Controllability and observability of 2-D systems

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Controllability and Observability of
2-D Systems
by
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Eindhoven, April 1978
The Netherlands
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Abstract

In this paper a necessary and sufficient condition for modal controllability (modal observability) of a 2-D system, as defined in [4] is obtained in terms of controllability (observability) of a system as is derived in [1]. Furthermore it is shown that modal controllability (modal observability) is a generic property.
1. Introduction

The state-space model for a 2-D system will be the model as has been proposed by Roesser in [5]

\[
\begin{bmatrix}
    x_{h+1,k} \\
a_{h+1,k}
\end{bmatrix} = \begin{bmatrix}
    A_1 & A_2 \\
    A_3 & A_4
\end{bmatrix} \begin{bmatrix}
    x_{h,k} \\
a_{h,k}
\end{bmatrix} + \begin{bmatrix}
    B_1 \\
    B_2
\end{bmatrix} u_{h,k};
\]

\[
y_{h,k} = \begin{bmatrix}
    C_1 & C_2
\end{bmatrix} \begin{bmatrix}
    x_{h,k} \\
a_{h,k}
\end{bmatrix}
\]

where \( x_{h,k} \in \mathbb{R}^n, a_{h,k} \in \mathbb{R}^m, u_{h,k} \in \mathbb{R}^p, y_{h,k} \in \mathbb{R}^r \).

The matrices have appropriate dimensions. We will take the usual D-matrix to be zero.

In [4] some state-space models for 2-D systems are compared and the authors argue in favor of Roesser's model. For this model the concept of modal controllability (modal observability) is defined in terms of left (right) coprimeness of two-variable polynomial matrices in the following way.

1.2. Definition. The system (1.1) is modally controllable iff

\[
\begin{bmatrix}
    zI - A_1 & -A_2 \\
    -A_3 & zI - A_4
\end{bmatrix} \text{ and } \begin{bmatrix}
    B_1 \\
    B_2
\end{bmatrix}
\]

are left coprime with respect to \( \mathbb{C}[s,z] \).

The system (1.1) is modally observable iff

\[
\begin{bmatrix}
    C_1 & C_2
\end{bmatrix} \text{ and } \begin{bmatrix}
    zI - A_1 & -A_2 \\
    -A_3 & zI - A_4
\end{bmatrix}
\]

are right coprime with respect to \( \mathbb{C}[s,z] \).

For the definition of coprimeness see [4].

In [3] the following theorem is obtained.

1.3. Theorem. The matrices

\[
\begin{bmatrix}
    zI - A_1 & -A_2 \\
    -A_3 & zI - A_4
\end{bmatrix} \text{ and } \begin{bmatrix}
    B_1 \\
    B_2
\end{bmatrix}
\]

are left coprime w.r.t. \( \mathbb{C}[s,z] \) iff they are left coprime w.r.t. \( \mathbb{C}(s)[z] \) and \( \mathbb{C}(z)[s] \).
It can easily be seen that in the case of real matrices we may take \( \mathbb{R}(s)[z] \) and \( \mathbb{R}(z)[s] \) in stead of \( \mathbb{C}(s)[z] \) and \( \mathbb{C}(z)[s] \) respectively. An analogous result can be obtained for modal observability.

In the following necessary and sufficient conditions for modal controllability (modal observability) are obtained in terms of rational matrices. Also a very simple sufficient condition will be derived.

2. Some definitions and concepts

\( \mathbb{R}[s] \) denotes the set of polynomials in the variable \( s \) with real coefficients.
\( \mathbb{R}[s,z] \) denotes the set of polynomials in the variables \( s \) and \( z \) with real coefficients.

\( \mathbb{R}^{xp}[s,z] \) denotes the set of \( r \times p \) matrices with entries in \( \mathbb{R}[s,z] \).
\( \mathbb{R}(s) \) denotes the set of real rational functions in \( s \).
\( \mathbb{R}(s)[z] \) denotes the set of polynomials in \( z \) with coefficients in \( \mathbb{R}(s) \).
\( \mathbb{R}^{xp}(s) \) denotes the set of \( r \times p \) matrices with entries in \( \mathbb{R}(s) \).
\( \mathbb{R}(s,z) \) denotes the set of real rational functions on \( s \) and \( z \).
\( \mathbb{R}^{xp}(s,z) \) denotes the set of \( r \times p \) matrices with entries in \( \mathbb{R}(s,z) \).

The elements of \( \mathbb{R}[s,z] \) can also be considered as polynomials in \( z \) with coefficients in \( \mathbb{R}[s] \), thus \( \mathbb{R}[s,z] = \mathbb{R}[s][z] \).

Analogously, \( \mathbb{R}^{xp}[s,z] = \mathbb{R}[s]^{xp}[z] \).

A polynomial \( q \in \mathbb{R}[s,z] \) seen as an element of \( \mathbb{R}[s][z] \) will be notated as \( \bar{q} \).

Analogously for \( P \) and \( \bar{P} \) where \( P \in \mathbb{R}^{xp}[s,z] \) and \( \bar{P} \in \mathbb{R}[s]^{xp}[z] \). Let \( T \in \mathbb{R}^{xp}(s,z) \), \( T \) can be written in the form \( P/q = \bar{P}/\bar{Q} \) where \( P, \bar{P}, q, \bar{Q} \) are as above. We will also use some of the above sets where \( \mathbb{R} \) is replaced by \( \mathbb{C} \) (the complex numbers).

2.1. Definition. An element \( T \in \mathbb{R}^{xp}(s,z) \) will be called a 2-D transfer matrix:
\( T \) is called a 2-D transfer function if \( r = p = 1 \).

2.2. Definition. \( T = \bar{P}/\bar{Q} \in \mathbb{R}^{xp}(s,z) \) is called proper if the degree in \( z \) of \( \bar{q}(z) \) is not less than the degree in \( z \) of \( \bar{P}(z) \).

A proper transfer matrix \( T \) is called causal if the degree in \( s \) of the coefficient of the highest power in \( z \) of \( \bar{q}(z) \) is not less than the degree in \( s \) of all other coefficients of \( \bar{q}(z) \) and the entries of \( \bar{P}(z) \).

Remark. If \( z \) is replaced by \( s \) in definition (2.2) and interchanged also in the next, a completely parallel theory can be obtained.
3. The results

In [1] it is shown that Roessers model can be obtained by realizing the so-called first level realization \((A(s), B(s), C(s), D(s))\) of a causal 2-D transfer matrix \(T(s, z)\). Here we have \(A(s) \in \mathbb{R}^{n \times n}(s)\), \(B(s) \in \mathbb{R}^{n \times p}(s)\), \(C(s) \in \mathbb{R}^{r \times n}(s)\), \(D(s) \in \mathbb{R}^{r \times p}(s)\). In fact these matrices are proper rational matrices themselves. For an interpretation of the first level realization see [1]. The above matrices play the role of the usual \(A, B, C, D\)-matrices in 1-D system theory.

Now suppose that \(T(s, z)\) is a causal 2-D transfer matrix and that (1.1) is a state space realization. Then we have:

\[
T(s, z) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} zI - A_1 & -A_2 \\ -A_3 & sI - A_4 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
\]

and if \((A(s), B(s), C(s), D(s))\) is a first level realization then we have:

\[
T(s, z) = C(s) [zI - A(s)]^{-1} B(s) + D(s).
\]

A possible first level realization is (see [1]):

\[
\begin{align*}
A(s) &= A_1 + A_2 [sI - A_4]^{-1} A_3, \\
B(s) &= B_1 + A_2 [sI - A_4]^{-1} B_2, \\
C(s) &= C_1 + C_2 [sI - A_4]^{-1} A_3, \\
D(s) &= C_2 [sI - A_4]^{-1} B_2.
\end{align*}
\]

By interchanging the role of \(s\) and \(z\) we can construct another first level realization. A possible first level realization is then:

\[
\begin{align*}
\bar{A}(z) &= A_4 + A_3 [zI - A_1]^{-1} A_2, \\
\bar{B}(z) &= B_2 + A_3 [zI - A_1]^{-1} B_1, \\
\bar{C}(z) &= C_2 + C_1 [zI - A_1]^{-1} A_2, \\
\bar{D}(z) &= C_1 [zI - A_1]^{-1} B_1.
\end{align*}
\]
Now suppose $A(s) \in \mathbb{R}^{n \times n}(s)$, $B(s) \in \mathbb{R}^{n \times p}(s)$.

3.1. Definition. $(A(s), B(s))$ is said to be controllable w.r.t. $\mathbb{R}(s)$ iff

$$\text{rank}[B(s)|A(s)B(s)|\ldots|A(s)^{n-1}B(s)|] = n$$

where the rank is considered over the field $\mathbb{R}(s)$.

Remark. It can easily be seen that the controllability condition is satisfied if the rank condition holds for some complex number $s$. (see [2]).

We can now state:

3.2. Theorem. The system (1.1) is modally controllable iff

1) $(A_1 + A_2[sI - A_4]^{-1}A_3, B_1 + A_2[sI - A_4]^{-1}B_2)$ is controllable w.r.t. $\mathbb{R}(s)$ and

2) $(A_4 + A_3[zI - A_1]^{-1}A_2, B_2 + A_3[zI - A_1]^{-1}B_4)$ is controllable w.r.t. $\mathbb{R}(z)$.

Proof. Consider the matrices

$$\begin{bmatrix} zI - A_1 & -A_2 & B_1 \\ -A_3 & sI - A_4 & B_2 \end{bmatrix} = \begin{bmatrix} I & -A_2 \\ 0 & sI - A_4 \end{bmatrix} \begin{bmatrix} zI - A(s) & 0 & B(s) \\ -[sI - A_4]^{-1}A_3 & I & [sI - A_4]^{-1}B_2 \end{bmatrix}$$

where $A(s) = A_1 + A_2[sI - A_4]^{-1}A_3$, $B(s) = B_1 + A_2[sI - A_4]^{-1}B_2$.

If $zI - A(s)$ and $B(s)$ have a nonunimodular left common factor then it is clear by the above factorization that also

$$\begin{bmatrix} zI - A_1 & -A_2 \\ -A_2 & sI - A_4 \end{bmatrix} \text{ and } \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

are not left coprime w.r.t. $\mathbb{C}(s)[z]$ and therefore (1.1) is not modally controllable. Failure of 2) gives the same result in an analogous way and therefore necessity has been proven. 1) together with 2) is also sufficient.

Suppose that 1) is true. Then $[zI - A(s)|B(s)]$ is right invertible. Therefore there exist matrices $L$ and $Q$ with entries in $\mathbb{R}(s)[z]$ such that:

$$[zI - A(s)]L + B(s)Q = I$$
Now it is straightforward to verify that

\[
\begin{bmatrix}
Z-I-A_1 & -A_2 \\
-A_3 & S-I-A_4
\end{bmatrix}
\begin{bmatrix}
L & I_A_2[S-I-A_4]^{-1} \\
\end{bmatrix}
\]

This implies left coprimeness of \(Z-I-A_1-A_3\) w.r.t. \((s)\) and in the same way \(2)\) implies left coprimeness w.r.t. \((z)\) and therefore modal controllability of (2.1) has been proven.

3.3. Theorem. If \((A_1,B_1)\) is a controllable pair then

\[
(A(s),B(s)) = (A_1 + A_2[S-I-A_4]^{-1}A_3, B_1 + A_2[S-I-A_4]^{-1}B_2)
\]

is a controllable pair w.r.t. \(\mathcal{R}(s)\).

Proof. From the right invertibility of \([B_1|A_1B_1|...|A_1^{-1}B_1]\) follows the right invertibility of \([B(s)|A(s)B(s)|...|A(s)^{-1}B(s)]\).

3.4. Theorem. Modal controllability is a generic property.

Proof. Suppose (1.1) is not modally controllable then by (3.2) and (3.3) \((A_1,B_1)\) or \((A_4,B_2)\) is not a controllable pair, from which it follows that the set of points where modal controllability fails is contained in a Zariski closed set in the parameter space \(\mathbb{C}^{(n+m)}(n+m+p)\).

Remark. By duality all the results of this paper are also valid for modal observability.

For example the observability counterpart of (3.2) becomes:

3.5. Theorem. The system (1.1) is modally observable iff

1) \((C_1 + C_2[sI-A_4]^{-1}A_3, A_1 + A_2[sI-A_4]^{-1}A_2)\) is observable w.r.t. \(\mathcal{R}(s)\) and
2) \((C_2 + C_1[zI-A_1]A_2, A_4 + A_3[zI-A_1]^{-1}A_2)\) is observable w.r.t. \(\mathcal{R}(z)\).

Here "observability w.r.t. \(\mathcal{R}(s) (\mathcal{R}(z))" is as usual defined as dual to "controllability w.r.t. \(\mathcal{R}(s) (\mathcal{R}(z))"."
4. Conclusions

In this paper an equivalent characterization of modal controllability has been obtained in terms of a so-called first level realization. It has been shown that modal controllability is a generic property. Also a very simple sufficient condition for modal controllability of (1.1) has been obtained, namely "(A₁, B₁) and (A₄, B₂) both are a controllable pair". By duality analogous results hold for modal observability.
5. References

COSOR Memorandum 77-16.

[2] F. Eising: "Realization of NSHP-Filters".
COSOR Memorandum 78-04.

