{L}_u$'s and the $\mathcal{C}_\alpha$'s. We only remark, that for every $u > 1$: $(\frac{1-p}{1-pz})^u \in \mathcal{L}_u$, but $\notin \mathcal{C}_\alpha$ for all $\alpha < 1$ (cf. (3.67)). However, we have the same sort of characterizations of $\mathcal{C}_\alpha$ and $\mathcal{L}_u$, at which the functions $\frac{P(z)}{P(\alpha z)}$ for $\mathcal{C}_\alpha$, and $P(z)^{1/u}$ for $\mathcal{L}_u$, play analogous roles. Compare therefore:

$$i) \quad \mathcal{C}_\alpha = \{ \text{pgf } P \mid \frac{P(\alpha)P(z)}{P(\alpha z)} \in \mathcal{C}_0 \}$$

$$\mathcal{L}_u = \{ \text{pgf } P \mid P^{1/u} \in \mathcal{C}_0 \}$$

$$ii) \quad \mathcal{C}_\alpha = \{ \text{pgf } P \mid 1 - \frac{P(\alpha z)}{P(z)} \text{ abs mon} \}$$

$$\mathcal{L}_u = \{ \text{pgf } P \mid 1 - (\frac{P_0}{P(z)})^{1/u} \text{ abs mon} \}$$

$$iii) \quad \mathcal{C}_\alpha = \{ \text{pgf } P \mid \frac{P(\alpha z)}{P(z)} [R_1(z) - \alpha R_1(\alpha z)] \text{ abs mon} \}$$

$$\mathcal{L}_u = \{ \text{pgf } P \mid P(z)^{-1/u} R_1(z) \text{ abs mon} \} .$$

Finally, we formulate some properties of $\mathcal{L}_u$, which are very similar to those of $\mathcal{C}_\alpha$.

Theorem 4.6. For all $u \geq 0$ we have

i) A pgf P , which is the limit of a sequence pgf's $P_n \in \mathcal{L}_u$, is in $\mathcal{L}_u$.

ii) If $P \in \mathcal{L}_u$, then $\frac{P(\gamma z)}{P(\gamma)} \in \mathcal{L}_u$ for all $\gamma \in [0, 1]$.

iii) If $P \in \mathcal{L}_u$, then $\frac{P(\gamma)P(z)}{P(\gamma z)} \in \mathcal{L}_u$ for all $\gamma \in [0, 1]$.

iv) If $P \in \mathcal{L}_u$, then $P(z)^\gamma \in \mathcal{L}_{\gamma u}$ for all $\gamma \geq 0$.

Proof.

- i) Let $P_n \in \mathcal{L}_u$. Then $P_n^{1/u} \in C_0$, so that, according to theorem 3.20, for $P = \lim_{n \rightarrow \infty} P_n$ we have

$$P^{1/u} = (\lim_{n \rightarrow \infty} P_n)^{1/u} = \lim_{n \rightarrow \infty} P_n^{1/u} \in C_0.$$

Hence $P \in \mathcal{L}_u$.

- ii) Let $P \in \mathcal{L}_u$. Then $P^{1/u} \in C_0$, so that, on account of (3.42), for all $\gamma \in [0, 1]$ we have

$$\left(\frac{P(\gamma z)}{P(\gamma)} \right)^{1/u} = \frac{P(\gamma z)^{1/u}}{P(\gamma)^{1/u}} \in C_0.$$

Hence $\frac{P(\gamma z)}{P(\gamma)} \in \mathcal{L}_u$ for all $\gamma \in [0, 1]$.

- iii) Let $P \in \mathcal{L}_u$. Then $P^{1/u} \in C_0$, so that, according to theorem 3.21, for all $\gamma \in [0, 1]$ we have

$$\left(\frac{P(\gamma)P(z)}{P(\gamma z)} \right)^{1/u} = \frac{P(\gamma)^{1/u}P(z)^{1/u}}{P(\gamma z)^{1/u}} \in C_0.$$

Hence $\frac{P(\gamma)P(z)}{P(\gamma z)} \in \mathcal{L}_u$ for all $\gamma \in [0, 1]$.

- iv) Let $P \in \mathcal{L}_u$. Then $P^{1/u} \in C_0$, so that for all $\gamma > 0$

$$(P(z)^\gamma)^{1/\gamma u} = P(z)^{1/u} \in C_0.$$

Hence $P(z)^\gamma \in \mathcal{L}_{\gamma u}$ for all $\gamma \geq 0$. □

Remark. If we define for a pgf P and $u > 0$

$$S_u(z) := 1 - \left(\frac{P_0}{P(z)} \right)^{1/u},$$

then, using characterization (4.6), we can also prove theorem 4.6 from the following relations

$$i) \quad S_u(z) = 1 - \left(\frac{P_0}{P(z)} \right)^{1/u} = 1 - \left(\lim_{n \rightarrow \infty} \frac{P_n(0)}{P_n(z)} \right)^{1/u} = \lim_{n \rightarrow \infty} S_u^{(n)}(z).$$

$$ii) \quad S_u^{(\gamma)}(z) = 1 - \left(\frac{P_0}{P(\gamma z)} \right)^{1/u} = S_u(\gamma z).$$

$$\begin{aligned} \text{iii)} \quad S_u^{(\gamma)}(z) &= 1 - \left(\frac{P(\gamma z)}{P(z)}\right)^{1/u} = \left(\frac{P(\gamma z)}{P_0}\right)^{1/u} \left\{ \left(\frac{P_0}{P(\gamma z)}\right)^{1/u} - \left(\frac{P_0}{P(z)}\right)^{1/u} \right\} = \\ &= \left(\frac{P(\gamma)}{P_0}\right)^{1/u} \left(\frac{P(\gamma z)}{P(\gamma)}\right)^{1/u} \{S_u(z) - S_u(\gamma z)\} . \end{aligned}$$

$$\text{iv)} \quad S_{\gamma u}^{(\gamma)}(z) = 1 - \left(\frac{P_0^\gamma}{P(z)^\gamma}\right)^{1/\gamma u} = 1 - \left(\frac{P_0}{P(z)}\right)^{1/u} = S_u(z) .$$

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