Linear image reconstruction by Sobolev norms on the bounded domain

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Linear Image Reconstruction by Sobolev Norms on the Bounded Domain^{*}

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Abstract. The reconstruction problem is usually formulated as a variational problem in which one searches for that image that minimizes a so called prior (image model) while insisting on certain image features to be preserved. When the prior can be described by a norm induced by some inner product on a Hilbert space the exact solution to the variational problem can be found by orthogonal projection. In previous work we considered the image as compactly supported in $\mathbb{L}_2(\mathbb{R}^2)$ and we used Sobolev norms on the unbounded domain including a smoothing parameter $\gamma > 0$ to tune the smoothness of the reconstruction image. Due to the assumption of compact support of the original image components of the reconstruction image near the image boundary are too much penalized. Therefore we minimize Sobolev norms only on the actual image domain, yielding much better reconstructions (especially for $\gamma \gg 0$). As an example we apply our method to the reconstruction of singular points that are present in the scale space representation of an image.

1 Introduction

One of the fundamental problems in signal processing is the reconstruction of a signal from its samples. In 1949 Shannon published his work on signal reconstruction from its equispaced ideal samples [17]. Many generalizations [16,18] and applications [3,13] followed thereafter.

Reconstruction from differential structure of scale space interest points, first introduced by Nielsen and Lillholm [15], is an interesting instance of the reconstruction problem since the samples are non-uniformly distributed over the image they were obtained from and the filter responses of the filters do not necessarily coincide. Several linear and non-linear methods [10,12,14,15] appeared in literature which all search for an image that (1) is indistinguishable from its original when observed through the filters the features were extracted with and (2) simultaneously minimizes a certain prior. We showed in earlier work [10] that if such a prior is a norm of Sobolev type on the *unbounded* domain one can obtain visually attractive reconstructions while retaining linearity. However, boundary problems degrade the reconstruction quality.

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Fig. 1. An illustration of the bounded domain problem: Features that are present in the center of the image are lost near its border, since they, in contrast to the original image f are not compactly supported on Ω

The problem that appears in the unbounded domain reconstruction method is best illustrated by analyzing Figure 1. The left image is a reconstruction from differential structure obtained from an image that is a concatenation of (mirrored) versions of Lena’s eye. One can clearly observe that structure present in the center of the image does not appear on the border. This can be attributed to the fact that, when features are measured close to the image boundary, they partly lay outside the image and are “penalized” by the energy minimization methods that are formulated on the unbounded domain. This is illustrated by the right image in Figure 1. We solve this problem by considering bounded domain Sobolev norms instead. An additional advantage of our method is that we can enforce a much higher degree of regularity than the unbounded domain counterpart (in fact we can minimize semi-norms on the bounded domain). Furthermore we give an interpretation of the 2 parameters that appear in the reconstruction framework in terms of filtering by a low-pass Butterworth filter. This allows for a good intuition on how to choose these parameters.

2 Theory

In order to prevent the above illustrated problem from happening we restrict the reconstruction problem to the domain $\Omega \subset \mathbb{R}^2$ that is defined as the support of the image $f \in \mathbb{L}_2(\mathbb{R}^2)$ from which the features $\{c_p(f)\}_1^P, c_p(f) \in \mathbb{R}$ are extracted. Recall that the $\mathbb{L}_2(\Omega)$ -inner product on the domain $\Omega \subset \mathbb{R}^2$ for $f, g \in \mathbb{L}_2(\Omega)$ is given by

$$(f, g)_{\mathbb{L}_2(\Omega)} = \int_{\Omega} \overline{f(x)} g(x) dx. \quad (1)$$

A feature $c_p(f)$ is obtained by taking the inner product of the p^{th} filter $\psi_p \in \mathbb{L}_2(\Omega)$ with the image $f \in \mathbb{L}_2(\Omega)$,

$$c_p(f) = (\psi_p, f)_{\mathbb{L}_2(\Omega)}. \quad (2)$$

An image $g \in \mathbb{L}_2(\Omega)$ is equivalent to the image f if they share the same features, $\{c_p(f)\}_1^P = \{c_p(g)\}_1^P$, which is expressed in the following equivalence relation for $f, g \in \mathbb{L}_2(\Omega)$.

$$f \sim g \Leftrightarrow (\psi_p, f)_{\mathbb{L}_2(\Omega)} = (\psi_p, g)_{\mathbb{L}_2(\Omega)} \text{ for all } p = 1, \dots, P. \quad (3)$$

Next we introduce the Sobolev space of order $2k$ on the domain Ω ,

$$\mathbb{H}^{2k}(\Omega) = \{f \in \mathbb{L}_2(\Omega) \mid |\Delta|^k f \in \mathbb{L}_2(\Omega)\}, \quad k > 0. \quad (4)$$

The completion of the space of $2k$ -differentiable functions on the domain Ω that vanish on the boundary of its domain $\partial\Omega$ is given by

$$\mathbb{H}_0^{2k}(\Omega) = \{f \in \mathbb{H}^{2k}(\Omega) \mid f|_{\partial\Omega} = 0\}, \quad k > \frac{1}{2}. \quad (5)$$

Now $\mathbb{H}_0^{2k, \gamma}(\Omega)$ denotes the normed space obtained by endowing $\mathbb{H}_0^{2k}(\Omega)$ with the following inner product,

$$(f, g)_{\mathbb{H}_0^{2k, \gamma}(\Omega)} = (f, g)_{\mathbb{L}_2(\Omega)} + \gamma^{2k} \left(|\Delta|^{\frac{k}{2}} f, |\Delta|^{\frac{k}{2}} g \right)_{\mathbb{L}_2(\Omega)} = (f, g)_{\mathbb{L}_2(\Omega)} + \gamma^{2k} \left(|\Delta|^k f, g \right)_{\mathbb{L}_2(\Omega)}, \quad (6)$$

for all $f, g \in \mathbb{H}_0^{2k}(\Omega)$ and $\gamma \in \mathbb{R}$.

The solution to the reconstruction problem is the image g of minimal $\mathbb{H}_0^{2k, \gamma}$ -norm that shares the same features with the image $f \in \mathbb{H}_0^{2k, \gamma}(\Omega)$ from which the features $\{c_p(f)\}_1^N$ were extracted. The reconstruction image g is found by an orthogonal projection, within the space $\mathbb{H}_0^{2k, \gamma}(\Omega)$, of f onto the subspace V spanned by the filters κ_p that correspond to the ψ_p filters,

$$\arg \min_{f \sim g} \|g\|_{\mathbb{H}_0^{2k, \gamma}(\Omega)}^2 = \mathbb{P}_V f, \quad (7)$$

as shown in previous work [10]. The filters $\kappa_p \in \mathbb{H}_0^{2k, \gamma}(\Omega)$ are given by

$$\kappa_p = (I + \gamma^{2k} |\Delta|^k)^{-1} \psi_p. \quad (8)$$

As a consequence $(\kappa_p, f)_{\mathbb{H}_0^{2k, \gamma}(\Omega)} = (\psi_p, f)_{\mathbb{L}_2(\Omega)}$ for $(p = 1 \dots P)$ for all f . Here we assumed that $f \in \mathbb{H}^{2k}(\Omega)$ however, one can get away with $f \in \mathbb{L}_2(\Omega)$ if ψ satisfies certain regularity conditions. The interested reader can find the precise conditions and further details in [7].

The two parameters, γ and k , that appear in the reconstruction problem allow for an interesting interpretation. If $\Omega = \mathbb{R}$ the fractional operator $(I + \gamma^{2k} |\Delta|^k)^{-1}$ is equivalent to filtering by the classical low-pass Butterworth filter [2] of order $2k$ and cut-off frequency $\omega_0 = \frac{1}{\gamma}$. This filter is defined as

$$B_{2k} \left(\frac{\omega}{\omega_0} \right) = \frac{1}{1 + \left| \frac{\omega}{\omega_0} \right|^{2k}}. \quad (9)$$

A similar phenomena was recently observed by Unser and Blu [20] when studying the connection between splines and fractals. Using this observation we can

Fig. 2. The filter response of a Butterworth filter. On the left γ is kept constant and the filter responses for different $k > 0$ are shown. On the right the order of the filter, $2k$, is kept constant and the filter responses for different $\gamma > 0$ are shown.

interpret equation (7) as finding an image, constructed by the ψ_p basis functions that, after filtering with the Butterworth filter of order $2k$ and with a cut-off frequency determined by γ , is equivalent (cf. equation (3)) to the image f . The filter response of the Butterworth filter is shown in Figure 3. One can observe the order of the filter controls how well the ideal low-pass filter is approximated and the effect of γ on the cut-off frequency.

2.1 Spectral Decomposition

For now we set $k = 1$ and investigate the Laplace operator on the bounded domain: $\Delta : \mathbb{H}_0^2(\Omega) \mapsto \mathbb{L}_2(\Omega)$ which is bounded and whose right inverse is given by the minus Dirichlet operator, which is defined as follows.

Definition 1 (Dirichlet Operator). *The Dirichlet operator $\mathcal{D}$ is given by*

$$g = \mathcal{D}f \Leftrightarrow \begin{cases} \Delta g = -f \\ g|_{\partial\Omega} = 0 \end{cases} \quad (10)$$

with $f \in \mathbb{L}_2(\Omega)$ and $g \in \mathbb{H}_0^2(\Omega)$.

The Green's function $G : \Omega \times \Omega \mapsto \mathbb{R}^2$ of the Dirichlet operator is given by

$$\begin{cases} \Delta G(\mathbf{x}, \cdot) = -\delta_{\mathbf{x}} \\ G(\mathbf{x}, \cdot)|_{\partial\Omega} = 0 \end{cases} \quad (11)$$

for fixed $\mathbf{x} \in \Omega$. Its closed form solution reads

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log \left| \frac{\text{sn}(x_1 + ix_2, \tilde{k}) - \text{sn}(y_1 + iy_2, \tilde{k})}{\text{sn}(x_1 + ix_2, \tilde{k}) - \overline{\text{sn}(y_1 + iy_2, \tilde{k})}} \right|. \quad (12)$$

Here $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \Omega$ and $\tilde{k} \in \mathbb{R}$ is determined by the aspect ratio of the rectangular domain Ω , sn denotes the well know Jacobi-elliptic function [9]. In Appendix A we derive equality (12), and show how to obtain $\tilde{k}$. Figure 3 shows a graphical representation of this non-isotropic Green's function for a square domain ($\tilde{k} \approx 0.1716$). Notice this function vanishes at its boundaries

and is, in the center of the domain, very similar to the isotropic fundamental solution on the unbounded domain [5]. In terms of regularisation this means the Dirichlet operator smoothens inwards the image but never “spills” over the border of the domain Ω .

Fig. 3. From left to right plots of the graph of $\mathbf{x} \mapsto G_{\mathbf{x}, \mathbf{y}}$, isocontours $G(\mathbf{x}, \mathbf{y}) = c$ and isocontours of its Harmonic conjugate $H(\mathbf{x}, \mathbf{y}) = \frac{-1}{2\pi} \arg \left(\frac{\operatorname{sn}(x_1 + ix_2, \tilde{k}) - \operatorname{sn}(y_1 + iy_2, \tilde{k})}{\operatorname{sn}(x_1 + ix_2, \tilde{k}) - \operatorname{sn}(y_1 + iy_2, \tilde{k})} \right)$

When the Dirichlet operator as defined in Definition 1 is expressed by means of its Green’s function, which is presented in equation (12),

$$(\mathcal{D}f)(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \quad f \in \mathbb{L}_2(\Omega), \quad \mathcal{D}f \in \mathbb{H}_0^2(\Omega) \quad (13)$$

one can verify that it extends to a compact, self-adjoint operator on $\mathbb{L}_2(\Omega)$. As a consequence, by the spectral decomposition theorem of compact self-adjoint operators [21], we can express the Dirichlet operator in an orthonormal basis of eigen functions. The normalized eigen functions f_{mn} with corresponding eigen values λ_{mn} of the Laplace operator $\Delta : \mathbb{H}_0^2(\Omega) \mapsto \mathbb{L}_2(\Omega)$ are given by

$$f_{mn}(x, y) = \sqrt{\frac{1}{ab}} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \quad \lambda_{mn} = - \left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \right) \quad (14)$$

with $\Omega = [0, a] \times [0, b]$. Since $\Delta \mathcal{D} = -I$, the eigen functions of the Dirichlet operator coincide with those of the Laplace operator (14) and its corresponding eigen values are the inverse of the eigen values of the Laplace operator.

2.2 Scale Space on the Bounded Domain

The spectral decomposition presented in the previous subsection, by (14), will now be applied to the construction of a scale space on the bounded domain [6].

Before we show how to obtain a Gaussian scale space representation of an image on the bounded domain we find, as suggested by Koenderink [11], the image $h \in \mathbb{H}^2(\Omega)$ that is the solution to $\Delta h = 0$ with as boundary condition that $h = f$ restricted to $\partial\Omega$. Now $\tilde{f} = f - h$ is zero at the boundary $\partial\Omega$ and can serve as an initial condition for the heat equation (on the bounded domain). A practical method for obtaining h is suggested by Georgiev [8]. Now $f_{mn}(x, y)$ is obtained by expansion of

$$\tilde{f} = \sum_{m,n \in \mathbb{N}} (f_{mn}, \tilde{f})_{\mathbb{L}_2(\Omega)} f_{mn} , \quad (15)$$

which effectively exploits the sine transform.

The (fractional) operators that will appear in the construction of a Gaussian scale space on the bounded domain can be expressed as

$$|\Delta|^{2k} f_{mn} = (\lambda_{mn})^k f_{mn} , \quad e^{-s|\Delta|} f_{mn} = e^{-s\lambda_{mn}} f_{mn} . \quad (16)$$

We also note that the κ_p filters, defined in equality (8), are readily obtained by application of the following identity

$$(I + \gamma^{2k} |\Delta|^k)^{-1} f_{mn} = \frac{1}{1 + \gamma^{2k} (\lambda_{mn})^k} f_{mn} . \quad (17)$$

Consider the Gaussian scale space representation¹ on bounded domain Ω

$$u_{\tilde{f}}^\Omega(x, y, s) = \sum_{m,n \in \mathbb{N}} e^{-(\lambda_{mn})s} (f_{mn}, \tilde{f})_{\mathbb{L}_2(\Omega)} f_{mn}(x, y) \quad (18)$$

where the scale parameter $s \in \mathbb{R}^+$. It is the unique solution to

$$\begin{cases} \frac{\partial u}{\partial s} = \Delta u \\ u(\cdot, s)|_{\partial\Omega} = 0 \text{ for all } s > 0 \\ u(\cdot, 0) = \tilde{f} \end{cases} . \quad (19)$$

The filter ϕ_p that measures differential structure present in the scale space representation $u_{\tilde{f}}^\Omega$ of $\tilde{f}$ at a point p with coordinates (x_p, y_p, s_p) , such that

$$\left(D^{\mathbf{n}_p} u_{\tilde{f}}^\Omega \right) (x_p, y_p, s_p) = \left(\phi_p, \tilde{f} \right)_{\mathbb{L}_2(\Omega)} , \quad (20)$$

is given by (writing multi-index $\mathbf{n}_p = (n_p^1, n_p^2)$)

$$\phi_p(x, y) = \sum_{m,n \in \mathbb{N}} e^{-(\lambda_{mn})s_p} (D^{\mathbf{n}_p} f_{mn})(x_p, y_p) f_{mn}(x, y) , \quad (21)$$

where we note that

$$(D^{\mathbf{n}_p} f_{mn})(x_p, y_p) = \sqrt{\frac{1}{ab}} \left(\frac{m\pi}{b} \right)^{n_p^2} \left(\frac{n\pi}{a} \right)^{n_p^1} \sin\left(\frac{m\pi y_p}{b} + \frac{\pi}{2} n_p^2 \right) \sin\left(\frac{n\pi x_p}{a} + \frac{\pi}{2} n_p^1 \right) ,$$

$\mathbf{x} = (x, y) \in \Omega$, $\mathbf{x}_p = (x_p, y_p) \in \Omega$ and $\mathbf{n}_p = (n_p^1, n_p^2) \in \mathbb{N} \times \mathbb{N}$.

¹ The framework in this paper is readily generalized to α -scale spaces in general (see e.g. [6]) by replacing $(-\lambda_{mn})$ by $(-\lambda_{mn})^{2\alpha}$.

2.3 The Solution to the Reconstruction Problem

Now we have established how one can construct a scale space on the bounded domain and shown how to measure its differential structure we can proceed to express the solution to the reconstruction problem (recall equations (7) and (8)) in terms of eigen functions and eigen values of the Laplace operator:

$$\mathbb{P}_V \tilde{f} = \sum_{p,q=1}^P G^{pq}(\phi_p, \tilde{f})_{\mathbb{L}_2(\Omega)} \kappa_q = \sum_{p,q=1}^P G^{pq} c_p(\tilde{f}) \kappa_q , \quad (22)$$

where G^{pq} is the inverse of the Gram matrix $G_{pq} = (\kappa_p, \kappa_q)_{\mathbb{H}_0^{2k,\gamma}(\Omega)}$ and the filters κ_p satisfy

$$\kappa_p(x, y) = \sum_{m,n \in \mathbb{N}} \frac{e^{-(\lambda_{mn}) s_p}}{1 + \gamma^{2k} (\lambda_{mn})^k} (D^{\mathbf{n}_p} f_{mn})(x_p, y_p) f_{mn}(x, y) . \quad (23)$$

This is the unique solution to the optimization problem

$$\arg \min_{\tilde{f} \sim g} \|g\|_{\mathbb{H}_0^{2k,\gamma}(\Omega)}^2 , \quad (24)$$

which was introduced in equation (7). Instead of Dirichlet boundary conditions one can impose Neumann-boundary conditions. In this case the eigenvalues λ_{mn} are maintained, the eigen functions are given by

$$f_{mn}(x, y) = \sqrt{\frac{1}{ab(1 + \delta_{m0})(1 + \delta_{n0})}} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) . \quad (25)$$

3 Implementation

The implementation of the reconstruction method that was presented in a continuous Hilbert space framework is completely performed in a discrete framework in order to avoid approximation errors due to sampling, following the advice: “*Think analog, act discrete!*” [19].

First we introduce the discrete sine transform $\mathcal{F}_S : l_2(I_N^{\mathcal{D}}) \mapsto l_2(I_N^{\mathcal{D}})$ on a rectangular domain $I_N^{\mathcal{D}} = \{1, \dots, N-1\} \times \{1, \dots, M-1\}$

$$(\mathcal{F}_S f)(u, v) = -\frac{2}{\sqrt{MN}} \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} \underbrace{\sin\left(\frac{iu\pi}{M}\right) \sin\left(\frac{jv\pi}{N}\right)}_{(\varphi_i \otimes \varphi_j)(u,v)} f(i, j) , \quad (26)$$

with $(u, v) \in I_N^{\mathcal{D}}$. Notice that this unitary transform is its own inverse and that

$$(\varphi_i, \varphi_j)_{l_2(I_N^{\mathcal{D}})} = \delta_{ij} , \quad (27)$$

so $\{\varphi_i \otimes \varphi_j \mid \begin{smallmatrix} i = 1, \dots, M-1 \\ j = 1, \dots, N-1 \end{smallmatrix}\}$ forms an orthonormal basis in $l_2(I_N^{\mathcal{D}})$.

The Gaussian scale space representation $u_f^{I_N^D}(i, j, s)$ of an image $f \in l_2(I_N^D)$ introduced in the continuous domain in equality (18) now reads

$$u_f^{I_N^D}(i, j, s) = (e^{s\Delta} f)(i, j) = -\frac{2}{\sqrt{MN}} \sum_{u=1}^{M-1} \sum_{v=1}^{N-1} \hat{f}(u, v) e^{-s\left(\frac{u^2}{M^2} + \frac{v^2}{N^2}\right)\pi^2} (\varphi_u \otimes \varphi_v)(i, j)$$

where $\hat{f}(u, v) = (\mathcal{F}_S f)(u, v)$. Differential structure of order $\mathbf{n}_p = (n_p^1, n_p^2) \in \mathbb{N} \times \mathbb{N}$ at a certain position $(i_p, j_p) \in I_N^D$ and at scale $s_p \in \mathbb{R}^+$ is measured by

$$\begin{aligned} (D^{\mathbf{n}_p} u_f^{I_N^D})(i_p, j_p, s_p) &= -\frac{2}{\sqrt{MN}} \sum_{u=1}^{M-1} \sum_{v=1}^{N-1} \hat{f}(u, v) e^{-s_p\left(\frac{u^2}{M^2} + \frac{v^2}{N^2}\right)\pi^2} \\ &\quad \left(\frac{u\pi}{M}\right)^{n_p^1} \left(\frac{v\pi}{N}\right)^{n_p^2} \sin\left(\frac{i_p u \pi}{M} + \frac{\pi}{2} n_p^1\right) \sin\left(\frac{j_p v \pi}{N} + \frac{\pi}{2} n_p^2\right). \end{aligned}$$

The filters ϕ_p , with $p = (i_p, j_p, s_p, \mathbf{n}_p)$ a multi-index, are given by

$$\begin{aligned} \phi_p(i, j, s) &= -\frac{2}{\sqrt{MN}} \sum_{u=1}^{M-1} \sum_{v=1}^{N-1} e^{-s_p\left(\frac{u^2}{M^2} + \frac{v^2}{N^2}\right)\pi^2} (\varphi_u \otimes \varphi_v)(i, j) \\ &\quad \left(\frac{u\pi}{M}\right)^{n_p^1} \left(\frac{v\pi}{N}\right)^{n_p^2} \sin\left(\frac{i_p u \pi}{M} + \frac{\pi}{2} n_p^1\right) \sin\left(\frac{j_p v \pi}{N} + \frac{\pi}{2} n_p^2\right) \end{aligned} \quad (28)$$

and the filters κ_p corresponding to ϕ_p read

$$\begin{aligned} \kappa_p(i, j, s) &= -\frac{2}{\sqrt{MN}} \sum_{u=1}^{M-1} \sum_{v=1}^{N-1} \frac{e^{-s_p\left(\frac{u^2}{M^2} + \frac{v^2}{N^2}\right)\pi^2}}{1 + \gamma^{2k} \left(\frac{u^2}{M^2} + \frac{v^2}{N^2}\right)^k} (\varphi_u \otimes \varphi_v)(i, j) \\ &\quad \left(\frac{u\pi}{M}\right)^{n_p^1} \left(\frac{v\pi}{N}\right)^{n_p^2} \sin\left(\frac{i_p u \pi}{M} + \frac{\pi}{2} n_p^1\right) \sin\left(\frac{j_p v \pi}{N} + \frac{\pi}{2} n_p^2\right). \end{aligned} \quad (29)$$

An element $G_{pq} = (\phi_p, \phi_q)_{l_2(I_N^D)}$ of the Gram matrix can, because of the orthonormality of the transform, be expressed in just a double sum,

$$\begin{aligned} G_{pq} &= -\frac{2}{\sqrt{MN}} \sum_{u=1}^{M-1} \sum_{v=1}^{N-1} \frac{e^{-(s_p + s_q)\left(\frac{u^2}{M^2} + \frac{v^2}{N^2}\right)\pi^2}}{1 + \gamma^{2k} \left(\frac{u^2}{M^2} + \frac{v^2}{N^2}\right)^k} \left(\frac{u\pi}{M}\right)^{n_p^1} \left(\frac{v\pi}{N}\right)^{n_p^2} \sin\left(\frac{i_p u \pi}{M} + \frac{\pi}{2} n_p^1\right) \\ &\quad \sin\left(\frac{j_p v \pi}{N} + \frac{\pi}{2} n_p^2\right) \left(\frac{u\pi}{M}\right)^{n_q^1} \left(\frac{v\pi}{N}\right)^{n_q^2} \sin\left(\frac{i_q u \pi}{M} + \frac{\pi}{2} n_q^1\right) \sin\left(\frac{j_q v \pi}{N} + \frac{\pi}{2} n_q^2\right). \end{aligned}$$

In order to gain accuracy we implement equality (3) by summing in the reverse direction and multiplying by γ^{2k} . Then we compute

$$\tilde{g} = \sum_{p, q=1}^P G^{pq} \gamma^{2k} c_p(f) \phi_q \quad (30)$$

and find the reconstruction image g by filtering $\tilde{g}$ by a discrete version of the 2D Butterworth filter of order $2k$ and with cut-off frequency $\omega_0 = \frac{1}{\gamma}$.

The implementation was written using the sine transform as defined in equality (26) where we already explicitly mentioned the transform can be written as

$$(\mathcal{F}_S f)(u, v) = -\frac{2}{\sqrt{MN}} \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (\varphi_i \otimes \varphi_j)(u, v) f(i, j). \quad (31)$$

Now we define the cosine transform $\mathcal{F}_C : l_2(I_N^N) \mapsto l_2(I_N^N)$ on a rectangular domain $I_N^N = \{0, \dots, N+1\} \times \{0, \dots, M+1\}$ in a similar manner

$$(\mathcal{F}_C f)(u, v) = \sum_{i=0}^{M+1} \sum_{j=0}^{N+1} (\tilde{\varphi}_i \otimes \tilde{\varphi}_j)(u, v) f(i, j). \quad (32)$$

where $(\tilde{\varphi}_i \otimes \tilde{\varphi}_j)(u, v) = \cos\left(\frac{\pi(i+\frac{1}{2})u}{M}\right) \sqrt{\frac{2-\delta_{u0}}{M}} \cos\left(\frac{\pi(j+\frac{1}{2})v}{N}\right) \sqrt{\frac{2-\delta_{v0}}{N}}$. These cosine basis functions $\{\tilde{\varphi}_i \otimes \tilde{\varphi}_j \mid \begin{smallmatrix} i=0, \dots, M+1 \\ j=0, \dots, N+1 \end{smallmatrix}\}$ form an orthogonal basis in $l_2(I_N^N)$ and can thus be used to transform the reconstruction method that was explicitly presented for the Dirichlet case into a reconstruction method based on Neumann boundary conditions.

4 Experiments

We evaluate the reconstruction by applying our reconstruction method to the problem that was presented in the introduction. The upper row of Figure 4 shows, from left to right, the image from which the features were extracted, a reconstruction by the unbounded domain method [10] (parameters: $\gamma = 50, k = 1$) and a reconstruction by the newly introduced bounded domain method using Dirichlet boundary conditions (parameters: $\gamma = 1000, k = 1$). Features that were used are up to second order derivatives measured at the singular points [4] of the scale space representation of f . One can clearly see that the structure that is missing in the middle image does appear when the bounded domain method is used. The bottom row of Figure 4 shows reconstructions from second order differential structure obtained from the singular points of the scale space of the laplacian of f . On the left the unbounded domain method was used with $\gamma = 100$ and $k = 1$, this leads to a reconstructed signal that has “spilled” too much over the border of the image and therefore is not as crisp as the reconstruction obtained by our newly proposed method using Dirichlet boundary conditions (parameters: $\gamma = 1000$ and $k = 1$). Due to this spilling the Gram matrix of the bounded domain reconstruction method is harder to invert since basis functions start to become more and more dependent, this problem gets worse when γ increases. Our bounded domain method is immune to this problem.

Fig. 4. Top left: The image f from which the features were extracted. Top center and right: reconstruction from second order structure of the singular points of f using the unbounded domain method[10] (parameters: $\gamma = 50, k = 1$) and the bounded domain method (parameters: $\gamma = 1000, k = 1$). Bottom row: unbounded domain (left) and bounded domain (right) reconstruction from singular points of the laplacian of f with $k = 1$ and γ set to 100 and 1000 respectively.

5 Conclusion

In previous work we considered the image as compactly supported in $L_2(\mathbb{R}^2)$ and we used Sobolev norms on the unbounded domain including a smoothing parameter $\gamma > 0$ to tune the smoothness of the reconstruction image. Due to the assumption of compact support of the original image components of the reconstruction image near the image boundary are too much penalized. Therefore we proposed to minimize Sobolev norms only on the actual image domain, yielding much better reconstructions (especially for $\gamma \gg 0$). We also showed an interpretation for the parameter γ and the order of the Sobolev space k in terms of filtering by the classical Butterworth filter. In future work we plan to exploit this interpretation by automatically selecting the order of the Sobolev space.

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A Closed form Expression of the Green's Function of the Dirichlet Operator

The Green's function $G : \Omega \times \Omega \mapsto \mathbb{R}^2$ of the dirichlet operator $\mathcal{D}$ (recall Definition 1) can be obtained by means of conformal mapping². To this end we first map the rectangle to the upper half space in the complex plane. By the Schwarz-Christoffel formula the derivative of the inverse of such a mapping is given by

$$\frac{dz}{dw} = -\frac{1}{\tilde{k}}(w-1)^{-\frac{1}{2}}(w+1)^{-\frac{1}{2}}(w-\frac{1}{\tilde{k}})^{-\frac{1}{2}}(w+\frac{1}{\tilde{k}})^{-\frac{1}{2}} = \frac{1}{\sqrt{1-w^2}} \frac{1}{\sqrt{1-\tilde{k}^2 w^2}},$$

where $w(\pm 1/\tilde{k}) = \pm a + ib$. As a result

$$z(w, \tilde{k}) = \int_0^w \frac{dt}{\sqrt{1-t^2}\sqrt{1-\tilde{k}^2 t^2}} \Leftrightarrow w(z) = sn(z, \tilde{k}),$$

where sn denotes the well-known Jacobi-elliptic function. We have $sn(0, \tilde{k}) = 0$, $sn(\pm a, \tilde{k}) = \pm 1$, $sn(\pm a + ib, \tilde{k}) = \pm(1/\tilde{k})$ and $sn(i(b/2), \tilde{k}) = i/\sqrt{\tilde{k}}$, where the elliptic modulus $\tilde{k}$ is given by

$$(b/a)z(1, \tilde{k}) = z(1, \sqrt{1-\tilde{k}^2}).$$

For example in case of a square $b/a = 2$ we have $\tilde{k} \approx 0.1715728752$.

The next step is to map the half plane onto the unit disk $\overline{B_{0,1}}$. This is easily done by means of a linear fractional transform

$$\chi(z) = \frac{z - sn(y_1 + i y_2, \tilde{k})}{z - sn(y_1 + i y_2, \tilde{k})}.$$

To this end we notice that $|\chi(0)| = 1$ and that the mirrored points $sn(y_1 + i y_2, \tilde{k})$ and $\overline{sn(y_1 + i y_2, \tilde{k})}$ are mapped to the mirrored points $\chi(sn(y_1 + i y_2, \tilde{k})) = 0$ and $\chi(\overline{sn(y_1 + i y_2, \tilde{k})}) = \infty$.

Now define $F : \mathbb{C} \rightarrow \mathbb{C}$ and $\mathbf{F} : \overline{\Omega} \rightarrow \overline{B_{0,1}}$ by

$$F = \chi \circ sn(\cdot, \tilde{k}), \text{ i.e. } F(x_1 + i x_2) = \frac{sn(x_1 + i x_2, \tilde{k}) - sn(y_1 + i y_2, \tilde{k})}{sn(x_1 + i x_2, \tilde{k}) - \overline{sn(y_1 + i y_2, \tilde{k})}} \quad (33)$$

$$\mathbf{F}(x_1, x_2) = (\text{Re}(F(x_1 + i x_2)), \text{Im}(F(x_1 + i x_2)))^T,$$

then $\mathbf{F}$ is a conformal mapping of $\overline{\Omega}$ onto $\overline{B_{0,1}}$ with $\mathbf{F}(\mathbf{y}) = \mathbf{0}$. As a result we have by the Cauchy-Riemann equations

$$\Delta_{\mathbf{F}(\mathbf{x})} = |\mathbf{F}'(\mathbf{x})|^{-1} \Delta_{\mathbf{x}}, \quad (34)$$

² Our solution is a generalization of the solution derived by Boersma in [1]

where the scalar factor in front of the right Laplacian is the inverse Jacobian:

$$|\mathbf{F}'(\mathbf{x})|^{-1} = (\det \mathbf{F}'(\mathbf{x}))^{-1} = \left(\left(\frac{\partial F_1}{\partial x}(\mathbf{x}) \right)^2 + \left(\frac{\partial F_2}{\partial x}(\mathbf{x}) \right)^2 \right)^{-1} = |F'(x_1 + ix_2)|,$$

for all $\mathbf{x} = (x_1, x_2) \in \Omega$.

Now $\tilde{G}(\mathbf{u}, \mathbf{0}) = \frac{-1}{2\pi} \log \|\mathbf{u}\|$ is the unique Greens function with Dirichlet boundary conditions on the disk $\overline{B_{\mathbf{0},1}} = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$ with singularity at $\mathbf{0}$ and our Green's function is given by $G = \tilde{G} \circ \mathbf{F}$, i.e.

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log |(\chi \circ sn(\cdot, \tilde{k}))(x_1 + ix_2)| = -\frac{1}{2\pi} \log \left| \frac{sn(x_1 + ix_2, \tilde{k}) - sn(y_1 + iy_2, \tilde{k})}{sn(x_1 + ix_2, \tilde{k}) - \overline{sn(y_1 + iy_2, \tilde{k})}} \right|. \quad (35)$$