Incorporating Prior Knowledge in Local Parametric Modeling for Frequency Response Measurements: Applied to Thermal/Mechanical Systems

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Abstract—Frequency response function (FRF) identification is a key step in experimental modeling of many applications, including mechatronic systems. Applying these techniques to systems where measurement time is limited leads to a situation where the accuracy of the identified model is deteriorated by transient dynamics. This article aims to develop an identification procedure that mitigates these transient dynamics by employing local parametric modeling techniques. To improve the modeling accuracy, prior knowledge is suitably incorporated in the procedure while at the same time allowing for rational parameterizations that maintain a closed-form solution. The prior knowledge is exploited in a relevant local frequency range using a specific Möbius transformation. Preexisting methods, including the commonly used local polynomial method, are recovered as a special case. The presented framework leads to accurate identification results in a simulation study as well as on experimental measurement data.

Index Terms—Frequency response function (FRF), precision mechatronics, system identification, thermomechanical.

I. INTRODUCTION

Frequency response function (FRF) identification is a key step in identification and control. Acquiring FRFs is often fast, inexpensive, and accurate and requires very limited user intervention. The obtained FRFs are used for many purposes, ranging from controller design by manual tuning [1], optimal synthesis [2], stability analysis, and interaction analysis [3] to parametric identification [4]. Also, these FRFs are used in many application domains, e.g., in mechanical systems with flexible dynamics [5], [6], thermal systems [7], electrical systems [8], and combustion systems [9].

Identification of FRFs has recently been substantially advanced by explicitly mitigating transient errors. Indeed, one of the tacit assumptions is that the system under test is in steady state, which is often not valid for experimental systems. Moreover, due to the slow dynamics in certain applications, e.g., thermal–mechanical systems, transients are increasingly relevant. In the local polynomial method (LPM) [10], the smoothness, in the frequency domain, of the transient response is exploited by locally approximating the transfer function by a polynomial function to estimate and remove the transient component. In [11], this is generalized toward the local rational method (LRM), which uses a rational function in the local approximation.

Although the general parameterization used in the LRM enables improved identification results, the LRM involves an optimization problem that introduces additional challenges. As a key advantage, the LRM is a more general parameterization, directly recovering the LPM as a special case, and the additional freedom in the parameterization allows to capture dynamics, especially lightly damped, more accurately (see [12] for a theoretical analysis of the local approximation error and [5] for experimental evidence). On the other hand, the rational parameterization leads to a nonconvex optimization problem, which is approximated in typical LRM approaches as in [13]. Further improvements to employ an iterative algorithm have mixed outcomes (see [5]). Furthermore, as a direct consequence, the variance results, which are valid for LPM, are only accurate for the LRM for sufficiently high signal-to-noise ratio (SNR) due to a bias effect.

This article aims to present a unified framework for FRF identification that employs a rational local parameterization, in conjunction with a closed-form optimizer to yield reliable variance expressions and improved estimation accuracy. This is achieved by exploiting prior knowledge on the system, characterized by pole locations in the complex plane, e.g., resonance frequency region in mechanical systems [5], or pole locations on the real axis [14], [15] for thermal systems.

In [16], VF is used to provide the local rational parameterization with estimated pole locations. The VF algorithm is adapted to work on transient input–output data and employs an iterative process to estimate the pole locations. In [17], a bootstrapped total least squares (LS) estimator is employed to obtain unbiased estimates for a local rational model. The approach in this article exploits the freedom of selecting those poles at a prescribed location. The presented approach is linear in the parameters, similar to the LPM, and therefore maintains the associated benefits, e.g., an analytical solution

and bias and variance expression. The approach utilizes orthonormal rational basis functions (OBFs), which are well studied [18]–[20] to form the basis for the local regression problem. Moreover, it is shown that it requires the development of OBFs with complex coefficients over the real line, which is in sharp contrast to the typical use of OBFs in system identification, where the support is typically over the imaginary axis or the unit disk. Indeed, local parametric methods (LPM/LRM) often tacitly represent FRFs of systems by complex-valued transfer functions that are evaluated over the real axis.

The main contributions of this article are given in the following.

\section*{C1:} A local parametric method is developed that uses a linearly parameterized basis, leading to an efficient optimization and closed-form solution, providing an accurate variance analysis yielding a reliable quality metric and exploiting prior knowledge.

\section*{C2:} Development of the necessary theory of OBFs using single complex pole parameterizations over the real line and transformation of prior knowledge to this domain.

\section*{C3:} Validation of the method by application on relevant systems, e.g., on resonant dynamics in a simulation study of a mechanical system and first-order dynamics in an experimental study on a high-tech industrial thermal setup.

This article is organized as follows. In Section II, FRF identification under transient conditions is investigated. In Section III, the local parametric method using the general parameterization considered is presented. In Section IV, a design for the local approximation basis is introduced together with a framework for orthonormal rational functions on the real line. A Möbius transformation is used to connect different components of the framework to appropriate prior knowledge and previous results. In Section V, the method is applied in a case study on relevant industrial applications, involving a simulation study involving lightly damped resonant dynamics and experiments on a high-tech industrial thermal setup. In Section VII, conclusions are presented.

\section*{II. PROBLEM FORMULATION}

\subsection*{A. Transients in FRF Identification}

In this section, the role of transients in FRF identification is investigated.

Consider a discrete-time linear time-invariant (LTI) single-input–single-output (SISO) system $G$ in an open-loop setting, as shown in Fig. 1, and an excitation input $u(n)$. Then, the output $y(n)$, for an infinite time interval $n \in (-\infty, \infty)$, is given by

$$y(n) = \sum_{k=-\infty}^{\infty} u(k)g(n-k) + v(n)$$

(1)

where $g(n)$ is the impulse response of $G$ and $v(n)$ is the noise contribution on the output. By applying the discrete Fourier transform (DFT)

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

(2)

on a finite interval, i.e., $n \in [0, 1, \ldots, N-1]$ of (1), it yields

$$Y(k) = G(\Omega_k)U(k) + T(\Omega_k) + V(k)$$

(3)

in the frequency domain, where $G(\Omega_k)$ is the FRF of the dynamic system. Also, $Y(k)$, $U(k)$, and $V(k)$ are the DFT of $y(n)$, $u(n)$, and $v(n)$, respectively. Here, $\Omega_k$ denotes a generalized frequency unit, e.g., in continuous time $\Omega_k = j\omega$ and in discrete time $\Omega_k = e^{j\omega T_s}$, where $T_s$ is the sample time and $k$ denotes the $k$th frequency bin, and the latter is used throughout.

In (3), $T(\Omega_k)$ accounts for the transients for both the system response $T_G(\Omega_k)$ and the noise $T_V(\Omega_k)$. The transient terms are directly related to the initial and final conditions of the system, i.e., the transition of an infinite to a finite interval such that the relation (3) is an exact representation. Also, the estimation error by neglecting the term $T(\Omega_k)$ is often referred to as leakage error [4]. An insightful interpretation (see also [11]) of $T(\Omega_k)$ can be made by using a state-space realization of the system $G$ in Fig. 1, e.g.,

$$x(n+1) = Ax(n) + Bu(n)$$

$$y(n) = Cx(n) + Du(n) + v(n)$$

(4)

where $x(n) \in \mathbb{R}^{n_x}$ is the state vector with $n_x$ the number of states. By then applying the DFT to (4), this directly connects to (3) through

$$G(\Omega_k) = C(e^{j\omega T_s} - I)^{-1}B + D$$

(5)

$$T(\Omega_k) = C(e^{j\omega T_s} - I)^{-1}(x(0) - x(N))e^{j\omega k}$$

(6)

where $G(\Omega_k)$ is the FRF of the LTI system and $T(\Omega_k)$ is the transient contribution in the frequency domain. This shows that $T(\Omega_k)$ and $G(\Omega_k)$ exhibit similar but not identical dynamics since they share the same poles and it is assumed that no pole/zero cancellations are present. However, the zeros of their transfer functions can be different.

\subsection*{B. Classical Approach}

The finite-time response in (3) contains the additional terms $T_G(\Omega_k)$ and $V(k)$. A classical approach to identify the system $G(\Omega_k)$ is to use the empirical transfer function estimate (ETF), i.e.,

$$\hat{G}(k) = Y(k)U^{-1}(k).$$

(7)

Analysis reveals that

$$G(\Omega_k) - \hat{G}(k) = T(\Omega_k)(k)U^{-1}(k) + V(k)U^{-1}(k).$$

(8)

Hence, the transients, in addition to the noise contribution $V(k)$, lead to an estimation error.
C. Transients in Different Applications

The result (8) reveals that transients lead to estimation errors of the true system $G(\Omega_k)$. From (5) and (6), the system dynamics, i.e., pole locations, largely determine the transient contribution.

The central idea in this article is that the application domain provides clear prior knowledge on the locations of these poles. In Fig. 2, pole locations are shown for a mechanical system, in Fig. 2(a), and a thermal system, in Fig. 2(b). An ad hoc solution that is often employed to cope with transient data is to remove the initial transient response from the identification data. Electromechanical systems, e.g., servo-positioning systems, often have lightly damped dynamics and small time constants. For these systems, the transient contribution in (3) can sometimes be substantial due to low damping. For other systems, e.g., thermomechanical systems [21], the time constants are significantly larger and the transient contributions constitute a dominant part of the output response. Waiting for the initial transient to settle requires an unacceptable increase in experiment time. FRF identification for a more general class of systems, e.g., including thermomechanical systems, requires a method capable of coping with data obtained under transient conditions.

In this article, an approach that explicitly considers the transient term $T(\Omega_k)$ to obtain an unbiased estimate of $G(\Omega_k)$ is presented. This eliminates the need to discard the initial transient data, thus achieving significant savings in the necessary experiment time to accurately identify the system.

D. Problem Formulation

The problem considered in this article is as follows. Given an input–output sequence, $u$ and $y$ where $y$ contains a transient response, provide an accurate nonparametric FRF estimate $\hat{G}(\omega)$ of the true system $G_0(\omega)$ accompanied with a reliable variance estimation $\hat{\sigma}_{\hat{G}}(\omega)$.

To address this, a new parameterization is employed, which allows for the estimation of transient responses by solving an optimization problem that has a closed-form solution, thereby providing an accurate estimate and accompanying variance expressions. Moreover, this method facilitates the incorporation of appropriate prior knowledge, enabling increased estimation accuracy.

III. LOCAL PARAMETRIC MODELING

In this section, a local parametric approach is presented with a rational parameterization that allows convex optimization, in particular a closed-form solution, leading to contribution C1. The outline of the section is as follows. First, the concept of local parametric modeling is presented. Second, the linear parameterization used in this article is presented. Third, the presented approach is connected to previous results on local parametric modeling, including the LPM [4] and the LRM [11]. In fact, these results are recovered as special cases.

A. Local Modeling

Local parametric methods construct approximations of $G(e^{i\omega_k})$ and $T(e^{i\omega_k})$ in (3) on a subset of points

$$\lambda = \{k - l, \ldots, k + l\} \subset \mathbb{N}$$

i.e., a local window of width $2l + 1$ with $l \in \mathbb{N}$, in the complex plane. This is shown in Fig. 3 where the first- and second-order local approximations are shown.

Remark 1: Throughout this article, the notation $\{k - l, \ldots, k + l\}$ is dropped, and $k$ indicates a single DFT-bin and $\lambda$ indicates the local window, i.e., $\lambda = \{k - l, \ldots, k + l\} \subset \mathbb{R}$.

The key mechanism underlying local parametric methods is the smoothness of both $G(\Omega_k)$ and $T(\Omega_k)$ in (3).

In particular, the input $U(k)$ is selected such that it is sufficiently exciting and “wild,” i.e., having a nonsparse and nonsmooth frequency spectrum in magnitude and/or phase. Signals that are particularly suited are (filtered) white noise or random-phase multisines. This ensures that $G(\Omega_k)$ can be distinguished from $T(\Omega_k)$ since the latter is not affected by the system input $U(k)$. For each frequency point $k$, a local parametric approximation is constructed over neighboring points $\lambda$, e.g., by polynomial functions in the LPM and rational functions in the LRM. Then, an estimate for the transient component at $\Omega_k$ is constructed using this local approximation such that it can subsequently be removed from $G(\Omega_k)$. This is shown in Fig. 3 for $G(k)$. The result for the transient $T(k)$ follows along conceptually similar lines (for more details on the local parametric method in general, see [21], [22]). In the remainder of this article, the emphasis lies on constructing a suitable local parametric approximation basis for the functions $G(\Omega_k)$ and $T(\Omega_k)$.

Remark 2: The local modeling approach presented in this article is still classified as a nonparametric identification approach. The local parametric models are solely used as an intermediate step to obtain the FRM at a single frequency point $k$, after which they are discarded.

B. Linearly Parameterized Rational Parameterization

Consider again (3) and let

$$G(\Omega_k) = \sum_{b=0}^{N_0} \theta_G^b(\Omega_k) \Psi(b, \lambda)$$

$$T(\Omega_k) = \sum_{b=0}^{N_0} \theta_T^b(\Omega_k) \Psi(b, \lambda)$$
such that locally the terms in (3) are approximated in a window $\lambda$ by an expansion of degree $N_b$ using general basis functions $\Psi(b, \lambda) \in \mathbb{C}$, where $\theta^p_\lambda(b) \in \mathbb{C}$ and $\theta^q_\lambda(b) \in \mathbb{C}$ are the local coefficients for $G(\Omega_k)$ and $T(\Omega_k)$, respectively. Note that $\Psi = \lambda^b$ in (10) for the well-known LPM. For each point $\lambda$, this local approximation yields

$$\hat{Y}(\lambda) = \left[ \theta_G(k) \, \theta_T(k) \right] \left[ \Psi(\lambda) \otimes U(\lambda) \right] \left[ \Psi(\lambda) \right]^H$$

(12)

where $\Theta(k)$ contains the local approximation coefficients for the basis functions contained in the regression matrix $K(\lambda) = K(U(\lambda), \Psi(\lambda))$. From this local approximation, only the center value, i.e., $G(\Omega_k)$ is used, as shown in Fig. 3. This local approximation is repeated to obtain an estimate of $G(\Omega_k)$ and $T(\Omega_k)$ for each frequency bin $k$.

**Remark 3:** The input $U(p)$ is assumed to vary sufficiently over the full input spectrum such that $G(\Omega_k)U(\lambda)$ can be distinguished from $T(\Omega_k)$ so $K(\lambda)$ does not lose rank, i.e., the input is sufficiently “wild.” This can be achieved by, e.g., broadband noise excitation or random-phase multisines [4]. The coefficients $\Theta(k)$ are found by solving the LS problem

$$\hat{\Theta}(k) := \arg \min_{\Theta} \sum_{p=k-l}^{k+l} |Y(p) - \Theta(k)K(p)|^2$$

(13)

which should be overdetermined. Then, $\hat{\Theta}$ is given by

$$\hat{\Theta}(k) = Y(\lambda)K(\lambda)^H(K(\lambda)K(\lambda)^H)^{-1}$$

(14)

where $K(\lambda)^H$ is the Hermitian transpose of the regression matrix $K(\lambda)$ in (12). Due to the closed-form solution, which resembles the LPM [4, Sec. 7.2.2], an estimate for the noise covariance matrix is given by

$$\hat{C}_n(k) = \frac{1}{q}(Y(\lambda) - \hat{Y}(\lambda))(Y(\lambda) - \hat{Y}(\lambda))^H$$

(15)

where $q$ is the degree of freedom of the residual $Y(\lambda) - \hat{Y}(\lambda)$, i.e., $q = 2l + 1 - 2(N_b + 1)$. The final covariance on the estimated FRF $\text{Cov}(\text{vec}(\hat{G}(k)))$ is then determined by an appropriate transformation, e.g., [4, Ch. 7].

Indeed, the covariance of the estimated FRF is given by

$$\text{Cov}(\text{vec}(\hat{G}(k))) = \hat{S}(N_b, k)$$

(16)

and $\text{vec()}$ denotes a stacking of the columns of the matrix and $n_b$ is the number of inputs of the estimated system.

**Remark 4:** The key point is that (15) holds for a linearly parameterized model (10), which is in sharp contrast to the LRM (see Section III-C), in which case variance estimations are typically biased. This is caused by the Levy approximation that introduced measurement data in the regressor matrix. The result (15) and (16) holds for any linearly parameterized basis, and the results in [4] are recovered as a special case.

The basis $\Psi$ is general and allows for user-chosen parameterizations. For instance, the basis $\Psi$ can be chosen to be a polynomial, rational, or fractional function of the window parameter $\lambda$. In Section II-C, it is shown how earlier approaches fit in the framework. Then, a new approach is presented, which enables the incorporation of prior knowledge.

**C. Connection to LRM**

A particular choice regarding the parameterization in (10) is to select rational polynomials for $\Psi$, which directly connects to previously used rational functions in the LRM [11]. To introduce the LRM, consider the optimization problem

$$\hat{\Theta}_{\text{LM}} := \arg \min_{\Theta_{\text{LM}}} \sum_{p=k-n}^{k+n} |Y(p) - \frac{N_2}{D_2}U(p) - M_2|_2^2$$

(18)

where

$$N_2 = \sum_{s=0}^{N_m} n_s(k) \lambda^s$$

(19)

$$M_2 = \sum_{s=0}^{N_d} m_s(k) \lambda^s$$

(20)

$$D_2 = 1 + \sum_{s=1}^{N_d} d_s(k) \lambda^s$$

(21)

In case of the typical LRM parameterization, the free parameters are $\Theta_{\text{LM}} = [n_0, \ldots, n_{N_b}, m_0, \ldots, m_{N_b}, d_1, \ldots, d_{N_d}] \in \mathbb{C}^{N_b+N_m+N_d}$, where $N_b$, $N_m$, and $N_d$ denote the order of the plant and transient numerator and common denominator, respectively. From (18), the LPM is directly recovered by setting $[d_{N_b} \cdots d_1] = 0$. For general $D_{\omega}$, (18) is nonlinear in the parameters, and generally, no closed-form solution similar to (14) exists. At least two approaches [11], [12], [23], [24] have been pursued to determine (18) and find $\Theta_{\text{LM}}$: 1) iteratively solving the LS problem, which has been applied in [23] with mixed results or 2) multiplying the criterion with $D_{\omega}$ as in the classical approach in [13], which introduces an
functions are a result of Gram–Schmidt orthogonalization of the basis in (22).

**Theorem 1:** Given \( \beta_b \in \mathbb{C} \) where \( b \in [0, N_b] \), i.e., the poles in (22). Let \( \Psi(\omega) \) be parameterized as

\[
\Psi_b(\omega) = \begin{cases} \sqrt{-2\text{Im}(\beta_0)} e^{-i\beta_0} & b = 0 \\ \sqrt{-2\text{Im}(\beta_b)} e^{-i\beta_b} \prod_{j=0}^{b-1} \frac{\omega + \beta_j}{\omega - \beta_j} & b \neq 0 \end{cases}
\]

and then

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} \Psi_n(\omega) \overline{\Psi_m(\omega)} d\omega = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}
\]

where \( \omega \in \mathbb{R} \), i.e., the basis \( \Psi \) is orthonormal on the real line. A proof of Theorem 1 is provided in the Appendix. The basis in Theorem 1 is novel since the orthonormal basis functions are considered on the real line, opposed to on the imaginary line.

The orthogonality of the functions \( \Psi \) facilitates the repetition of poles to expand the basis, improving the approximation of the true FRF [25]. In fact, for a function \( f(\omega) \) in the Hardy space \( H_2 \) for arbitrary \( \epsilon > 0 \) and for sufficiently large \( m \), an element \( g(\omega) \in \left\{ \Psi_n(\omega) \right\}_{n=1}^{m} \) can be found such that \( ||f - g||_2 \leq \epsilon \), i.e., the basis \( \Psi(\omega) \) is complete in \( H_2 \). The basis \( \Psi \) is complete if the following theorem holds.

**Theorem 2:** The model set spanned by the basis functions \( \{\Psi_n(\omega)\}_{n=0}^{\infty} \) is complete in all of the spaces \( H_p, 1 < p < \infty \) if and only if

\[
\sum_{k=1}^{\infty} \frac{-\text{Im}(\beta_k)}{1 + |\beta_k|^2} = \infty.
\]

A proof for this is presented in the Appendix.

The condition (25) in Theorem 2 is mild and fulfilled by (23) if the poles \( \beta_k \) are not on the real line. Finally, the orthogonality of the basis potentially yields improved numerical conditioning for the approximation problem [26].

**Remark 5:** The basis in Theorem 1 is orthogonal when evaluated over the real line as in (24). In the local window \( \lambda \), this orthogonality can be lost since a discrete set of points is considered that no longer spans the complete real line. A solution for this could be found in line with [27] that considers a discrete data-dependent basis to improve conditioning. The conditioning [28] of (23) improves with the window size for \( \lim_{\lambda \to N} \), where \( N \) is the total amount of samples in a measurement.

**V. EMPLOYING PRIOR KNOWLEDGE: MOBIUS TRANSFORMATION**

In this section, it is shown how to incorporate prior knowledge from different domains using a specific Möbius transformation, constituting contribution C2. Moreover, this transformation is used to connect the presented framework to previous results from the literature.

A Möbius transformation is a conformal mapping defined on the extended complex plane \( \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \) of the form

\[
f(z) = \frac{az + b}{cz + d}
\]
where \( z \in \mathbb{C}_\infty \) and \( a, b, c, d \in \mathbb{C} \) satisfy \( ad - bc \neq 0 \).

A well-known special case of this transformation is the bilinear transform, i.e., Tustin’s method [29].

Remark 6: In previous local parametric modeling methods, discrete-time frequency-domain data, i.e., an FRF evaluated on points on the unit disk \( e^{j\omega} \), is approximated by local models that are a function of \( k \in \mathbb{N} \) on the real line, i.e., the DFT bins, as shown in Fig. 3. The transformation from \( s = j\omega \) or \( z = e^{j\omega} \) to the \( \omega \) domain is often implicit, but it is essential when including prior system knowledge, as is done in this article.

A. Prior Knowledge

In Section IV, a suitable basis \( \Psi \) for the local approximation (10) is presented, i.e., (23), where \( \beta_k \) is selected \( \text{a priori} \). However, prior knowledge, e.g., system poles, is often defined in terms of the Laplace variable \( s \) or the \( Z \) transform variable \( z \), where the frequency response is obtained by substituting \( s = j\omega \) or \( z = e^{j\omega} \), respectively. In contrast, the functions (22) for which the poles are defined as \( \beta \) are evaluated on the real line \( \omega \). Consequently, prior knowledge regarding the poles of the physical system \( G(\Omega_k) \) cannot directly be incorporated in the local function (22). The appropriate prescribed poles \( \beta \) for (23) are transformed using the forthcoming results from prior knowledge in the continuous domain \( \alpha \) or discrete domain \( \gamma \), by employing the M"obius transformations shown in Fig. 4.

a) Continuous Time \( \alpha \): Given some knowledge on the system poles in continuous time domain, i.e., \( \Omega_k = s \), an equivalent prior is obtained for the \( \omega \) domain by using \( \beta = -j\alpha \) such that their FRF is equal, e.g., \( \Psi(\beta)|_\omega \equiv G(\alpha)|_{j\omega} \).

b) Discrete Time \( \gamma \): In the discrete-time domain \( \Omega_k = z \), a similar approach is applied by mapping the pole locations using a bilinear transformation \( \beta = -j(2/\Delta)(z - 1/z + 1)\gamma \) that is known as a Cayley transformation. However, care has to be taken since, for \( \Psi(\beta)|_\omega \equiv G(\gamma)|_{e^{j\omega}} \), it holds that \( \omega \neq \hat{\omega}, \) i.e., the frequency \( \hat{\omega} \) is distorted with respect to the original frequency \( \omega \). If aliasing is particularly relevant, a future extension may be to use the impulse invariant transformation in favor of the bilinear transform.

Remark 7: To compensate for the frequency distortion when including discrete-time domain prior knowledge, several approaches can be used. First, the warping can be ignored as is commonly done in LPM techniques for discrete-time systems. This could be used if the prior knowledge is mainly located at lower frequencies, where \( \omega \approx \hat{\omega} \). Second, prewarping in the bilinear transformation can be employed for each local frequency window \( \omega \) such that locally the reconstruction is exact. Third, the frequency axis can be adjusted by using \( \omega = (2/T)\tan(\hat{\omega}(T/2)) \) such that the reconstruction is exact over the full frequency range. Note that this results in non-equidistant frequency points in the \( \omega \) domain [29].

Summarizing, by applying a suitable M"obius transformation, any available prior knowledge can directly be incorporated into the basis \( \Psi \) in the \( \omega \) domain.

B. Connection to Existing Results

In this section, the presented framework is connected to previous parameterizations in the literature. Applying the M"obius transform \( -j \) to the LRMP in (23) results in the well-known continuous-time Takenaka–Malmquist [26] basis. Moreover, if the M"obius transform \( -j \) is combined with a bilinear transform, e.g., the Tustin approximation, the discrete-time Takenaka–Malmquist [30] functions are obtained. The generalization in (23) simplifies to the well-known Laguerre functions by taking all \( \beta_k \in \mathbb{C}, \text{Re}(\beta) = 0 \) such that the rational functions model a system containing first-order real-valued poles. Selecting \( \beta_{k+1} = -\hat{\beta}_k \in \mathbb{C} \) such that all complex poles appear in real positive/negative pairs results in the Kautz basis functions. In Fig. 5, the M"obius transformation is used to connect the presented basis to existing results found in the literature [19], [26], [30].

VI. CASE STUDY

In this section, the theory from Section IV with the design guidelines from Section V is applied in a case study: both in a
simulation study of a mechanical system and an experimental study of a thermal system.

Remark 8: Throughout this section, it is assumed that accurate prior knowledge is available, e.g., through finite-element model (FEM) or initial experiments. In the case of uncertainty in the prior knowledge, mechanisms, such as pole repetition or iterative pole placement, can aid in improving the estimation accuracy. A comprehensive sensitivity analysis toward uncertainty in the prescribed pole locations would provide additional value to the presented method. This sensitivity analysis is outside the scope of the current research.

A. Procedure

In the case study, the following procedure is employed to construct the FRF $\hat{G}$ and variance estimate $\hat{C}_v$.

Procedure 1 (FRF Identification): Require: Appropriate excitation signal, e.g., white noise or random-phase multisine.

1: Perform simulation/experiment
2: if Prior knowledge then $\beta \leftarrow$ according to Fig. 4
3: else $\beta = 0$
4: end if
5: procedure LRMP($U(k), Y(k), \beta$)
6: for $k \in [0, \ldots, N]$ do
7: Construct (13) to obtain $\hat{\Theta}(k)$
8: Calculate $\hat{G}(k)$ and/or $\hat{T}(k)$.
9: Calculate the variance estimate $\hat{C}_v$
10: end for
11: end procedure

In this context, an appropriate excitation signal is system dependent, e.g., for a thermal system, an offset to the input is often required since a negative heat flux input is infeasible using conventional actuators.

B. Simulation Study: Mechanical System

In this section, the method presented in Section IV is applied to a mechanical system with resonant behavior in a simulation study. The true system is given by

$$G_0(s) = \frac{\Omega_1^2}{s^2 + 2\zeta\Omega_1 + \Omega_1^2} + \frac{\Omega_2^2}{s^2 + 2\zeta\Omega_2 + \Omega_2^2}$$

characterized by the natural frequencies $\Omega_1 = 5$ [rad/s] and $\Omega_2 = 4\Omega_1$ and damping coefficient $\zeta = 0.05$. The system $G_0(s)$ is then discretized using zero-order hold with a sample time $T_s = 1/20$ [s] to obtain $G_0(z)$. The dynamics of the
true system is characterized by the system poles that are \( \gamma = [0.76 \pm 0.6i, 0.98 \pm 0.16i] \) in discrete time.

The system \( G_0(z) \) is excited using two periods of a random-phase multisine of 60 [s] that is defined as

\[
\text{Definition 1:}
\]

\[
u(n) = \sum_{k=1}^{N} A_k \sin(2\pi f_k n / N + \phi_k)
\]

where \( n \) is a specific discrete sample, \( N \) is the total number of samples, \( A_k \) is the amplitude of the sinusoidal signal at frequency \( f_k \), \( \phi_k \) is a uniformly distributed random phase on \([0, 2\pi)\) such that \( \mathbb{E}(e^{i\phi_k}) = 0 \).

Then, an FRF estimate \( \hat{G}_0(\omega) \) is obtained using the classical ETFE approach, the traditional LPM, and the new LRMP approach. The total number of basis functions in the proposed method is \( N_b = 3 \). Initially, the prescribed poles \( \beta \) are set to 0, i.e., \( \beta = [0, 0, 0] \), by doing so the LPM is recovered as a special case of the LRMP. This results in an estimation error, as shown in Fig. 6(a). Here, the LPM and LRMP both obtain a significantly better estimation of \( G_0(z) \) when compared to the ETFE. Moreover, it is observed that the LPM and LRMP obtain similar results.

By then applying the LRMP with appropriate prior knowledge, e.g., by employing the transformations in Fig. 4 on the poles \( a, \gamma \), an improved estimation error is obtained. In Fig. 6(b), the prescribed poles include a single complex pole \( \gamma (1) \) at the first resonance of \( G_0(z) \). This results in an improved estimation error at the first resonance frequency. By expanding the basis \( \Psi \) to include a single complex pole at each resonance, the estimation error is decreased further, as shown in Fig. 6(c).

### C. Uncertainty in the Prior Knowledge

The approach presented in this article relies on accurate prior knowledge to reduce the estimation error. In Section VI-B, exact prior knowledge is available on the pole locations. In most practical applications, accurate prior knowledge is available through initial experiments or FEMs. Results of a preliminary investigation into the effect of uncertainty in the prior knowledge on the estimation accuracy of the method proposed in this article are shown in Fig. 7.

The analysis is performed by repeating the procedure of Section VI-B while perturbing the prior knowledge on the first pole location by complex perturbations within a radius of varying size. As a performance metric, the maximum value of the relative estimation error is compared between the LPM and the proposed method. The median result of 100 simulations for varying perturbation sizes and three different systems with varying damping ratios is shown in Fig. 7. It illustrates that the method proposed in this article is insensitive to large perturbations in the prior knowledge for systems with a large damping ratio. It also shows that for lightly damped systems, increasingly accurate prior knowledge is required to improve the estimation accuracy. For application of the proposed method to very lightly damped systems, a more elaborate sensitivity analysis is recommended and outside the scope of this article.

### D. Experimental Study: Thermal System

In this section, an experimental case study on a precision motion system, considered in a thermal control context, is presented.

1) **Thermal System:** The experimental setup used in this article is shown in Fig. 8(a). A 2-D schematic overview of the setup, including the relevant components and sensor location, is shown in Fig. 8(b). In the original application, the system under test is a high-precision linear motion stage moving in the \( y \)-direction, used in optical inspection equipment. In this article, thermal aspects of this setup are investigated. Transient effects in these types of systems are often dominant, and hence, the approach in Section III is expected to be highly applicable.

To isolate the thermal aspects of the system, the linear motor stator is removed, and its rotor, the coils, is maintained. The linear motors are then used as a thermal excitation source by passing a current through the coils, thereby heating them.

2) **Transient Response:** To facilitate the presentation, a single temperature measurement is used as shown in Fig. 8(b), yielding an SISO system. The system is excited using a random-phase multisine limited to 0.1 Hz, with a peak of 5 W centered around an offset of 5 W, since only heating is possible. Measurements are sampled at \( F_s = 1 \) [Hz] since the dynamics are predominantly low-frequent. The periodic excitation has a period length of \( L = 1 \) h, that is repeated \( P = 48 \) times, yielding a total data set of \( F_s L P = 172800 \) samples. The system response is presented in Fig. 9, and it shows the temperature over a 48-h period.
The experimental data as shown in Fig. 9 are separated into two subrecords, i.e., sets 1 and 2. Set 1 contains the first two periods, i.e., 2 h, of the response data that include the strong transient behavior due to the offset in the excitation signal. Set 2 contains the two periods as shown in the magnified plot in Fig. 9, i.e., 2 h starting at hour 20, that contains a minimal amount of transient response and can be used as a validation data set. Furthermore, it is seen that the small and relatively fast dynamic response is superimposed onto a much larger and slower transient response caused by initial excitation and the 24-h, e.g., day/night, cycle of the environment. The prior knowledge that is used in the proposed method is a distribution of $N_b = 3$ poles at $[10^{-4}, 10^{-3}, 10^{-2}]$ Hz. To validate the accuracy of the estimation, a validation FRF $G_0$ is obtained by averaging over 20 periods between hours 10 and 30. Since the true system is unknown, this FRF $G_0$ is taken as ground truth where the other methods are compared to.

By applying the classical approach, the ETFE, and the presented method, the LRMP, results shown in Fig. 10 are obtained. The results show the estimated plant $\hat{G}(\omega)$ for both methods and the estimated transient component $\hat{T}(\omega)$. Clearly, the first subrecord contains a strong transient response; therefore, the ETFE yields a biased and poor estimate of $G_0$. Moreover, the covariance estimate $\text{cov}(\hat{G})$ using the ETFE appears to also be biased by the transient. The presented approach is able to estimate the FRF more accurately since it is
close to \( G_0 \). The second subrecod contains significantly less transient contribution, allowing the ETFE to also accurately estimate the FRF. Although the variance \( \text{cov}(\tilde{G}) \) still appears slightly biased since it still deviates from the results obtained through the proposed method. Relying on the second subrecod for FRF estimation requires a significant time investment since an additional 20 h of experimental time is required to obtain the results.

By applying the approach presented in this article, a significant reduction in experimental time is achieved since the FRF can be estimated by measuring 2 h, by using the first subrecod, opposed to 22 h required for the classical approach.

VII. CONCLUSION

Incorporation of prior knowledge in conjunction with explicit transient estimation leads to improved FRF estimation for a large class of systems, including thermal and mechanical systems. Indeed, the transient response often present in measurements from these systems can cause a biased FRF estimate when employing classical approaches. Recent advancements in FRF identification employ local modeling techniques to estimate and remove these transients from the response. The framework presented in this article enables fast and accurate FRF estimation of a wide class of systems with a reliable quality metric, i.e., covariance expressions. This is achieved by utilizing a unified approach to local parametric modeling. It presents a local rational parameterization while maintaining a closed-form solution by using prescribed poles.

Previous methods, e.g., the LPM, also assume a prior, albeit implicit, by placing the poles at \( \infty \). The method presented in this article leverages the available freedom in the pole locations to increase the estimation accuracy. An approach is provided that leverages appropriate Möbius transformations to incorporate prior knowledge from different domains and applications in the local parameterization. The presented approach yields high-fidelity models that enable the application of advanced design, analysis, and control procedures.

APPENDIX I

APPENDIX//PROOFS

A. Proof to Theorem 1

Proof: Given the results in [26], where it is shown that the continuous-time Takanaka–Malmquist functions, e.g.,

\[
B_n(s) = \frac{\sqrt{2 \text{Re}[\alpha_n]} \prod_{i=0}^{n-1} s - \alpha_i}{s + \alpha_n} \tag{29}
\]

are orthonormal with respect to

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} B_n(j\omega) \overline{B_m(j\omega)} d\omega = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \tag{30}
\]

and it suffices to show that \( \Psi(\beta, \omega) \equiv B(\alpha, j\omega) \). Given \( \alpha = x + jy \) then \( \beta = -j\alpha = y - jx \), resulting in

\[
\Psi(\beta, \omega) = \sqrt{2\pi j} \prod \frac{\omega + y + jx}{\omega + y - jx} \prod \frac{\omega + y + jx}{\omega + y - jx} \tag{31}
\]

which concludes the proof.

B. Proof to Theorem 2

Proof: Given the results in [26], where it is shown that (29) is complete if and only if

\[
\sum_{n=1}^{\infty} \text{Re}[\alpha_n] \equiv |\alpha|^2. \tag{34}
\]

It is shown that \( \Psi(\beta, \omega) \equiv B(\alpha, j\omega) \). Therefore, it is sufficient to show that (25) is equivalent to (34), where \( \beta = -j\alpha \). Since \( |\alpha|^2 \equiv |\beta| \) and \( \text{Re}[\alpha] = \text{Re}[\beta] = -\text{Im}[\beta] \), this equivalence is straightforward, which concludes the proof.

REFERENCES

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