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A SKEW HADAMARD MATRIX OF ORDER 36

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Hadamard matrices exist for infinitely many orders $4m, m \geq 1$, $m$ integer, including all $4m < 100$, cf. [3], [2]. They are conjectured to exist for all such orders. Skew Hadamard matrices have been constructed for all orders $4m < 100$ except for 36, 52, 76, 92, cf. the table in [4]. Recently Szekeres [6] found skew Hadamard matrices of the order $2(p' + 1) \equiv 12 \pmod{16}$, $p$ prime, thus covering the case 76. In addition, Blatt and Szekeres [1] constructed one of order 52. The present note contains a skew Hadamard matrix of order 36 (and one of order 52), thus leaving 92 as the smallest open case.

The unit matrix of any order is denoted by $I$. The square matrices $Q$ and $R$ of order $m$ are defined by their only nonzero elements

$$q_{i, i+1} = q_{m-1, 1} = 1, \quad i = 1, \cdots, m-1; \quad r_{i, m-i+1} = 1, \quad i = 1, \cdots, m$$

We have

$$Q^m = I, \quad R^2 = I, \quad QR = QT R.$$  

Any square matrix $A$ of order $m$ is symmetric if $A = AT$, skew if $A + AT = 0$, circulant if $AQ = QA$. Hence, for circulant $A$ we have

$$A = \sum_{i=0}^{m-1} a_i Q^i, \quad RA = A^T R.$$  

Any square matrix $H$ of order $4m$ is skew Hadamard if its elements are 1 and $-1$ (we write $+$ and $-$) and

$$HH^T = 4mI, \quad H + H^T = 2I.$$

**Theorem 1.** If $A, B, C, D$ are square circulant matrices of order $m$, if $A$ is skew, and if

$$AA^T + BB^T + CC^T + DD^T = (4m - 1)I,$$

then

$$H = \begin{bmatrix}
    A + I & BR & CR & DR \\
    -BR & A + I & -D^T R & C^T R \\
    -CR & D^T R & A + I & -B^T R \\
    -DR & -C^T R & B^T R & A + I
\end{bmatrix}$$

satisfies $HH^T = 4mI, \quad H + H^T = 2I$.

**Proof.** By straightforward verification.

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REMARK. If, in addition, B, C, and D are symmetric, then $H$ may be written in terms of the quaternion matrices $K_4$, $L_4$, $M_4$ and the Kronecker product $\otimes$ as follows:

$$H = I_4 \otimes (I + A) + K_4 \otimes BR + L_4 \otimes CR + M_4 \otimes DR,$$

hence looking much like a Williamson-type matrix, cf. [7].

**Theorem 2.** There exist skew Hadamard matrices of orders 36 and 52.

**Proof.** We apply theorem 1 with the following circulant matrices of order 9:

$$A = (0 + + - + - + - - -), \quad B = (+ - + + - - + + -), \quad C = (- - + + + + + + -), \quad D = (+ + - + + - + + +).$$

By inspection the skew $A$ and the symmetric $B, C, D$ are seen to satisfy the hypotheses. Hence a skew Hadamard matrix of order 36 is obtained. Secondly, we consider the following circulant matrices of order 13:

$$A = (0 + + + - + + - + - -), \quad B = (- + - + + - - + + +), \quad C = D = (- - + + + + + + + + +).$$

Application of theorem 1 to $A, B, C, D$ yields a skew Hadamard matrix of order 52 since

$$AA^T = 15I - J + 2B, \quad BB^T = 12I - J - 2B, \quad CC^T = DD^T = 12I + J.$$

REMARK. The positive elements of $B$ indicate the quadratic residues mod 13. The matrix of order 26

$$\begin{bmatrix} B + I & C \\ C^T & -B - I \end{bmatrix}$$

is an orthogonal matrix with zero diagonal, cf. [2] p. 1007. The matrix $A$ describes the unique tournament of order 13 having no transitive subtournament of order 5, which was recently found by Reid and Parker [5].

**References**