Engineering toolbox for structural-acoustic design

Applied to MRI-scanners


This research was financed by the Dutch Technology Foundation (STW).
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Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Eindhoven,
op gezag van de Rector Magnificus, prof. dr. M. Rem,
voor een commissie aangewezen door het College voor Promoties
in het openbaar te verdedigen op
woensdag 31 januari 2001 om 16:00 uur

doors

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Virtual prototyping avant-la-lettre:

It is absolutely immaterial to me whether I run my turbine in thought or test it in my shop.

– Nikola Tesla (1856–1943)
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Summary

This thesis covers numerical tools for low noise design of MRI scanners, as used in clinical practice. In combination with the previously published thesis of Kuipers (1999), it describes the results of a project entitled "Acoustic modelling and design of MRI scanners", that was funded by the Dutch Technology Foundation (STW). The MRI scanner is a typical example of a device that, due to its underlying operational principles, generates excessive noise as a side effect. If the current operational principles are unchanged, sound level reductions of 40 dB or more are desirable, compared to the scanners of today.

Such reductions require the combination of a number of extensive actions. Firstly, these actions concern the noise from the gradient coil that escapes through the tunnel in which the patient is located. Also of importance, is the sound that originates from noise and vibration in the gradient coil, and that is transferred to the magnet housing. Another noise source is the inductive excitation of the magnet housing by the gradient coil. The technical realisation of such extensive actions, however, is restricted because of limitations in space, applicable materials and maximum allowable vibrational displacements of the gradient coil.

The goal of the aforementioned STW-project is the development of design tools for the optimisation of gradient coils, in order to develop low noise scanners. The software based on the Finite Element Method and on the Boundary Element Method, that was available at the start of the project, was not suitable for this purpose. The required frequency range of a few kHz already leads to excessive analysis times for every single design alternative. This prevented the application of such software as a tool for design and optimisation. In the thesis of Kuipers (1999), a method is developed for the efficient calculation of gradient coil noise radiation. By taking advantage of axisymmetry and by using a special formulation of the radiation operator, an enormous CPU time reduction is obtained.

The goal of this thesis is the development of a much more extensive numerical tool. The previously developed acoustical module is integrated into this tool. The newly developed additions apply to the mechanical excitation, to the coil vibrations, and to the optimisation of coil structures.

The Lorentz forces, that excite the coil, are described in terms of a spatial Fourier series. From this description, symmetry conditions of the excitation are derived. Next, an efficient numerical method for the analysis of the coil vibrations is implemented, which exploits the global axisymmetry and the symmetry of the excitation. Existing techniques are improved, which has led to shorter CPU times and to a wider range of applicability. Also, research has been conducted into the
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modelling of frequency dependent fibre reinforced materials and structures with glue layers. Now, quick estimates of the effects of design changes on the noise levels can be made with the help of these numerical models. Geometry, material behaviour, mechanical excitation and kinematic boundary conditions can be parameterised at will. This is demonstrated using a number of such parameter studies, including an investigation into the effects of Lorentz force balancing. Therefore, by applying the developed tools, the possibilities for noise reduction within a certain design space can be explored. Such information facilitates well-founded decisions about actions that are needed to achieve a certain reduction.

Parameters studies in this thesis show that the acoustical behaviour of certain structures can change considerably after the application of small design changes. Most optimisation methods, however, are not effective when the system responses fluctuate rapidly. For that reason, research is conducted into a method that works under such circumstances. This has resulted in the adaptation and implementation of a mid-range method, which is further enhanced by coupling it to a strategy in which multiple goals can be optimised simultaneously. The final result is an optimisation tool that better agrees with the needs that emerge from engineering practice. The method is also fit for use in applications other than low noise design.

The axisymmetric models constructed in this thesis are tested restrictedly by measuring an MRI scanner in operation. Using a laser-Doppler vibrometer, the velocity distribution on part of the inner surface of a gradient coil is sampled. The measurement setup is designed such that the strong sound fields and magnetic fields have negligible influence on measurement accuracy. One of the goals of the measurements is to learn more about the vibrational behaviour of the gradient coil.

Measurement results indicate that the coil does not behave like an axisymmetric structure. Partly for that reason, responses calculated using axisymmetric models deviate from the measured response. Also, the measurements reveal behaviour that is typical for devices with a high degree of structural complexity. Therefore, it is likely that the many details that are lacking from the axisymmetric models give rise to the discrepancies between calculated and measured responses. Modelling all these details, however, would require too much time, both for modelling and for the actual analyses. It is advised, therefore, to direct future research towards the development of efficient numerical tools in which details can be smeared out in the form of local uncertainties. Nonetheless, there is also a degree of similarity between calculated and measured responses. Therefore, the idealised models that are used in this thesis, produce a number of useful design hints.

In closing, it can be stated that, in their present form, the developed tools can
already be employed in evaluating the feasibility of new design concepts. Also, they are sufficiently efficient and versatile to quickly become familiar with the structural-acoustic behaviour of coil models that already contain more details. The developed tools are directly applicable for the low noise design of homogeneous or layered axisymmetric structures, other than MRI scanners. A relevant example is the optimisation of axisymmetric enclosures.
Summary
Samenvatting

Dit proefschrift handelt over rekengereedschap voor het geluidarm ontwerpen van MRI-scanners, die in medische klinieken worden gebruikt. Samen met een eerder verschenen proefschrift (Kuijpers, 1999) beschrijft het de resultaten van het STW-project "Modelvorming en ontwerpregels voor geluidbewust ontwerpen van MRI systemen". De MRI-scanner is een voorbeeld van een apparaat, dat door zijn werking principieel als bijverschijnsel extreem veel hinderlijk geluid veroorzaakt. Bij handhaving van de huidige werkingsprincipes zou het wenselijk zijn om op een duur geluidniveauminderingen van 40 dB of meer te realiseren in vergelijking met de scanners van nu. Zulke niveauverlagingen vereisen het combineren van een aantal vergaande ingrepen. Deze betreffen allereerst het geluid dat vanaf de gradiëntspoel via de tunnel waarin de patiënt ligt naar buiten komt. Ook speelt het geluid dat naar buiten komt vanaf de gradiëntspoel via geluid- en trillingsoverdracht naar de magneetconstructie een rol. Daarnaast ontstaat er nog geluid door het opwekken van inductieve aanslag van de magneetconstructie vanuit de gradiëntspoel. De technische uitvoering van zulke vergaande ingrepen is echter aan sterke beperkingen onderhevig voor wat betreft beschikbare ruimte, bruikbare materialen en toegestane trillingsverplaatsingen van de spoelconstructie.

Het doel van het bovengenoemde STW-project is het ontwikkelen van rekengereedschappen voor het optimaliseren van gradiëntspoelconstructies ten behoeve van geluidarme scanners. De bij de aanvang van het project reeds beschikbare softwarepakketten voor de Eindige Elementen en Randelementen Methoden zijn hiervoor niet geschikt. Alleen al het te bestrijken frequentiegebied van enkele kHz leidt tot excessief lange rekentijden voor elke afzonderlijke constructievariant. Hierdoor was het in de praktijk uitgesloten dergelijke software in te zetten als gereedschap voor ontwerpen en optimaliseren. In het proefschrift van Kuijpers (1999) is een methode ontwikkeld voor het efficiënt berekenen van de geluidafstaling van gradiëntspoelen. Door gebruik te maken van axisymmetrie en van een speciale formulering van de afstralingsoperator is een enorme reductie van rekentijden verkregen.

Het doel van dit proefschrift is het ontwerpen van veel uitgebreider numeriek gereedschap. Hierin wordt de eerder ontwikkelde akoestische module geïntegreerd. De in dit proefschrift beschreven nieuwe onderdelen hebben achtereenvolgens betrekking op de krachtaanstoting, op de spoeltent en op optimalisatie van spoelconstructies.

De beschrijving van de Lorentzkrachten, die de spoel aanstoten, gebeurt in de vorm van een ruimtelijke Fourier reeks. Hieruit zijn symmetrie condities van de
belasting afgeleid. Vervolgens is er een efficiënt rekenmodel voor de spoeltrillingen geïmplementeerd, waarbij gebruik wordt gemaakt van de globale axisymmetrie van de spoelconstructie en van de symmetrie in de belasting. Bestaande technieken zijn hierbij verbeterd, wat heeft geleid tot kortere rekentijden en een bredere toepasbaarheid. Ook is hierin eigen onderzoek verwerkt naar de modeltering van frequentieafhankelijke versterkte kunststoffen en van gelijmde constructies. Met deze rekenmodellen is het nu mogelijk om snel een afscatting te maken van het effect van een ontwerpwijziging op de geluidspopductie. Geometrie, materiaalgrend, mechanische belasting en kinetische randvoorwaarden kunnen daarbij naar believen worden geparameteriseerd. Dit wordt gedemonstreerd met een aantal van zulke parameterstudies. Daarbij is onder andere gekeken naar de effecten van het balanceren van de Lorentzkrachten. Met het gereedschap is het dus mogelijk om studies te maken van de mogelijkheden die er zijn voor geluidvermindering binnen een bepaalde ontwerpruimte. Aan de hand van zulke informatie is het mogelijk om beter gedundeerd beslissingen te nemen die nodig zijn om een bepaalde reductie te kunnen bewerkstelligen.

Uit de parameterstudies in dit proefschrift blijkt dat het akoestisch gedrag van sommige apparaten sterk kan veranderen in reactie op kleine ontwerpwijzigingen. De meeste optimaliseringsmethoden zijn echter niet effectief bij dergelijk sterk fluctuerend gedrag van responsies. Daarom is onderzoek gedaan naar een methode die wel geschikt is onder deze omstandigheden. Dit heeft geleid tot de aanpassing en implementatie van een mid-range methode. Deze is daarnaast nog gekoppeld aan een strategie waarin meerdere doelen gelijktijdig kunnen worden geoptimaliseerd. Het resultaat is een optimaliseringsgereedschap dat beter aansluit bij de praktijkbehoefte. De methode is ook goed toepasbaar buiten het gebied van geluidspopnomenen.

De in dit proefschrift ontwikkelde axisymmetrische modellen zijn in beperkte zin toegesteld aan de hand van metingen aan een in bedrijf zijnde scanner. Met behulp van een laser-Doppler vibrometer is daarbij de snelheidsverdeling op een deel van het binnenoppervlak van een gradieentspoel in kaart gebracht. De meetopstelling was zodanig ontworpen dat de aanwezige sterke geluids- en magnetische velden verwaarloosbare invloed hebben op de meetnauwkeurigheid. De metingen hebben tevens tot doel om meer kennis op te doen over het trillingsgedrag van de gradieentspoel.

Uit de meetresultaten blijkt dat de gemeten spoel zich niet gedraagt als een axisymmetrische constructie. Op een aantal punten wijken de berekenende responsies nogal af van de gemeten respons. Daarnaast is op te maken dat de spoel kenmerken vertoont die typerend zijn voor complexe constructies. Het is daarom zeker niet ondenkbaar dat de vele details, die in de modellen ontbreken, aanleiding geven tot de verschillen tussen berekende en gemeten respons. Het modelleren van deze details zal echter weer leiden tot ondoenlijk veel tijd die besteed zou
moeten worden aan modellering en aan berekeningen. Het is daarom aan te bevelen om het onderzoek voort te zetten naar de ontwikkeling van efficiënt reken gereedschap waarin details kunnen worden uitgesmeerd in de vorm van lokale onzekerheden. Niettemin bestaan er ook overeenkomsten tussen berekende en gemeten responsies. De gebruikte geïdealiseerde modellen geven daarom toch allerlei nuttige ontwerpformatie.

Afsluitend kan worden gesteld, dat het ontwikkelde gereedschap in huidige vorm reeds prima dienst kan doen in haalbaarheidsonderzoeken van nieuwe concepten. Ook is het voldoende efficiënt en veelzijdig om relatief snel vertrouwd te geraken met het structureel-akoestisch gedrag van spoelconstructies waarin al meer details zijn gemodelleerd. Het ontwikkelde rekengereedschap is ook direct toepasbaar bij het geluidarm ontwerpen van andere homogène of gelaagde axisymmetrische constructies dan de MRI scanner. Een relevant voorbeeld is het optimaliseren van axisymmetrische omkastingen.
1.1 Background

Although the Industrial Age is over, its aftermath is increasingly taking its toll on our environment. Urbanisation and mass production have led to machines and devices being present everywhere in daily life. Although they serve to aid in our work or for our amusement, their production, use and disposal often have their unpleasant side effects. Acoustic noise is one of the down sides of mechanisation, often leading to annoyance. As a consequence, market pressure and legislation dictate quiet products. This has resulted in a prime consideration of acoustic noise control in a number of industries today, most notably in aircraft and automotive industries which have otherwise technically matured. This has also become the case for MRI scanners, which is the concern of this thesis.

MRI: a sound source

MRI (Magnetic Resonance Imaging) is a non-invasive technique used to display internal structures. In medicine, MRI-scanners such as shown in figure 1.1 are applied as a diagnostic tool, able to make images of bones, tissue and blood flow. While image quality and operational speed have been advanced to their limits, sound radiation has remained a problem. As it will grow to excessive levels in future devices, the noise problem has become a main issue in MRI research and development.

In MRI, images are made by aligning the spins of the nuclei of certain atoms in a body by a magnetic field, disturbing the alignment and interpreting the returning signals. The spin alignment is achieved by a large superconducting magnet (0.5-1.5 T). By sending a radio-frequent (RF) pulse, the alignment is disturbed. After this pulse, the spin axes realign, causing a decrease of transverse magnetisation. As the amount of this decrease depends on the kind of tissue in the body, the internal structure can be reconstructed by measuring and decoding the transverse
magnetisation signal at various points in the imaging volume. The locations of
these points are controlled by superimposing a gradient on the static magnetic
field, which changes the spinning frequency of the nuclei. Only those atom nuclei
whose spinning frequencies equal the RF-frequency will be excited and respond.

The gradients are generated by sending electrical currents through a set of coils,
collectively known as the gradient coil. These currents result in Lorentz forces as
they traverse the strong static field of the superconducting magnet, causing the
gradient coil to vibrate. This is the main source of the loud hammering sound
that has become characteristic for MRI scanners.

Even small vibrations can lead to audible sound because of the sensitivity of our
hearing. The human ear is capable of detecting sounds that displace the eardrum
by only one-tenth of the diameter of an hydrogen molecule (Kinsler et al., 1982). In
air, for a young person with normal hearing at 2 kHz, this corresponds to
sound pressure fluctuations of approximately $2 \cdot 10^{-5}$ Pa and sound intensities
of $10^{-12}$ W/m². As the audible intensities range approximately from this lower
bound to $10$ W/m², a logarithmic scale is used in acoustics. Usually decibels
are applied to denote the magnitudes of physical quantities. Another reason for
doing so is that an increase in sound intensity by a factor of ten (an increase of
10 dB) is perceived as a doubling of the loudness. As the perceived loudness is
frequency dependent, a weighted decibel scale is a better way to express sound
levels in environmental noise problems. The most common correction is to use
so-called A-weighted levels, abbreviated to dB(A).
A decade ago, MRI sound pressure levels ranged (Hurwitz et al., 1989) from 84 to 103 dB (82 to 93 dBA) for a number of different manufacturers. Although these levels do not result in permanent hearing damage because of the short exposure time, they are annoying and lead to temporary hearing loss in 43% of patients undergoing MR imaging without ear protection (Brummet et al., 1988). Since then, design changes aimed at improving imaging speed and quality have led to even higher levels. Also, acoustic noise has side-effects such as imaging disturbances due to stimulation of auditory sensory portions of the brain in the temporal lobes (Cho et al., 1998). This situation is undesirable and even potentially dangerous for patients and medical staff, therefore being an important marketing issue. High-intensity sound may degrade performance of certain tasks and lead to irritation ranging from annoyance to rage (Ward, 1997). At levels of 130 dB sound becomes painful and may even affect the functions of internal organs. Permanent hearing loss is caused by short exposures to high sound levels or by repeated exposure to more moderate levels. The specific effects of sound largely depend on individual differences. Despite many studies, however, particular characteristics such as age, sex and experience have never been demonstrated (Ward, 1997) to correlate with susceptibility to hearing damage.

Research project

While the noise problem has been a concern for a long time, solutions have been less obvious. For instance, while the sound levels are found to increase with stronger static magnetic field strengths for a particular scanner, scanners from different manufacturers do not correlate with static field strength (Hurwitz et al., 1989). For a 0.35 T scanner a noise level of 103 dB is found, while for another 1.5 T device a lower level of 94 dB is reported, both during gradient-echo imaging. A different design can thus have substantial influence on the sound production. However, computational tools and knowledge needed to analyse sound problems for a device as complex as MRI scanners and for other noise producing structures have not been available. In order to close this gap, a research project has been started at the Eindhoven University of Technology in cooperation with Philips Medical Systems and funded by the Dutch Technology Foundation (STW). The main goal of this project is to supply computer-aided tools and modelling directives to support engineers in including structural-acoustics in the design stages. This effort has been split in two closely related parts. Kuijpers (1999) has developed efficient computational tools for acoustic analyses of (nearly) axisymmetric structures and investigated their application in a doctoral thesis. That work is complemented here with the development and application of numerical models, efficient numerical structural analysis methods and optimisation strategies. Choices made in the development of these tools are based on the specifics of the MRI noise problem, which will be discussed in the next
section. As many features of this problem can be found in other applications as well, most of the results from the project can be employed more broadly.

1.2 Identification of the MRI noise problem

The first step in noise reduction is to reveal all aspects that are relevant to the sound problem. This includes a description of the responsible sources and the paths through which sound is transferred to the receiver, as well as of the frequency bands in which sound problems occur. As the Philips Gyroscan T5A MRI-scanner will be used in this thesis as the object of study, mechanisms and circumstances related to its sound production are identified here. This information is essential in deciding which actions are expected to be effective for noise reduction, and what kind of engineering tools will be required. Appropriate measures that can be or already have been applied will be discussed in the next section. The parts of the scanner that are relevant to the sound problem are shown in figure 1.2.

![Diagram of MRI scanner parts](image)

(a) Open view, without casing  
(b) Cross-section, including casing

**Figure 1.2:** Schematic representations of the parts relevant to the acoustic radiation.

The sound paths that lead from the source to the receiver (patients and medical staff) are shown schematically in figure 1.3. For an MRI-scanner, all sound radi-
ation originates from a single source: the electrical currents in the gradient coil. These cause Lorentz forces in the gradient coil itself, but they also induce eddy currents in the cryo magnet and in the cryo magnet housing.

In the windings of the gradient coil, the electrical currents generate fluctuating Lorentz forces up to 1250 Hz. These cause vibrations which, in turn, lead to acoustic radiation from the coil's surfaces in the same frequency range. The gradient coil supports and the RF-coil, which are connected to the gradient coil, are mechanically excited.

In metallic parts of the housing and in the coils of the cryo magnet, eddy currents are induced by the fluctuating magnetic gradient fields. These currents also result in Lorentz forces, causing vibrations in the cryo magnet. The gradient coil supports are connected to the cryo magnet housing and to the patient bed bridge. These are therefore also excited by the Lorentz forces in the gradient coil.

Radiated sound waves from the gradient coil and the cryo magnet housing cause the scanner's casing to vibrate and to radiate sound externally. The casing is also excited mechanically since it is connected directly to vibrating parts: the RF-coil and the cryo magnet housing. Other contributions to the noise are the parts of the internally generated sound that escape through openings in the casing, and sound radiation from the patient bed.

Investigations have indicated (Kooymans et al., 1993, 1994) that the gradient coil and the cryo magnet contribute equally to the total sound radiation. Since elimi-
1. Introduction

ininating only one of them would lead to a noise reduction of only 3 dB, both of these sources need to be treated. As part of the vibrations in the cryo magnet are the direct result of vibrations in the gradient coil, it makes sense to start with alterations in the gradient coil. For these actions to be effective, however, radiation resulting from the cryo magnet has to be reduced also at some stage.

The frequency band of the noise problem ranges (Kooymans et al., 1994) from 250 to 1250 Hz. Because of the linear behaviour of the system, the maximum frequency in the frequency spectrum of the excitation determines the upper bound of the vibrational spectrum and therefore the noise frequency band. These relatively high frequencies are caused by the short rise times of the pulse shapes of the signals used in controlling the gradient coils.

1.3 Noise control in MRI-scanners

As was seen in the previous section, all acoustic noise that is perceived by patients and medical staff is radiated from the patient bed, from the casing and through openings in the casing. Applying absorbing materials to the casing and closing the openings has resulted in reductions of up to 6 dB.

Additional measures may be aimed at reducing the sources of the casing vibrations. As was seen, the currents in the gradient coil windings are the source of all of the vibrations in the scanner. Reducing the amplitudes of these currents, however, is not an option as that would deteriorate the imaging quality. The next best measure would then be to reduce the effects caused by the currents, as they indirectly excite structural parts. Amplitudes of the Lorentz forces are partly determined by the strength of the main magnetic field, which, like the amplitudes of the currents, can not be lowered. Also of influence to Lorentz force amplitudes is the geometry of the coils, which contributes to the spatial distribution of the excitation. Altering the geometry of the windings, however, also affects the produced magnetic gradient fields. Nevertheless, it might be possible to design a different coil set that has a more favourable Lorentz force distribution while generating gradient fields of comparable quality.

Improving the shielding of gradient fields outside of the gradient coil will reduce Lorentz forces in the cryo magnet. Lorentz forces caused by eddy currents would be proportionally reduced.

Mechanically decoupling the gradient coil from the cryo magnet is possible by disconnecting the gradient coil supports from the cryo magnet. As an alternative, active isolation (Bies and Hansen, 1996) can be applied to reduce vibrations
on strategic places in the structure. This can be used to prevent structure-borne noise from the vibration source to reach other parts of the structure which could otherwise also radiate sound. In MRI scanners, this technique can be applied to block the transmission path up to 5 kHz between the gradient coil and other parts of the scanner such as the cryo magnet and the patient bed.

In Active Structural Acoustic Control (ASAC), structurally radiated sound is directly controlled by active structural inputs. The controlled system should have a lower radiation efficiency than the uncontrolled system. A reduction of sound levels is sometimes accompanied by an increase in structural vibration levels. Fuller et al. (1996) report a reduction of interior noise in a jet aircraft fuselage of 10 dB at a 170 Hz excitation. Besides controller design, ASAC requires solving problems such as determining locations and number of sensors and actuators. For application in the gradient coil, this approach has been deemed impractical. While being theoretically applicable, the relatively large number of required actuators renders the control problem too complex (van Schothorst, 1998).

An additional option is to reduce sound after it has been radiated by applying active noise control. A secondary sound field (commonly known as anti-noise) is created that together with the primary sound field causes a net cancellation. This is especially successful in the low-frequency range. Goldman et al. (1989) used this technique in MRI, in combination with passive attenuation. An average reduction of 14 dB is reported, of which 11 dB results from active noise control inside the patient’s ear covers and the rest from the ear covers themselves. The sound cancellation system worked optimally up to 500 Hz.

In order to have the most substantial effect, noise control has to be incorporated early in the design process. An example of introducing a different design concept to MRI scanners for spin-echo imaging was introduced by Cho et al. (1998). The pulses of the 𝑥 and 𝑦 gradient coils are replaced by a constant DC voltage, using mechanically rotating coils. A reduction of 21 dB is reported, although it should be noted that in their case radiation by the 𝑧 coil is small. This is not always the case, so the net result would be smaller.

While some of these previous attempts have been successful, they have not led to silent MRI scanners yet. To be at an acceptable level, the noise production has to be decreased by at least 40 or 50 dB, and preferably even more. This requires more fundamental approaches, and a combination of both active and passive methods. The next section will go further into the matter of passive methods, as that is the focus of this project.
1.4 Passive noise control

Passive noise control involves altering the design or its surroundings, without resorting to actively controlled inputs. All efforts are taken at the design stages. It is therefore instructive to see which factors may influence the vibrational and acoustic behaviour of structures.

A structure has certain properties that are relevant to its vibrational behaviour, which can be expressed in terms of structural modes and eigenfrequencies. These properties are constant characteristics as long as the structure itself does not change. Which of these modes appear in the vibrational response, and with what relative contribution, depends on the spatial distribution and the frequency spectrum of the excitation. These structural properties depend on the structure's shape, on the distribution of mass and material modulus properties, and on the kinematic boundary conditions, as shown in figure 1.4.

![Diagram showing the relationship between mechanical properties, structural analysis, acoustic analysis, and acoustic radiation.]

**Figure 1.4:** Properties relevant to passive noise control.

Like the properties that are related to the vibrational response, a structure also has properties that define its acoustic radiation characteristics with respect to total radiated acoustic power. These acoustic radiation properties can be defined
in terms of radiation modes and radiation efficiencies. Radiation modes represent normal velocity distributions at the radiating surfaces, while their corresponding efficiencies determine how well these mode shapes can radiate sound. Acoustic properties depend solely on the part of the structure's outer surface that is in contact with the acoustic medium, often called the wet surface.

All of the mechanical properties thus influence the acoustic radiation of a structure. Passive noise control can now be defined as modifying a structure’s vibrational and/or acoustic properties by changing its mechanical properties such that the sound radiation is reduced.

General guidelines that offer directions of how to change these properties can not be posed, except maybe for the most simple structures such as flat plates. Increasing stiffness, for example, may lead to reductions of vibration levels in a certain frequency band. This, however, may cause the structure to radiate more efficiently in that frequency band or lead to increased vibration amplitudes at other frequencies. Also, the mechanical properties are interrelated; changing the mass of a structure, for instance, requires changing the geometry or using other materials. As it is generally impossible to foresee the effects of alterations of mechanical properties by experience, designers require predictive tools.

1.5 Research objective

Passive noise control requires the prediction of the effects of one or more design changes to acoustic radiation. The goal of this thesis is to develop a numerical tool for this purpose, and to construct models for the MRI gradient coil. This objective is obtained by:

- constructing models that are as efficient as possible but give useful predictions with respect to acoustic radiation,
- evaluating and integrating necessary numerical analysis tools,
- developing more efficient tools where necessary,
- using the developed tools to investigate the effects of structural inhomogeneities and fabrication inaccuracies,
- evaluating methods that partially automate the optimisation of structures in structural acoustics.

This approach will now be explained in more detail.
1.6 Scope

In this thesis, the effects of material properties, internal geometry, inhomogeneities and fabrication inaccuracies on the vibration and on the externally radiated sound power levels of the gradient coil are examined. Although the gradient coil is not the only sound source, it is the source that will be hardest to reduce. Due to its structural complexity, it is difficult to model its vibrational behaviour. Also, by optimising the gradient coil with respect to sound radiation, vibrations will probably be reduced in other parts of the scanner. However, care has to be taken as they may be increased as well. If the sound radiation caused by the gradient coil can not be reduced, measures aimed at other components of the scanner will have marginal effect on the total sound radiation.

Although the MRI gradient coil is the object of study, many of the results will also be applicable to other problems in structural acoustics.

Modelling mechanical properties

The effects of changing the surface velocity distribution are investigated by varying the mechanical properties of the gradient coil except for the wet surface. Effects of changing the wet surface on the radiation properties of the MRI-scanner have already been studied by Kuijpers (1999). In this thesis the radiation properties will be held constant, using the outer geometry of the current design without the casing and the RF-coil (see figure 1.2).

Because of the structural complexity, and the required efficiency of the modelling process, it is of importance to know which of the structural features are relevant to the acoustic radiation. At sound levels of 40 dB(A) or higher, the human ear is insensitive to differences smaller than 1 dB between 100 and 1000 Hz (Kitsler et al., 1982). At other frequencies differences up to 2 dB can hardly be perceived. Such deviations in the predictions of the actual radiation are therefore considered irrelevant. The influence of some of the features may also be neglected compared to deviations due to imperfections. These imperfections can for instance be caused by manufacturing variations, and can be expected to occur more frequently when the structure is more complex.

The gradient coil is constructed from metals, polymers and composite materials. Because the vibration amplitudes are small, the behaviour of these materials can be represented by assuming linearly elastic and viscoelastic behaviour. Most composites, such as fibre reinforced materials, are handled by using relatively straightforward rules for constructing equivalent constitutive relations. As viscoelastic behaviour usually leads to frequency dependency, the matrix systems
1. Introduction

used to compute the model deflections become frequency dependent which has negative consequences to computational speed. The relevance of incorporating the frequency dependency of materials in the models will therefore be investigated.

**Frequency range and Fourier expansions**

In MRI, the signals that are used to control the magnetic gradients depend on the kind of imaging that is required in a given situation. Therefore, a scanner has a range of possible scan sequences, most of which are block shaped. In order to design scanners that produce less noise, the total sound level in the operational frequency range, which is up to 1250 Hz, will have to be reduced. Higher operational frequencies are anticipated by performing analyses up to 2000 Hz. This means that the models will have to be accurate at high frequencies. A full scale accurate 3D-model of the gradient coil, however, will take prohibitive amounts of CPU-time if it is to be applied in a design environment. By using Fourier expansions in the analysis of axisymmetric problems, significant time reductions are achieved without loss of accuracy, both in structural and acoustic analyses.

**Effects of inhomogeneities**

While using Fourier expansions does not pose a problem for the acoustic model of the gradient coil radiation because of the axisymmetric radiating surface, the gradient coil is not axisymmetric internally. Structural analyses could therefore fail, which means that much attention has to be paid to the structural model. The speed of Fourier analyses, however, justifies an investigation into its performance in nearly-axisymmetric structures. Inhomogeneities in radial and axial directions can be modelled more easily using a Finite Element implementation for discretisation.

**Effects of fabrication inaccuracies**

The engineer is faced with uncertainties when dealing with structural-acoustics. Firstly, a model has inherent inaccuracies. Secondly, two structures are never identical due to spreadings in fabrication. It is therefore relevant to investigate the influence of uncertainties to the response. When fabrication variations are substantial, the effects of design changes will also show a certain variation.

Measurements have indicated that sound pressure levels of different gradient coils manufactured according to the same specifications differ by several decibels. This has two consequences: the effects of changing some of the structure's
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features may be small compared to variations due to (random) imperfections, and changing some of the features may influence the structure's sensitivity to imperfections. Because of the importance of these imperfections, they will be considered throughout.

Optimisation of the design

Using the developed models and analysis methods, the performance of numerical optimisation in structural acoustics are evaluated. The effects of inhomogeneities and inaccuracies will be incorporated in the design strategies. When optimisation methods are applied in structural acoustics, the algorithms typically spend time in analysing designs while trying to improve a design within a 1 dB range before converging. As stated above, such small differences in acoustic response are not or hardly noticeable. As this unnecessarily consumes analysis time, a method that has a more favourable convergence behaviour will be sought for.

1.7 Outline

In the next Chapter, the equations that govern the vibrational and acoustic response are presented. These will be discretised using methods based on a finite element formulation and Fourier expansions, resulting in uncoupled matrix systems for the structural and the acoustic analysis. The meaning and benefits of radiation modes are explained.

Models for the structural analysis of the MRI gradient coil are developed and validated in Chapter 3. It starts with a more detailed description of the gradient coil. The spatial distributions of the Lorentz forces generated in its windings are derived. After presenting the vibration measurements on the coil's radiating surface, several models of increasing complexity are constructed. The importance of incorporating inhomogeneities is evaluated.

Before numerical optimisation will be applied, the sensitivity of the response to changes in different mechanical properties is evaluated in Chapter 4 using parameter studies. Effects of fabrication inaccuracies to the vibrational and acoustic response are also investigated.

With the availability of analysis tools and numerical models, numerical optimisation methods are employed. Their application in structural acoustics is covered in Chapter 5.
Finally, in Chapter 6 the main results and conclusions are summarised and discussed. Also, directions for further improving the models and extending the applicability of the analysis and optimisation methods for the MRI-scanner and other areas are given.
1. Introduction
2.1 Introduction

Successful integration of low noise designs procedures for MRI-scanners largely depends on the efficiency of the tools that are applied to simulate the structural-acoustic response of the system. A single analysis should only take a limited time to complete, while at the same time it has to be accurate and provide enough flexibility to handle features in the structure that are of importance to acoustic radiation.

The gradient coil can roughly be viewed as a layered cylindrical system, where the layers can have different lengths and consist of different materials. Strictly speaking, it can not be regarded as an axisymmetric structure. For initial models, however, assuming axisymmetry for the geometry and material distribution can be very instructive.

In this Chapter, several methods are presented that are suitable for preliminary structural and acoustic analyses of the existing gradient coil. Their application is not limited to such structures. They can be used to evaluate the structural and acoustic response of axisymmetric structures that consist of anisotropic linear elastic and/or linear viscoelastic (composite) materials and that are subjected to arbitrary mechanical surface tractions. Most of the methods that will be discussed here are efficient but accurate analysis techniques, to which new modifications are introduced that either shorten the required analysis times even further, or extend their range of applicability.

In the next section, the differential equations and boundary conditions are summarised that describe the vibration and acoustic radiation of structures. Constitutive equations that characterise the behaviour of linear viscoelastic materials are presented in section 2.3. Also, it is explained how these constitutive relations
can be used to construct equivalent material parameters for composites. This is followed by the derivation of matrix equations by discretising the differential equations for the structure together with the constitutive equations. These matrix equations can be solved to obtain the structural response. Efficient acoustic analysis methods for this project have been developed by Kuipers (1999), and they are treated briefly in section 2.4.

2.2 Governing equations

In this section, the differential equations that describe the acoustic radiation by a structure that is excited by harmonic external forces and emerge in a fluid will be presented. The structure with specific mass $\rho(\vec{x})$ and domain $\Omega_s$ displayed in figure 2.1 has a displacement field $\vec{u}(\vec{x}, t)$. The system is assumed to behave linearly, so a harmonic excitation leads to a harmonic response

$$\bar{q}(\vec{x}, t) = \text{Re} \left( \bar{q}(\vec{x}) e^{j\omega t} \right) \quad \text{on } \Gamma_q, \quad \bar{u}(\vec{x}, t) = \text{Re} \left( \bar{u}(\vec{x}) e^{j\omega t} \right) \quad \text{on } \Omega_s \quad (2.1)$$

In the absence of body forces, the displacement field is described by the equations of motion:

$$\omega^2 \rho(\vec{x}) \bar{u}(\vec{x}) + \nabla \cdot \bar{S} = 0 \quad \text{on } \Omega_s \quad (2.2)$$

![Figure 2.1: Domains and relevant quantities of an external radiation problem.](image-url)
2. Analysis methods for axisymmetric structures

where $\mathbf{S}$ is a Cauchy stress tensor that relates stresses in the domain $\Omega_\alpha$ to the deformations and therefore to the displacement field. On a part $\Gamma_u$ of the boundary, the displacement amplitudes may be prescribed, which is denoted by the kinematic boundary condition:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_{\Gamma_u}(\mathbf{x}) \quad \text{on } \Gamma_u$$ (2.3)

On a part or on the entire surface of the structure, external surface traction amplitudes are prescribed:

$$q(\mathbf{x}) = q_{\Gamma_q}(\mathbf{x}) \quad \text{on } \Gamma_q$$ (2.4)

Acoustic pressure amplitudes $p(\mathbf{x})$ in the fluid domain $\Omega_\alpha$ are governed by the Helmholtz equation:

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0 \quad \text{on } \Omega_\alpha$$ (2.5)

where $p(\mathbf{x})$ is the acoustic pressure amplitude, and $k$ is the free field wavenumber $k = \omega/c_0$, with $c_0$ being the speed of sound in the acoustic fluid. The acoustic pressure amplitudes are related to velocity amplitudes in the acoustic domain by the momentum equation:

$$j \rho_0 \omega \mathbf{v}(\mathbf{x}) + \nabla p(\mathbf{x}) = 0 \quad \text{on } \Omega_\alpha$$ (2.6)

where $\rho_0$ is the constant equilibrium density of the fluid. Structural vibrations and acoustic pressures are interrelated, expressed by an interface condition that is imposed on their common boundary $\Gamma_a$:

$$\mathbf{v}(\mathbf{x}) \cdot \mathbf{n} = j \omega \mathbf{v}(\mathbf{x}) \cdot \mathbf{n} = v_n(\mathbf{x}) \quad \text{on } \Gamma_a$$ (2.7)

where $\mathbf{n}$ is the surface normal, and $v_n(\mathbf{x})$ is the surface normal velocity amplitude. This relation expresses that the vibrating structure can only move fluid particles at the interface in a direction normal to their common surface. Vice versa, the structure only experiences forces by the acoustic fluid in that direction. The displacement field of the structure and the acoustic pressures therefore have to be determined from a set of coupled equations. However, acoustic pressures in air are typically very small, and their influence on the behavior of the structure can be neglected if the structure is heavy. For the gradient coils, it can therefore safely be assumed that the structure behaves as in vacuum. The acoustic analysis can thus be uncoupled from the structural analysis, and the two problems will be solved separately.

Except for the most trivial cases, the above equations cannot be solved analytically. Instead, approximations are computed after discretisation of the equations. Before the differential equations (2.2) can be discretised, however, an explicit relationship between the Cauchy stress tensor and the displacement field is required.
2. Analysis methods for axisymmetric structures

Only displacements that lead to deformations cause stresses, so only gradients of the displacements appear in such relations. Therefore, the resulting equations of motion will be ordinary differential equations in terms of the displacement amplitudes.

In the following section, an expression for the Cauchy stress tensor will be presented that is valid for both linear elastic and linear viscoelastic material behaviour. A discretisation of the equations of motion will be derived using a Fourier series approximation for the circumferential coordinate direction, and a finite element procedure in the other coordinate directions.

2.3 Methods for the structural analysis

2.3.1 Material behaviour

As was presented in the previous section, structural vibrations \( \ddot{u} \) are to be derived from the differential equations of motion:

\[
\omega^2 \rho \ddot{u}(\vec{x}) + \nabla \cdot \mathbb{S} = 0 \quad \text{on } \Omega_s
\]

(2.8)

in which \( \rho \) is the specific mass in a material point, and \( \mathbb{S} \) is the Cauchy stress tensor. The relationship between stresses and the displacements depends on the specific material, and is given by a constitutive relation that models the material’s behaviour using a relation between stresses and strains. A second relationship is required that relates displacements to strains. The mechanical properties of most real materials can be described by assuming viscoelastic behaviour (Malkin, 1994). Viscoelasticity is a combination of characteristics of liquids (viscous dissipative losses) and solids (storage of elastic energy). Viscoelastic behaviour can be considered as a delayed development of stresses and deformations in time. An expression for the Cauchy stress tensor for these materials will now be presented.

Constitutive model for linear viscoelastic materials

In a viscoelastic material, stress is a function of the strain history. Its behaviour is strain rate dependent, and energy is dissipated when the material experiences deformations. A general constitutive relation for viscoelastic materials can be expressed as:

\[
P(\vec{x}, t) = G(\mathbb{E}(\vec{x}, \tau) | -\infty \leq \tau \leq t)
\]

(2.9)
where \( \mathcal{P} \) is the second Piola-Kirchhoff stress tensor, and \( \mathcal{E} \) is the Green-Lagrange strain tensor. When stresses only depend on the most recent changes in the strains (fading memory) and the strains are zero for \( \tau \leq 0 \), relation (2.9) can be written as:

\[
P(\bar{x}, t) = \int_0^t 4\mathcal{R}(\bar{x}, \mathcal{E}(\bar{x}, t - \tau)) : \mathcal{D}(\bar{x}, \tau) \, d\tau
\]

where \( 4\mathcal{R} \) is a fourth-order tensor of relaxation functions. A material is called linear viscoelastic if \( 4\mathcal{R} \) does not depend on the strain tensor \( \mathcal{E} \). When additionally the deformations and displacements are small, the constitutive relation is described by the Boltzmann formulation:

\[
\mathcal{S}(\bar{x}, t) = \int_0^t 4\mathcal{R}(\bar{x}, t - \tau) : \mathcal{D}(\bar{x}, \tau) \, d\tau
\]

(2.11)

where \( \mathcal{D} \) is the infinitesimal strain tensor:

\[
\mathcal{D}(\bar{x}, \tau) = \mathcal{D}(\bar{\mathcal{u}}(\bar{x}, \tau)) = \frac{1}{2} \left( \left( \nabla \bar{\mathcal{u}}(\bar{x}, \tau) \right) + \left( \nabla \bar{\mathcal{u}}(\bar{x}, \tau) \right)^T \right)
\]

(2.12)

Since the constitutive model is being derived for the harmonic case, the strain tensor \( \mathcal{D} \) can be written as:

\[
\mathcal{D}(\bar{\mathcal{u}}(\bar{x}, \tau)) = \mathcal{D}(\bar{\mathcal{u}}(\bar{x})) e^{j\omega \tau}
\]

(2.13)

and the Boltzmann formulation (2.11) becomes:

\[
\mathcal{S}(\bar{x}, t) = j \omega \int_0^t 4\mathcal{R}(\bar{x}, \xi) : \mathcal{D}(\bar{\mathcal{u}}(\bar{x})) e^{j\omega(1-\xi)} \, d\xi
\]

(2.14)

where \( \xi = t - \tau \). The integration interval can be extended since the strains are zero outside of the interval, leading to the Fourier transform:

\[
\mathcal{S}(\bar{x}, t) = j \omega \int_{-\infty}^{\infty} 4\mathcal{R}(\bar{x}, \xi) e^{-j\omega \xi} \, d\xi : \mathcal{D}(\bar{\mathcal{u}}(\bar{x})) e^{j\omega t}
\]

(2.15)

The constitutive relation can then be written as:

\[
\mathcal{S}(\bar{x}, t) = 4\mathcal{G}(\bar{x}, \omega) : \mathcal{D}(\bar{\mathcal{u}}(\bar{x}, t))
\]

(2.16)

where \( 4\mathcal{G} \) is the complex modulus tensor, which for linear viscoelastic materials is related to the Fourier transform \( \mathcal{F}(4\mathcal{R}) \) of the relaxation tensor:

\[
4\mathcal{G}(\bar{x}, \omega) = j \omega \mathcal{F}(4\mathcal{R}(\bar{x}, t))
\]

(2.17)
Alternatively, the constitutive model can be expressed as:

\[ S(\vec{x}) e^{j\omega t} = \mathbf{4G}(\vec{x}, \omega) : \mathbf{D}(\vec{u}(\vec{x})) e^{j\omega t} \tag{2.18} \]

or, by dropping the exponents:

\[ S(\vec{x}, \omega) = \mathbf{4G}(\vec{x}, \omega) : \mathbf{D}(\vec{u}(\vec{x})) \tag{2.19} \]

Compressible elastic materials can be seen as a special case of the class of viscoelastic materials, where viscous effects can be neglected. The stresses can be expressed in similar form:

\[ S(\vec{x}) = \mathbf{4H}(\vec{x}) : \mathbf{D}(\vec{u}(\vec{x})) \tag{2.20} \]

in which \( \mathbf{4H} \) is a real constant fourth-order elasticity tensor. In the frequency domain, constitutive relations for elastic and viscoelastic can thus be written in similar form for harmonic cases. This is known as the elastic-viscoelastic correspondence principle (Gibson, 1994). The material tensor is real for elastic materials, and complex and frequency dependent for viscoelastic materials.

### 2.3.2 Composites

While polymers are suitable in noise reduction because of their good damping properties, they provide little stiffness. Therefore, they are often reinforced, for instance by embedding fibres in a polymer matrix, at the cost of impairing the damping properties. Constitutive relationships can be derived for a large class of composites by relatively straightforward rules (e.g., Tsai and Hahn, 1980; Eduljee and McCullough, 1993), called rules-of-mixtures. This is possible when material inhomogeneities are small compared to the smallest wavelength of deformation that is of interest. In the micro-mechanical models used in the derivation of the rules-of-mixtures, linear elasticity usually is assumed. Due to the elastic-viscoelastic correspondence principle that was discussed in the previous subsection, these relations are also valid for linear viscoelastic materials (Gibson, 1994). Frequency dependent behaviour for a composite can therefore be determined from the frequency dependent properties of (some of) the constituents. The equivalent material properties then have to be evaluated in the frequency domain. The longitudinal complex modulus of a fibre-reinforced composite, for instance, can be determined from (Gibson, 1994):

\[ G_L(\omega) = v_f G_f(\omega) + v_m G_m(\omega) \tag{2.21} \]

where \( G_f \) is the longitudinal complex modulus of the fibre, \( G_m \) is the complex modulus of the isotropic matrix material, and \( v_f \) and \( v_m \) are the respective volume fractions of the constituents. Other equivalent material properties can be
2. Analysis methods for axisymmetric structures

derived using similar expressions. In this work, the Tsai-Halpin relations are used, including the modified rule-of-mixture equations for transverse and shear moduli. These relations are summarised in Appendix A.

Material tailoring

Since the (visco)elastic properties of a composite can be influenced by changing volume fractions of the constituents or by modifying the orientation of reinforcement materials, the behaviour of a material can be tailored. Examples of such material tailoring in the gradient coil will be presented in Chapters 4 and 5. Measurements by Suárez et al. (1986) have shown that material loss factors of a fibre-reinforced material also can be influenced by varying fibre orientation. Stiffness, damping and mass can therefore be optimised for a structure.

2.3.3 Fourier finite elements

In this section, the differential equations that govern the structural response will be transformed into a set of linear matrix equations that can be used to obtain the displacement field. A new derivation is thereby introduced that leads to an extra efficiency improvement. The formulation will be valid for structures that are geometrically axisymmetric, and for which the material distribution is homogeneous in circumferential direction:

\[ \rho(x) = \rho(r, z), \quad \mathbf{G}(x, \omega) = \mathbf{G}(r, z, \omega) \]  

(2.22)

where \( \mathbf{G} \) is a fourth-order complex modulus tensor, valid for anisotropic linear viscoelastic material behaviour. Surface tractions are assumed to be time harmonic, and are allowed to have arbitrary spatial distribution. Kinematic boundary conditions may be applied, but have to be known in advance for the full circumference for reasons that will be explained later.

The differential equations of motion can now be written as:

\[ \omega^2 \rho(r, z) \ddot{u}(r, \theta, z) + \nabla \cdot \mathbf{S}(r, \theta, z, \omega) = 0 \quad \text{on } \Omega_u \]  

(2.23)

with boundary conditions:

\[ \ddot{u}(r, \theta, z) = \ddot{u}e(r, \theta, z) \quad \text{on } \Gamma_u \quad \ddot{q}(r, \theta, z) = \ddot{q}e(r, \theta, z) \quad \text{on } \Gamma_q \]  

(2.24)

and constitutive relation:

\[ \mathbf{S}(r, \theta, z, \omega) = \mathbf{G}(r, z, \omega) : \mathbf{D}(\ddot{u}(r, \theta, z)) \]  

(2.25)
where $\mathbf{D}$ is an infinitesimal strain tensor:

$$
\mathbf{D}(\bar{\mathbf{u}}(r, \theta, z)) = \frac{1}{2} \left( \left( \nabla \bar{\mathbf{u}}(r, \theta, z) \right)^T + \left( \nabla \bar{\mathbf{u}}(r, \theta, z) \right) \right)
$$

(2.26)

The displacement amplitude field can be discretised using finite elements, as shown in figure 2.2 (a) for a ring-shaped structure. Within an element, displacements are then approximated by

$$
\bar{\mathbf{u}}(r, \theta, z) \approx \bar{\mathbf{e}}^T \sum_{i=1}^{n_c} N_i(r, \theta, z) \mathbf{u}_i
$$

(2.27)

where $\bar{\mathbf{e}}^T = [\bar{e}_r \, \bar{e}_\theta \, \bar{e}_z]$, $n_c$ is the number of nodes in the element and $N_i$ is a finite element interpolation function for node $i$. The column matrix $\mathbf{u}_i$ contains the displacement amplitude components at that node. By storing the interpolation functions in a matrix $\mathbf{N}$, the discretisation can be written as:

$$
\bar{\mathbf{u}}(r, \theta, z) \approx \bar{\mathbf{e}}^T \mathbf{N}(r, \theta, z) \mathbf{u}
$$

(2.28)

where $\mathbf{u}^T = [\mathbf{u}_1^T \ldots \mathbf{u}_{n_c}^T ]$. Alternatively, for axisymmetric structures, the circumferential variations of the displacement amplitudes can be expanded into Fourier series. The displacements are split into a part $\bar{\mathbf{u}}^s$ and a part $\bar{\mathbf{u}}^d$ that contain sine and cosine terms. A truncated Fourier series approximates the displacement field, in combination with a finite element discretisation for the other coordinate.
directions:

\[
\vec{u}(r, \theta, z) \approx \epsilon^T \sum_{m=0}^{n_f} \mathbf{N}(r, z) \begin{bmatrix} u_m^r \cos(m \theta) + u_m^\alpha \sin(m \theta) \\ u_m^\alpha \sin(m \theta) + u_m^r \cos(m \theta) \\ u_m^z \cos(m \theta) + u_m^z \sin(m \theta) \end{bmatrix}
\] (2.29)

where \( u^r \) and \( u^\alpha \) contain Fourier coefficients of the nodal displacement amplitudes. The specific choice for the ordering of sine and cosine terms facilitates the determination of the gradients of the displacements in the determination of an expression for the strains. Using Fourier expansions in combination with FEM, a ring can then be discretised using a single element, as shown in figure 2.2(b). The number of nodes needed for the finite element discretisation is now reduced considerably. The number of degrees of freedom per node, however, has increased from three to \( 6n_f \). Fortunately, as derived in Appendix B, using the Fourier expansion in the discretisation of the differential equations of motion leads to a set of \( n_f + 1 \) uncoupled systems:

\[
(-\omega^2 \mathbf{M} + \mathbf{K}_m(\omega)) \mathbf{u}_m = \mathbf{f}_m \quad m = 0, \ldots, n_f
\] (2.30)

where

\[
\mathbf{u}_m = \begin{bmatrix} u_m^r \\ u_m^\alpha \\ u_m^z \end{bmatrix}, \quad \mathbf{f}_m = \begin{bmatrix} f_m^r \\ f_m^\alpha \\ f_m^z \end{bmatrix}
\] (2.31)

so each node has six degrees of freedom per Fourier harmonic. Prescribing kinematic boundary conditions is handled by prescribing their Fourier transforms:

\[
\mathbf{u}_0(r, z) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{u}(r, \theta, z) \, d\theta \quad \text{on } \Gamma_u
\] (2.32)

\[
\mathbf{u}_m^r(r, z) = \frac{1}{\pi} \int_0^{2\pi} \begin{bmatrix} u_r(r, \theta, z) \cos(m \theta) \\ u_\theta(r, \theta, z) \sin(m \theta) \\ u_z(r, \theta, z) \cos(m \theta) \end{bmatrix} \, d\theta \quad \text{on } \Gamma_u \text{ for } m > 0
\] (2.33)

\[
\mathbf{u}_m^\alpha(r, z) = \frac{1}{\pi} \int_0^{2\pi} \begin{bmatrix} u_r(r, \theta, z) \sin(m \theta) \\ u_\theta(r, \theta, z) \cos(m \theta) \\ u_z(r, \theta, z) \sin(m \theta) \end{bmatrix} \, d\theta \quad \text{on } \Gamma_u \text{ for } m > 0
\] (2.34)

where \( \mathbf{u}_0 = \mathbf{u}_0^r + \mathbf{u}_0^\alpha \). Kinematic boundary conditions over only part of the circumference can therefore not be prescribed; the Fourier integral can not be evaluated for those cases.

The excitation column matrices \( \mathbf{f}_m^r \) and \( \mathbf{f}_m^\alpha \) containing the nodal forces are obtained from discretising the Fourier expansions of the external surface loads (see
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Appendix B:

\[
\begin{align*}
\mathbf{f}_m^a &= \int \mathbf{N}^T(r, z) \mathbf{q}^a_m(r, z) \, r \, dl(r, z) \\
\mathbf{f}_m^b &= \int \mathbf{N}^T(r, z) \mathbf{q}^b_m(r, z) \, r \, dl(r, z)
\end{align*}
\]

(2.35)

(2.36)

where \(\mathbf{q}^a_m\) and \(\mathbf{q}^b_m\) are Fourier coefficients of the distributed surface tractions, that can be determined in the same way as the kinematic boundary conditions:

\[
\begin{align*}
\mathbf{q}_0(r, z) &= \frac{1}{2\pi} \int_0^{2\pi} \mathbf{q}(r, \theta, z) \, d\theta \quad \text{on } \Gamma_q \\
\mathbf{q}^a_m(r, z) &= \frac{1}{\pi} \int_0^{2\pi} \begin{bmatrix} q_r(r, \theta, z) \cos(m \theta) \\ q_\theta(r, \theta, z) \sin(m \theta) \\ q_z(r, \theta, z) \cos(m \theta) \end{bmatrix} \, d\theta \quad \text{on } \Gamma_f \text{ for } m > 0 \\
\mathbf{q}^b_m(r, z) &= \frac{1}{\pi} \int_0^{2\pi} \begin{bmatrix} q_r(r, \theta, z) \sin(m \theta) \\ q_\theta(r, \theta, z) \cos(m \theta) \\ q_z(r, \theta, z) \sin(m \theta) \end{bmatrix} \, d\theta \quad \text{on } \Gamma_f \text{ for } m > 0
\end{align*}
\]

(2.37)

(2.38)

(2.39)

Solving the equations of motion (2.30) for a limited number of Fourier harmonics is highly efficient compared to a full three-dimensional FEM analysis. By using further improvements and in special problem cases, the efficiency can be improved even more, as will be demonstrated below.

One-time assembly

The stiffness matrices \(\mathbf{K}_m\) are always completely reassembled for every Fourier harmonic of interest. However, by introducing an alternative derivation for the Fourier elements, as is done in Appendix B, it is shown that the strain column matrix can be expressed as a Fourier series in terms of the Fourier coefficients of the nodal displacements and in which the Fourier index \(m\) appears explicitly:

\[
\mathbf{\varepsilon}(r, \theta, z) = \sum_{m=0}^{n_f} (\tilde{\mathbf{\varepsilon}}_m(r, z) \cos(m \theta) + \hat{\mathbf{\varepsilon}}_m(r, z) \sin(m \theta))
\]

(2.40)

where:

\[
\begin{align*}
\tilde{\mathbf{\varepsilon}}_m(r, z) &= \left( \mathbf{P}(r, z) + m \mathbf{Q}(r, z) \right) \mathbf{u}^a_m + \left( \mathbf{R}(r, z) + m \mathbf{S}(r, z) \right) \mathbf{u}^b_m \\
\hat{\mathbf{\varepsilon}}_m(r, z) &= \left( \mathbf{R}(r, z) - m \mathbf{S}(r, z) \right) \mathbf{u}^a_m + \left( \mathbf{P}(r, z) - m \mathbf{Q}(r, z) \right) \mathbf{u}^b_m
\end{align*}
\]

(2.41)

(2.42)
with the matrices in the right-hand sides defined in Appendix B. Continuing the
derivation using these expressions leads to a relation for the stiffness matrix that
also explicitly depends on \( m \):

\[
K_m = \begin{bmatrix}
A_0 & B_0 \\
B_0 & A_0 \\
\end{bmatrix} + m \begin{bmatrix}
A_1 & B_1 \\
-B_1 & -A_1 \\
\end{bmatrix} + m^2 \begin{bmatrix}
A_2 & B_2 \\
B_2 & A_2 \\
\end{bmatrix}
\]

(2.43)

Once the matrices \( A_i, B_i, i = 1, 2, 3 \) have been assembled, the stiffness matrix for
any Fourier harmonic can be obtained by matrix additions. Since the assembly
process is computationally expensive, this offers a significant CPU-time reduc-
tion. The method has been implemented in Matlab, and a validation is presented
in Appendix B.

Note that when there are viscoelastic materials in the structure, the submatrices
are frequency dependent also. In these cases, when solving the structural prob-
lem, the loop over the Fourier harmonics has to be nested within the loop over
the frequencies in order to profit from this formulation.

Reduction for isotropy and special orthotropy

Certain forms of orientations of material anisotropy lead to additional reductions
in CPU time. This depends on the structure of the material matrix. The fourth-
order material tensor has 81 components, not all of which are independent. Since
both the stress and strain tensors are symmetric, there are six independent stress
and strain components, which are related by at most 36 material constants. The
constitutive model can therefore be written in matrix form as:

\[
\sigma = G \varepsilon
\]

(2.44)

where \( G \) a \( 6 \times 6 \) material matrix, and \( \sigma \) and \( \varepsilon \) are a stress column matrix and a
strain column matrix respectively.

Orthotropic (orthogonally anisotropic) materials, which have three mutually or-
thogonal planes of elastic symmetry, have elements in the elasticity matrix that
are zero:

\[
G = \begin{bmatrix}
G_{11} & G_{12} & G_{13} & 0 & 0 & 0 \\
G_{12} & G_{22} & G_{23} & 0 & 0 & 0 \\
G_{13} & G_{23} & G_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{66} \\
\end{bmatrix}
\]

(2.45)

Isotropic materials behave independently of orientation. Only two independent
material constants remain: a modulus of elasticity and Poisson’s ratio. The struc-
2. Analysis methods for axisymmetric structures

ture of the material matrix is the same as for orthotropic materials:

\[
G = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & 0 & 0 & 0 \\
\nu & 1 - \nu & \nu & 0 & 0 \\
\nu & \nu & 1 - \nu & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} - \nu & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} - \nu \\
\end{bmatrix}
\tag{2.46}
\]

When the principal material coordinates coincide with the axes of the coordinate system in which the structural problem is defined, and the material itself is orthotropic, the problem is said to be specially orthotropic. This is of particular interest here, since for isotropic and specially orthotropic cases, the matrices \( B_0, B_1 \) and \( B_2 \) in (B.73) vanish because of the structure of the material matrix. As a result, the nodal displacement amplitudes \( u^n \) and \( u^b \) can be solved independently:

\[
\begin{align*}
(-\omega^2 M + K_m^a) u_m^n &= f_m^n \\
(-\omega^2 M + K_m^b) u_m^b &= f_m^b
\end{align*}
\tag{2.47}
\tag{2.48}
\]

where the stiffness matrices can be expressed as:

\[
\begin{align*}
K_m^a &= A_0 + m A_1 + m^2 A_2 \\
K_m^b &= A_0 - m A_1 + m^2 A_2
\end{align*}
\tag{2.49}
\tag{2.50}
\]

Solving these two smaller matrix equations is more economical than solving the larger single matrix equation (B.73). For isotropy and special orthotropy, the problem can therefore be solved more efficiently than for generally orthotropy or anisotropy.

2.3.4 Frequency-dependent stiffness matrix

As was shown in the previous subsection, viscoelastic material behaviour causes the stiffness matrix to become frequency dependent. At every frequency of interest, the stiffness matrix therefore has to be reassembled using the material data at that frequency. Another disadvantage is that modal analyses cannot be performed. Several methods are available in literature to overcome these problems (e.g. Golla and Hughes, 1985; Yiu, 1993; Lesieutre and Bianchini, 1995). Below, an improvement to Yiu’s method is introduced which extends its use to anisotropic materials.

It can be shown using thermodynamic theory (Schapery, 1974) that for any anisotropic linear viscoelastic material for which the elastic moduli are positive def-
inite, the relaxation modulus matrix can be expressed as:

\[ R(t) = R_0 + \sum_{k=1}^{n_k} R_k e^{-t/\tau_k} \]  

(2.51)

where the matrices \( R_k, \ k = 0, \ldots, n_k \) are real and constant matrices, and \( \tau_k, \ k = 1, \ldots, n_k \) are relaxation times, which are positive material time constants. Each of the terms in the right-hand side of (2.51) is required to be symmetric, but they are not restricted by thermodynamics to be positive semidefinite or definite. From a Laplace transformation of (2.51), it follows that:

\[ R(\omega) = \frac{1}{j \omega} R_0 \sum_{k=1}^{n_k} \frac{j \omega \tau_k}{j \omega \tau_k + 1} R_k \]  

(2.52)

The complex modulus matrix can then be written as:

\[ G(\omega) = j \omega R(\omega) = R_0 + \sum_{k=1}^{n_k} \frac{j \omega \tau_k}{j \omega \tau_k + 1} R_k \]  

(2.53)

For a homogeneous structure, the matrix system of equations of motion can now be written as:

\[ \left( -\omega^2 M + K_0 + \sum_{k=1}^{n_k} \frac{j \omega \tau_k}{j \omega \tau_k + 1} K_k \right) u = f \]  

(2.54)

which gives the explicit frequency dependency of the system. Yiu (1993) introduces additional variables \( z_k \), called dissipation coordinates, in order to obtain an equivalent matrix equation in which the system matrices no longer depend on the frequency:

\[ \left( -\omega^2 M^{(\omega)} + j \omega D^{(\omega)} + K^{(\omega)} \right) w = f^{(\omega)} \]  

(2.55)

where:

\[ w = \begin{bmatrix} u \\ z_1 \\ \vdots \\ z_{n_k} \end{bmatrix}, \quad f^{(\omega)} = \begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]  

(2.56)

and \( D^{(\omega)} \) is a damping matrix. Yiu defined the dissipation coordinates as:

\[ z_k \equiv \frac{1}{j \omega \tau_k} u \]  

(2.57)
The method is valid for a structure consisting of an isotropic homogeneous linear viscoelastic material. Here, it will be shown that, by using a modified definition of the dissipation coordinates, the applicability of the method can be extended to anisotropic materials. For that purpose, the original definition (2.57) is changed to:

\[
\frac{1}{j \omega \tau_k + 1} K_k \mathbf{u} = K_0 \mathbf{z}_k
\]  

(2.58)

Substituting this definition into (2.54) results in:

\[
(-\omega^2 \mathbf{M} + K_0) \mathbf{u} + \sum_{k=1}^{n_k} j \omega \tau_k K_0 \mathbf{z}_k = \mathbf{f}
\]  

(2.59)

The definition equation (2.58) of the dissipation coordinates can be rearranged to:

\[
K_k \mathbf{u} - K_0 \mathbf{z}_k - j \omega \tau_k K_0 \mathbf{z}_k = 0
\]  

(2.60)

or:

\[
j \omega \tau_k K_0 \mathbf{z}_k = K_k \mathbf{u} - K_0 \mathbf{z}_k
\]  

(2.61)

which gives, after substitution into (2.59):

\[
\left(-\omega^2 \mathbf{M} + K_0 + \sum_{k=1}^{n_k} K_k\right) \mathbf{u} - \sum_{k=1}^{n_k} K_0 \mathbf{z}_k = \mathbf{f}
\]  

(2.62)

Using equations (2.60) and (2.62), the matrices in (2.55) can be expressed as:

\[
\mathbf{M}^{(w)} = \begin{bmatrix}
M & 0 & \ldots & 0 \\
0 & 0 & \ldots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  

(2.63)

\[
\mathbf{D}^{(w)} = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & \tau_1 K_0 & 0 & \ldots & 0 \\
\vdots & 0 & \tau_2 K_0 & 0 & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & 0 & \tau_{n_k} K_0
\end{bmatrix}
\]  

(2.64)

\[
\mathbf{K}^{(w)} = \begin{bmatrix}
K_0 + \sum_{k=1}^{n_k} K_k & -K_1 & -K_2 & \cdots & -K_{n_k} \\
-K_1 & K_0 & 0 & \cdots & 0 \\
-K_2 & 0 & K_0 & 0 & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
-K_{n_k} & 0 & 0 & 0 & K_0
\end{bmatrix}
\]  

(2.65)
The formulation results in a matrix system containing matrices that are constant, real and symmetric. Performing a modal analysis on a structural model with strong material damping is now enabled. A disadvantage of the method is that the material model parameters in (2.51) have to be fitted to measured material data. Also, the size of the matrices has increased considerably as \( n_k \) has to be large enough in order to obtain a good fit. The method therefore is efficient only when the response has to be determined at many frequencies. Using an efficient modal analysis algorithm, the modes in terms of the displacements \( \mathbf{u} \) and the dissipation coordinates \( z_k \) can be extracted. From these modes, the response can be determined at any frequency within the range for which the modes were computed.

2.4 Methods for the acoustic analysis

Just as for the structural analysis, efficient methods have been developed for the acoustic calculations (Kuipers, 1999). Those techniques that are used in this thesis will be shortly reviewed here. Fourier expansions are used in discretising the differential equations in circumferential direction, just as for the structural problem. For the acoustic analysis, however, this is done in combination with boundary elements instead of finite elements. Applying the boundary element method (BEM) is preferred because it is more efficient than FEM for external radiation since in BEM only the radiating surfaces have to be modelled. A recently developed alternative, the infinite element method, rivals the efficiency of BEM in external radiating problems. Fourier infinite elements, however, are not available at the time of writing.

There is no unambiguous measure to assess acoustic radiation. Different criteria can be used, such as a sound pressure level or the radiated sound power level. For the gradient coil, Kuipers (1999) has demonstrated that when a design is modified, these two quantities react similarly to the changes, and can be considered to be equivalent measures. The sound power level is the most suitable choice for the criterion, because the sound power can be determined more efficiently than sound pressures in numerical analyses. In the next section, it is shown how Fourier-BEM can be applied to come from structural displacement amplitudes to radiated sound power.
2.4.1 Fourier boundary elements

For discretisation problems in which the domain is homogeneous, and the governing equations form a linear system, a boundary element method can be applied. As its name suggests, only the boundaries of the domain have to be discretised. An acoustic analysis tool based on Fourier-BEM, called dArC, has been developed which is fast compared to 3D-BEM and FEM analyses (Kuijpers, 1999).

The differential equations that describe the relations between velocities and acoustic pressures in an acoustic fluid were presented in section 2.2. Discretisation of these equations using Fourier-BEM leads to a set of linear equations in matrix form Kuijpers (1999):

\[ A_m(\omega) p_m = B_m(\omega) v_m \]  

(2.66)

where the matrices \( A_m \) and \( B_m \) have to be assembled for each Fourier harmonic and for each frequency for which an acoustic analysis has to be performed. The column matrix \( v_m \) contains the Fourier coefficients of the surface normal velocity amplitudes for Fourier harmonic \( m \). These velocity amplitudes are determined from nodal surface normal displacement amplitudes at the radiating surface, as computed in a structural analysis. Solving this system would lead to the Fourier coefficients of the nodal sound pressure amplitudes \( p_m \) at the surface. From these surface pressures, the pressure in any point in the acoustic domain can be determined. Alternatively, the radiated sound power can be obtained (Kuijpers, 1999):

\[ \mathcal{P}(\omega) = \sum_{m=0}^{n_f} \mathcal{T}_m(\omega), \quad \mathcal{T}_m(\omega) = \frac{1}{2} v_m^H C_m(\omega) v_m \]  

(2.67)

where an overline denotes temporal averaging, and \( C_m \) is the power coupling matrix:

\[ C_m(\omega) = \text{Re} \left( \left( A_m^{-1}(\omega) B_m(\omega) \right)^T N \right) \]  

(2.68)

in which \( A_m \) and \( B_m \) are the matrices from (2.66). The matrix \( N \) is defined as:

\[ N = \int_{\Gamma_a} \varphi \varphi^T \, d\Gamma_a \]  

(2.69)

where \( \varphi \) is a column matrix containing the basis functions that are used to discretise the pressure and the surface normal velocity.

In assembling the system matrices for the acoustic analysis, a number of circumferential integrals have to be computed that cause numerical problems. Kuijpers
et al. (1997) recognised that these integrals are Fourier transforms, and devised an
efficient algorithm to evaluate them. This has improved the efficiency of Fourier-
BEM compared to earlier implementations.

The radiated sound power can be determined for any velocity distribution, but
that would require the storage of the power coupling matrices $C_m$ for each Fourier
harmonic $m$ and for each frequency of interest. In the next subsection, it is shown
that a considerable reduction in memory requirement can be achieved by applying
the radiation modes formulation.

### 2.4.2 Radiation modes formulation

Of the reduction methods presented in this Chapter, the most powerful one is the
radiation modes formulation. Radiation modes form an orthogonal modal basis
that decomposes the velocity distribution into independently radiating compo-
ments with respect to the total acoustic power.

The radiated sound power at a certain frequency depends both of the amplitu-
de and the shape of the surface normal velocity distribution. The radiation
efficiency $\sigma$ is a commonly used measure in structural-acoustics for quantifying
how well a certain shape is able to radiate at a particular frequency. It is defined
as the ratio of the power per unit area radiated by the source, and the power ra-
diated by a reference sound source that vibrates uniformly with the same mean
square velocity as the source and that radiates plane waves:

$$
\sigma(\omega) = \frac{\overline{P}(\omega) / S}{\rho_0 c_0 \langle v^2 \rangle} 
$$

(2.70)

where $\overline{P}/S$ is the power per unit area radiated by the source, and $\rho_0 c_0 \langle v^2 \rangle$ is the
power per unit area radiated by the reference source. The radiation efficiency,
which can exceed unity, depends on frequency; a certain shape may be able to
radiate well at some frequencies but poorly at other frequencies. In discrete form,
(2.70) can be expressed as:

$$
\sigma(\omega) = \frac{1}{\rho_0 c_0} \frac{v^H C(\omega) v}{v^H N v} 
$$

(2.71)

where $v^H$ denotes the hermitian (complex conjugate transpose) of the normal
surface velocity amplitude column matrix $v$. This relation is equivalent to a gen-
eralised eigenvalue problem (Cunefare, 1991):

$$
C(\omega) v = \lambda(\omega) N v, 
$$

(2.72)
where $\lambda = \rho_0 c_0 \sigma$. The resulting real eigenvalues $\lambda_k$ are related to the modal radiation efficiencies $\sigma_k$ associated with each radiation mode:

$$\sigma_k(\omega) = \frac{\lambda_k(\omega)}{\rho_0 c_0}$$  (2.73)

The eigenvectors $\psi_k$ are normalised with respect to the matrix $N$, resulting in:

$$\Psi^H N \Psi = I, \quad \Psi^H C \Psi = \Lambda$$  (2.74)

where $\Psi = [\psi_1 \psi_2 \ldots \psi_n]$, $I$ is the identity matrix and $\Lambda$ is a diagonal matrix containing the eigenvalues $\lambda_k$. The radiation modes can now be used as an orthogonal basis to expand the surface normal velocities:

$$v = \sum_{k=1}^{n_r} \psi_k \zeta_k = \Psi \zeta$$  (2.75)

where $\zeta_k$ are modal contribution coefficients. After pre-multiplying this equation by $\Psi^H N$:

$$\Psi^H N v = \Psi^H N \Psi \zeta$$  (2.76)

and by using the normalisation in (2.74), an expression for the modal contribution coefficient column matrix is obtained:

$$\zeta = \Psi^H N v$$  (2.77)

Substitution of the surface normal velocity expansion (2.75) into (2.67):

$$\overline{P} = \frac{1}{2} v^H C v = \frac{1}{2} \zeta^H \Psi^H C \Psi \zeta$$  (2.78)

leads, again by using the normalisation in (2.74), to an expression for the sound power in terms of the modal contribution coefficients and the radiation modes eigenvalues:

$$\overline{P}(\omega) = \frac{1}{2} \zeta^H \Lambda(\omega) \zeta$$  (2.79)

This implies that the radiated sound power can be determined almost instantly once the radiation modes have been computed. Since the radiation modes depend only on the acoustic geometry and on frequency, they can be reused as long as the acoustic geometry remains unchanged. As a consequence, evaluating the effects that modifications to the structural design have on the radiated power requires only a single full numerical acoustic analysis.

There are cases where only part of the outer geometry is radiating. Also, an analyst may be interested in the contribution of a certain part of the geometry to
the total sound radiation. Kuijpers (1999) showed that computational efforts in
determining the radiation modes can be reduced. Instead of extracting the modes
from the full matrix system, a smaller system can be derived to which a modal
analysis method can be applied. This way, an additional CPU time reduction is
achieved for the analysis in this project, as the larger part of the casing of the
MRI-scanner will be assumed rigid.

2.5 Summary

A number of analysis methods have been presented that are implemented in the
Structural Acoustics Toolbox, applicable in passive noise reduction of axisym-
metric structures in light fluids such as air. Structural analyses may include
anisotropic linear viscoelastic material behaviour which facilitates the modelling
of polymeric materials. Composite materials such as fibre-reinforced polymers
can therefore be incorporated in the models. Frequency dependence of the mod-
uli and of material damping can be modelled correctly.

For some of the discussed methods, improvements have been introduced to make
them highly efficient for use in a design environment:

- In Fourier-FEM complete reassembly of the stiffness matrix for every Fourier
  harmonic is not necessary, as the matrix can be expressed explicitly in terms
  of the Fourier index;
- The speed of the determination of circumferential integrals in Fourier-BEM
  is increased by a fast Fourier integration technique;
- Using the radiation modes formulation for subsystems results in CPU time
  reductions when only part of the outer geometry is radiating.

Also, the applicability of some of the methods has been extended:

- Fourier-FEM is shown to be applicable for general anisotropy;
- Yiu’s method for converting a matrix system of equations of motion that
  explicitly depends on frequency to a system containing constant matrices
  is extended to be valid for general anisotropy instead of isotropy only.

Improvements related to the acoustic analysis methods have been developed
by Kuijpers (1999).
2. Analysis methods for axisymmetric structures
3.1 Introduction

Structural-acoustic models should be able to predict the acoustic radiation with sufficient accuracy. Compared to a complex structure, modelling the homogeneous acoustic medium is relatively easy. A linear numerical acoustic analysis is known to give satisfactory results if the input, the normal surface velocity distribution, is correct. Emphasis therefore has to be placed on model construction for the vibrational behaviour of the structure.

For complex structures, like the MRI gradient coil, there are many factors that influence the vibrational behaviour. Geometry, material properties, boundary conditions and excitation all have to be included in a model. Including all details of a complex structure in a model often takes too much time and effort in constructing the model and in the numerical analysis. This is certainly the case in a design environment were parameter studies or numerical optimisation require many analyses to be performed.

In order to reduce modelling and computational efforts, the questions to be answered are which structural properties are of primary importance to the prediction of the vibrational behaviour and the radiation of the gradient coil, and what are the underlying physical mechanisms. When the effects of structural complexity are understood, details that only marginally influence the radiation can safely be omitted from the models. This kind of information also increases confidence in the predictive capabilities of the models after design changes have been applied.

A possible strategy is to systematically remove details from a model and to observe the differences in the response. A reverse approach is followed here; more detail is gradually added to an initially simple model. The individual effects of these features can then be identified more clearly, which leads to a better un-
understanding. Firstly, homogeneous models are developed, in which all inhomogeneities are averaged in each individual substructure. Using these relatively simple models, the effects of overall changes in material properties and geometry can be studied efficiently due to high computational speed. Such models have also been used by Kuipers (1999), and have led to new and useful insights into the acoustic behaviour of internally radiating cylinders. The homogeneous models will also be used to study the effects of anisotropy.

Smearing out inhomogeneities, however, only is allowed when the scales of the inhomogeneities are small compared to wavelengths occurring in the frequency range of interest. It can be applied successfully when, for instance, homogenising the behaviour of fibre reinforced materials, where fibre thicknesses are small. Structure-borne waves will be (partially) reflected at every inhomogeneity they encounter. The speed of these waves is affected by the material properties through which they travel. These effects will not occur in homogenised models, and will lead to deviations in the predicted response. The modelling errors in the vibrational behaviour due to homogenisation will therefore be examined in three stages in this Chapter. Firstly, only inhomogeneities in radial and axial directions will be studied, so the models remain axisymmetric. Secondly, the effects of inhomogeneities in circumferential directions are examined. The newly developed Fourier elements which allow non-axisymmetric material behaviour, as described in the previous Chapter, will be used for this purpose. Lastly, inhomogeneities in circumferential and other directions are modelled in combination.

In section 3.2 a detailed description of the gradient coil is given, and the Lorentz force distribution is derived. As the outer geometry will be the same for all models, a single numerical acoustic analysis suffices in which the radiation modes are determined, which is the topic of section 3.3. An upper bound for the number of terms in the Fourier expansions in the structural analyses and for measurements of the vibrational behaviour is derived from this acoustic analysis. Then, in section 3.4, measurements on the vibrational behaviour of the gradient coil are described. The results of these measurements are examined for clues about the kind of phenomena that occur in the actual structure. They are also compared to the responses of the models that are developed in section 3.5. The main purpose of this combination of problem analysis, numerical acoustic analysis, vibration measurement and stepwise structural model construction is to gain insight into the mechanisms that are of importance to sound radiation, as well as how to model and analyse them, while finding a balance between accuracy and computational speed. The findings are summarised and discussed in section 3.6.
3. Model construction of MRI gradient coils

3.2 The MRI gradient coil

Creating a model that enables the simulation of the vibrational behaviour of the gradient coil requires detailed information about its construction. The mechanical properties of the materials that are used in the different parts of the coil system and how these parts are structurally connected are potentially relevant to the surface velocity response. A more detailed study of the coil construction is also necessary to reveal the spatial distribution of the excitation caused by the Lorentz forces. Another reason for this examination is to understand the functionality of each of the features. This is vital in deciding what kind of design variables can be assigned to the models for the parameter studies in Chapter 4 and the numerical optimisation runs in Chapter 5.

3.2.1 Gradient coil construction

The most essential parts of a gradient coil system are the conducting windings that induce magnetic gradient fields when a voltage is applied. These gradient fields are used to control the imaging location in the body of the patient (Hashemi and Bradley Jr., 1997). For each Cartesian coordinate axis there is a separate coil; these are termed x-coil, y-coil and z-coil. Their construction is shown schematically in figure 3.1. The coils also produce magnetic fields outside the gradient tube, causing eddy currents in other parts of the scanner and disturbing electronic devices. A second set of three coils is therefore applied at the outer surface of the gradient tube in which the currents always have opposite direction relative to the inner coils. The combined magnetic fields of the coils produce linear fields in the imaging volume while partly shielding the fields outside of the gradient

![Figure 3.1: Current flow directions in the MRI gradient coils.](image-url)
3. Model construction of MRI gradient coils

coil.

Figure 3.2: Schematic structure of the Philips gradient coil system.

Figure 3.3: Schematic cross-sections of the Philips gradient coil system.

Because the use of metals in the gradient coil other than the copper windings would induce additional coil vibrations and imaging disturbances, other materials such as polymers and glass fibres are applied in the coil system that is studied here. In order to supply stiffness to the tube, a carrier tube made of fibre-reinforced epoxy resin is used as the main component, as shown in figure 3.2. The carrier tube itself is bolted to aluminium supports that are connected to the cryo magnet. The outer coils are located at its outer surface. The inner coils and the carrier tube are separated by longitudinal profiles. The openings between the profiles, which consist of fibre-reinforced polyester, function as air cooling ducts. This cooling is necessary because of the heat generated by the electrical currents in the windings. The profiles are glued to the carrier tube by a layer of epoxy
resin. The copper windings are held in place by embedding them in epoxy resin, which also supplies mutual isolation as a protection to short-circuiting.

This coil system does not leave much room for design changes. Its inner diameter is constrained since there has to be enough room inside the bore for the patient, the RF-coil and the casing. Increasing the inner radius or the coil system’s total thickness would lead to a drastic increase in costs. Altering the patterns of the windings deteriorates the linearity of the magnetic gradient fields, and therefore the imaging quality, unless the new winding patterns would be carefully designed. This might however be beneficial for noise reduction, as the winding patterns determine the spatial distribution of the excitation force fields, as will be explained next.

### 3.2.2 Lorentz excitation

The electrical currents in the coil windings traverse the strong static field of the main magnet. A current in a magnetic field results in a Lorentz force \( \vec{F} \) whose direction and amplitude are given by (Ling et al., 1995):

\[
d\vec{F} = I \left( d\vec{i} \times \vec{B} \right)
\]

where \( I \) is the strength of the current, \( d\vec{i} \) denotes the orientation of the conductor and \( \vec{B} \) is the magnetic field strength vector. The magnet generates a strong field \( B_r \) in the axial direction and a smaller radial component \( B_r \):

\[
\vec{B}(r, z) = B_r(r, z) \hat{e}_r + B_z(r, z) \hat{e}_z
\]

The axial dependence of the magnetic field at the inner and the outer coil is shown in figure 3.4. In a cylindrical coordinate system, the Lorentz force com-
components can be written as:
\[
\begin{align*}
    dF_r &= I B_z \, dr \\
    dF_\theta &= I (B_r \, dz - B_z \, dr) \\
    dF_z &= -I B_r \, d\theta
\end{align*}
\] (3.3) (3.4) (3.5)

As was shown in figure 3.1, the conductors are either oriented in axial or in circumferential direction (\( dl_r = 0 \)). The Lorentz force relations therefore reduce to:

\[
\begin{align*}
    dF_r &= I B_z \, r \, d\theta \\
    dF_\theta &= I B_r \, dz \\
    dF_z &= -I B_r \, r \, d\theta
\end{align*}
\] (3.6) (3.7) (3.8)

Because the axial magnetic field component \( B_z \) is large compared to the radial component \( B_r \), the Lorentz forces have a strong radial component and smaller contributions in circumferential and axial directions. These forces result in a distributed surface loading on each conductor, since the conductors have rectangular cross-sections. If, for each individual circumferential conductor, the magnetic field strength is assumed to be constant, the surface load is constant over each circumferential conductor:

\[
\begin{align*}
    \tau_r &= \frac{1}{w} \frac{dF_r}{r \, d\theta} = \frac{I}{w} B_z, \\
    \tau_z &= -\frac{1}{w} \frac{dF_z}{r \, d\theta} = -\frac{I}{w} B_r.
\end{align*}
\] (3.9)
where $w$ is the width of the conductor, and an overbar denotes spatial averaging. For axial conductors, the surface load is constant along the circumferential direction but dependent on the axial coordinate,

$$p_\theta(z) = \frac{1}{w} \frac{dF_0}{dz} = \frac{I}{w} B_z(z)$$  \hspace{1cm} (3.10)

The surface loading amplitude on a particular conductor therefore only depends on the current strength $I$, on the width of the conductor $w$ and on the local magnetic field strength $B_z$. The distributed loads have to be transformed into Fourier series before they can be applied to the Fourier element models. As the patterns of the $z$-coil windings are different from the patterns of the $x$-coil and the $y$-coil, their Fourier transformations will be treated separately below. Also, the symmetry conditions following from the coil structure and the load distributions will be derived.

**Excitation by the z-coil**

The $z$-coil, constructed of circular conductors, only has currents flowing in circumferential direction, as shown in figure 3.1. Since the magnetic field’s main component is in the axial direction, an axisymmetric mainly radial Lorentz force distribution is generated. The orientations of these forces are shown in figure 3.5. Radial components of the main field result in additional axisymmetric forces in axial direction. By looking at the current vectors, the circumferential current distribution seems to be antisymmetric with respect to the $x = 0$ plane and the $y = 0$ plane. Circumferential $z$-coil currents, however, have the same direction in each quadrant of the $x$-$y$ plane, as shown in table 3.2. In this table, the relative direction of the components of electrical current vectors, magnetic field vectors and force vectors are denoted by plus and minus signs for each quadrant. The relative directions of the force components are obtained by ‘multiplying’ the signs of the associated signs in the corresponding quadrants of the current and magnetic field components. The resulting Lorentz force distribution is symmetric with respect to the $x = 0$ plane and the $y = 0$ plane, as is also shown in figure 3.5. Due to antisymmetry of the current flow directions in the $z = 0$ plane, the force distribution is antisymmetric in that plane (a symmetric axial force distribution has opposite signs). Only axially antisymmetric vibrations would therefore be excited in an axially symmetric structure.

The circumferential distribution of the $z$-coil excitation will now be expanded into a Fourier series:

$$p(r, \theta, z) = \sum_{m = 0}^{\infty} \begin{bmatrix} p_{r, m}^e(r, z) \cos(m \theta) \\ p_{r, m}^e(r, z) \sin(m \theta) \\ p_{z, m}^e(r, z) \cos(m \theta) \\ p_{z, m}^e(r, z) \sin(m \theta) \end{bmatrix}$$  \hspace{1cm} (3.11)
3. Model construction of MRI gradient coils

![Diagram of Lorentz forces](image)

**Figure 3.5:** Lorentz force distributions for the z-coil. 
- : electric current vector,
- : magnetic field vector,
- : force vector.

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**Table 3.2:** Derivation of symmetry in Lorentz force distribution for the z-coil.
3. Model construction of MRI gradient coils

The arrangement of the sine and cosine terms follow from the derivations of the equations of motion in Chapter 2. Only $v_m^a$ terms will be excited in an isotropic or specially orthotropic axisymmetric structure by $p_m^a$ excitations, while only $v_m^b$ terms are excited by $p_m^b$ excitations. The coefficients of the expansion are obtained from the Fourier integrals:

$$p_m^a(r, z) = \frac{1}{\alpha_m} \int_0^{2\pi} \begin{bmatrix} p_r(r, \theta, z) \cos(m \theta) \\ p_\theta(r, \theta, z) \sin(m \theta) \\ p_z(r, \theta, z) \cos(m \theta) \end{bmatrix} d\theta$$  \hspace{1cm} (3.12)

$$p_m^b(r, z) = \frac{1}{\alpha_m} \int_0^{2\pi} \begin{bmatrix} p_r(r, \theta, z) \sin(m \theta) \\ p_\theta(r, \theta, z) \cos(m \theta) \\ p_z(r, \theta, z) \sin(m \theta) \end{bmatrix} d\theta$$  \hspace{1cm} (3.13)

where:

$$\alpha_m = \begin{cases} 2\pi & \text{if } m = 0 \\ \pi & \text{if } m > 0 \end{cases}$$  \hspace{1cm} (3.14)

Since for each z-coil conductor the load distribution is uniformly distributed in circumferential direction, its transformation is straightforward. After substituting relations (3.9) for the excitation amplitudes on a single z-coil conductor into (3.12) and (3.13), the components of the Fourier coefficients can be written as:

$$p_r^a = \frac{1}{\alpha_m} \frac{T_r I}{w} \int_0^{2\pi} \cos(m \theta) d\theta, \hspace{1cm} p_r^b = \frac{1}{\alpha_m} \frac{T_r I}{w} \int_0^{2\pi} \sin(m \theta) d\theta$$  \hspace{1cm} (3.15)

$$p_\theta^a = 0, \hspace{1cm} p_\theta^b = 0$$  \hspace{1cm} (3.16)

$$p_z^a = \frac{1}{\alpha_m} \frac{T_r I}{w} \int_0^{2\pi} \cos(m \theta) d\theta, \hspace{1cm} p_z^b = \frac{1}{\alpha_m} \frac{T_r I}{w} \int_0^{2\pi} \sin(m \theta) d\theta$$  \hspace{1cm} (3.17)

The sine and cosine integrals reduce to:

$$\int_0^{2\pi} \sin(m \theta) d\theta = 0, \hspace{1cm} \int_0^{2\pi} \cos(m \theta) d\theta = \begin{cases} 2\pi & \text{if } m = 0 \\ 0 & \text{if } m > 0 \end{cases}$$  \hspace{1cm} (3.18)

Therefore, only the $m = 0$ Fourier harmonic $p_0^a$ is nonzero for the excitation of each z-coil conductor:

$$p_0^a = \frac{1}{w} \left[ \frac{T_r I}{T_r} \right], \hspace{1cm} p_0^a = 0 \hspace{1cm} \text{for } m > 0, \hspace{1cm} p_m^b = 0$$  \hspace{1cm} (3.19)
3. Model construction of MRI gradient coils

Figure 3.6: Axial distributions of the z-coil excitation for the $m = 0$ Fourier harmonic. (Different scales have been used for the axial and radial components.)
The axial distributions of the $m = 0$ Fourier excitation, which in this case equals the total excitation, are shown in figure 3.6. Currents in the outer coils always have opposite direction with respect to the currents in the inner coils. Excitations in the outer coils therefore have opposite direction with regard to the inner coils. Also, different conductor patterns result in different load distributions.

**Excitation by the transverse coils**

The transverse coils, as the x-coil and the y-coil are collectively known, have conductors that guide electrical currents either in axial or in circumferential direction (see figure 3.1). Currents in axial direction result in forces in circumferential direction due to radial main field components, as shown in figure 3.7(b). The conductor patterns of the transverse coils are symmetric in the $x = 0$ plane, the $y = 0$ plane and the $z = 0$ plane. Currents, however, have opposite directions in some of the planes (see figure 3.7 and tables 3.3 and 3.4); for the x-coil, current directions are antisymmetric in the $x = 0$ plane and symmetric in the $y = 0$ plane. The y-coil is identical to the x-coil, but rotated by 90° about the longitudinal axis, resulting in reverse symmetry conditions in the $x = 0$ plane and the $y = 0$ plane. The current distributions of both coils are symmetric in the $z = 0$ plane. For the load distributions, this leads to symmetry conditions shown in tables 3.3 and 3.4. In table 3.5, the symmetry conditions for the transverse coils are summarised. Radial and axial components of the x-coil excitation are symmetric in the $x = 0$ plane, and antisymmetric in the $y = 0$ plane. The circumferential component has opposite symmetry. The y-coil causes opposite

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*Table 3.3:* Derivation of symmetry in Lorentz force distribution for the x-coil.
Figure 3.7: Lorentz force distributions for the x-coil. The distributions for the y-coil are identical, but rotated by $90^\circ$ about the z-axis. $\rightarrow$: electric current vector, $\longrightarrow$: magnetic field vector, $\rightarrow\rightarrow$: force vector.
### Table 3.4: Derivation of symmetry in Lorentz force distribution for the y-coil.

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### Table 3.5: Symmetry conditions for load distributions of the transverse coils. S: symmetric; A: antisymmetric.

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<tr>
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3. Model construction of MRI gradient coils

... symmetry in its excitation. Excitations of both transverse coils are symmetric with respect to the $z = 0$ plane.

When the circumferential distribution of the excitation is Fourier transformed, only Fourier components with the same symmetry conditions as in the excitation will be nonzero, as can be concluded from figure 3.8. For the x-coil, this means

![Diagram of x-coil components]

**Figure 3.8**: Symmetry of sine and cosine components of Fourier harmonics.

that the Fourier series of the circumferential load distribution:

$$
p(r, \theta, z) = \sum_{m=0}^{N} \begin{bmatrix} p_{m-1}^a(r, z) \cos(m \theta) \\ p_{m-1}^b(r, z) \sin(m \theta) \\ p_{m}^a(r, z) \cos(m \theta) \\ p_{m}^b(r, z) \sin(m \theta) \end{bmatrix} + \begin{bmatrix} p_{m-1}^a(r, z) \cos(m \theta) \\ p_{m-1}^b(r, z) \sin(m \theta) \\ p_{m}^a(r, z) \cos(m \theta) \\ p_{m}^b(r, z) \sin(m \theta) \end{bmatrix} \tag{3.20}
$$

consists of only odd harmonics, because the remaining vanish according to the Fourier integrals (3.12) and (3.13):

$$
p_{\text{x-coil}}(r, \theta, z) = \sum_{m \text{ is odd}} \begin{bmatrix} p_{m}^a(r, z) \cos(m \theta) \\ p_{m}^b(r, z) \sin(m \theta) \\ p_{m}^a(r, z) \cos(m \theta) \\ p_{m}^b(r, z) \sin(m \theta) \end{bmatrix} \tag{3.21}
$$
Since the y-coil patterns are identical to the x-coil patterns but rotated by 90° about the z-axis, the Fourier transform also contains odd Fourier harmonics, but with reversed symmetry conditions (see figure 3.8):

\[
p_{y\text{-coil}}(r, \theta, z) = \sum_{m \text{ is odd}} \begin{bmatrix} p_{m_1}^y (r, z) \sin(m \theta) \\ p_{m_2}^y (r, z) \cos(m \theta) \\ p_{m_3}^y (r, z) \sin(m \theta) \end{bmatrix}
\]  
(3.22)

Mathematical proof is presented in Appendix C, where the Fourier coefficients for the excitation by the transverse coils are derived. Use has been made of geometric symmetry of the conductor patterns as shown in figure 3.9. For each set of four conductors, a Fourier coefficient can be derived from the geometry information in the first quadrant. For \( m > 0 \), this leads for such set of four conductors in the x-coil to:

\[
p_{m}^{x}(z) = \frac{2(1 - (-1)^m)}{m \pi w} \begin{bmatrix} \overline{B}_x (\sin(m \theta_2) - \sin(m \theta_1)) \\ \overline{B}_z (\cos(m \theta_2) - \cos(m \theta_1)) \\ -\overline{B}_r (\sin(m \theta_2) - \sin(m \theta_1)) \end{bmatrix}, \quad p_{m}^{y} = 0
\]  
(3.23)

where \( \theta_1 \) and \( \theta_2 \) are the circumferential coordinates of the end points of the conductor in the first quadrant, as defined in figure C.1(a). The Fourier expansion
3. Model construction of MRI gradient coils

for a set of four conductors is, therefore:

\[
P_{x\text{-coil}}(\theta, z) = \sum_{m=1}^{N} \frac{2(1 - (-1)^m)I}{m \pi w} \begin{bmatrix} B_{z} \left( \sin(m \theta_2) - \sin(m \theta_1) \right) \cos(m \theta) \\ B_{r}(z) \left( \cos(m \theta_2) - \cos(m \theta_1) \right) \sin(m \theta) \\ -B_{r} \left( \sin(m \theta_2) - \sin(m \theta_1) \right) \cos(m \theta) \end{bmatrix}
\]  

(3.24)

which is in agreement with relation (3.21). For the y-coil, the derivation in Appendix C shows:

\[
P_{y\text{-coil}}(\theta, z) = \sum_{m=1}^{N} \sin\left(\frac{1}{2} m \pi\right) \frac{2(1 - (-1)^m)I}{m \pi w} \begin{bmatrix} -B_{z} \left( \cos(m \theta_2) - \cos(m \theta_1) \right) \\ B_{r}(z) \left( \sin(m \theta_2) - \sin(m \theta_1) \right) \cos(m \theta) \\ B_{r} \left( \cos(m \theta_2) - \cos(m \theta_1) \right) \sin(m \theta) \end{bmatrix}
\]  

(3.25)

and:

\[
P_{y\text{-coil}}(\theta, z) = \sum_{m=1}^{N} \sin\left(\frac{1}{2} m \pi\right) \frac{2(1 - (-1)^m)I}{m \pi w} \begin{bmatrix} -B_{z} \left( \cos(m \theta_2) - \cos(m \theta_1) \right) \\ B_{r}(z) \left( \sin(m \theta_2) - \sin(m \theta_1) \right) \cos(m \theta) \\ B_{r} \left( \cos(m \theta_2) - \cos(m \theta_1) \right) \sin(m \theta) \end{bmatrix}
\]  

(3.26)

The total Fourier transform can be obtained from a summation of all conductors in the first quadrant. In order to obtain a measure for the Fourier contents of the excitation, an excitation level is defined as:

\[
L_f = 10 \log \left( \frac{\overline{P}}{P_{\text{ref}}} \right)
\]  

(3.27)

Where \((\cdot)\) and \(\overline{\cdot}\) denote temporal and spatial averages, respectively.

For odd Fourier harmonics, the components of this excitation level are shown in figure 3.10. As can be seen, the radial forces dominate, particularly the \(m = 1\) harmonic. This exceeds the other Fourier harmonics by approximately 20 dB or more. At all Fourier harmonics, circumferential components seem negligible compared to the other components.

Conclusions

Due to the symmetry conditions of the gradient coil, the spatial Fourier transform of the excitation contains a selected number of harmonics. Only \(m = 0\) harmonics are present in the z-coil excitation. The transverse coil load distributions only
Figure 3.10: Excitation levels at odd Fourier harmonics for the transverse coils. Back: radial component; middle: axial component; front: circumferential component.

contain odd Fourier harmonics, dominated by the $m = 1$ and the $m = 3$ harmonics. As the structural mode shapes of perfectly axisymmetric structures are sinusoids as well, the vibrational response of such structures to the Lorentz excitation can only contain the mentioned Fourier harmonics. When using Fourier analysis methods, performing z-coil analyses for $m > 0$ or transverse coil analyses for even values of $m$ is unnecessary. This also follows from symmetry considerations. For non-axisymmetric structures, all Fourier harmonics in a given range would have to be included in the analysis, since structural modes would consist of multiple Fourier harmonics.

Axial symmetry of the gradient system also enables model reduction in the finite element discretisation, since only half of the structure has to be modelled. A single analysis will suffice after appropriate boundary conditions have been applied at the $z = 0$ plane.

Excitation level differences between inner and outer coils are observed. It has been shown (Chapman and Mansfield, 1995; Mansfield et al., 1995; Petropoulos and Morich, 1995) that gradient coil noise can be reduced by balancing the forces. The possible effects of force balancing will be examined using parameter studies in section 4.4.
3.3 Acoustic analysis of the gradient coil

The acoustic analysis is performed in this project before any structural analysis or measurement. This is enabled by using the radiation modes formulation. These radiation modes fully describe the acoustic behaviour of the structure with respect to radiated power, and can be determined from the wet geometry of the structure alone. It is not only possible but also useful here to perform the acoustic analysis first. Making use of the radiation modes formulation has two advantages here: the acoustic analysis only has to be performed once, and the highest Fourier mode of interest can be determined. As the wet geometry is not subject to changes, the radiation modes are valid for all models in this thesis. From the modal radiation efficiencies an upper bound for the Fourier expansions can be derived. As was shown by Termeer (1997) and Kuijpers (1999), higher Fourier harmonics in the vibrations become efficiently radiating distributions at higher frequencies. As a consequence, only a finite set of Fourier harmonics is relevant to the total sound power below the upper bound of the frequency range of interest. Therefore, those higher Fourier harmonics that do not contribute to the sound radiation in the frequency range of interest do not have to be included in the structural analyses.

The radiation modes are computed using bArd (1998), an acoustic analysis code that uses Fourier boundary elements, and that is part of the Structural-Acoustics Toolbox. It is possible to use a boundary element formulation because the acoustic medium is homogeneous and is governed by a linear differential equation.

3.3.1 Model for the acoustic analysis

The acoustic model involves a description of the geometry of the radiating surface and the acoustic properties of the air. Only part of the gradient coil’s surface is in contact with the air, as shown in figure 3.11. For the acoustic radiation, the outer surfaces of other parts of a device are relevant as well. Although the magnet is not included in the structural analysis, it does influence the acoustic radiation. The BEM mesh, shown in figure 3.12, contains 84 quadratic elements, with a total of 168 nodes. Only 73 of these nodes belong to the radiating part of the geometry. Another reduction technique, called the radiation modes formulation for subsystems (Kuijpers, 1999), can therefore be applied. This means that a matrix system of equations is used in the determination of the radiation modes that is smaller than the total system and involves only degrees of freedom at the nodes that are part of the vibrating section of the outer geometry.

The radiation modes at 50 logarithmically spaced discrete frequencies within the range of 100 to 2000 Hz are computed. The size of the elements are de-
Figure 3.11: Scanner geometry that is relevant to the acoustic radiation by the gradient coil.

Figure 3.12: Boundary element mesh for the acoustic analysis.
3. Model construction of MRI gradient coils

determined using the minimum wavelength of interest in the acoustic domain:
\[ \lambda_{\text{min}} = \frac{c_0}{f_{\text{max}}} \]
where \( c_0 \) is the speed of sound, which in air is 343 m/s at 20 °C. The minimum wavelength, at 2000 Hz, is approximately 17.2 cm. By choosing a minimum element length of 6 cm, there are at least six nodes per wavelength. At each frequency, the first 12 modes are determined for each of the first 15 Fourier harmonics \( (m = 0, 1, \ldots, 14) \). Two examples are displayed in figure 3.13; note the high radiation efficiencies that can occur inside ducts. As explained by Kuijpers (1999), they occur because of resonances caused by partial reflections of acoustic waves near the duct openings.

An important conclusion that is derived from studies of the acoustic behaviour of ducts is, that it is indeed necessary to perform an acoustic analysis (Termeer, 1997; Kuijpers, 1999). This finding is not as obvious as it may seem. For flat plates, for instance, above a given frequency any normal surface velocity distribution leads to efficient radiation. As was seen, this is not the case for radiation inside ducts since both efficient and inefficient radiation modes exist at any given frequency. As a consequence, reducing velocity levels at a certain frequency does not necessarily lead to a proportional reduction in acoustic power levels. In fact, an increase may even be obtained.

3.3.2 Range of Fourier harmonics

In axisymmetric linear structural systems, each Fourier harmonic is associated with structural modes, and Fourier harmonics that appear in the response are also present in the excitation. Therefore, the Fourier harmonics that appear in
the structural response can be predicted by a spatial Fourier transform of the excitation.

The maximum radiation efficiencies for each Fourier harmonic are plotted in figure 3.14. The first Fourier harmonic \( m = 0 \) is radiating efficiently at low frequencies. From this plot, it can be seen that Fourier harmonics above \( m = 14 \) are not relevant to sound radiation by the gradient coil in the frequency range of interest, and do not have to be incorporated in structural Fourier analyses. These results will also be used in determining the sample point distribution for surface vibration measurements in the following section.

### 3.4 Measurements

Useful information on features that are of importance to modelling the gradient coil can be obtained from the vibrational behaviour of the structure itself. Therefore, the surface velocities at the inner surface of the inner coil have been measured. For these measurements the coil supports have been removed, and the coil was placed on air cushions. This prevented the coil from being excited by vibrations in the magnet, and simplifies modelling efforts since the non-axisymmetric
supports can not be captured as boundary conditions in the Fourier models.

3.4.1 Measurement set-up

Since many measurement spots are needed for a representation of the velocity distribution, a good positioning system for the spots is required. A laser-Doppler vibrometer is considered to be a good choice for this purpose. By using a laser beam, the velocity amplitude on a small spot can be measured directly and contactless. The vibrometer is positioned at some distance from the scanner, and aimed at a prism which directs the beam to the measurement spot on the coil's surface (see figure 3.15). The coil then reflects the beam, which returns to the vibrometer in which a phase difference is measured from which the velocity is determined. The prism is contained in a gantry that includes a tube which is hollow at one end, so the laser beam can enter. The tube's length axis coincides with the gradient coil's length axis. By rotating the tube or by translating it along this axis, the location of the measurement spot can be controlled. For the rotation, a system is used with fixed intervals of 9°, that has been designed and manufactured at Philips Laboratories for positioning a microphone for sound measurements within the scanner's bore. The gantry has been designed to fit in the positioning system. Because of the gradient coil symmetry conditions, it is assumed that it suffices to scan only a quarter of the total surface: half the surface in axial direction, and half the surface in circumferential direction.

With the positioning system that is applied here, \( \frac{180°}{9} = 20 \) intervals can be measured in circumferential on half of the circumference. As high Fourier harmonics are less efficient radiators than low harmonics, and the low Fourier harmonics are expected to be dominantly present in the structural response, only the har-
monics up to \( m = 7 \) are estimated to be relevant. A minimum of approximately 6 sample points per wavelength are then available.

The gradient coil is excited by a frequency sweep between 100 and 4000 Hz. This signal is generated by a PC equipped with a DIFA measuring system, and sent to the gradient coil amplifiers, as shown schematically in figure 3.16. As soon as

the signal is applied, the measurement signal from the laser-Doppler device is collected, and an average over eight measurements is stored. For each spot, this is done for the x-coil, y-coil and z-coil separately. The laser beam is then directed to the next spot by rotating or translating the tube containing the prism.

The following devices have been used:

- The vibrometer used for measuring the surface normal velocities is a Polytec OFV 302 laser vibrometer. Jung (1994) tested the accuracy of this device. From his results it can be concluded that measurement errors for the vibrometer will be below 2%.

- The vibrometer controller is a Polytec OFV 3000.

- The control and acquisition system is a 12 channel DIFA measuring system (type DSA 220-12).

- A Kistler accelerometer (type 8692B50), is used for monitoring prism vibrations.
3. Model construction of MRI gradient coils

Design of the gantry

An MRI-scanner room is a harsh environment for performing measurements. Its magnet attracts all magnetic objects in its neighbourhood, while the magnet gradient fields induce eddy currents and therefore Lorentz forces in metallic parts of measurement devices. Also, the magnetic fields may disturb electrical and electronic devices, and the acoustic pressure fields cause devices to vibrate. The tube-prism configuration has been designed (Paar, 1998) to function in this environment. The following design criteria were imposed for the gantry:

- The velocity amplitude of the gantry at the prism may not exceed a value of 10% of the velocity amplitude at the measurement spot. This ensures a minimum difference of 20 dB between the prism velocity levels and the coil velocity. In that case, the maximum relative measurement error because of prism vibrations will be approximately 1 dB.

- The inner radius of the tube has a minimum admissible value, which depends on the tube’s stiffness. This restriction is needed to prevent the bending tube from obstructing the laser beam.

- The positioning system has to be able to rotate and translate about its length axis.

In order to evaluate the effects of eddy currents and acoustic pressure fields on the gantry, two different studies were made (Paar, 1998). In the first study, an analytical beam model was used to determine the effects of eddy currents on a metal tube. Any electrically conducting object in or near the scanner will experience eddy currents because of the alternating magnetic fields that are generated by the gradient coils. The combination of these eddy currents and the high static magnetic field result in Lorentz forces. It was concluded that velocity levels in a steel tube with an outer diameter of 30 mm would exceed the coil velocity levels by 8 dB or more. Therefore, no metallic parts were used in the gantry arm.

Eventual choices for the gantry arm’s dimensions and the materials from which it was constructed were further motivated by the second study. A semi-analytical model was constructed, based on structural-acoustics and on reciprocity. This fairly complicated model was used to derive a relationship between a beam’s vibrational velocity and a diffuse acoustic pressure field. Experiments have shown the model’s reliability, so it was used to design a suitable gantry arm. The required stiffness is provided by using polymeric materials reinforced by glass fibres. Stiffness can be further increased by a larger tube thickness, but a larger outer diameter also results in stronger excitation, while the inner diameter has a minimum. Using parameter studies, a compromise was found which satisfied all design criteria.
3.4.2 Measurement results

The velocity levels of the measured frequency responses are shown in figure 3.17(a). These levels, expressed in decibels, are determined by spatial and temporal averaging of the squared surface normal velocities $v_n$ at the radiating surface, and taking the logarithm:

$$L_v = 10 \log \left( \frac{\langle v_n^2 \rangle}{v_{\text{ref}}^2} \right)$$  \hspace{1cm} (3.28)

where $\langle \cdot \rangle$ and $\bar{\cdot}$ denote spatial and temporal averages respectively. Usually, a standard value is taken for the reference velocity $v_{\text{ref}}$. Since the actual values presented here are confidential information, an arbitrary reference will be used consistently.

![Graphs showing velocity levels](image)

(a) scanned surface  \hspace{1cm} (b) half of scanned surface

**Figure 3.17:** Measured and corrected frequency response function. ---: x-coil excitation; ----: y-coil excitation; -----: z-coil excitation.

The velocity distributions reveal a spot on the coil’s surface where the velocity amplitudes are relatively high in a broad frequency range. This is most probably due to a local structural defect in the measured coil. This spot is located at one half of the scanned surface. The velocity levels of the undamaged part are shown in figure 3.17(b). In order to determine the effect of the defect on vibration levels, the velocity levels of the entire scanned surface are compared to the levels of the ‘undamaged’ part in figure 3.18. The defect particularly influences the response
Figure 3.18: Comparison between velocity levels of the entire scanned surface and levels of the part without defect. 
- : entire scanned surface; 
--- : half of the scanned surface.
by the y-coil, which is located closest to the radiating surface: it causes deviations up to 7 dB over a broad frequency range, as shown in figure 3.18(b). The frequency responses for all three excitation cases show similar trends. Transverse coil excitations cause maximum response at about frequency 8, while the z-coil excitation shows no decrease above that frequency.

In the lower frequency range, a number of resonance peaks can be detected. Table 3.6 lists peak frequencies found in the measurement results, together with the dominating circumferential Fourier harmonic in the deflection shape. Because of

<table>
<thead>
<tr>
<th>Excitation by</th>
<th>Dominating Fourier harmonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-coil</td>
<td>y-coil</td>
</tr>
<tr>
<td>2.05</td>
<td>2.05</td>
</tr>
<tr>
<td>2.16</td>
<td>2.14</td>
</tr>
<tr>
<td>2.65</td>
<td>2.69</td>
</tr>
<tr>
<td>2.90</td>
<td>2.87</td>
</tr>
<tr>
<td>3.49</td>
<td>3.50</td>
</tr>
</tbody>
</table>

*Table 3.6:* Normalised measured eigenfrequencies of the gradient coil in the lower frequency range.

the defect, it is not possible to identify resonances in the mid-frequency range, while in the upper range the modal density is too high.

Figure 3.19 shows the deflection shapes at the first resonance frequency in the measured range. The same global shape is found at this frequency for all three excitation cases. The gradient coil does not behave like an axisymmetric structure. As was seen in subsection 3.2.2, the response would contain only odd Fourier harmonics, which is not the case. Also, for an axisymmetric structure, the x-coil response would be the same as the y-coil response, but rotated by 90° about the z-axis. This behaviour is found only at some frequencies, but not, for instance, at $f = 2.05$ as shown in figure 3.19. Due to circumferential inhomogeneities, single Fourier components do not occur in isolation. This effect is particularly visible in the z-coil response. While the Fourier spectrum of the excitation only has an $m = 0$ component, many other harmonics are found in the response.
Figure 3.19: Measured deflection shapes at the inner coil surface at $f = 2.05$. 
3. Model construction of MRI gradient coils

3.5 Gradient coil modelling

As was seen in subsection 3.2.1, the coil is constructed from several composite materials. This section is concerned with determining a correct way of modelling their dynamical behaviour. All of the previously obtained information from the analysis of the excitation, from the computation of the radiation modes and from the surface vibration measurements will now be used in the initial stages of model construction. The analysis toolbox will be applied here to generate additional information. Its efficiency allows for a number of different model studies to be performed within a short time frame. The main purpose of the models will be to examine the effects of design changes on sound power radiation. It is therefore not the primary goal to fit model parameters to measurement data. Instead, the toolbox is used to gain understanding of the dynamical and acoustical phenomena in the gradient coil. This will provide a more solid ground on which future models of new coil designs can be based.

In evaluating the models, the results can be presented in a number of ways to aid in the interpretation. Velocity levels and sound power levels are useful in displaying the model’s behaviour over frequency and detecting resonance frequencies in the structural response. Sound power levels are defined similarly to the velocity levels in (3.28) as:

\[ L_w = 10 \log \left( \frac{P}{P_{\text{ref}}} \right) \]  

(3.29)

Besides these levels, instructive comparisons are made between the vibrational shapes at the radiating surface of the model and the measured shapes. Finally, by plotting the deflection shapes and deformations of an entire coil model, it can be seen if and/or where local deformations occur, thus providing more insight into the coil’s behaviour. These local deformations will be shown by visualising the strain distribution in the coil models.

Table 3.7 summarises the material properties as used in the models for the composite constituents that are present in the gradient coil. Since damping properties of the non-polymeric materials are negligible compared to those of the polymers, their material loss factors are set to zero.

Like in the determination of the radiation modes, structural analyses are performed at 50 discrete frequencies, logarithmically spaced between normalised frequencies 0.80 and 16. Only the Fourier harmonic \( m = 0 \) is computed for z-coil excitation, and odd harmonics for transverse coil excitation. After each structural analysis, sound power levels are evaluated using the stored radiation modes. All calculations are performed using the Structural-Acoustics Toolbox.
3. Model construction of MRI gradient coils

<table>
<thead>
<tr>
<th>material</th>
<th>Storage modulus $E'$ [N/m$^2$]</th>
<th>Material loss factor $\eta$ [-]</th>
<th>Poisson's ratio $\nu$ [-]</th>
<th>Specific mass $\rho$ [kg/m$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>copper</td>
<td>$1.17 \times 10^{11}$</td>
<td>0</td>
<td>0.30</td>
<td>8941</td>
</tr>
<tr>
<td>E-glass fibre</td>
<td>$7.2 \times 10^{10}$</td>
<td>0</td>
<td>0.22</td>
<td>2540</td>
</tr>
<tr>
<td>epoxy resin</td>
<td>$3.4 \times 10^9$</td>
<td>0.02</td>
<td>0.35</td>
<td>1250</td>
</tr>
</tbody>
</table>

Table 3.7: Material properties for the constituents as used in the analyses.

3.5.1 Homogeneous models

Isotropic materials

The most straightforward model that will be examined in this Chapter is a cylindrical model consisting of five homogeneous isotropic layers; see figure 3.3 and table 3.1 on page 39. Each layer is homogenised simply by averaging the material properties according to their respective volume fractions. In the next section, this model will be compared to a model in which more realistic anisotropic material properties are prescribed.

Four of the gradient coil layers consist of composites, and estimated volume fractions for each layer are gathered in table 3.8. For the transverse coil excitation, the Fourier harmonics $m = 1, 3, 5, 7$ are included.

<table>
<thead>
<tr>
<th>Layer(s)</th>
<th>Constituents</th>
<th>Fibre orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 5  coils</td>
<td>95% copper, 5% epoxy resin</td>
<td></td>
</tr>
<tr>
<td>2        profiles</td>
<td>90% glass fibre, 10% epoxy resin</td>
<td>0°</td>
</tr>
<tr>
<td>3</td>
<td>epoxy resin</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>70% glass fibre, 30% epoxy resin</td>
<td>$-45^\circ, +45^\circ$</td>
</tr>
</tbody>
</table>

Table 3.8: Constituents of the gradient coil composites. Layer numbers refer to figure 3.3, and percentages denote volume fractions. Fibre orientations are with respect to the z-axis.

Velocity levels and sound power levels obtained from the analysis results are shown in figure 3.20. In the sound power levels, the peaks can be observed that are caused by acoustic resonances near cut-on frequencies. For the transverse coil excitation, the contributions of the individual Fourier harmonics are shown in figure 3.21. Both in the structural and in the acoustic response, the $m = 7$ harmonic contribution is more than 20 dB below the total response in the entire fre-
Figure 3.20: Velocity levels and sound power levels for a model with isotropic homogeneous layers. 

- : transverse coil excitation; : : z-coil excitation.

Figure 3.21: Calculated velocity levels and sound power levels of the separate Fourier harmonics. 

frequency range. Higher harmonics can therefore safely be ignored in these models. From the sound power levels, it can be seen how the number of relevant Fourier harmonics increases with increasing frequency. The deflection shapes at the first two structural resonance frequencies are displayed in figure 3.22 for comparison with the results from later models. These figures show the deformed meshes of

![Deflection shapes](image)

**(a)** $m = 3, f = 5.66$

**(b)** $m = 3, f = 7.22$

**Figure 3.22:** Calculated deflection shapes at resonance frequencies for the isotropic model (transverse coil excitation). Darker gray denotes higher local deformation.

the coil geometry (compare to figure 3.3). Nodal displacements have been magnified in order to visualise the vibrational shapes. For the studies in this Chapter, it is of interest to observe which parts of the model structure deform the most. Large deformations in any internal layer cause displacements at the outer surfaces, even even the outer layers do not deform. Deformations are not suitable indicators for local deformations, since a structure can experience large (whole-body) displacements without deforming at all. Therefore, local deformations are indicated in these plots by showing the average strains as grayscale values. In figure 3.22(b), for instance, it can now be seen that the coil mainly undergoes deformations in layers 2 and 3. This would be hard to tell from looking at the displacements alone.

The model results are compared to those of the measurements in figure 3.23. The predicted velocity levels are 30 to 40 dB below the measured levels. Also the measurements reveal a number of resonance frequencies well below the first resonance frequency found in the model responses. Therefore, this simple model is certainly not adequate for engineering design purposes.
Anisotropic material behaviour

In the previous model, all of the material properties for the composite were determined by simply averaging the properties of their constituents. As was seen in subsection 2.3.4, transverse moduli and shear moduli can not be determined correctly in such ways. In the model of this section, for layers 2 and 3 modified rule-of-mixtures equations as developed by Tsai and Hahn (1980) are applied, which are summarised in Appendix A.

Polymeric materials generally exhibit more damping in shear deformation than in elongation. Shear moduli and therefore shear damping were not approximated well in the previous simple model. In the present anisotropic model damping is more pronounced, as can be seen by a decrease of velocity levels at the first two resonance peaks in figure 3.24 compared to figure 3.20. Resonance frequencies have shifted, but the deflection shapes are globally the same as in the simple model (compare figure 3.22 to figure 3.25). Therefore, including anisotropy did not lead to substantial differences in the acoustic response, except for the z-coil excitation near the upper frequency bound. The more accurate determination of equivalent material properties has not resulted in a better correspondence with the measurement results, as can be seen in figure 3.26.

Figure 3.23: Comparison between response levels obtained from measurements and from the simple model. —— model; —— measurement.
3. Model construction of MRI gradient coils

![Graphs showing velocity and sound power levels](image)

**Figure 3.24:** Calculated velocity levels and sound power levels for a model with anisotropic homogeneous layers. **- - -** : transverse coil excitation; **--** : z-coil excitation.

![Deflection shapes](image)

**Figure 3.25:** Calculated deflection shapes at resonance frequencies for the anisotropic model (transverse coil excitation). Darker gray denotes higher local strains.
Softening

The previous results indicate that the overall stiffness of the models is overestimated: predicted velocity levels are much lower than the measured ones, and predicted resonance frequencies occur at higher frequencies. There are mainly uncertainties about the ways in which the coil layers and the profile layers have been modeled. The thick-walled fibre-reinforced layer is most probably described correctly. Therefore, two extreme cases will now be studied:

- Modelling the coil layers as existing of epoxy resin as far as stiffness is considered, but with specific mass as in the previous models. This is to mimic the possibility that the coils deform mainly at the epoxy resin locations, while the copper windings perform rigid-body motions.
- Weakening the profile layer to account for the relatively soft glue that is used to attach the inner coil layer to the profile layer.

Because of the expected stiffness decrease due to these modifications, odd Fourier harmonics up to $m = 11$ are now included in the analyses. The response levels for these two cases are shown in figures 3.27 and 3.28. As expected, the levels are higher compared to those of the earlier models, and resonances are found at
3. Model construction of MRI gradient coils

Figure 3.27: Calculated velocity levels and sound power levels for a model with anisotropic homogeneous layers and softened coil layers. ••••: transverse coil excitation; - - - - : z-coil excitation.

Figure 3.28: Calculated velocity levels and sound power levels for a model with anisotropic homogeneous layers and softened profile layer ••••: transverse coil excitation; - - - - : z-coil excitation.
lower frequencies. Therefore, better agreement with the measurements is now obtained, as is displayed in figure 3.29. Deflection shapes show that deformations occur mainly in the softened parts. The global deflection shapes at the first resonance frequency have changed from those of the earlier models: almost uniform deflections are observed at the inner coil surface. Such vibrational patterns are also found in the measurements, as shown in figure 3.19.

In the model response with the softened coils, local oscillations at the coil layers occur, as can be seen in figures 3.30 and 3.31. Similar effects can be distinguished in the measurements (figure 3.19).

**Extra glue layer**

The previous findings motivate the next step in adding more complexity to the model: inserting an extra layer between the inner coil layer and the longitudinal profiles to account for the glue. Effective material properties for the coil and profile layers are determined as in the first anisotropic model. Both the new glue layer and the existing layer between the profiles and the carrier tube are modelled as low-modulus layers.

The response of this model is compared to the measurement results in figure 3.34.
Figure 3.30: Deflection shapes at resonance frequencies for the anisotropic model (transverse coil excitation) with softened coil layers. Darker gray denotes higher local deformation.

Figure 3.31: Deflection shapes at the inner coil surface for the model with softened coil layers, excited by a transverse coil. \(---\) : model; \(---\) : measurement.
3. Model construction of MRI gradient coils

![Graphs showing velocity level vs frequency for transverse and 2-coil excitations]

**Figure 3.32:** Comparison between response levels obtained from measurements and from the homogeneous anisotropic model with softened profile layer. **---** : model; **- - -** : measurement.

![Deflection shapes at resonance frequencies](image)

**(a) m = 3, f = 1.88**

**(b) m = 3, f = 3.47**

**Figure 3.33:** Deflection shapes at resonance frequencies for the anisotropic model (transverse coil excitation), softened profile layer. Darker gray denotes higher local deformation.
The degree of agreement between model predictions and measurements is comparable to those of the softening models. As the effective material properties used in this model can be physically motivated, the results are strong indications that the soft glue used to attach the profiles to the other layers largely determine the high velocity amplitudes at the radiating surface. As can be seen in figure 3.35, the coil deformations are largely concentrated at the soft glue layers.

### 3.5.2 Inhomogeneous layers

In all of the previous models, the gradient coil layers were homogenised. Strictly speaking, however, homogenisation is only allowed when certain conditions are satisfied. The length scales of the inhomogeneities should be small compared to the minimum wavelength of interest, and the inhomogeneities should be sufficiently diffusely distributed.

Some of the layers in the gradient coil contain inhomogeneities that incorporate length scales which may influence vibrational behaviour and thereby the acoustic radiation. The effects of such inhomogeneities in the coil layers are studied in this subsection. The coil layers are modelled as ring-shaped inclusions, made of copper, embedded in epoxy resin. The extra glue layer has also been included.
Predicted results for this model are compared to the measured velocity levels in figure 3.36. Although details in the response differ from the previous model, the results are roughly the same. Since it is the more global vibration distribution that determines the sound radiation, such global variations have to be predicted correctly. Therefore, it may suffice to model the conductors as homogeneous layers, because the overall stiffness is mainly determined by the soft glue layers, as figure 3.37 indicates.

3.6 Summary and discussion

The developed tools are highly suitable as an addition to available knowledge about the vibrational behaviour of the MRI gradient coil. By offering the possibility to perform model studies fast, a better understanding of what is observed in the measurements is gained, especially in combination with a detailed analysis of the excitation. By using only a few models, essential information about the gradient coil’s vibrational behaviour has been revealed in a structured way.

Although the calculated velocity levels differ from the measured levels, the model results are qualitatively similar in a number of ways. It has been shown
Figure 3.36: Comparison between measurements and the model with inhomogeneous coil layers for transverse coil excitation. —— : model; —— : measurement.

(a) $m = 3, f = 2.26$

(b) $m = 3, f = 3.26$

Figure 3.37: Deflection shapes at resonance frequencies for the model with extra glue layer and inhomogeneous coil layers. Darker gray denotes higher local deformation.
that the soft glue that is used to attach the longitudinal profiles may well be the
cause for low-frequency resonances and high velocity levels. Like in the measure-
ments, \( m = 3 \) Fourier harmonics dominate the response in the lower frequency
region. The measurements, however, reveal a number of resonances that were
not observed in the model responses. The analysis of the excitation showed that
transverse coil excitations do not contain even Fourier harmonics. Such harmoni-
cs are, however, strongly present in the gradient coil response, from which it can
be concluded that the gradient coil does not behave like an axisymmetric struc-
ture, and consequently can not be modelled as such. Missing Fourier harmonics
may be the reason that the model responses are generally below the measured
levels.

The studies in this Chapter should be seen as a first step in modelling the gra-
dient coil. In order to model inhomogeneities, an extension to the Fourier ele-
ments used in this thesis are under development, and will be published in a
forthcoming paper. In the new Fourier analysis method, the material homo-
geneties are transformed into Fourier series also, resulting in coupled equations
for the Fourier harmonics. This is expected to be more efficient than using full
three-dimensional models.

When an MRI scanner is in service, the gradient coil is not radiating directly. But
since it is one of the main sources of the sound problem, it is necessary to either
modify its design and/or to apply vibration control. For both purposes, reliable
gradient coil models are required.

In order to evaluate model reliability, measurements from more gradient coils
would be required. Data from different coil types in particular can be useful
to assess a parameterised model's performance. The measurement set-up with
a laser-Doppler vibrometer as used in this project has proved to be useful. Al-
though labour-intensive, the normal surface velocity distribution of a gradient
can be measured in a few days. Automating the positioning system is ex-
pected to be problematic, because the set-up fails if the prism is not lined out
correctly with respect to the laser beam, and manual adjustments are sometimes
necessary.

As a certain amount of scatter is common in the vibrational response of complex
systems, it is desired to predict deviations that are to be expected for a particular
design. Satisfactory tools for such analyses, however, are currently not available,
certainly not in commercially available software. An ideal solution would be
to assign stochastic parameters to differential equations. Such equations would
have to be solved using robust computational stochastic methods based on the
finite element method. A considerable challenge is the determination of correct
values for variations of the stochastic model parameters. In the next Chapter,
deterministic parameter studies will be performed. One of the purposes of these
3. Model construction of MRI gradient coils

studies will be to evaluate model sensitivities: detecting parameters that cause considerable deviations in the acoustic response when being changed slightly. Such deterministic model studies can be a first step towards the development of stochastic methods in structural-acoustics.
Parameter studies for MRI gradient coils

4.1 Introduction

In this Chapter, several parameter studies are presented that reveal how design modifications relate to changes in the structural and acoustic response. In a parameter study, one or more variables are assigned to a model, and the model is analysed at multiple combinations of design variable values. In structural-acoustics, several types of parameters can be used as design variables. These can be categorised into variables related to the geometry, to material properties, to the kinematic boundary conditions and to the excitation. The limits of the variables span the design space, and each design point represents a combination of discrete values for the design variables. Typically, many analyses have to be performed in a parameter study. This has become viable due to the efficiency of the numerical tools developed in this project. Particularly the reusability of the already computed radiation modes is a great advantage.

The parameter studies in this Chapter are performed for several reasons. Firstly, they will uncover the magnitudes of the reductions that can be achieved by passive noise control of MRI gradient coils. Parameters that only merely influence sound radiation can then be omitted in later optimisations. A second reason for the parameter studies, is to investigate what physical mechanisms determine the quality of a design. Understanding these mechanisms leads to a better insight on what is possible in passive noise reduction for the gradient coil, and what kind of difficulties are to be expected. Lastly, the parameter studies are used to assess the suitability of the developed tools in performing them, and to show how the tool can be applied efficiently.
Fundamentals of internally radiating ducts

Kuijpers (1999) has performed parameter studies on different and simpler gradient coil models. By studying the radiation modes, sound power levels, sound pressure levels, velocity levels and radiation efficiencies, he revealed mechanisms that are fundamental to the internal sound radiation of cylindrical structures. Understanding these mechanisms is key in comprehending the effects of structural design changes to the acoustic response. As already discussed in section 2.4, the sound power radiation at a certain frequency is determined both by the shape and the amplitude of the surface normal velocity distribution. The radiation efficiency of a particular shape depends on the vibrational frequency. As each design modification leads to changes in both the amplitudes and the shape of the surface vibrations at each frequency, the acoustic effect of a modification is a combination of two mechanisms. A modification that results in lower velocity levels may also result in improved radiation efficiencies and therefore in higher sound power levels. As a result, velocity levels and sound power levels are not linearly correlated, and the total sound power level as a function of a design variable generally behaves irregularly.

Choices for the parameter studies

The parameter studies in this Chapter are performed using models that better resemble the actual coil behaviour than the ones used in Kuijpers (1999). One of the goals is to determine the magnitudes of sound power reductions that can be obtained. The design variables that are assigned to the models are therefore not chosen arbitrarily, but are motivated by results from the previous Chapter in which gradient coil models were constructed. One of the conclusions was that the soft glue layers seem to contribute significantly to the sound radiation. In section 4.2, the result of using bonding layers with higher moduli is examined. Once the glue layers have been replaced by a better alternative, other parts of the gradient coil may be modified to gain additional reductions. The influence of the thickness of the fibre reinforced carrier layer is therefore studied in section 4.3.

As was observed in section 3.2.2, the Lorentz forces at the inner and outer coils have different distributions and amplitudes. A topic that has received some attention in literature is Lorentz force balancing. In this method, the gradient coil is modified such that the net force is minimal. The Structural Acoustics Toolbox is used in section 4.4 to assess the performance of force balancing using a numerical parameter study. For the first time, the gradient coil dynamics are taken into account in such design alteration efforts.

Although the developed tools offer great potential in passive noise reduction studies, they have to be used with care. In section 4.5, it is shown how faulty
conclusions can easily be drawn using an example in which the fibre orientation in the carrier layer are varied.

4.2 Variation of the glue modulus

Results from the previous Chapter indicate that the soft glue layers account for much of the high overall velocity and sound power levels. Using a parameter study, the effect of improving the bonding between layers will now be examined. By analysing 40 models that each have a different value for the modulus of the glue material, the relationship between the bonding and the sound radiation can be observed. The modulus is varied between $E = 3.4 \times 10^8$ (which is the reference design) and $E = 3.4 \times 10^{11}$ [N/m²].

Figure 4.1 shows the results of the parameter study. The reductions of sound power levels and of the velocity levels are plotted against the glue modulus. Positive values for the sound power denote improvements over the original design. A highly irregular relationship between the modulus and level reductions is found. Velocity level reductions and sound power level reductions, however, follow a similar trend. Improvements in the sound power radiation are generally seen, whereas velocity level reductions are typically smaller. At a number
of design points, an increase in velocity levels is accompanied by a decrease in sound power levels. These results can be explained by examining the responses of individual designs. As an example, in figure 4.2 the frequency response for \( E = 2.0 \cdot 10^{10} \text{ [N/m}^2\text{]} \) is compared to the response at the reference design. The reference design exhibits a high radiation efficiency at \( f = 15.1 \) (see figure 4.2(c)). This high efficiency, together with the high velocity levels in the upper frequency range, leads to a sound power level peak in figure 4.2(b) at that frequency.

As can be seen in figure 4.1, at \( E = 2.0 \cdot 10^{10} \text{ [N/m}^2\text{]} \) both the total velocity level and the total power level are increased with respect to the reference design. Figure 4.2(a) shows two consequences for stiffening the glue layers that were to be expected: velocity levels generally decrease, and resonance frequencies shift upwards. As a result of changing the structure, however, a structural mode is introduced at \( f = 13.3 \) that is efficiently excited by the gradient coil excitation. This results in a high peak in the velocity levels at that frequency. Besides being excited strongly by the Lorentz forces, the shape of the new mode that dominates the structural response at \( f = 13.3 \) also leads to a higher radiation efficiency, as can be concluded from figure 4.2(c). These two factors account for a high peak in the sound power levels, and therefore result in an increase in the total sound power level.

Also noticeable is that in the lower frequency range the radiation efficiency has reached its maximum for the design with the higher-modulus glue layer. As a consequence, the reduction in sound power levels is cancelled completely in the lower frequency regions. This underlines that radiation efficiency always has to be taken into account in structural-acoustic design. For sound reduction in complex structures, conclusions should never be based on the velocity responses alone.

### 4.3 Geometry variation

As a second parameter study, the thicknesses of two of the layers are varied. Only one parameter is used: the thickness of the carrier layer. By imposing that the total thickness of the structure has to remain constant, the profile layer thickness changes along, and is therefore parameterised using the same variable as for the carrier layer thickness. The glue layer between the inner conductor layer and the profile layer is omitted for this parameter study. The normalised carrier layer thickness is varied linearly between \( t = 4.0 \cdot 10^{-3} \) and \( t = 0.044 \), and analysed at 20 uniformly spaced design points.
Figure 4.2: Comparison between the calculated responses of two designs in the glue modulus parameter study. 

- : reference; 
  - : $E = 2.0 \cdot 10^{10}$ [N/m$^2$]; 
  - : maximum radiation efficiency.
4. Parameter studies for MRI gradient coils

Varying the internal geometry of a structure does not influence the radiative properties of a structure, so that here also the previously calculated radiation modes can be reused to evaluate the sound power radiation. The result of the parameter study is shown in figure 4.3. Level reductions are plotted relative to the reference design with normalised carrier layer thickness \( t = 0.020 \). The total velocity level is hardly influenced by the thickness variations. At two design points in the parameter range a sudden change of 2 dB is observed. At the same points decreases in total sound power levels occur: 7 dB around \( t = 0.015 \) and 8 dB around \( t = 0.032 \). Again, an explanation can be found by examining the responses at individual designs. In figure 4.4, two designs near the level jumps around \( t = 0.015 \) are compared. The velocity levels in figure 4.4(a) show only minor differences between the two designs. Both have a high peak at normalised frequency \( f = 11.8 \). At this frequency, however, figure 4.4(c) displays a difference in radiation efficiency of 16 dB between the two designs. A dominating peak in the sound power levels is therefore observed for the design with carrier layer thickness \( t = 0.017 \). The same effect occurs around \( t = 0.032 \).

As this example clearly shows, slight changes in the structure may lead to large variations in the radiation efficiencies, and therefore in the radiated sound power. Such sensitivities greatly hamper structural-acoustic design. Even manufacturing inaccuracies can cause variations in the acoustic response between similar structures. It is therefore of importance to not only search for a design with a minimal sound power radiation, but also to find a region in the design space...
Figure 4.4: Comparison between the calculated responses of two designs in the layer thickness parameter study. \( t = 0.014 \); \( t = 0.017 \)
which is less sensitive to design variations. Small deviations from the optimal design might otherwise cause unpredictable results. In the next Chapter, a possible solution is presented in which the influences of sensitivities and uncertainties are incorporated in a new numerical optimisation strategy.

4.4 Lorentz force balancing

One of the possible measures to reduce gradient coil noise is known in literature as force balancing. The forces generated by the inner and outer coils have opposite direction. By modifying the locations of the coils or by changing the conductor patterns of one or both of the coil sets, a more favourable excitation may be obtained.

Petropoulos and Morech (1995) aimed to achieve a zero net force. They argue that when the forces at the inner coil and at the outer coil cancel each other, coil vibrations and therefore sound radiation will be reduced. A zero net force, however, only suppresses static deflections totally. The net force is reduced considerably, but no experimental or numerical verification for the effects on the response is performed. Dynamic forces that are equal in magnitude but have opposite directions do result in structural vibrations when they are applied at different parts of the structure. Chapman and Mansfield (1995) recognise that the performance of Lorentz force balancing depends on the mechanical coupling between the coil sets. They propose to balance forces locally by reducing Lorentz forces at the outer coil. A third coil is added to restore the original function of the second coil: shielding eddy currents. This third coil can optionally be Lorentz force balanced also. The radius of the force shielding coil is optimised to reduce the net forces while retaining eddy current shielding and magnetic gradient field linearity. Noise reductions of up to 30 dB are reported, particularly at low frequencies.

As was seen from the parameter studies in the foregoing sections, a reduction in velocity amplitudes does not necessarily lead to noise reduction. The developed toolbox offers the possibility to review the workings of force balancing using numerical studies. Instead of evaluating optimum Lorentz force balancing in terms of zero net forces, the gradient coil can be optimised directly in terms of noise reduction.

The parameter study that is presented in this section will contain multiple design variables. Besides varying the ratios of the amplitudes of the inner and outer coil excitations, the total stiffness of the structure is also varied. That way, several questions can be answered: does force balancing reduce velocity levels when dynamics are taken into consideration, does this reduction lead to a significant
reduction in the sound power levels, and how does its performance depend on mechanical properties of the structure.

Contrary to other design variables, evaluating the response for changes in the excitation does not require a full new FEM analysis. Only the right-hand side of the matrix system of equations changes. Solving this system involves a matrix decomposition that consumes the largest part of the computational efforts. For multiple load cases this decomposition only has to be performed once since the matrix does not change.

In the following parameter study, force balancing will be simulated by varying the amplitudes of the outer coil excitations. This way, an even more efficient method of analysis can be applied. Since the system is linear, the normal velocity response is the sum of the coil responses when excited by the inner and outer coil separately. If the amplitudes of the outer coil Lorentz forces are multiplied by a factor \( \alpha \), then the velocity amplitudes will also change by that factor. The response for the combined coil excitations can therefore be expressed as:

\[
v_i = v_i + \alpha v_o, \quad (4.1)
\]

where \( v_i \) is a column matrix containing the nodal surface normal velocity amplitudes for the acoustic mesh, that are determined from a structural analysis with inner coil excitation only. Similarly, \( v_o \) contains velocity amplitudes for outer coil excitation for \( \alpha = 1 \). Sound power levels are determined using the radiation modes (see section 2.4):

\[
\mathbf{T} = \frac{1}{2} \mathbf{\zeta}^H \Lambda \mathbf{\zeta}, \quad \mathbf{\zeta} = \mathbf{\Psi}^H \mathbf{N} v
\]

where \( \mathbf{N} \) is the shape function matrix, and \( \mathbf{\Psi} \) is a matrix containing the eigenvectors of the radiation modes. For the inner coil excitation response, the outer coil excitation response and the total response, the modal participation factors are:

\[
\mathbf{\zeta}_i = \mathbf{\Psi}^H \mathbf{N} v_i \quad (4.3)
\]

\[
\mathbf{\zeta}_o = \mathbf{\Psi}^H \mathbf{N} v_o \quad (4.4)
\]

\[
\mathbf{\zeta}_t = \mathbf{\zeta}_i + \alpha \mathbf{\zeta}_o \quad (4.5)
\]

Using these expressions, the radiated sound power can be evaluated for each of these cases:

\[
\mathbf{T}_i = \frac{1}{2} \mathbf{\zeta}_i^H \Lambda \mathbf{\zeta}_i \quad (4.6)
\]

\[
\mathbf{T}_o = \frac{1}{2} \mathbf{\zeta}_o^H \Lambda \mathbf{\zeta}_o \quad (4.7)
\]

\[
\mathbf{T}_t = \frac{1}{2} \mathbf{\zeta}_t^H \Lambda \mathbf{\zeta}_t + \frac{1}{2} \alpha \mathbf{\zeta}_i^H \Lambda \mathbf{\zeta}_o + \frac{1}{2} \alpha \mathbf{\zeta}_o^H \Lambda \mathbf{\zeta}_t + \frac{1}{2} \alpha^2 \mathbf{\zeta}_o^H \Lambda \mathbf{\zeta}_o
\]

\[
= \mathbf{T}_i + \alpha \text{Re} \left( \mathbf{\zeta}_i^H \Lambda \mathbf{\zeta}_o \right) + \alpha^2 \mathbf{T}_o \quad (4.8)
\]
For any given gradient coil design, the optimum value for the force amplitude factor $\alpha$ is given by:

$$\frac{\partial T_t}{\partial \alpha} = 0$$  \hspace{1cm} (4.9)

which leads to:

$$\alpha_{\text{opt}} = -\frac{1}{2} \text{Re} \left( \mathbf{\zeta}_i^H \mathbf{\zeta}_o \right) \frac{1}{T_0}$$  \hspace{1cm} (4.10)

This optimum always is a minimum in the total sound power level $T_t$, since

$$\frac{\partial^2 T_t}{\partial \alpha^2} = 2T_0 > 0$$  \hspace{1cm} (4.11)

For $\alpha > 0$, the term:

$$\text{Re} \left( \mathbf{\zeta}_i^H \mathbf{\zeta}_o \right) = \text{Re} \left( \sum_{k=1}^{n} \rho_k \alpha \sigma_k \mathbf{\zeta}_i^* \mathbf{\zeta}_i \right)$$  \hspace{1cm} (4.12)

has to be negative in order to reduce the radiated sound power. This is accomplished if the most efficient radiation modes that are present in both of the separate responses have opposite polarity, so that $\text{sign}(\mathbf{\zeta}_i) = -\text{sign}(\mathbf{\zeta}_o)$. Force balancing can therefore be seen as noise cancellation before the noise is radiated. Efficient radiation modes have to be suppressed, which is, to a certain degree, allowed to be accompanied by increases in the modal participation factors for non-efficient radiation modes. It is therefore not strictly necessary to reduce the velocity levels.

As depicted in figure 4.5, increasing the stiffness of the glue layer does not lead to noise reduction if the structure is excited by the inner coil or by the outer coil individually. Only in the higher stiffness range small reductions are found.

The force amplitude factor $\alpha$ in (4.1) is now varied between $\alpha = 0$ (no outer coil excitation) and $\alpha = 2$ (doubled outer coil excitation amplitudes). Figure 4.6(b) shows the dependence of the total sound power level on the glue layer modulus and the force amplitude factor simultaneously. The level reductions relative to the reference design ($E = 3.4 \cdot 10^9$, $\alpha = 1.0$) are plotted against the design variables. As a general trend, higher sound power level reductions are achieved when the excitation amplitude factor for the outer coil is near its reference value. The optimum, a noise reduction of 14 dB, is located at $E = 2.4 \cdot 10^{11}$ and $\alpha = 0.90$. Local optima are also encountered. The worst design has a sound power level increase of 10 dB.
4. Parameter studies for MRI gradient coils

Figure 4.5: Total sound power level reductions and velocity level reductions as a function of glue layer modulus for inner coil excitation and for outer coil excitation. 
- : total sound power level reductions; 
- : total velocity level reductions.
Figure 4.6: Effect of force balancing and glue layer modulus variation on the total velocity level and on the total sound power level with respect to the reference design.
4. Parameter studies for MRI gradient coils

There is no proportional relationship between improved coil stiffness and the effect of force balancing. Lorentz forces were already balanced well for the optimal design; noise reductions are mainly due to the changes in the glue layer modulus. At some local optima, however, particularly in the lower ranges of the modulus values, $\alpha$ is decreased. For a glue layer modulus of about $E = 1 \cdot 10^{11}$ Lorentz force balancing has more effect, as is displayed in figure 4.7.

![Graph](image)

**Figure 4.7:** Influence of varying the amplitude of the outer coil excitation on the response. The level reductions are relative to the reference model.

- : total sound power level reduction; --- : total velocity level reduction.

Lorentz forces balancing can lead to noise reductions, provided that forces in the original design were not balanced well, and that the mechanical properties cause the gradient coil to be susceptible to changes in the excitation. It is therefore always worthwhile to take force balancing into account, as changing the mechanical properties may also change the force balance optimum.

### 4.5 Pitfalls in using the analysis tools

Not every improvement in a plot of the results of a parameter study is attributable to physical mechanisms. Due to incorrect application of the tools, apparent changes in noise production may be caused by numerical effects. In this section, this is illustrated by evaluating the influence of fibre orientation in the carrier layer.
Instead of the bi-directional plies in the original design, a uni-directional orientation is applied here, and the orientation angle is varied between $\gamma = 0^\circ$ and $\gamma = 90^\circ$. Twenty design points are generated and analysed. The total velocity level reductions and total sound power level reductions are plotted against the fibre angle in figure 4.8. The design for which the fibre angle is $0^\circ$ is used as a reference. Between the optimal design ($\gamma = 57^\circ$) and the worst design ($\gamma = 71^\circ$) the difference in total sound power level is 4 dB. At about $\gamma = 68^\circ$ a sudden change of almost 3.5 dB is observed in the sound power response while the total velocity level changes by less than 1 dB.

Closer examination of the frequency responses shows that neither the structural resonance frequencies nor the radiation efficiencies change significantly between the designs. The differences in predicted sound power levels between the designs are caused by corresponding differences in peak levels in the velocity levels. The structure is lightly damped, resulting in high, narrow peaks. Slight design changes cause slight changes in resonance frequencies. Since the frequency responses are evaluated at discrete frequencies, the calculated peak values are likely to differ more between individual designs than the actual peak values. It is therefore advisable to always check the results of a parameter analysis and search for numerical artifacts such as the frequency range discretisation in the example above.

Wrong conclusions can also be drawn when conclusions are based on the lev-
els of the frequency responses, and not realising that the frequency spectrum of the excitation does not cover the entire frequency range of the calculations. In the examples above, it has been assumed that the excitation covers the entire frequency range. Present gradient coils, however, are primarily excited up to frequency 8. Incorporating only that frequency range in the evaluation of the total sound power levels would lead to entirely different conclusions.

### 4.6 Summary and discussion

Four different kinds of parameter studies have been performed, showing the versatility of the Structural Acoustics Toolbox. The results indicate that significant sound power reductions can be achieved for the gradient coil by passive noise control. Reductions of up to 14 dB have been encountered in the parameter studies by varying a single design variable.

It was shown that the effect of Lorentz force balancing depends on mechanical properties of the gradient coil. For the model, the original design was almost balanced optimally already. More sophisticated models will have to decide if this is also true for the actual gradient coil. Also, only the force amplitude of the outer coil has been varied. A better alignment of the force distributions at the inner and outer coils might lead to design improvements.

The physical mechanisms underlying sound radiation inside cylindrical ducts were described by Kuipers (1999). These mechanisms completely explain all phenomena observed in the parameter studies. Using the tool, all relevant physical quantities such as velocity levels, sound power levels and radiation efficiency needed to understand the parameter studies result can be presented graphically. Care should be taken, however, when interpreting the results, since predicted noise reductions may be attributable to numerical artifacts.

The design variables in the presented parameter studies can not be categorised into parameters that either influence the sound power levels or not. Instead, regions in the design space are observed in which acoustic radiation is either more sensitive to design changes than in other regions. From a design perspective, it is attractive to find stable regions, so that variations between individual products can be reduced. By performing parameter studies, such regions can easily be identified. A disadvantage of parameter studies, however, is that the number of required analyses grows exponentially when more design variables need to be included. Using numerical optimisation is then far more efficient, as will be demonstrated in the next Chapter.
4. Parameter studies for MRI gradient coils
Five

Numerical optimisation

5.1 Introduction

Computer simulations are applied to the design of increasingly complex products. Improving such products means that a number of parameters will have to be changed such that a number of conflicting design criteria are satisfied. This is a task that can hardly be performed without a systematic approach. In this Chapter, it is shown how available computer algorithms can be applied for product improvements with respect to sound radiation. The most generally applicable and efficient methods will be selected, based on considerations such as design practice, problem characteristics, modelling issues and manufacturing variations.

In structural acoustics, finding an optimal design requires substantial resources. Only when the behaviour of all designs for a product in the set of possible designs is known, the optimal design can be located. Therefore, computing an optimal design directly is fundamentally impossible for almost all practical engineering applications. What can be evaluated is the response of a model of a particular design. The way to find an optimum is to take samples from the design space using simulation software, or by building a number of prototypes. Because taking a single sample already requires considerable resources, usually the number of samples that can be used is limited. In the following, it is assumed that using simulations is the only viable option for the type of product that is to be improved, and that a sufficiently accurate model is available.

The importance of finding the optimal design in engineering applications can easily be overstressed. In many practical cases, the performance of an optimal design will be only marginally better compared to a number of other designs that are not optimal in a mathematical sense. This marginal difference has little practical value, as it may well be overshadowed by modelling and manufacturing inaccuracies. Finding the “true” optimum is then a waste of resources. A better and more realistic goal is to strike a balance between product improvement and computational effort. The question, then, is how to improve a design as much as
possible using a limited amount of computational resources.

In order to select the most suitable algorithms, it is not necessary to evaluate all algorithms on their performance for a set of different problems. Such an evaluation would be of limited value since it would not give any information on why any algorithm performs better than others. There would be no way of predicting how that algorithm would perform on a new problem. Instead, the underlying methodologies of different classes of optimisation algorithms will be studied here. By weighting the assumptions on which the methods are based against the characteristics of the design problems and to practical considerations, it will be argued which methods can be expected to fail or to perform poorly. A class of suitable methods can then be distilled. Criteria for a suitable optimisation strategy are summarised in section 5.2.

In the past 60 years, different kinds of optimisation approaches have been developed. For detailed texts on numerical optimisation in relation to engineering, the reader is referred to e.g. Haťka and Gürdal (1992), Statnikov and Matusov (1995) and Vanderplaats (1999). The following discussion will only cover the approach known as parametric optimisation, because of its generality. Parametric optimisation has some additional benefits as well, as outlined in sections 5.3 and 5.4. A choice for the most suitable method, within the set of parametric methods, will then be made in section 5.5, based on the criteria and on the examination of the optimisation strategies. The performance of the chosen method is evaluated in section 5.6. Lastly, it will be explained how deterministic optimisation methods can be applied to problems that involve stochastic models.

5.2 Criteria for selecting an engineering optimisation method

An optimisation approach that is to be practically useful for application in structural acoustics should satisfy several criteria. The discussion in section 5.1 leads to the demands listed below. In particular, this list focuses on the selection of an optimisation approach that agrees with engineering practice:

- The total number of design trials should be small, since a single simulation consumes significant computational resources. A large part of the total design time is necessary for model construction and for the formulation of the optimisation problem. This limits the time that can be reserved for evaluating all the simulations.
• The optimisation software should be able to cooperate with available simulation software. It should not be necessary to alter the simulation code in order to realise a coupling between different modules. In practical situations, software is obtained from a third-party vendor, so modification of the code is not an option, because the source code is inaccessible.

• Multiple criteria have to be optimised simultaneously. Industrial design usually has a multidisciplinary character, which means that there will be different, conflicting design goals. When designing new MRI scanners, for instance, sound radiation is to be reduced, but so is cost price, while imaging speed and quality should be high. These are conflicting demands, and a compromise is to be found.

• Infeasible solutions cannot be accepted. A number of designs will not meet all of the design criteria, and these should be avoided by the algorithms. Also, an optimisation method should not search in regions in the design space in which the simulation models are invalid.

• A design that leads to small variances in behaviour between nominally identical products should be preferred over a design that leads to large variances, if those solutions perform approximately equally well in other respects. An optimisation strategy should be able to discriminate between such designs.

• There should be a mechanism by which the user can control the balance between product improvement and invested resources. Product features have to be improved as much as possible, but within a certain budget, computational resources and time frame. Finding an optimum in the mathematical sense is not required.

• Algorithms have to operate successfully in problems with irregular behaviour. As reported in literature, and as observed in the parameter studies of Chapter 4, quantities related to structural-acoustics often respond highly nonlinear to design changes.

An approach that addresses most of these demands is described in Statnikov and Matusov (1995). Their Parameter Space Investigation (PSI) method is generally applicable, but it does not meet one of the most important demands since the number of analyses that has to be employed is in the order of thousands. Methods that are more efficient in this regard are formed by combining parametric optimisation and function approximation concepts. In their present form, however, these approaches do not satisfy all of the criteria listed above. In the next two sections, the concepts behind parametric optimisation and approximate
optimisation are explained. This is followed by a discussion of how such an approach can be augmented in order to satisfy most or all of the imposed criteria, and which approach is the best choice for application in structural acoustics.

5.3 Parametric optimisation

Parametric optimisation is widely applicable. From the viewpoint of the optimisation algorithm, the simulation of the model is a black box. Any model configuration is represented by a set of design parameters $x_i$, ($i = 1, \ldots, n_d$), and the algorithm operates by repeatedly evaluating the model at a certain design point $x$, as visualised schematically in figure 5.1.

![Figure 5.1: Principle of parametric optimisation.](image)

Parametric optimisation can therefore be described as finding a set of parameter values $x_{\text{opt}}$ that result in optimal performance. This can be applied to design problems related to structural acoustics by relating design parameters to a parameterised model, such as those in Chapter 4. An optimum $x_{\text{opt}}$ is defined as the values of design parameters in $x$ where the objective function $F(x)$ has a minimum. Any computable scalar quantity, such as the radiated power level, can be used as the objective function. Also, constraint functions $G_j(x)$ can be defined to keep certain system properties within bounds.

5.3.1 General optimisation problem formulation

In parametric optimisation, the general problem formulation can be stated mathematically by (Fletcher, 1986):

\begin{align*}
\text{minimise} \quad & F(x) \quad (5.1) \\
\text{subject to} \quad & G_j(x) \leq c_j, \quad j = 1, \ldots, n_c \quad (5.2) \\
\text{and to} \quad & x_i^{(l)} \leq x_i \leq x_i^{(u)}, \quad i = 1, \ldots, n_d \quad (5.3)
\end{align*}
in which \( c_j \) are the bounds for the constraint functions, and \( x_i^{(l)} \) and \( x_i^{(u)} \) are the lower and upper bounds on the design parameters, known as side-constraints. In practical applications, bounds will have to be placed on the design parameters in order to prevent infeasible values such as negative thickness. The side constraints (5.3) can also be used to limit the optimisation to the design space in which the models are assumed to be valid. For instance, when a thin-shell theory is used, the thickness variables can be set appropriately so the theory remains valid throughout the optimisation.

When both the objective function and the constraint functions depend linearly on the design parameters, the optimisation problem is called a Linear Programming (LP) problem. When the objective function is a quadratic function, a Quadratic Programming (QP) problem arises. Both of these formulations can fairly easily be solved by standard algorithms (Haftka and Gürdal, 1992). In most cases, however, the objective function and the constraint functions will be nonlinear functions of the design parameters, and a Nonlinear Programming (NP) problem has to be solved. These are usually solved iteratively, in which a direction of search is determined in each iteration by solving an LP, QP or an unconstrained sub-problem (see e.g. Haftka and Gürdal (1992)).

Many such algorithms have been developed, each of which has its strengths and weaknesses. Although usually the algorithms are able to operate without user intervention, setting up a “correct” optimisation problem formulation and selecting a suitable optimisation method requires expertise. Formulating the optimisation problem is highly problem dependent, and a general consideration is not within the scope of the present discussion. An integral approach to engineering optimisation is covered in Papalambros and Wilde (2000), treating model construction, problem formulation and optimisation issues. For the MRI scanner, for instance, a suitable criterion for the acoustic radiation has to be selected. For the gradient coil, Kuijpers (1999) has shown how the developed simulation tools can be used effectively to assess an appropriate acoustic criterion. The outcome of this study was that using the total radiated power level is largely equivalent to using an average pressure level, and that using the total radiated power level is computationally more efficient. When applying numerical optimisation to the actual design of MRI scanners, other criteria that are also of importance but not related to sound radiation are to be selected. Issues such as cost price, imaging quality, and size, each pose their limits on certain quantities, and these quantities have to be determined. In the next section, it is explained how to use numerical optimisation when such conflicting design goals arise in the resulting formulation.
5.3.2 Multi-objective optimisation

In practical applications, usually multiple objectives have to be optimised. Mathematically, this can be posed by changing the optimisation formulation to:

\[
\begin{align*}
\text{minimise} \quad & F(x) \\
\text{subject to} \quad & G_j(x) \leq c_j, \quad j = 1, \ldots, n_c \\
\text{and to} \quad & x_i^{(l)} \leq x_i \leq x_i^{(u)}, \quad i = 1, \ldots, n_d
\end{align*}
\]

in which the performance vector \( F \) is a column matrix of objective functions. The performance vector is a mapping from the parameter space into the objective space, as shown graphically in figure 5.2 for a two-dimensional case. The objective space is not always a closed volume; multiple "islands" may exist. Each of the individual objective functions will have minima at different design points. Therefore, a new definition of optimality is required. Any design point, from which one of the objectives can be decreased by changing the design parameters without increasing any of the other objectives, can not be considered optimal. An optimality criterion that is therefore often applied is that of non-inferiority. Any design point \( x_p \), which has no point in its vicinity for which any objective can be decreased without increasing any of the other objectives, is called non-inferior or Pareto optimal (Statnikov and Matusov, 1995). These solutions are located on the boundary of the feasible region, as illustrated in figure 5.2. Multi-objective optimisation is therefore concerned with finding the set of non-inferior solution points, and selecting one of these points. Selection of a particular solution can be done by giving more importance to some objectives, or, in other words, by weighting the objectives. Two methods in which weight factors can be associated with the objective will now be explained.

![Figure 5.2: Mapping from parameter space into objective function space.](image-url)
The Weighted Sum method

A popular way to apply multi-objective optimisation is to convert the column of objectives to a single objective using a weighted sum:

$$\text{minimise} \quad f(x) = \sum_{k=1}^{n_o} w_k F_k(x)$$

(5.7)

in which the weighting factors $w_k$ have to be determined by the designer to express the relative importance of each objective function. Dimensions of the weighting factors can be chosen such that the resulting objective function $f(x)$ is dimensionless. The concept of the Weighted Sum method is shown graphically in figure 5.3. The line $L$ in the objective space is determined by $w^T F(x) = C$.

![Figure 5.3: Geometrical representation of the Weighted Sum method.](image)

so its slope is determined by the weighting factors. For $C = 0$, the line $L$ crosses the origin. The optimisation can be interpreted as finding a value for $C$ for which the line $L$ touches the boundary of the feasible region. An optimum is found at the point, or at one of the points, where $L$ touches the feasible region. At these points, the weighted sum $f(x)$, which equals $C$, has the smallest value for which all constraints are satisfied.

This formulation, however, suffers from a serious flaw. For some problems, part of the parameter space can not be reached by an optimisation algorithm, as is illustrated in figure 5.3. The region between points $A$ and $B$, in this case, is unavailable. Another difficulty of the Weighted Sum method, is that the weighting factors do not correspond directly to the relative importance of the objectives. The relation between a weighting factor and an objective only becomes clear after the optimisation. As a result, designers are inclined to change the weights...
after an optimisation, and to perform multiple optimisation runs with different sets of parameters (Statnikov and Matusov, 1995).

The Goal Attainment method

A different multi-objective optimisation procedure that does not have the problems of the Weighted Sum method, and that is related more closely to engineering practice, is the Goal Attainment method of Gembicki (1974). For each objective $F_k$, a design goal $F_k^*$ is chosen. When, for example, both the radiated sound power, denoted by $F_1(x)$, and the mass, denoted by $F_2(x)$, of an MRI gradient coil have to be reduced, goals could be defined for these objectives, such as $F_1^* = 90$ dB and $F_2^* = 640$ kg. (This example will be worked out in section 5.6.)

In the Goal Attainment method, the objectives are required to satisfy:

$$F_k(x) \leq F_k^* + \alpha_k \gamma, \quad \alpha_k \geq 0, \quad k = 1, \ldots, n_y \tag{5.8}$$

This way, the designer does not have to set strict goals, because the formulation allows each objective to be under-achieved by a value of $\alpha_k \gamma$. The concept of the Goal Attainment method is illustrated graphically in figure 5.4. The inequalities (5.8) limit the values of the objective functions. They act as additional constraints, and reduce the feasible region, such as visualised by the gray area in figure 5.4. It is not required that the goals can be achieved within the feasible region, which allows the values for $F_k^*$ to be chosen arbitrarily. The factor $\gamma$ is dimensionless, and each coefficient $\alpha_k$ has the same dimension as its associated

![Figure 5.4: Geometrical representation of the Goal Attainment method. The goal point is outside of the feasible region, and the goals are under-achieved at the optimum.](image-url)
objective $F_k$. By reducing the factor $\gamma$, the size of the feasible region is reduced also. Ultimately, the intersection point of the attainment constraints will coincide with the boundary of the original feasible region. That point is the point that satisfies the original constraints and that is as close as possible to the goal point $F^*$ and is on the line $L_1$. The slope of the intersection line is controlled by the values of the coefficients $\alpha_k$.

Figure 5.4 shows two other lines, $L_2$ and $L_3$, that represent two other choices of coefficient values. They would lead to different solutions, but each solutions will be unique. Choosing a relatively small value for a coefficient $\alpha_k$ causes the corresponding objective $F_k(x)$ to be closer to the objective, so the relative importance of each objective can be expressed by selecting appropriate values for the coefficients $\alpha_k$. Since it is more intuitive to set higher weights to objectives that have more importance, alternative weighting factors can be defined from which the coefficients $\alpha_k$ can be derived. A set of dimensionless weighting factors $w_k$, that are proportional to the relative importance of each objective, can be established, such that:

$$\alpha_k = F_k^* \left(1 - \frac{w_k}{100}\right) \quad (5.9)$$

where each weighting factor $w_k$ is expressed as a percentage:

$$0 < w_k < 100\%, \quad k = 1, \ldots, n_g \quad (5.10)$$

such that:

$$\sum_{k=1}^{n_g} w_k = 100\% \quad (5.11)$$

It can be observed that, unlike in the Weighted Sum method, the entire feasible region is accessible by the Goal Attainment method. Note that the coefficients $\alpha_k$ are not allowed to be negative here, as that would cause objectives to be increased.

Over-achievement of the goals is possible by allowing $\gamma$ to be negative. This situation arises, for instance, when the goal point is located inside the feasible region, as is illustrated in figure 5.5. Again, the objective functions are reduced by decreasing $\gamma$, but this time the solution is as far as possible from the goal point.

For any set of design parameters $x$, it follows from (5.8), that the factor $\gamma$ is determined by:

$$\gamma \geq \frac{F_k(x) - F_k^*}{\alpha_k} \quad (5.12)$$
which means that the inequalities (5.8) are satisfied if:

\[
\gamma = \max_k \frac{F_k(x) - F^*_k}{\alpha_k}
\]  

(5.13)

A better achievement (smaller under-achievement or larger over-achievement) of the goals is obtained if \( \gamma \) is reduced. Therefore, a new optimisation problem formulation can be introduced (Gembicki, 1974) in which \( \gamma \) is minimised:

\[
\begin{align*}
\text{minimise} & \quad \gamma \\
\text{subject to} & \quad F_k(x) - \alpha_k \gamma \leq F^*_k, \quad k = 1, \ldots, n_g \\
& \quad G_j(x) \leq c_j, \quad j = 1, \ldots, n_c \\
& \quad x_i^{(l)} \leq x_i \leq x_i^{(u)}, \quad i = 1, \ldots, n_d
\end{align*}
\]  

(5.14)\text{ and to } (5.15)\text{ and to } (5.16)\text{ and to } (5.17)

which is known as the Goal Attainment formulation, and is in the form of a standard Nonlinear Programming problem. The parameter \( \gamma \) can be considered as an extra design parameter next to the original design parameters in \( x \) from the multi-objective optimisation problem formulation. It is also the objective function in the Goal Attainment formulation; the objective functions from the original formulation have been converted into additional constraints by using the inequalities (5.8).

The advised way of using the Goal Attainment method is to set ambitious goals, such that the solution will be under-achieved. Weighting factors that are related directly to the relative importance of each goal can then be defined using (5.9)
and (5.10). This way, the Goal Attainment formulation better corresponds to engineering practice than the general formulation. Not only is it possible to denote multiple design criteria to be optimised, it also allows specification of design goals in the formulation. Using the weighting factors, the relative importance of each objective can be defined correctly. This formulation, however, does not render the optimisation process more efficient. An approach that will lead to more efficiency, and that can be applied both to the general and to the Goal Attainment formulation, will be discussed next.

5.4 Approximate optimisation

5.4.1 The concept

Constrained optimisation problems can only be solved directly when the constraints are linear functions of the design parameters, and the objective function is linear (Linear Programming problem) or quadratic (Quadratic Programming problem). As discussed earlier in section 5.3.1, any other problem is to be classified as a Nonlinear Programming (NP) problem. Sequential Quadratic Programming (SQP), for instance, is a popular method in which in each iteration a QP sub-problem is constructed. Each sub-problem is constructed from function data at a single point \( \mathbf{x}_e \), usually from function values and first-order gradients.

NP methods, however, generally require a substantial number of simulation analyses. To a large extent, this can be overcome by introducing an intermediate step between the simulations and the optimisation algorithm, as shown in figure 5.6. The function evaluations are now used to construct explicit functions, and an optimisation algorithm is applied to these explicit functions which can be quickly evaluated. As these explicit functions are approximations to the actual functions, this practice is known as approximate optimisation. The original optimisation problem is replaced by a problem in which the functions are replaced by approximations:

\[
\begin{align*}
\text{minimise} & \quad \overline{F}(\mathbf{x}) \\
\text{subject to} & \quad \overline{G}_j(\mathbf{x}) < c_j \quad j = 1, \ldots, n_c \\
\text{and to} & \quad \underline{x}_i^{(l)} \leq x_i \leq \overline{x}_i^{(u)} \quad i = 1, \ldots, n_d
\end{align*}
\]

(5.18) (5.19) (5.20)

Note that function approximations can be applied to the multi-objective formulation just as well.

Reductions in computing time can be achieved if the number of points necessary
to construct the approximations is smaller than the number of points that would be required by applying the optimisation algorithms directly. Since evaluating the approximation functions is fast, efficiency is no longer an issue when selecting an optimisation algorithm, and the most robust method can be chosen.

The optimum found in the approximate problem is likely to be different from the solution of the original problem. This optimum is therefore often used as a starting point for a next iteration. By using a subregion of the design space in each iteration, the approximations will become more accurate. This approach of iteratively constructing approximations and shifting and resizing the search region is known as sequential approximate optimisation.

Function approximation can be applied independent of the type of optimisation algorithm or of the type of simulations. Therefore, an additional advantage of this approach is that the function approximation software forms an interface between optimisation algorithms and simulation software. Any simulation software can be integrated in an optimisation environment without alterations, as long as it features non-interactive input.

In terms of computational efficiency and ease of implementation, application of approximate optimisation has obvious benefits over applying optimisation algorithms directly. The discussion of selecting a suitable approach therefore
should be focused on selecting the most appropriate function approximation approach. In figure 5.6, three kinds of function approximation approaches are distinguished: local, mid-range and global methods. To enable the selection of the most suitable class of methods for structural-acoustics, the characteristics of these approximate optimisation approaches will now be discussed, based on the list of demands of section 5.2.

5.4.2 Local approximation methods

Local methods employ approximations that are constructed from function values and first-order gradients in a single design point. Therefore, these approximations are valid only in a small subregion of the total search region. In their simplest form, linear approximations are used, so in each cycle an LP problem has to be solved.

The region in which the approximations are assumed to be valid is bounded by so-called move-limits, as depicted in figure 5.7 by dashed lines. An optimisation algorithm is applied to the region of the parameter space within the move-limits of the current optimisation cycle. This can be accomplished by simply setting the side-constraints to the move-limit values. For cycle number $r$, this can be expressed as:

$$x_{(r)}^{(l)} \leq x \leq x_{(r)}^{(u)}$$ (5.21)

After each cycle, it has to be decided how to change these move-limits. Usually, the search region is placed such that the starting point is in the centre. The size of
the search region is often motivated by the quality of the approximations in the previous cycle. When the optimisation process is considered to be converging, the search region is decreased in size to obtain accurate approximations in order to fine-tune the optimum. In order to keep the optimisation process automatic, rules for updating the move-limits have to be implemented. Implementing a local approximation approach therefore requires a move-limit strategy. A disadvantage of using relatively small search regions, is that the process converges to the nearest local optimum.

5.4.3 Global approximation methods

Global methods are on the opposite side of the spectrum of function approximation methods. A relatively large number of points is used to fit high-order functions to the data, such as third-order polynomials. As this approach is not sequential, no move-limit strategy is required. The applicability, however, is limited by the number of design parameters, since the number of required design points increases exponentially with the number of design parameters.

Figure 5.8 illustrates the smoothing ability of global approximations. The approximation is constructed by sampling the strongly nonlinear function at a few points, and fitting a quadratic function through these points. Applying any optimisation algorithm to the approximation function would result in an optimum close to the global optimum. Such methods are sometimes used as an initial step in the optimisation process. They produce a rough location of an optimum, that can be used as an initial point for a local method.
The application of approximate optimisation based on multiple points is thus able to capture global trends in the design space, that would otherwise be obscured by smaller local oscillations. Global approximation methods have been applied in structural-acoustics in the tuning of carillon bells by Schoofs (1987) and by Roozen-Kroon (1992).

5.4.4 Mid-range approximation methods

Methods that combine most of the strengths of local and global methods, are known as mid-range approximation methods. They are sequential methods, but larger search regions can be applied since data from multiple points are used to construct the approximations, as is illustrated in figure 5.9. In every optimisation cycle, data from multiple points are used, instead of just the current starting point, as is done in local methods. Because the data on which the approximations are fitted comes from points that are distributed across the design space, the approximations are valid in a larger region compared to local methods.

The major drawback of these methods is that they require a highly sophisticated move-limit strategy. Besides determining how to move and resize the search region, a number of other decisions have to be made as well, all of which influence its performance. This results in a potentially large number of parameters in the move-limit strategy. Unless most of these parameters can be given a fixed value, working with these methods becomes overly complicated.

Mid-range methods are dynamic in the sense that their behaviour may vary in
different stages. Initially, high accuracy of the approximations is not required and the search regions can be large. In the final stages, the search region can be chosen small. As more data becomes available from previous cycles, approximation models with more parameters can be applied which enhances the approximation accuracy.

For the present discussion, it is necessary to distinguish two classes of mid-range methods: single-point path methods and multi-point path methods. These will therefore be covered separately.

**Single-point path methods**

In local methods, all information from previous cycles is discarded. When data from earlier cycles is stored and used in later steps, more accurate approximations and larger search regions can be applied. Such methods are termed *single-point path* methods, since a single new point is added in each cycle.

Examples of such methods can be found in Haftka et al. (1987) and in Fadel et al. (1990). There would be insufficient data available for model fitting if only function values would be used, so single-point methods require the use of gradients.

**Multi-point path methods**

Using data from multiple design points in sequential approximate optimisation can be taken to the next level by generating multiple new design points in each cycle. This approach is particularly useful when the use of gradient information is not viable.

This approach was first applied by Toropov (1989). Instead of using the extra points to compute finite difference gradients, these points were placed further apart. As was explained in section 5.4.3, approximations based on multiple points can capture trends in the design space, and this is what multi-point path methods do as well. The move-limit strategy is aimed at following these trends in successive cycles.

It has been shown (Kessels et al., 1998) that multi-point path mid-range methods perform well in the field of structural-acoustics. The gradient coil is modelled as a three-layer cylinder, with the inner and outer layer consisting of epoxy. For the middle layer, a relatively heavy undamped stiff material is used. The thicknesses of the two innermost layers are taken as design variables, while the total thickness is held constant. For demonstration purposes, the design space is evaluated first using a parameter study, as presented in figure 5.10. A minimum in the total radiated A-weighted sound power level is sought. As a starting point for the
optimisation, the local optimum in figure 5.10 is chosen. The applied multi-point path mid-range method manages to escape from this local optimum, and to converge to the global optimum using 4 cycles. A total of 13 analyses were required. As a second example, the mass of the structure will be minimised, while the total sound power level is constrained to 92 dB(A). The heavy layer is replaced by a fiber-reinforced epoxy layer, and the fiber volume ratio is used as a third design variable.

After the optimisation, the mass is reduced by 42 kg (11%). A total of 21 analyses are required in 5 cycles. Note that the number of required simulations increases linearly with the number of design parameters, and not exponentially as in global methods. Further, multi-point path methods exhibit the smoothing ability of global methods, and are less likely to get trapped in a local optimum than local methods do.
5.5 Optimisation strategy for sound reduction problems

In some function approximation methods, gradients are used as part of the data on which the explicit functions are fitted. As was observed in Chapter 4, the sound power radiation can fluctuate rapidly as a function of design parameters. In some regions of the design space, gradient information is therefore valid only in the immediate vicinity of the point in which the gradients are computed. The use of gradients will therefore lead to poor approximations in most of the search region. This motivates the exclusion of gradient information in constructing approximations.

Hambric (1995) has evaluated the efficiency of applying approximation methods in structural-acoustics. The mass and manufacturing cost of a submerged cylinder with end caps are minimised while constraining the externally radiated noise. His approximation method can not be categorised as either a local, mid-range or global method as defined above, but it resembles a single-point path mid-range approach. First-order and partial second-order Taylor series are applied as approximation functions for the constraints in the entire design space. Both the function value and the first-order and second-order gradients in the starting point are used to construct the approximations. The constraints are then incorporated as a kind of penalty function in the objective function. An unconstrained optimisation algorithm is used to find the optimum. This optimum is then used as a new starting point, and this process is repeated until convergence. Data from previous cycles are used to improve the accuracy of the approximations. Hambric uses gradients (computed using finite differences) in the construction of approximations, but correctly observes that this may be misleading because of the fluctuating constraint function behaviour. This approximate optimisation approach is evaluated by comparing it to the direct application of optimisation algorithms. Hambric’s evaluation indicates that the use of approximations is more effective since less explicit function evaluations have to be performed, while the optima found are generally closer to the global optimum.

Using global methods is viable when the number of design parameters is small, but some fine-tuning using a method from the other classes would be necessary. Global methods, therefore, become inapplicable when there are more than a few variables, which will often be the case in structural acoustic optimisation. The efficiency of local approximations is largely independent of the number of design parameters, but these would immediately get trapped in a local minimum.

Based on the evaluation above, it can be concluded that multi-point path mid-range methods seem to have the right properties for optimisation in structural-acoustics. They are able to escape from local minima since the search region can be large in the first cycles. Gradient information is not required, and irregular
behaviour of objective functions and constraints is smoothed. An optimum can be found to any desired degree of accuracy by narrowing the size of the search region in the final cycles. In order to enable this behaviour, however, an elaborate move-limit strategy is needed. Etman (1997) has developed such a strategy, and successfully applied it to the optimisation of multibody systems. The same strategy has also been applied by van Houten (1998) to a variety of problems.

5.6 Applications

5.6.1 Implemented optimisation method

A multi-point path mid-range method has been implemented in MATLAB, which has resulted in the Approximate Optimization Toolbox (Kessels, 2000). This toolbox serves as the middle column in figure 5.6, and uses optimisation routines (the left column) from the Optimization Toolbox (Coleman et al., 1999). An enhanced version (Kessels, 1994) of Toropov’s method (see section 5.4.4 and Toropov et al. (1993)) has been implemented. Both the robustness of the optimisation process and the computational stability of model fitting have been improved. The move-limit strategy of Etman (1997) has been adopted. This strategy is more extensive than those in other mid-range applications, and incorporates different behaviour in different stages of the optimisation run. In the initial cycles, approximation errors will be relatively large, and much support is given to such situations. When approximation errors are considered to be too large, for instance, the last optimum is rejected, and the last cycle is restarted from the previous optimum, but with a smaller-sized search region. In the final stage of the optimisation, changes in the objective function are small or the process tends to oscillate between two solutions. When oscillations occur, the search region is reduced considerably, which ends the oscillatory behaviour and speeds up convergence.

The performance of this optimisation approach will now be demonstrated using models of the MRI gradient coil that were developed in Chapter 3, and applied in the parameter studies of Chapter 4.

5.6.2 Single-objective optimisation

As a first application, the total radiated sound power will be minimised, using the carrier layer thickness and the glue modulus of the gradient coil as design parameters. These parameters have been varied, each in turn, in sections 4.2 and 4.3 to study their effect on sound radiation. For the parameter study involving the
carrier layer thickness, 40 analyses had been used, and the model was analysed for 20 different values of the glue modulus. A parameter study to investigate both effects simultaneously would therefore, in this case, require 800 analyses.

In the optimisation, the total mass of the gradient coil is constrained to 700 kg. The logarithm of the glue modulus is used as the first design parameter \( x_1 \), and the scaled carrier layer thickness is used as the second parameter \( x_2 \). The Approximate Optimisation Toolbox takes care of linear scaling of the design variables automatically. This is necessary to ensure robustness of both model fitting and the optimisation process itself. The reason that the logarithm of the glue modulus is used, is to contract its range into one order of magnitude. The scaled design parameters are bounded using the side-constraints:

\[
\begin{align*}
8.53 & \leq x_1 \leq 11.53 \\
0.004 & \leq x_2 \leq 0.044
\end{align*}
\]

which are the same bounds as used in the parameter studies. The optimisation is started from design parameter values \( x_1 = 10 \) and \( x_2 = 0.02 \). At this starting point, the radiated sound power is 103 dB, and the mass is 660 kg. The results of the optimisation are shown in figure 5.11. In each cycle, the mass is smaller than or equal to 700 kg. A sound power reduction of 13 dB is obtained after 5 cycles, taking a total of 16 structural FEM calculations. The increase in the objective function in cycle 2 is allowed by the move-limit strategy, as this is sometimes necessary to reach a better solution in another region of the parameter space. Note that the number of analyses required to obtain this reduction is less than the

![Figure 5.11](image_url)
number of analyses used in either of the single-variable parameter studies, which amounted to 20 and 40 respectively, and far less than the total of 800 analyses that would have been required for a combined parameter study. This illustrates the efficiency of methods based on searching through the design space, compared to more rigorous but expensive scanning approaches.

5.6.3 Multi-objective optimisation

The sound power level reduction in the previous example was obtained by increasing the thickness of the relatively heavy carrier layer, resulting in an increase of the total mass of the coil. As an example of multi-objective optimisation, both the sound radiation and the mass will now be treated as objectives. The goals are set as 90 dB for the sound radiation, and 640 kg for the mass of the coil. As was explained in section 5.3.2, the relative importance of the objectives can be expressed by assigning weighting factors. Here, the second objective is given more importance by setting the weighting factors to $w_1 = 35\%$ and $w_2 = 65\%$. (This is equivalent to setting $\alpha_1 = 58$ dB and $\alpha_2 = 230$ kg.)

The same starting point as in the previous example is used. As can be seen from the optimisation history in figure 5.12, the goals are almost met after four cycles, taking 13 structural FEM analyses. Using the mass as a second objective

![Graph showing the history of multi-objective optimisation example](image)

**Figure 5.12**: History of the multi-objective optimisation example. **-** : total sound power level; **--** : total mass.

has forced the optimisation process to lower the sound radiation by reducing the thickness of the carrier layer. The total weighting has been reduced by al-
most 20 kg, and this is an improvement of 45 kg compared to the optimum of the constrained optimisation example.

5.6.4 Balancing result and resources

One of the demands listed in section 5.2, is that there should be a way to limit the invested resources if the use of extra resources would lead to marginal improvements of the objectives. Allowing an optimisation algorithm to spend several cycles on reducing levels within a 1 dB range is not useful in most cases, as this improvement is hardly noticeable. The ratio between results and expenses will then be poor. This can be prevented by selecting appropriate convergence parameters. In the implemented approach, convergence is achieved when the relative change of the objective function between successive cycles is smaller than a certain criterion value $\beta$, provided that the quality of the approximations was sufficiently high in the last cycle. Choosing a higher value for $\beta$ causes the optimisation process to terminate earlier. When sound levels near an optimum are about 100 dB, for instance, setting $\beta$ to 0.01 will cause the optimisation tool to stop when the improvement between two cycles is less than 1 dB. For other optimum levels, $\beta$ should be adapted in order to obtain the same convergence behaviour.

The effect of changing the convergence criterion is demonstrated using the examples above. The results for the example in which the mass was formulated as a constraint, are summarised in table 5.1. It can be observed that choosing a larger

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Number of cycles</th>
<th>Number of analyses</th>
<th>Sound power level [dB]</th>
<th>Reduction [dB]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.010</td>
<td>3</td>
<td>10</td>
<td>92.3</td>
<td>10.7</td>
</tr>
<tr>
<td>0.002</td>
<td>5</td>
<td>16</td>
<td>89.6</td>
<td>13.4</td>
</tr>
<tr>
<td>0.001</td>
<td>7</td>
<td>22</td>
<td>89.1</td>
<td>13.9</td>
</tr>
</tbody>
</table>

Table 5.1: Effect of changing the convergence criterion $\beta$ for the constrained optimisation example, using a standard optimisation formulation.

value for $\beta$, the number of required cycles, and therefore the number of expensive FEM analyses, is reduced. Setting the convergence criterion too high, however, can lead to premature convergence. For $\beta = 0.01$, the obtained sound power level reduction is considerable, but an extra reduction of more than 2 dB would have been attained for $\beta = 0.002$, consuming 6 additional analyses. Lowering this criterion any further would result in only marginal improvement, while the increase in the number of analyses is relatively large.
5. Numerical optimisation

For the multi-objective optimisation example, the convergence parameter $\beta$ is related to the first objective. Almost identical results are found for the values of $\beta$ in table 5.2. The number of required analyses, on the other hand, doubles when $\beta$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Number of cycles</th>
<th>Number of analyses</th>
<th>Sound power level [dB]</th>
<th>Mass [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.010</td>
<td>4</td>
<td>13</td>
<td>90.5</td>
<td>642</td>
</tr>
<tr>
<td>0.005</td>
<td>8</td>
<td>25</td>
<td>90.3</td>
<td>642</td>
</tr>
<tr>
<td>0.003</td>
<td>9</td>
<td>28</td>
<td>90.0</td>
<td>643</td>
</tr>
</tbody>
</table>

Table 5.2: Effect of changing the convergence criterion $\beta$ for the multi-objective optimisation example.

is lowered from 0.01 to 0.005. These studies suggest that it is practical to choose a relatively high initial value for the convergence criterion. After the optimisation is completed, it can then be decided, based on the results, whether the optimisation is to be continued with a lower value of $\beta$. In practice, such decisions will be motivated by the resources that are rested to complete the design process. If an optimisation run finishes within the given time limits, the optimisation can be continued to see if a further improvement can be obtained. The level of improvement is therefore influenced both by the efficiency of the optimisation approach in terms of required simulation analyses, and by the computational efficiency of the simulation analyses themselves.

5.7 Worst-case optimisation of stochastic systems

All optimisation routines discussed so far assume that the processes to be optimised are of a deterministic nature. In reality, however, products will show some randomness in their behaviour. No two products will be completely identical. Small structural differences sometimes lead to noticeable differences in behaviour. For the engineer, it is important to keep these variations within specified bounds. This type of effect plays a significant role in the acoustic radiation of products with some level of structural complexity. The relatively large variation of interior noise in nominally identical cars is a typical example.
5.7.1 Effects of structural complexity on sound radiation

Structural complexity has a number of possible consequences for the structural dynamic behavior not observed in simpler homogeneous structures. Firstly, internal local resonances may lead to apparent energy dissipation on a global scale, even in the absence of material damping (Pierce et al., 1995). This effect occurs when the driving frequency is near an eigenfrequency of an internal oscillator, and is larger when the oscillator’s actual damping is smaller. As a result, this apparent damping effect is frequency-dependent. It is not caused by dissipative effects, but, roughly, by local vibrations of masses that take up kinetic energy that is not transferred to other parts of the structure. When a global eigenfrequency coincides with eigenfrequencies of one or more local oscillators, the global vibration amplitude at that frequency seems to be damped. Correctly modeling the material damping is then of little importance, especially when the structure is made up of materials with light intrinsic damping.

Secondly, coupling of scales occurs. Inhomogeneities on meso-scales cause effects in structural vibrations on length scales that are of the same order of magnitude as the wavelengths that occur in the frequency range of interest. Mode shapes become more complex, combining waves on different length scales. This is demonstrated by modeling the response of cylinder with ring-shaped inclusions of a heavier and stiffer material, as shown in figure 5.13, in two ways. The cylinder is excited by a uniformly distributed radial load at the outer surface. In

![Figure 5.13: Cross-section of the cylinder with ring-shaped inclusions.](image)

the first model the details are included, while in the second model these inclusions are smeared out to obtain a homogeneous cylinder comparable in weighting and overall stiffness. The vibrational responses of the two cases are shown in figure 5.14. The cylinder model with smeared-out inhomogeneities results in less eigenfrequencies and lower overall velocity levels. This clearly shows that simple averaging of inhomogeneities on meso-scales leads to erroneous models. Apparent damping effects and coupling of scales thus cause the response to be different than in a similar structure without such complexities. Both the shapes and am-
The first two effects have been observed in the measurements on the vibrational behavior of the gradient coil of an MRI-scanner, as presented in 3.4.2. The frequency response curves of the measured velocity levels show many peaks and higher overall levels compared to the layered models in which the inhomogeneities are smeared out in each layer. Also, the structure seems to be damped much stronger than can be explained from the material properties alone. Modeling all details in a deterministic model would not be practical for intended use in
structural-acoustic design. Additional measurements on coils built according to the same specifications show differences in acoustic levels up to a few dB. Therefore, using models with smeared-out properties would be an attractive alternative, and this will be elaborated on next.

5.7.2 Including variations in standard optimisation problems

In standard optimisation formulations, some characteristic is optimised subject to a number of constraints. Product variations are usually neglected. Although stochastic optimisation formulations have been developed (Marti, 1997; Victor, 1997), implemented versions of these methods are not readily available yet. It is, however, possible to incorporate variations into a standard optimisation problem, as will be shown below. Readily available routines can then be used to handle such problems.

When it is possible to calculate response variations, this valuable information should be incorporated into the optimisation of the behaviour of a product. Such variations can be included in the optimisation formulation in two ways. One possibility is to use additional constraint functions that keep the variations within bounds (Kessels et al., 1999):

\[
\text{minimise} \quad E(F(x)) + \text{Var}(F(x)) \quad (5.24)
\]

subject to \( E(G_j(x)) \leq c_j, \quad j = 1, \ldots, n_c \quad (5.25) \)

and to \( \text{Var}(G_j(x)) \leq d_j \quad (5.26) \)

and to \( x_i^{(l)} \leq x_i \leq x_i^{(u)}, \quad i = 1, \ldots, n_d \quad (5.27) \)

where \( E(\cdot) \) denotes the expected value. Splitting the constraint limits into an expected term and a variance term is convenient for keeping the variations of constraints within bounds when necessary, which would be allowed to grow if:

\[
E(G_j) + \text{Var}(G_j) \leq c_j \quad (5.28)
\]

would be used. This may double the number of constraints, but that usually does not deteriorate the efficiency of the optimization algorithms. Constraint values are now restricted to:

\[
E(G_j) + \text{Var}(G_j) \leq c_j + d_j \quad (5.29)
\]

When the design parameters are deterministic, such an optimisation problem is deterministic also and can be solved using existing methods. As the upper bounds of response functions are used in the design evaluation criterion, relatively small differences in response between different designs are likely to disappear in the objective function. An additional advantage of the proposed worst-case approach is then that the optimization algorithm will not waste time in trying to find a design which is only marginally better.
Another way to incorporate response variations in the optimisation problem formulation is to minimise the variations by treating them as objective functions. Using a multi-objective optimisation approach would then be a natural choice. The problem can then be formulated like:

\[
\begin{align*}
\text{minimise} & \quad E(F(x)) \\
\text{and} & \quad \text{Var}(F(x)) \\
\text{and} & \quad \text{Var}(G_j(x)) & j = 1, \ldots, n_c \\
\text{subject to} & \quad E(G_j(x)) \leq c_j, & j = 1, \ldots, n_c \\
\text{and to} & \quad x_i^{(l)} \leq x_i \leq x_i^{(u)}, & i = 1, \ldots, n_d
\end{align*}
\]

(5.30)  
(5.31)  
(5.32)  
(5.33)  
(5.34)

If desired, multiple objective functions could be added to this formulation as well. When the Goal Attainment method is used to solve this problem, a goal can be specified for each variance, and each variance can be given relative importance by setting appropriate weight factors. This way, the variance of the sound radiation, for instance, can be given a 1 dB variance goal, regardless if it is used as an objective or as a constraint. This would mean that a variance of 1 dB is allowed within a sample of fabricated specimens of the product that is being designed. The weighting factor assigned to the variance of the sound radiation can then be used to reflect the importance of meeting this goal, relative to other variances and to the objectives.

### 5.8 Summary and discussion

The design of complex products, that have to satisfy conflicting demands from multiple disciplines, is not transparent for designers. Product improvements are therefore hard to obtain using design methods in which the designers have to rely on experience, rules-of-thumb or on test results from prototypes. The use of numerical prediction and optimisation tools can offer a viable alternative, that can be used to systematically search for a design with better performance.

Optimisation can be regarded primarily as a mathematical technique. As such, optimisation algorithms are often employed in ways that do not agree well with engineering practice. By setting up a list of criteria for a more suitable approach, the most optimal strategy for structural-acoustics has been selected among available techniques. Using the Goal Attainment method in combination with a multi-point path mid-range approximation approach is then the most obvious choice, as this satisfies all of the imposed criteria in section 5.2.

The applicability of the chosen optimisation strategy has been demonstrated. It
was shown that this approach enables product improvements using a limited amount of invested resources. The implementation of approximation concepts forms a physical interface between optimisation algorithms and simulation software, which simplifies their combined application, and leads to a flexible, modular design tool. Optimisation problems can be formulated in a way that resembles the formulation of design criteria for products in multidisciplinary environments, and multiple objectives can be improved simultaneously in a meaningful way. It was also verified that the approach provides a mechanism to balance product improvement and invested resources. The multi-point path mid-range strategy does not use gradient information, but uses multiple points instead, which smoothes the irregular behaviour of response functions that is common in structural-acoustics.

Even variances in response functions can be handled by the optimisation approach. Commercial simulation software, however, can not provide such information yet. Applying analysis methods and numerical optimization techniques based on stochastics would result in a powerful approach to the design of complex structures. It would offer a closer relationship to engineering practice than deterministic methods could, since it inherently encompasses fabrication inaccuracies and modeling uncertainties, while it may be possible to efficiently model complex structures. Further development and implementation of stochastics-based methods would therefore greatly enrich numerical design tools.

Although an efficient and powerful optimisation approach has now become available for application in structural-acoustics, the design of complex products has not become trivial. Before such a tool can be applied, a reliable model of the product has to be constructed, that has to be accurate within a sufficiently large range of the design parameters. This is particularly difficult for complex products, and may lead to computationally expensive models. Also, formulating the problem that the optimisation tool has to solve is a critical task, as this largely determines the success of the operation. Both the model construction and the optimisation problem formulation require the engineer’s creativity, as well as skills that have to be acquired through experience. The efficient simulation tools developed in this project form an ideal starting point for gaining such experience, as the models, although relatively simple, exhibit many phenomena that are typical also for more complex three-dimensional models.
Conclusions, discussion and recommendations

In this thesis, a computational toolbox is developed for the structural-acoustic analysis and optimisation of MRI scanners. The toolbox includes powerful acoustical analysis methods that have been developed in a companion doctoral thesis (Kuijpers, 1999). In section 6.1, conclusions are drawn from the findings in the current thesis. Next, in section 6.2, the results will be discussed. In the final section, directions for the use of the developed toolbox are given, and recommendations are made about possible extensions that can further broaden the applicability of the toolbox. Further research that may be required for future expansions is also considered.

6.1 Conclusions

6.1.1 Numerical tools for structural vibration analysis

Complete three-dimensional models would currently impose a prohibitively high computational burden for use in industrial design. Therefore, more efficient Fourier elements have been applied. An efficiency improvement to the Fourier finite element formulation has been introduced, which shows that the time required for the assembly of the system matrices can be reduced considerably, without loss of accuracy. This is accomplished by expressing the stiffness matrix explicitly in terms of the Fourier harmonic $m$. Using this approach has resulted in a tool that, for the MRI gradient coils, is efficient up to a few kHz.

Further, the presented Fourier-FEM formulation is shown to be applicable for generally anisotropic materials, while the usual formulations are derived such that they are applicable for isotropic configurations only. Symmetric and antisymmetric terms in the Fourier expansion are then coupled. Therefore, in the plane in which the structure is axisymmetric, geometrically axisymmetric structures with anisotropic materials can respond in partially antisymmetric vibra-
tions when excited by symmetric mechanical excitations. In cases of material isotropy or special orthotropy, the proposed formulation leads to a simplified system of matrix equations that can be solved more efficiently, since the symmetric and antisymmetric Fourier terms are then uncoupled.

For the modeling of the dynamic behaviour of polymeric constituents, of which there are many in the gradient coil, linearly viscoelastic theory is implemented. A consequence of including polymers in a model, is that the stiffness matrix becomes frequency dependent. Using a modal structural analysis is not viable in that case. In a direct analysis, on the other hand, the stiffness matrix has to be reassembled at each frequency of interest. Methods exist that convert the matrix system of equation into a form that explicitly depends on frequency. One of these techniques, developed by Yiu, is extended in this thesis to be valid for general anisotropic cases, since the original formulation is valid for isotropic cases only. A drawback of such methods, however, is that the size of the system matrices is increased considerably, leading to an explosive increase in the CPU time required for solving the system. Another difficulty is that only a single viscoelastic material may be present in the model. Sometimes, however, the frequency dependence of materials can be neglected, as will be discussed further in section 6.2.1.

6.1.2 Modeling and measuring gradient coil vibrations

For the application of Fourier-FEM, the Lorentz forces produced by the gradient coil are expanded into Fourier series. From these series, it is observed that the force excitations by the transverse coils contain only odd Fourier components \(m = 1, 3, 5, \ldots\). The amplitudes of the first of these, \(m = 1\), are most pronounced, and the radial components of the Lorentz forces are significantly higher than the axial and circumferential components. The \(z\)-coil excitation on the structure is purely axisymmetric \((m = 0)\). In axial direction, the excitation is symmetric for the transverse coil and antisymmetric for the \(z\)-coil.

Symmetry in the gradient coil and in its excitation enable the model size to be reduced. Only half of the coil has to be modelled in axial direction, if appropriate boundary conditions are prescribed at the plane of symmetry.

Fourier-FEM is particularly efficient when the number of Fourier harmonics that has to be included in the analysis, is not too large. In the frequency range of interest, only selected Fourier harmonics have to be included in the analyses of the gradient coil models. An upper bound is given by the acoustic radiation properties of the coil. A radiation modes analysis shows that, above \(m = 14\), the coil will never radiate efficiently.
Measurements have been performed to gain more insight into the structural behaviour of an operational gradient coil. Velocity amplitudes in a grid of 19 by 21 points have been measured. This grid covers a quarter of the inner surface of a gradient coil. Using accelerometers is too cumbersome because of the many measurement spots, and a laser-Doppler vibrometer is applied instead. A gantry holding a prism is used for positioning the measurement spot. It was anticipated that the eddy currents and the sound field, generated by the MRI scanner, cause vibrations in the setup that affect the measurements. Therefore, the gantry is designed such that the velocity amplitudes of the gantry vibrations are well below the velocity amplitudes of the gradient coil. This has been verified during the measurements. Measurement data shows that the gradient coil does not behave like an axisymmetric structure. This is most clear from the non-axisymmetric response of the axisymmetric z-coil excitation.

Discrepancies are found between calculated and measured responses. During successive model refinements, these discrepancies reduce, but still a good agreement was not obtained. Calculated predictions are well below measured data, and the modal density in the measured data is higher, particularly in the upper regions of the frequency range. For further discussions of the computations and the measurements, see section 6.2.2.

6.1.3 Parameter studies and optimisation

The efficient tools have allowed a number of gradient coil parameter studies to be performed without user intervention, and within a small time frame. Such studies reveal features that can be considered to be common to many sound radiating structures. Parameter studies from Chapter 4 reveal that the total sound power level responds highly irregularly to changes in design parameters. Sound power level fluctuations of 10 dB or more are found in the parameter studies.

In one of the parameter studies that are performed in this thesis, Lorentz force balancing is applied. The concept of Lorentz force balancing is to modify the inner and outer coil sets, such that the their respective forces cancel each other. In earlier studies, found in literature, the Lorentz force distributions were balanced with respect to structural vibrations. Here, the first numerical study is performed in which the effects on the radiated noise are evaluated. For this purpose, force balancing is achieved simply by multiplying the amplitudes of one of the coil sets by a factor. It is shown that, in this case, the radiated sound power can be expressed explicitly in terms of the multiplication factor. The optimum value for the factor can be derived analytically from this expression. The study shows that, when force balancing is applied for noise level reduction, velocity levels are not necessarily reduced. Besides the multiplication factor, the modulus of the glue
layer between the carrier layer and the longitudinal profiles was parameterised. In this manner, the effect of a different mechanical coupling between the inner and outer coil sets has been evaluated. Calculations show that the effects of force balancing are highly dependent on the value of the glue layer modulus.

Parameter studies reveal much information on which design decisions can be made. When the number of design parameters increases, however, the number of required analyses rises exponentially. Parameter studies, then, are no longer viable. An alternative is the application of numerical optimisation algorithms. Direct application of these methods, however, imposes a number of drawbacks. These can largely be circumvented by the application of approximations. For structural acoustics, multi-point path mid-range methods are shown to be particularly suitable. An optimisation problem formulation that is more closely related to engineering than the standard formulation is offered by the application of the Goal Attainment method. Multiple design objectives can be now specified, in a way that resembles the formulation of design criteria.

The optimisation module in the developed toolbox is versatile, and applicable for almost any engineering problem. Considerable design improvements can be obtained using a modest amount of sample points, which is vital, since each analysis is expensive in engineering applications. The optimisation module offers additional benefits as well. Computation of gradients is not necessary, so analysis software does not have to supply them. Related to this, is that the behaviour of the objective and constraint functions is not required to be smooth for the method to be successful. Further, the approximation routines serve as an interface between the analysis software and the optimisation algorithms, facilitating the coupling of different kinds of analysis software to the optimisation tool. Therefore, it can also be used to reduce MRI sound radiation, while simultaneously improving or constraining other criteria that are relevant to the functionality or to the marketability of the scanners.

6.2 Discussion

6.2.1 Numerical simulation tools

The developed analysis methods have resulted in efficient tools, that allow fast calculations on broad-banded problems, even on a standard desktop PC. The Fourier finite elements can be used for the analysis of axisymmetric structures. The extension to anisotropic material behaviour, in combination with linear viscoelasticity, has enabled material tailoring for tuning the vibrational response of
axisymmetric structures. This is an important feature for compact structures in which there is not much space for changing the shapes of individual components, as is the case for MRI scanners. Instead, properties of composites, such as fiber orientation or fiber volume rate, can be optimised in order to reduce noise levels.

The inclusion of polymers introduces frequency dependent material behaviour. When a polymer is included in a composite, however, the frequency dependence is reduced in a relative sense, since the polymer typically has a low modulus compared to the other constituents. Frequency dependence of the composite may therefore be neglected in some cases. This results in a more efficient analysis as the stiffness matrix does not have to be reassembled at every frequency of interest.

### 6.2.2 Comparison of computed and measured data

When velocity levels, as a function of frequency, from measured data are compared to the calculated structural response of the gradient coil models, two main differences can be observed. Firstly, in all of the models, the calculated response is below the measured response. Secondly, the measured data shows a higher modal density than the calculations. A possible explanation for these observations may be found by taking the structural complexity of the coil into account. A relatively high modal density is typical for structures that are strongly inhomogeneous. These inhomogeneities can consist of complicated structural details, of imperfections introduced during fabrication, and of small defects such as delaminations that develop during use. Particularly when these inhomogeneities are unevenly distributed, their effect on the overall vibrational response can be considerable. Also, the presence of Fourier harmonics in the measured coil response that are absent in the excitation, can be explained from inhomogeneities. Homogenisation, as it has been applied in the present study, is only appropriate when inhomogeneities are evenly distributed.

It is advisable to perform additional measurements on other gradient coils. This would provide more information about differences in behaviour between coils, and be of importance for the development of more accurate coil models. The application of detailed three-dimensional models in design, however, is discouraged for two reasons. Firstly, such models are not suitable because of the required computational costs. Even if the current rate of the increase of computer power is sustained, it will take almost a decade for three-dimensional models to become as efficient as Fourier models are today. Secondly, as long as the real causes of the discrepancies between computations and measurements are not known, more elaborate numerical models are not necessarily improved models. The complexity of structures like the gradient coil cause variations in structural and acous-
tic behaviour between nominally identical products. The accuracy gained from more detailed models would fall within the variation bandwidth. Most probably, a more meaningful and more powerful approach, therefore, is the application of stochastic models, which allow for the inclusion of uncertainties in the computations.

Measurements from different types of coils can also be used to further investigate when Fourier models are applicable. Their use is not restricted to perfectly axisymmetric structures, but homogenisation has to be possible in circumferential direction. Comparisons between Fourier models and measurements from gradient coils with a more solid construction might show a better agreement. Based on this reasoning, it might follow that a more solid construction that the one that is investigated would have considerably lower velocity levels, and therefore may also be more quiet.

6.2.3 Parameter studies and optimisation

From the parameter studies, it appears that force balancing does not lead to significant noise reduction for the current gradient coil design. The effects of force balancing, as applied here, however, have been performed with a simplified model of the gradient coil. Results might be different if a more accurate model is used. Also, only the amplitudes of one of the coils was varied in this study. Better results may be achieved when the spatial force distribution shapes are altered as well. Changing the Lorentz force distributions requires changing the conductor patterns. Inevitably, the quality of the magnetic gradient fields will be affected. A possible solution is to optimise the conductor patterns with respect to gradient field linearity, magnetic field shielding and force balancing simultaneously. That could be accomplished by integrating code for electro-magnetic analysis in the toolbox. Further, force balancing does not have to be used purely for sound reduction. It can also be applied for reduction of vibrations in the coil, that could otherwise cause vibrations in other components of the scanner. This would have to be imposed as an extra design criterion, as reducing the sound radiation does not guarantee velocity level reductions in itself.

Regardless of what kind of models have to be used for the accurate prediction of the gradient coil noise levels, the developed optimisation strategy is suitable for finding design improvements. Parametric optimisation can be applied when the global concept for the design has been established. Using these techniques with inaccurate models, however, is of no use. The optimum design that is found from such an exercise holds no relation to the actual product. For the detailed optimisation of gradient coil induced noise, this means that more detailed models have to be developed. It can be expected that such models will lead to sub-
6. Conclusions, discussion and recommendations

6.3 Recommendations

6.3.1 Guidelines for application of the developed tools

The developed toolbox consists of structural and acoustical analysis tools, an approximate optimisation module, and routines that automate the interfacing between these modules. In this thesis, the toolbox has been applied to analyse and optimise models of the gradient coil. Calculated analysis results have been used in conjunction with measurement data, to gain more insight into the structural behaviour of the coil. The way the toolbox is used here, should be seen as a first step. Using the toolbox can be useful for designers as well as for researchers, but the way in which they would typically use it will be different. Therefore, guidelines for designers and for researchers will be discussed separately below.

Guidelines for designers

Using numerical design tools is an acquired skill, and new users should familiarise themselves first by starting with simple examples. The efficient Fourier models provide excellent ways of learning the intricacies of noise reduction relatively quickly. For new applications, the toolbox offers the possibility of scanning the design space. An estimate for the noise reductions that are possible in a given situation can be evaluated efficiently. Based on such data, well-founded decisions can be made about further resources that are to be invested on design improve-
ments with respect to noise reduction. If the possibilities for improvements are too small, a different concept for the design may have to be considered.

For accurate predictions of the gradient coil vibrations, different models or even different methods of analysis may have to be developed, which is a task for researchers. In design environments, however, the existing acoustical analysis tool and the approximate optimisation method can be used in their present form. Once a sufficiently accurate and efficient structural model has been developed, detailed parameter studies can be performed. The analysis tools can be used to examine which design parameters have significant influence on sound radiation. For the gradient coils, it is especially worthwhile to examine what can be achieved by material tailoring, as that does not require any space. That is advantageous compared to using sizing parameters, since there is hardly any unused space within MRI scanners.

Finally, numerical optimisation can be applied using the parameters that are known to influence sound radiation. The number of design parameters should preferably be small, since the optimisation method needs to analyse more design points as the number of design parameters increases. Design specifications should then be carefully translated into an optimisation problem formulation. Information gathered during the parameter studies can be of help in deciding which quantities should be used as objectives, and how the objectives are to be weighted in order to express their relative importance. Suitable constraints can also be extracted from design specifications. Also, bounds have to be placed on the design parameters, so that these will remain within the range in which the model is assumed to be valid. From the formulated problem, the optimisation tool can then be applied to find a better design. Depending on the results and on available time, additional optimisation runs may be performed to see if further improvements can be found.

Using the toolbox in the MRI noise reduction would benefit from the inclusion of some other tools. For instance, integrating the software that is available for the design of the gradient coil’s conductor patterns offers additional advantages. With this software, the magnetic field that is generated by the gradient coil can be simulated. With the help of this software, the conductor patterns can be optimised in terms of gradient field linearity and force balancing simultaneously.

Further, the toolbox is suitable for applications other than MRI scanners. Many sound radiating structures or structural components are (nearly) axisymmetric, such as pipes, airplane fuselages, electric motors and cylindrical enclosures. Particularly, sound radiation by electric motors show many similarities to sound radiation by MRI gradient coils, and the tools can be applied in much the same way. Along the line, a similar approach can be followed for other applications. The order and nature of activities in which the tools are applied would generally
be the same: model construction and refinement, followed by parameter studies, and finally optimisation. Details of the ways in which the toolbox is applied in these stages, however, would be different for other applications. In any case, it is advisable to use the acoustical analysis tool to investigate if it is necessary to evaluate the acoustic response in the entire frequency range of interest. Some structures always radiate efficiently at higher frequencies. In those cases, sound reduction can be achieved by reducing velocity levels, and the evaluation of noise levels can be omitted. Using the radiation modes formulation, such an analysis can be performed irrespective of structural details.

Guidelines for researchers

In its current form, the toolbox can be used for performing numerical experiments on sound radiation by axisymmetric structures. In order to clearly define the applicability of Fourier models, it should be investigated for which kind of structures they provide adequate predictions. Additional measurements are necessary to answer such questions. The toolbox can also be used as a basis for the development of extensions, which could contribute to its general usefulness.

For applications in MRI, the next step is the development of more accurate models than the ones used in this thesis. If Fourier models turn out to be inadequate for the structural analyses, efficient alternative models will have to be developed. Attempts to improve models by including ever more details may very well prove to be a dead end. It would lead to high costs, both in terms of modelling efforts and in required CPU time. But, more importantly, such models do not take the uncertainties into account that seem to be typical for complex structures. Development of more accurate tools, therefore, should be aimed both on new analysis methods as well as on better models.

Another enhancement of the current toolbox can be obtained by adding a module for active control. If accurate models are available, the toolbox can then be used to simulate active noise control techniques, which is of advantage for studying the effectiveness of vibration isolation methods.

Whether Fourier models can be used to predict gradient coil noise levels with sufficient accuracy is not completely clear yet. Such models may perform well in cases with a more compact gradient coil design. Measurement results from different kinds of coils are necessary to point this out. These limitations, however, only apply to the structural analysis tool. The acoustical analysis tool is accurate as long as the radiating surface is axisymmetric, so it has a wider range of applicability than the structural tool. Its ability to compute radiation modes is of paramount importance for gradient coil design, since it is likely that changes in the outer geometry are not desirable when the design stage is reached in which
noise optimisation is applied.

The applicability of the developed tools for MRI scanners is not restricted to the gradient coil. In order to obtain sufficient sound reductions, the coil needs to be isolated for vibrations, as well as insulated for sound. As structural modifications for insulation will most likely be axisymmetric, the toolbox can be applied to model both the structural and the acoustical behaviour. Since it is likely that new noise control techniques have to be developed, the toolbox offers efficient ways to evaluate the effects of newly conceived concepts, using simplified models.

### 6.3.2 Directions for further research and expansions

The single most important way of improving the usefulness of numerical tools in structural acoustics, is the development of stochastic analysis tools. Two ways in which an engineer could benefit from such an enhancement can be distinguished. Firstly, it would help the engineer in taking variations in material behaviour and fabrication inaccuracies into account during the design process. Secondly, structural details could be smeared out in the models by assigning averages and variances to parameters of a coarser model. In the computations, more uncertainty in the model’s response are acceptable, provided that the CPU-times can be reduced sufficiently. For applications in which products exhibit structural complexity, this would make numerical tools become viable earlier. Current developments in which the variances of the stochastic parameters are linearised will not suffice, since only small variances can be handled when such methods are used. For a general framework, numerical operations on stochastic variables will have to be developed and implemented, such as multiplication and integration. This concerns not only the structural analysis tool, but the acoustical analysis tool as well, since it will have to allow stochastic velocity amplitudes as input. Minor changes to other modules in the toolbox would also be necessary, such as for the graphical presentation of stochastic results. The optimisation module, however, can be used in its present form, as is explained in Chapter 5.

The current Fourier-FEM formulation can be enhanced by transforming the material distributions in circumferential direction as well. This results in less efficient but, most probably, more accurate predictions. Only the structural analysis tool would have to be adapted for this purpose, since all other parts of the tool are unaffected. Such an enhancement could be a useful step in studying the effects of inhomogeneities in circumferential direction.

The usefulness of the tool can further be expanded by the inclusion of complex radiation modes, which enables solving coupled radiation problems for axisymmetric bodies submerged in a heavy fluid. Another possibility would be to extend the toolbox with control algorithms, such as the Matlab Control Toolbox.
Possibly in combination with the implementation of material models for piezo materials, active noise reduction methods could then be simulated as well.

A considerable part of the Structural Acoustics Toolbox is responsible for bookkeeping, aimed at establishing a coupling between structural and acoustic analysis tools. The purpose of this is to automate tasks that, if carried out by users, would be tedious and error prone. The current implementation is aimed at Fourier models, but similar routines could be implemented for full three-dimensional models. Such routines are necessary for running multiple analyses without user intervention.

The optimisation tools can be extended by supporting the use of discrete design parameters and integer design parameters. The number of longitudinal profiles in the gradient coil, for instance, could then be varied in the search for a better design.
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Modified rule-of-mixtures equations for transverse and shear moduli

When using numerical models for vibrational analysis, the engineer has to specify data that represents material behaviour. Sometimes, such data can be found in the manufacturer’s catalogue or in literature. When material tailoring is applied, however, the material data depends on some design parameters. This can only be done if some formula is available that can be used to determine equivalent material data. For composites such as fibre reinforced epoxies, relations are available that are based on micromechanics and on empirical data. The most widely used semiempirical equations were developed by Halpin and Tsai (1969) (also see Tsai and Hahn (1980)), and these are applied in this thesis as well.

In these relations, the fibre material is assumed to be isotropic. Equivalent material properties can be determined from the formulas in table A.1. Subscript \( x \) denotes longitudinal (parallel to the fibres) properties, subscript \( y \) denotes transverse properties, subscript \( f \) denotes fibre properties, subscript \( m \) denotes matrix material properties, and \( E_s \) is the shear modulus. Values for the parameter \( \xi \) for shear modulus \( E_s \), bulk modulus \( k \) and transverse shear modulus \( G_y \) can be determined from mechanical properties of the constituents:

\[
\xi_s = \frac{1}{2} \left( 1 + \frac{G_m}{G_f} \right) \quad (A.1)
\]

\[
\xi_k = \frac{1}{2(1-\nu_m)} \left( 1 + (1 - 2\nu_f)\frac{G_m}{G_f} \right) \quad (A.2)
\]

\[
\xi_G = \frac{1}{4(1-\nu_m)} \left( 3 - 4\nu_m + \frac{G_m}{G_f} \right) \quad (A.3)
\]

Using these parameters, the composite transverse modulus and the longitudinal shear modulus are obtained, together with other mechanical properties. The transverse Young’s modulus is given by:

\[
E_y = \frac{4 k_y G_y}{k_y + m G_y} \quad \text{where} \quad m = \frac{1 + 4 k_y \nu_f^2}{E_x} \quad (A.4)
\]
\[
X = \frac{1}{\nu_f + \xi \nu_m} \left( \nu_f X_f + \xi \nu_m X_m \right)
\]

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Table A.1: Formulas for equivalent material properties.
Fourier finite elements

In a Fourier finite element method, a differential equation is discretised by a combination of a discrete Fourier expansion and finite elements. A necessary condition for the application of Fourier-FEM is that the problem domain and the solution are spatially periodic in at least one of the coordinate directions. For axisymmetric structures, the response always has to be periodic in circumferential direction in order to be continuous.

Fourier elements are often thought to be applicable only for structures that have isotropic material behaviour. Also, the Fourier finite element formulation is always derived such that the stiffness matrix has to be completely reassembled for each Fourier harmonic. Here it will be shown that these stiffness matrices can be expressed explicitly in terms of the Fourier index $m$, and that Fourier elements are applicable for any form of anisotropy.

The following derivation will be valid for structures that are geometrically axisymmetric. Any kind of anisotropy is allowed, although some special forms of anisotropy will be shown to lead to a more efficient analysis method. The material distribution, however, has to be axisymmetric, and linear (visco)elasticity is assumed here. The excitation may have arbitrary distribution.

First, the relevant equations are presented. Harmonic structural vibrations are governed by the differential equations of motion:

$$\omega^2 \rho(\bar{x}) \ddot{u}(\bar{x}) + \nabla : S(\bar{x}, \omega) = 0 \quad \text{on } \Omega_s$$

where $\rho$ is the specific mass, $S$ is a Cauchy stress tensor and $\omega$ is the angular frequency. In a cylindrical coordinate system, the gradient operator $\nabla$ is defined as:

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

The amplitudes $\bar{u}$ of the displacement field can be prescribed on (part of) the surface, and distributed surface tractions $\bar{q}$ may be present. These are expressed by the boundary conditions:

$$\bar{u}(\bar{x}) = \bar{u}_{\Gamma_u}(\bar{x}) \quad \text{on } \Gamma_u \quad \bar{q}(\bar{x}_{\Gamma_q}) = \bar{q}_{\Gamma_q}(\bar{x}) \quad \text{on } \Gamma_q$$
For linear viscoelastic materials, the Cauchy stress tensor can be written as (see section 2.3.1):

\[ S(\vec{x}, \omega) = \mathcal{G}(\vec{x}, \omega) : \mathcal{D}(\vec{u}(\vec{x})) \]  

(B.4)

where \( \mathcal{G} \) is a fourth-order complex modulus tensor and \( \mathcal{D} \) is an infinitesimal strain tensor:

\[ \mathcal{D}(\vec{u}(\vec{x})) = \frac{1}{2} \left( \left( \nabla \vec{u}(\vec{x}) \right) + \left( \nabla \vec{u}(\vec{x}) \right)^c \right) \]  

(B.5)

Multiplying by an arbitrary but suitable weighting function \( \vec{w}(\vec{x}) \), integration over the domain \( \Omega_s \), application of Gauss’ Theorem and integration by parts gives the weak form of (B.1):

\[ -\omega^2 \int_{\Omega_s} \rho(\vec{x}) \vec{w}(\vec{x}) \cdot \vec{u}(\vec{x}) \, d\Omega_s + \int_{\Omega_s} (\nabla \vec{w}(\vec{x}))^c : S(\vec{x}, \omega) \, d\Omega_s = \int_{\Gamma_s} \vec{w}(\vec{x}) \cdot \vec{q}(\vec{x}) \, d\Gamma_q \]  

(B.6)

For axisymmetric structures, the circumferential variations of the displacements and the excitation can be expanded into a Fourier series. The displacements are split into a part \( \vec{u}^r \) and a part \( \vec{u}^h \). A truncated Fourier series approximates the displacements:

\[ \vec{u}(r, \theta, z) \approx \vec{e}^T N(r, z) \sum_{n=0}^{N} \begin{bmatrix} u_n^r \cos(n \theta) + u_n^h \sin(n \theta) \\ u_n^r \sin(n \theta) + u_n^h \cos(n \theta) \\ u_n^r \cos(n \theta) + u_n^h \sin(n \theta) \end{bmatrix} \]  

(B.7)

where \( N \) is a function that represents the radial and axial dependence of \( \vec{u}(r, \theta, z) \).

### 2.1 Stiffness term

By substituting equation (B.7) into (B.5), the components of the infinitesimal strain tensor \( \mathcal{D} \) can be determined. In matrix form, the strains can be expressed in terms of the column matrices \( \vec{u}_n^a \) and \( \vec{u}_n^b \) that contain the Fourier coefficients of the displacement amplitudes. Using the strain-displacement matrices \( B_n^a \) and \( B_n^b \):

\[ \varepsilon(r, \theta, z) = \sum_{n=0}^{N} \left( B_n^a(r, \theta, z) \vec{u}_n^a + B_n^b(r, \theta, z) \vec{u}_n^b \right) \]  

(B.8)
where $g$ is a column matrix containing the six independent strain components. In a cylindrical coordinate system, the strain-displacement matrix $B_n^a$ is:

$$B_n^a(r, \theta, z) = \begin{bmatrix}
\frac{\partial N}{\partial r} \cos(n \theta) & 0 & 0 \\
\frac{N}{r} \cos(n \theta) & n \frac{N}{r} \cos(n \theta) & 0 \\
0 & 0 & \frac{\partial N}{\partial z} \cos(n \theta) \\
-\frac{n}{r} N \sin(n \theta) & \left( \frac{\partial N}{\partial r} - \frac{N}{r} \right) \sin(n \theta) & 0 \\
\frac{\partial N}{\partial z} \cos(n \theta) & 0 & \frac{\partial N}{\partial r} \cos(n \theta) \\
0 & \frac{\partial N}{\partial z} \sin(n \theta) & -\frac{n}{r} N \sin(n \theta)
\end{bmatrix}$$

(B.9)

The further derivation will be simplified by splitting this matrix into a part that only contains cosine terms, and a part that only has sine terms:

$$B_n^a(r, \theta, z) = \hat{B}_n^a(r, z) \cos(n \theta) + \tilde{B}_n^a(r, z) \sin(n \theta)$$

(B.10)

where

$$\hat{B}_n^a(r, z) = \begin{bmatrix}
\frac{\partial N}{\partial r} & 0 & 0 \\
\frac{N}{r} n & \frac{n}{r} N & 0 \\
0 & 0 & \frac{\partial N}{\partial z} \\
0 & 0 & 0
\end{bmatrix} \quad \tilde{B}_n^a(r, z) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-\frac{n}{r} N & \frac{\partial N}{\partial r} - \frac{N}{r} & 0 \\
0 & 0 & 0 \\
0 & \frac{\partial N}{\partial z} & -\frac{n}{r} N
\end{bmatrix}$$

(B.11)

Key to the efficiency improvement for the Fourier finite element analysis is the observation that the resulting strain-displacement matrices linearly depend on the Fourier harmonic index $n$, and rewriting them as:

$$\hat{B}_n^a(r, z) = P + n Q$$

(B.12)

$$\tilde{B}_n^a(r, z) = R - n S$$

(B.13)
where:

\[
P = \begin{bmatrix}
\frac{\partial N}{\partial r} & 0 & 0 \\
N & 0 & 0 \\
0 & 0 & \frac{\partial N}{\partial z} \\
0 & 0 & 0 \\
0 & \frac{\partial N}{\partial r} & \frac{N}{r} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad R = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{\partial N}{\partial r} & -\frac{N}{r} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  

(B.14)

\[
Q = \frac{N}{r} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad S = \frac{N}{r} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(B.15)

A similar derivation for the matrix \(B^b_n\) in (B.8) gives:

\[
B^b_n(r, \theta, z) = \hat{B}^b_n(r, z) \cos(n \theta) + \bar{B}^b_n(r, z) \sin(n \theta)
\]  

(B.16)

where the matrices \(\hat{B}^b_n\) and \(\bar{B}^b_n\) can be expressed in terms of the same matrices as the previous strain-displacement matrices:

\[
\hat{B}_n^b(r, z) = R + nS \quad \text{(B.17)}
\]

\[
\bar{B}_n^b(r, z) = P - nQ \quad \text{(B.18)}
\]

Substituting the obtained relations for the strain-displacement matrices in relation (B.8) gives for the strain column matrix:

\[
\varepsilon(r, \theta, z) = \sum_{n=0}^{N} \left( (P + nQ) \cos(n \theta) + (R - nS) \sin(n \theta) \right) u_n^a + \\
\sum_{n=0}^{N} \left( (R + nS) \cos(n \theta) + (P - nQ) \sin(n \theta) \right) u_n^a
\]  

(B.19)

The strain column matrix can then be written as a Fourier series:

\[
\varepsilon(r, \theta, z) = \sum_{n=0}^{N} (\varepsilon_n(r, z) \cos(n \theta) + \varepsilon_n(r, z) \sin(n \theta))
\]  

(B.20)
where
\[
\varepsilon_n(r, z) = (P + nQ)u^a_n + (R + nS)u^b_n
\]  \hspace{1cm} (B.21)
\[
\varepsilon_n(r, z) = (R - nS)u^a_n + (P - nQ)u^b_n
\]  \hspace{1cm} (B.22)
which provides a relationship between strains and displacement for any Fourier harmonic in terms of the Fourier coefficients of the displacement amplitudes, and in which the Fourier index \( n \) appears explicitly.

As a choice for the weighting function, a form similar to the expression for the displacement field is used, but with only a single term from the sum in a Fourier series:
\[
\bar{w}(r, \theta, z) = e^T N(r, z) \begin{bmatrix} u^a_\theta \cos(m \theta) + u^a_\phi \sin(m \theta) \\ u^a_\phi \sin(m \theta) + u^a_\theta \cos(m \theta) \\ u^b_\theta \cos(m \theta) + u^b_\phi \sin(m \theta) \end{bmatrix}
\]  \hspace{1cm} (B.23)

This weighting function will be split in two parts that both satisfy the boundary conditions:
\[
\bar{w}^a(r, \theta, z) = e^T N(r, z) \begin{bmatrix} u^a_\theta \cos(m \theta) \\ u^a_\phi \sin(m \theta) \\ u^a_\phi \cos(m \theta) \end{bmatrix}
\]  \hspace{1cm} (B.24)
and:
\[
\bar{w}^b(r, \theta, z) = e^T N(r, z) \begin{bmatrix} u^b_\theta \sin(m \theta) \\ u^b_\phi \cos(m \theta) \\ u^b_\phi \sin(m \theta) \end{bmatrix}
\]  \hspace{1cm} (B.25)

These parts will be inserted one by one into (B.6) leading to two sets of equations. The gradient of the weighting function \( \bar{w}^a \) is similar to the gradient of the displacement field \( \bar{w} \):
\[
\nabla \bar{w}^a(r, \theta, z) = \mathcal{D}(\bar{w}^a(r, \theta, z))
\]  \hspace{1cm} (B.26)

We can then evaluate one of the terms in weak form of the equations of motion as:
\[
T_1 = \int_0^{2\pi} \left( \nabla \bar{w}^a(r, \theta, z) \right)^e : \mathbb{S}(r, \theta, z, \omega) \, d\theta
\]  \hspace{1cm} (B.27)

Substituting (B.4) and (B.26) into this expression and imposing a homogeneous material distribution gives:
\[
T_1 = \int_0^{2\pi} \mathcal{D}(\bar{w}^a(r, \theta, z)) : \mathbb{S}(r, \theta, z) \, d\theta
\]  \hspace{1cm} (B.28)
B. Fourier finite elements

Using:

$$D(\vec{u}^a(r, \theta, z)) = B_m^a \vec{u}^a$$  \hspace{1cm} (B.29)

results in:

$$T_1 = \vec{u}^a T \int_0^{2\pi} B_m^a T(r, \theta, z) \vec{G}(r, z, \omega) \vec{\varepsilon}(r, \theta, z) \, d\theta = T_{11} + T_{12}$$  \hspace{1cm} (B.30)

where $\vec{G}$ is a symmetric material matrix, and:

$$T_{11} = \vec{u}^a T \int_0^{2\pi} \sum_{n=0}^{N} B_m^a T(r, \theta, z) \vec{G}(r, z, \omega) \vec{\varepsilon}_n(r, z) \cos(n \theta) \, d\theta$$  \hspace{1cm} (B.31)

$$T_{12} = \vec{u}^a T \int_0^{2\pi} \sum_{n=0}^{N} B_m^a T(r, \theta, z) \vec{G}(r, z, \omega) \vec{\varepsilon}_n(r, z) \sin(n \theta) n \, d\theta$$  \hspace{1cm} (B.32)

By using (B.10) the first of these terms can be written as:

$$T_{11} = \vec{u}^a T \sum_{n=0}^{N} \left( \begin{array}{c} B_m^a(r, z) \\ \end{array} \right)^T \vec{G}(r, z, \omega) \vec{\varepsilon}_n(r, z) \int_0^{2\pi} \cos(m \theta) \cos(n \theta) \, d\theta +$$

$$\vec{u}^a T \sum_{n=0}^{N} \left( \begin{array}{c} B_m^a(r, z) \\ \end{array} \right)^T \vec{G}(r, z, \omega) \vec{\varepsilon}_n(r, z) \int_0^{2\pi} \sin(m \theta) \cos(n \theta) \, d\theta$$  \hspace{1cm} (B.33)

Most of the integrals within the sums vanish, since:

$$\int_0^{2\pi} \cos(m \theta) \cos(n \theta) \, d\theta = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n > 0 \\ 0 & \text{if } m \neq n \end{cases}$$  \hspace{1cm} (B.34)

$$\int_0^{2\pi} \sin(m \theta) \cos(n \theta) \, d\theta = 0$$  \hspace{1cm} (B.35)

The term $T_{11}$ therefore reduces to:

$$T_{11} = \begin{cases} 2\pi \vec{u}^a T \left( \begin{array}{c} \vec{B}_0^a(r, z) \\ \end{array} \right)^T \vec{G}(r, z, \omega) \vec{\varepsilon}_0(r, z) & \text{if } m = 0 \\ \pi \vec{u}^a T \left( \begin{array}{c} \vec{B}_m^a(r, z) \\ \end{array} \right)^T \vec{G}(r, z, \omega) \vec{\varepsilon}_m(r, z) & \text{if } m > 0 \end{cases}$$  \hspace{1cm} (B.36)
Similarly, the term $T_{12}$ evaluates to:

$$T_{12} = \begin{cases} 0 & \text{if } m = 0 \\ \pi \omega^a \left( \mathbf{B}^a_m(r, z) \right)^T G(r, z, \omega) \varepsilon_m(r, z) & \text{if } m > 0 \end{cases} \tag{B.37}$$

By using relations (B.12), (B.13), (B.21) and (B.22), for $m = 0$ this leads to:

$$T_1 = 2\pi \omega^a \left( P^T GP u^a_0 + P^T GR u^b_0 \right) \tag{B.38}$$

while for $m > 0$:

$$T_1 = \pi \omega^a \left( P^T GP + R^T GR \right) u^a_m +$$
$$m \pi \omega^a \left( P^T GQ + Q^T GP - R^T GS - S^T GR \right) u^b_m +$$
$$m^2 \pi \omega^a \left( Q^T GQ + S^T GS \right) u^b_m +$$

$$T_{12} \text{ and } T_{13} \text{ are normally zero. The second integral in the left-hand side of the equations of motion (B.6) can now be written explicitly in terms of } m, \text{ since:}$$

$$\int_\Omega (\mathbf{\bar{v}} \cdot \mathbf{\bar{u}}) : \mathbb{S} \, d\Omega = \int \int T_1 (r, z, \omega) r \, dr \, dz \tag{B.40}$$

For isotropic and specially orthotropic materials, the structure of the material matrix $G$ is such that a number of matrices in (B.38) and in (B.39) disappear:

$$P^T GR = 0 \quad R^T GP = 0 \tag{B.41}$$
$$P^T GS = 0 \quad S^T GP = 0 \tag{B.42}$$
$$R^T GQ = 0 \quad Q^T GR = 0 \tag{B.43}$$
$$Q^T GS = 0 \quad S^T GQ = 0 \tag{B.44}$$

For $m = 0$, $T_1$ reduces to:

$$T_1 = 2\pi \omega^a \left( P^T GP u^a_0 \right) \tag{B.45}$$

and for $m > 0$:

$$T_1 = \pi \omega^a \left( P^T GP + R^T GR \right) u^a_m +$$
$$m \pi \omega^a \left( P^T GQ + Q^T GP - R^T GS - S^T GR \right) u^b_m +$$
$$m^2 \pi \omega^a \left( Q^T GQ + S^T GS \right) u^b_m \tag{B.46}$$
so only the terms containing \( u_m^a \) remain.

Now the other weighting function (B.25) is selected, which reads:

\[
\mathbf{w}^b(r, \theta, z) = e^T \mathbf{N}(r, z) \begin{bmatrix} u_b^a \sin(m \theta) \\ u_b^b \cos(m \theta) \\ u_b^c \sin(m \theta) \end{bmatrix}
\]  
(8.47)

A term \( T_2 \) has to be evaluated:

\[
T_2 = \int_0^{2\pi} \left( \nabla \mathbf{w}^b(r, \theta, z) \right)^T : \mathbf{S}(r, \theta, z) \, d\theta
\]  
(8.48)

which is highly similar to the previous derivation of \( T_1 \). For \( m = 0 \), this leads to:

\[
T_2 = 2\pi \mathbf{w}^b \left( \mathbf{R}^T \mathbf{G} \mathbf{P} \mathbf{u}_0^a + \mathbf{R}^T \mathbf{G} \mathbf{R} \mathbf{u}_0^b \right)
\]  
(8.49)

and for \( m > 0 \):

\[
T_2 = \pi \mathbf{w}^b \left( \mathbf{P}^T \mathbf{G} \mathbf{R} + \mathbf{R}^T \mathbf{G} \mathbf{P} \right) \mathbf{u}_m^a + \\
\pi \mathbf{w}^b \left( \mathbf{S}^T \mathbf{G} \mathbf{P} - \mathbf{P}^T \mathbf{G} \mathbf{S} \right) \mathbf{u}_m^a + \\
m \pi \mathbf{w}^b \left( \mathbf{S}^T \mathbf{G} \mathbf{Q} + \mathbf{Q}^T \mathbf{G} \mathbf{S} \right) \mathbf{u}_m^a + \\
\pi \mathbf{w}^b \left( \mathbf{P}^T \mathbf{G} \mathbf{P} + \mathbf{R}^T \mathbf{G} \mathbf{R} \right) \mathbf{u}_m^b + \\
m \pi \mathbf{w}^b \left( \mathbf{R}^T \mathbf{G} \mathbf{S} + \mathbf{S}^T \mathbf{G} \mathbf{R} - \mathbf{P}^T \mathbf{G} \mathbf{Q} - \mathbf{Q}^T \mathbf{G} \mathbf{P} \right) \mathbf{u}_m^b + \\
m^2 \pi \mathbf{w}^b \left( \mathbf{Q}^T \mathbf{G} \mathbf{Q} + \mathbf{S}^T \mathbf{G} \mathbf{S} \right) \mathbf{u}_m^b
\]  
(8.50)

For isotropic and specially orthotropic materials, this becomes for \( m = 0 \):

\[
T_2 = 2\pi \mathbf{w}^b \mathbf{R}^T \mathbf{G} \mathbf{R} \mathbf{u}_0^0
\]  
(8.51)

while for \( m > 0 \):

\[
T_2 = \pi \mathbf{w}^b \left( \mathbf{P}^T \mathbf{G} \mathbf{P} + \mathbf{R}^T \mathbf{G} \mathbf{R} \right) \mathbf{u}_m^a - \\
m \pi \mathbf{w}^b \left( \mathbf{P}^T \mathbf{G} \mathbf{Q} + \mathbf{Q}^T \mathbf{G} \mathbf{P} - \mathbf{P}^T \mathbf{G} \mathbf{S} - \mathbf{S}^T \mathbf{G} \mathbf{R} \right) \mathbf{u}_m^a + \\
m^2 \pi \mathbf{w}^b \left( \mathbf{Q}^T \mathbf{G} \mathbf{Q} + \mathbf{S}^T \mathbf{G} \mathbf{S} \right) \mathbf{u}_m^a
\]  
(8.52)

### 2.2 Mass term

Discretisation of the mass term in the equations of motion (B.6) is more straightforward than the discretisation of the stiffness term. In a cylindrical coordinate
system, the integral is:

\[ T = \int_{\Omega} \rho(r, z) \mathbf{u}(r, \theta, z) \cdot \mathbf{\bar{u}}(r, \theta, z) r \, dr \, d\theta \, dz \]  

(B.53)

By substituting the Fourier expansion of the displacement amplitudes and expression (B.24) for the weight function \( w \), this integral can be written as:

\[ T^a = w^a T^a \int \int \rho(r, z) \mathbf{N}^T(r, z) \mathbf{N}(r, z) r \, dr \, dz \int_0^{2\pi} \begin{bmatrix} u_m^a \cos(m \theta) \cos(n \theta) \\ u_m^a \sin(m \theta) \sin(n \theta) \end{bmatrix} d\theta \]

which is equivalent to:

\[ T^a = \alpha_m w^a T^a \int \int \rho(r, z) \mathbf{N}^T(r, z) \mathbf{N}(r, z) r \, dr \, dz u_m^a \]  

(B.55)

where:

\[ \alpha_m = \begin{cases} 2\pi & \text{for } m = 0 \\ \pi & \text{for } m > 0 \end{cases} \]  

(B.56)

By using expression (B.24) for the weight function \( w \), the mass term is:

\[ T^b = w^b \int \int \rho(r, z) \mathbf{N}^T(r, z) \mathbf{N}(r, z) r \, dr \, dz \int_0^{2\pi} \begin{bmatrix} u_m^b \sin(m \theta) \sin(n \theta) \\ u_m^b \cos(m \theta) \cos(n \theta) \end{bmatrix} d\theta \]

or:

\[ T^b = \alpha_m w^b T^b \int \int \rho(r, z) \mathbf{N}^T(r, z) \mathbf{N}(r, z) r \, dr \, dz u_m^b \]  

(B.57)

2.3 Mechanical surface load term

The distributed surface load vector can be written as:

\[ \bar{q}(r, \theta, z) = \bar{e}^T q(r, \theta, z) \]  

(B.59)

Again, the weight function (B.24) is used first. The inner product of the weighting function and the excitation vector, which appears in the equation of motion (B.6),
can then be expressed as:

\[
\mathbf{\bar{w}}^a \cdot \mathbf{\bar{q}} = \mathbf{w}^a \mathbf{T}^T (r, z) 
\begin{bmatrix}
q_r (r, \theta, z) \cos (m \theta) \\
q_\theta (r, \theta, z) \sin (m \theta) \\
q_z (r, \theta, z) \cos (m \theta)
\end{bmatrix}
\]  

(B.60)

Integrating this term over the circumferential direction gives:

\[
\int_0^{2\pi} \mathbf{\bar{w}}^a (r, \theta, z) \cdot \mathbf{\bar{q}} (r, \theta, z) \ r \ d\theta = \mathbf{w}^a \mathbf{T}^T (r, z) \ r \int_0^{2\pi} \begin{bmatrix}
q_r (r, \theta, z) \cos (m \theta) \\
q_\theta (r, \theta, z) \sin (m \theta) \\
q_z (r, \theta, z) \cos (m \theta)
\end{bmatrix} d\theta \\
= \alpha_m \mathbf{w}^a \mathbf{T}^T (r, z) \mathbf{q}^a_m (r, z)
\]  

(B.61)

where:

\[
\alpha_m = \begin{cases} 
2\pi & \text{for } m = 0 \\
\pi & \text{for } m > 0
\end{cases}
\]  

(B.62)

and \( \mathbf{q}^a_m \) is a column matrix containing the Fourier coefficients of the load for the \( m \)th harmonic:

\[
\mathbf{q}^a_m (r, z) = \frac{1}{\alpha_m} \int_0^{2\pi} \begin{bmatrix}
q_r (r, \theta, z) \cos (m \theta) \\
q_\theta (r, \theta, z) \sin (m \theta) \\
q_z (r, \theta, z) \cos (m \theta)
\end{bmatrix} d\theta
\]  

(B.63)

The surface integral term in the equations of motion (B.6) can now be determined in terms of Fourier coefficients of the distributed surface load by integrating (B.61) along the surface contour in the \( r-z \) plane:

\[
\int_{\Gamma_u} \mathbf{\bar{w}}^a (\mathbf{x}) \cdot \mathbf{\bar{q}} (\mathbf{x}) \ d\Gamma_u = \alpha_m \mathbf{w}^a \mathbf{T} \int \mathbf{N}^T (r, z) \mathbf{q}^a_m (r, z) \ r \ dl (r, z)
\]  

(B.64)

Similarly, when expression (B.25) is used as a weighting function:

\[
\mathbf{q}^b_m (r, z) = \frac{1}{\alpha_m} \int_0^{2\pi} \begin{bmatrix}
q_r (r, \theta, z) \sin (m \theta) \\
q_\theta (r, \theta, z) \cos (m \theta) \\
q_z (r, \theta, z) \sin (m \theta)
\end{bmatrix} d\theta
\]  

(B.65)

and:

\[
\int_{\Gamma_u} \mathbf{\bar{w}}^b (\mathbf{x}) \cdot \mathbf{\bar{q}} (\mathbf{x}) \ d\Gamma_u = \alpha_m \mathbf{w}^b \mathbf{T} \int \mathbf{N}^T (r, z) \mathbf{q}^b_m (r, z) \ r \ dl (r, z)
\]  

(B.66)
2.4 Discretised equations of motion

All terms in the equations of motion (B.6) have now been discretised in circumferential direction using Fourier expansions. Each of these terms contain weighting functions, so these can be dropped. After collecting all terms, the following matrix equation appears:

\[
(-\omega^2 M + K_m) u_m = f_m 
\]

(B.67)

where:

\[
\begin{bmatrix}
u_m^a \\
u_m^b 
\end{bmatrix}, \quad
\begin{bmatrix}
f_m^a \\
f_m^b 
\end{bmatrix} 
\]

(B.68)

The nodal force amplitudes \(f_m^a\) and \(f_m^b\) are determined from the amplitudes of the distributed surface loads \(q\):

\[
f_m^a = \int N^T(r, z) q_m^a(r, z) r \, dl(r, z) 
\]

(B.69)

\[
f_m^b = \int N^T(r, z) q_m^b(r, z) r \, dl(r, z) 
\]

(B.70)

The mass matrix \(M\), which is independent of the Fourier index \(m\), consists of two identical smaller mass matrices \(M_0\):

\[
M = \begin{bmatrix} M_0 & 0 \\ 0 & M_0 \end{bmatrix} 
\]

(B.71)

where

\[
M_0 = \int \int \rho(r, z) N^T(r, z) N(r, z) r \, dr \, dz 
\]

(B.72)

The stiffness matrix \(K_m\) looks like:

\[
K_m = \begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} + m \begin{bmatrix} A_1 & B_1 \\ -B_1 & -A_1 \end{bmatrix} + m^2 \begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} 
\]

(B.73)
The matrices $A_i$ and $B_i$ have to be determined from:

\[
A_0 = \int \int \left( P^T G P + R^T G R \right) r \, dr \, dz \quad (B.74)
\]

\[
A_1 = \int \int \left( P^T G Q + Q^T G P - R^T G S - S^T G R \right) r \, dr \, dz \quad (B.75)
\]

\[
A_2 = \int \int \left( Q^T G Q + S^T G S \right) r \, dr \, dz \quad (B.76)
\]

\[
B_0 = \int \int \left( P^T G R + R^T G P \right) r \, dr \, dz \quad (B.77)
\]

\[
B_1 = \int \int \left( P^T G S - S^T G P \right) r \, dr \, dz \quad (B.78)
\]

\[
B_2 = \int \int \left( S^T G Q + Q^T G S \right) r \, dr \, dz \quad (B.79)
\]

All integrals are in terms of $r$ and $z$, and can be determined by numerical integration using a finite element formulation.

Note that for isotropic or specially orthotropic material behaviour, the $B_i$ matrices vanish. Then, two uncoupled matrix equations appear, one for the nodal displacement amplitudes $u^a_m$, and one for $u^b_m$:

\[
\begin{align*}
(-\omega^2 M_0 + K^a_m) u^a_m &= f^a_m \\
(-\omega^2 M_0 + K^b_m) u^b_m &= f^b_m
\end{align*}
\quad (B.80)\quad (B.81)
\]

where:

\[
K^a_m = A_0 + m A_1 + m^2 A_2 \quad (B.82)
\]

\[
K^b_m = A_0 - m A_1 + m^2 A_2 \quad (B.83)
\]

### 2.5 Validation

The Fourier-FEM method, as it was presented above, has been implemented. In order to validate the derivation and the implementation, a number of resonance frequencies as predicted by the method are compared to the theoretical results of the Flügge shell theory (Egle and Soder, 1969), and to measurement results by Egle and Bray (1968).

An isotropic, thin-walled axisymmetric shell, simply supported at both ends, is modeled using the parameters in table B.1. The results from the Fourier-FEM...
### Table B.1: Dimensions and material properties of the thin-walled cylindrical shell

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>0.2982 [m]</td>
</tr>
<tr>
<td>Radius</td>
<td>0.1482 [m]</td>
</tr>
<tr>
<td>Thickness</td>
<td>5.08 ( \times 10^{-4} ) [m]</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>2.034 ( \times 10^{11} ) [N/m²]</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.285 [-]</td>
</tr>
<tr>
<td>Density</td>
<td>7.8442 ( \times 10^{3} ) [kg/m³]</td>
</tr>
</tbody>
</table>

### Table B.2: Comparison between theoretical, numerical (Fourier-FEM) and experimental resonant frequencies for a homogeneous, simply supported isotropic cylinder for different values of Fourier coefficient \( m \) and number of half axial wavelengths \( n \). a: Flügge shell theory (Egle and Söder, 1969); b: Fourier-FEM (this thesis); c: measurements (Egle and Bray, 1968).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n=1 )</th>
<th>( n=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( a )</td>
<td>( b )</td>
</tr>
<tr>
<td>1</td>
<td>3271</td>
<td>3270</td>
</tr>
<tr>
<td>2</td>
<td>1862</td>
<td>1863</td>
</tr>
<tr>
<td>3</td>
<td>1102</td>
<td>1103</td>
</tr>
<tr>
<td>4</td>
<td>706</td>
<td>706</td>
</tr>
<tr>
<td>5</td>
<td>498</td>
<td>498</td>
</tr>
<tr>
<td>6</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>7</td>
<td>381</td>
<td>380</td>
</tr>
<tr>
<td>8</td>
<td>417</td>
<td>416</td>
</tr>
<tr>
<td>9</td>
<td>489</td>
<td>487</td>
</tr>
<tr>
<td>10</td>
<td>584</td>
<td>582</td>
</tr>
<tr>
<td>11</td>
<td>696</td>
<td>694</td>
</tr>
<tr>
<td>12</td>
<td>823</td>
<td>820</td>
</tr>
</tbody>
</table>
are compared to theoretical results and to measurement results in table B.2. As can be seen, the computed results are in excellent agreement with the theoretical predictions.
Fourier transform of the Lorentz excitation

Contrary to the Fourier expansion of the \( z \)-coils, the Fourier expansion of the transverse coils can not be determined easily. It is, however, possible to determine their Fourier coefficients analytically, as will be shown below. See Chapter 3 for more details on the transverse coils.

The spatial Fourier expansion of a distributed excitation can be written as:

\[
p(r, \theta, z) = \sum_{m=0}^{N} \begin{bmatrix} p_{a,m}^b(r, z) \cos(m \theta) \\ p_{a,m}^b(r, z) \sin(m \theta) \\ p_{a,m}^b(r, z) \cos(m \theta) \\ p_{a,m}^b(r, z) \sin(m \theta) \end{bmatrix}
\]  

(C.1)

Since each of the transverse coils generates an antisymmetric excitation in circumferential direction, the Fourier coefficients \( p_{a,0}^b \) and \( p_{a,1}^b \) will be zero. The other coefficients \( (m > 0) \) of this expansion are obtained from the Fourier integrals:

\[
p_{a,m}^b(r, z) = \frac{1}{\pi} \int_0^{2\pi} \begin{bmatrix} p_r(r, \theta, z) \cos(m \theta) \\ p_\theta(r, \theta, z) \sin(m \theta) \\ p_z(r, \theta, z) \cos(m \theta) \\ p_\theta(r, \theta, z) \sin(m \theta) \end{bmatrix} d\theta
\]  

(C.2)

\[
p_{b,m}^b(r, z) = \frac{1}{\pi} \int_0^{2\pi} \begin{bmatrix} p_r(r, \theta, z) \cos(m \theta) \\ p_\theta(r, \theta, z) \sin(m \theta) \\ p_z(r, \theta, z) \cos(m \theta) \\ p_\theta(r, \theta, z) \sin(m \theta) \end{bmatrix} d\theta
\]  

(C.3)

As illustrated in figure C.1 for a set of four conductors, the transverse coil conductor patterns are geometrically symmetric.

As on each of these conductors the excitation is constant, and they have equal amplitudes, the Fourier integrals can be derived using the geometry of the first quadrant. For the \( x \)-coil, the cosine transform of the radial or the axial compo-
\[\hat{\rho}_m = \frac{1}{\pi} \int_0^{2\pi} p(\theta) \cos(m\theta) \, d\theta\]
\[= \frac{p_l}{\pi} \int_{\theta_1}^{\theta_2} \cos(m\theta) \, d\theta - \frac{p_l}{\pi} \int_{-\pi}^{-\theta_1} \cos(m\theta) \, d\theta - \frac{p_l}{\pi} \int_{\pi+\theta_1}^{\pi+\theta_2} \cos(m\theta) \, d\theta + \frac{p_l}{\pi} \int_{-\theta_2}^{-\theta_1} \cos(m\theta) \, d\theta\]  \hspace{1cm} (C.4)

where \(p_l\) is the load amplitude on the conductor in the first quadrant. Only cosine integrals remain, and the first of these can be evaluated analytically as:

\[\frac{p_l}{\pi} \int_{\theta_1}^{\theta_2} \cos(m\theta) \, d\theta = \frac{p_l}{m\pi} (\sin(m\theta_2) - \sin(m\theta_1))\]  \hspace{1cm} (C.5)
The second integral can be expressed as:

\[
\frac{p_1}{\pi} \int_{\pi - \theta_1}^{\pi - \theta_2} \cos(m \theta) \, d\theta = \frac{p_1}{m \pi} \left( \sin(m(\pi - \theta_1)) - \sin(m(\pi - \theta_2)) \right)
\]

\[
= (-1)^m \frac{p_1}{m \pi} \left( \sin(-m \theta_1) - \sin(-m \theta_2) \right)
\]

\[
= (-1)^m \frac{p_1}{m \pi} \left( \sin(m \theta_2) - \sin(m \theta_1) \right)
\]

(C.6)

where \(\sin(m \pi - \alpha) = (-1)^m \sin(-\alpha) = -(\alpha)^m \sin(\alpha)\) has been used. Similarly, the third integral in (C.4) expands to:

\[
\frac{p_1}{\pi} \int_{\pi + \theta_1}^{\pi + \theta_2} \cos(m \theta) \, d\theta = \frac{p_1}{m \pi} \left( \sin(m(\pi + \theta_1)) - \sin(m(\pi + \theta_2)) \right)
\]

\[
= (-1)^m \frac{p_1}{m \pi} \left( \sin(m \theta_2) - \sin(m \theta_1) \right)
\]

(C.7)

and the last one is:

\[
\frac{p_1}{\pi} \int_{-\theta_1}^{-\theta_2} \cos(m \theta) \, d\theta = \frac{p_1}{m \pi} \left( \sin(-m \theta_1) - \sin(-m \theta_2) \right)
\]

\[
= \frac{p_1}{m \pi} \left( \sin(m \theta_2) - \sin(m \theta_1) \right)
\]

(C.8)

By adding the four integrals, the total X-coil cosine transform is obtained:

\[
\hat{p}_m = \frac{2(1 - (-1)^m)p_1}{m \pi} \left( \sin(m \theta_2) - \sin(m \theta_1) \right)
\]

(C.9)

The sine transforms for the circumferential component of the X-coil excitation can be expressed as:

\[
\hat{p}_m = \frac{1}{\pi} \int_0^{2\pi} p(\theta) \sin(m \theta) \, d\theta
\]

\[
= -\frac{p_1}{\pi} \int_{\theta_1}^{\theta_2} \sin(m \theta) \, d\theta - \frac{p_1}{2\pi} \int_{-\theta_2}^{-\theta_1} \sin(m \theta) \, d\theta
\]

\[
+ \frac{p_1}{\pi} \int_{\pi - \theta_1}^{\pi - \theta_2} \sin(m \theta) \, d\theta + \frac{p_1}{\pi} \int_{\pi + \theta_1}^{\pi + \theta_2} \sin(m \theta) \, d\theta
\]

(C.10)
so that again four integrals have to be determined, which are similar to the integrals in the cosine transforms. The first of these reads:

\[
\frac{p_l}{\pi} \int_{\theta_1}^{\theta_2} \sin(m \theta) \, d\theta = -\frac{p_l}{m \pi} (\cos(m \theta_2) - \cos(m \theta_1)) \tag{C.11}
\]

The second integral becomes:

\[
\frac{p_l}{\pi} \int_{-\theta_2}^{-\theta_1} \sin(m \theta) \, d\theta = -\frac{p_l}{m \pi} (\cos(m(\pi - \theta_1)) - \cos(m(\pi + \theta_2)))
\]

\[
= -(-1)^m \frac{p_l}{m \pi} (\cos(-m \theta_1) - \cos(-m \theta_2)) = (-1)^m \frac{p_l}{m \pi} (\cos(m \theta_2) - \cos(m \theta_1)) \tag{C.12}
\]

where \(\cos(m \pi + \alpha) = (-1)^m \cos(\alpha)\) has been used. The third integral gives:

\[
\frac{p_l}{\pi} \int_{\pi + \theta_1}^{\pi + \theta_2} \sin(m \theta) \, d\theta = -\frac{p_l}{m \pi} (\cos(m(\pi + \theta_2)) - \cos(m(\pi + \theta_1)))
\]

\[
= -(-1)^m \frac{p_l}{m \pi} (\cos(m \theta_2) - \cos(m \theta_1)) \tag{C.13}
\]

and finally:

\[
\frac{1}{\pi} \int_{-\theta_2}^{-\theta_1} p_l \sin(m \theta) \, d\theta = -\frac{p_l}{m \pi} (\cos(-m \theta_1) - \cos(-m \theta_2))
\]

\[
= \frac{p_l}{m \pi} (\cos(m \theta_2) - \cos(m \theta_1)) \tag{C.14}
\]

which completes the evaluation of the integrals for the x-coil sine transformation, and gives:

\[
\tilde{p}_m = \frac{2(1 - (-1)^m) p_l}{m \pi} (\cos(m \theta_2) - \cos(m \theta_1)) \tag{C.15}
\]

from which it can be seen that the Fourier coefficients vanish if \(m\) is even.

For the y-coil, the cosine transforms for the circumferential excitation component
can be split into:

\[
p_m = \frac{1}{\pi} \int_0^{2\pi} p(\theta) \cos(m \theta) \, d\theta
\]

\[
= \frac{p_1}{\pi} \int_{\frac{\pi}{2} - \theta_1}^{\frac{\pi}{2} + \theta_2} \cos(m \theta) \, d\theta - \frac{p_1}{\pi} \int_{\frac{\pi}{2} - \theta_1}^{\frac{\pi}{2} + \theta_2} \cos(m \theta) \, d\theta - \frac{p_1}{\pi} \int_{\frac{\pi}{2} - \theta_1}^{\frac{\pi}{2} + \theta_2} \cos(m \theta) \, d\theta
\]

(C.16)

Again, the four integrals have to be expanded. The first of these can be written as:

\[
\frac{p_1}{\pi} \int_{\frac{\pi}{2} - \theta_1}^{\frac{\pi}{2} - \theta_2} \cos(m \theta) \, d\theta = \frac{p_1}{m \pi} \left( \sin(m(\frac{\pi}{2} - \theta_1)) - \sin(m(\frac{1}{2} \pi - \theta_2)) \right)
\]

(C.17)

By using the relation:

\[
\sin(\frac{1}{2}m \pi - m \alpha) = \sin(\frac{1}{2}m \pi) \cos(m \alpha) - \cos(\frac{1}{2}m \pi) \sin(m \alpha)
\]

(C.18)

the following is obtained:

\[
\sin(m(\frac{\pi}{2} - \theta_1)) - \sin(m(\frac{1}{2} \pi - \theta_2)) = \\
- \sin(\frac{1}{2}m \pi)(\cos(m \theta_2) - \cos(m \theta_1)) - \cos(\frac{1}{2}m \pi)(\sin(m \theta_2) - \sin(m \theta_1))
\]

(C.19)

The second integral in the right-hand side of (C.16) leads to:

\[
\frac{1}{\pi} \int_{\frac{\pi}{2} - \theta_1}^{\frac{\pi}{2} + \theta_2} \cos(m \theta) \, d\theta = \frac{p_1}{m \pi} \left( \sin(m(\frac{\pi}{2} + \theta_2)) - \sin(m(\frac{1}{2} \pi + \theta_1)) \right)
\]

(C.20)

The sum of sine terms is similar to (C.34):

\[
\sin(m(\frac{1}{2} \pi + \theta_2)) - \sin(m(\frac{1}{2} \pi + \theta_1)) = \\
\sin(\frac{1}{2}m \pi)(\cos(m \theta_2) - \cos(m \theta_1)) + \cos(\frac{1}{2}m \pi)(\sin(m \theta_2) - \sin(m \theta_1))
\]

(C.21)
For the third integral:

\[
\frac{1}{\pi} \int_{\frac{3\pi}{2} - \theta_1}^{\frac{3\pi}{2} - \theta_2} p_I \cos(m\theta) \, d\theta = \frac{p_I}{m \pi} (\sin(m(\frac{3\pi}{2} - \theta_1)) - \sin(m(\frac{3\pi}{2} - \theta_2))) \tag{C.22}
\]

the sine terms can be rewritten using

\[
\sin\left(\frac{3}{2}m \pi - m \alpha\right) = \sin\left(m \pi + \frac{1}{2}m \pi - m \alpha\right) = (-1)^m \sin\left(\frac{1}{2}m \pi - m \alpha\right)
\]

Since

\[
\sin\left(\frac{1}{2}m \pi - m \alpha\right) = \sin\left(\frac{1}{2}m \pi\right) \cos(m\alpha) - \cos\left(\frac{1}{2}m \pi\right) \sin(m\alpha)
\]

the sine terms equal:

\[
\sin\left(\frac{3}{2}m \pi - m \alpha\right) = (-1)^m \sin\left(\frac{1}{2}m \pi\right) \cos(m\alpha) - (-1)^m \cos\left(\frac{1}{2}m \pi\right) \sin(m\alpha)
\]

such that:

\[
\sin\left(m\left(\frac{3}{2}\pi - \theta_1\right)\right) - \sin\left(m\left(\frac{3}{2}\pi - \theta_2\right)\right) =
\]

\[
(\sin\left(\frac{1}{2}m \pi\right) \cos\left(m\theta_2\right) - \cos\left(\frac{1}{2}m \pi\right) \sin\left(m\theta_1\right)) +
\]

\[
(-1)^m \cos\left(\frac{1}{2}m \pi\right) \left(\sin\left(m\theta_2\right) - \sin\left(m\theta_1\right)\right)
\]

The last integral:

\[
\frac{1}{\pi} \int_{\frac{3\pi}{2} + \theta_1}^{\frac{3\pi}{2} + \theta_2} p_I \cos(m\theta) \, d\theta = \frac{p_I}{m \pi} (\sin(m(\frac{3\pi}{2} + \theta_1)) - \sin(m(\frac{3\pi}{2} + \theta_2))) \tag{C.27}
\]

can be treated similarly:

\[
\sin\left(\frac{3}{2}m \pi + m \alpha\right) = (-1)^m \sin\left(\frac{1}{2}m \pi\right) \cos(m\alpha) + (-1)^m \cos\left(\frac{1}{2}m \pi\right) \sin(m\alpha)
\]

leading to:

\[
\sin\left(m\left(\frac{3}{2}\pi + \theta_2\right)\right) - \sin\left(m\left(\frac{3}{2}\pi + \theta_1\right)\right) =
\]

\[
(\sin\left(\frac{1}{2}m \pi\right) \cos\left(m\theta_2\right) - \cos\left(\frac{1}{2}m \pi\right) \sin\left(m\theta_1\right)) +
\]

\[
(-1)^m \cos\left(\frac{1}{2}m \pi\right) \left(\sin\left(m\theta_2\right) - \sin\left(m\theta_1\right)\right)
\]

\(\tag{C.29}\)
The Fourier integral (C.16) expands to:

\[
\hat{p}_m = \frac{2(1 - (-1)^m) \sin(\frac{1}{2}m \pi)}{m \pi} \left( \cos(m \theta_2) - \cos(m \theta_1) \right)
\]

(C.30)

which is the same as the sine transform for the x-coil, except for the term \(- \sin(\frac{1}{2}m \pi)\). For \(m\) is odd, this term is alternatingly unity and minus unity.

The sine transforms for the radial and axial excitation of the y-coil remain:

\[
\hat{p}_m = \frac{1}{\pi} \int_0^{2\pi} p(\theta) \sin(m \theta) \, d\theta
\]

\[
= \frac{p_I}{\pi} \int_{\frac{1}{2}\pi - \theta_1}^{\frac{1}{2}\pi - \theta_2} \sin(m \theta) \, d\theta + \frac{p_I}{\pi} \int_{\frac{1}{2}\pi + \theta_1}^{\frac{1}{2}\pi + \theta_2} \sin(m \theta) \, d\theta - \frac{p_I}{\pi} \int_{\frac{1}{2}\pi - \theta_1}^{\frac{1}{2}\pi + \theta_2} \sin(m \theta) \, d\theta - \frac{p_I}{\pi} \int_{\frac{1}{2}\pi - \theta_2}^{\frac{1}{2}\pi + \theta_1} \sin(m \theta) \, d\theta
\]

(C.31)

\[
\frac{p_I}{\pi} \int_{\frac{1}{2}\pi - \theta_1}^{\frac{1}{2}\pi - \theta_2} \sin(m \theta) \, d\theta = -\frac{p_I}{m \pi} \left( \cos(m(\frac{1}{2}\pi - \theta_1)) - \cos(m(\frac{1}{2}\pi - \theta_2)) \right)
\]

(C.32)

By using the relation:

\[
\cos(\frac{1}{2}m \pi - m \alpha) = \cos(\frac{1}{2}m \pi) \cos(m \alpha) + \sin(\frac{1}{2}m \pi) \sin(m \alpha)
\]

(C.33)

the following is obtained:

\[
\cos(m(\frac{1}{2}\pi - \theta_1)) - \cos(m(\frac{1}{2}\pi - \theta_2)) = -\cos(\frac{1}{2}m \pi) (\cos(m \theta_2) - \cos(m \theta_1)) - \sin(\frac{1}{2}m \pi) (\sin(m \theta_2) - \sin(m \theta_1))
\]

(C.34)

The second integral in the right-hand side of (C.16) leads to:

\[
\frac{1}{\pi} \int_{\frac{1}{2}\pi - \theta_1}^{\frac{1}{2}\pi + \theta_2} p_I \sin(m \theta) \, d\theta = -\frac{p_I}{m \pi} \left( \cos(m(\frac{1}{2}\pi + \theta_2)) - \cos(m(\frac{1}{2}\pi + \theta_1)) \right)
\]

(C.35)
The sum of sine terms is similar to (C.34):

\[
\cos(m(\frac{1}{2} \pi + \theta_2)) - \cos(m(\frac{1}{2} \pi + \theta_1)) = \\
\cos(\frac{1}{m} \pi) \cos(m \theta_2) - \cos(m \theta_1)) - \sin(\frac{1}{m} \pi) \sin(m \theta_2) - \sin(m \theta_1))
\]

(C.36)

For the third integral:

\[
\frac{1}{\pi} \int_{\frac{3}{2} \pi - \theta_1}^{\frac{3}{2} \pi - \theta_2} p_1 \sin(m \theta) \, d\theta = -\frac{P_1}{m \pi} \left( \cos(m(\frac{3}{2} \pi - \theta_1)) - \cos(m(\frac{3}{2} \pi - \theta_2)) \right)
\]

(C.37)

the cosine terms can be rewritten using

\[
\cos(\frac{3}{2} m \pi - m \alpha) = \cos(m \pi + \frac{1}{2} m \pi - m \alpha) = (-1)^m \cos(\frac{1}{2} m \pi - m \alpha)
\]

(C.38)

Since

\[
\cos(\frac{1}{3} m \pi - m \alpha) = \cos(\frac{1}{3} m \pi \cos(m \alpha) + \sin(\frac{1}{3} m \pi \sin(m \alpha)
\]

(C.39)

the cosine terms equal:

\[
\cos(\frac{3}{2} m \pi - m \alpha) = \\
(-1)^m \cos(\frac{1}{3} m \pi \cos(m \alpha) + (-1)^m \sin(\frac{1}{3} m \pi \sin(m \alpha)
\]

(C.40)

such that:

\[
\cos(m(\frac{3}{2} \pi - \theta_1)) - \cos(m(\frac{3}{2} \pi - \theta_2)) = \\
-(-1)^m \cos(\frac{1}{3} m \pi) \cos(m \theta_2) - \cos(m \theta_1)) + \\
(-1)^m \sin(\frac{1}{3} m \pi) \sin(m \theta_2) - \sin(m \theta_1))
\]

(C.41)

The last integral:

\[
\frac{1}{\pi} \int_{\frac{3}{2} \pi + \theta_2}^{\frac{3}{2} \pi + \theta_1} p_1 \sin(m \theta) \, d\theta = -\frac{P_1}{m \pi} \left( \cos(m(\frac{3}{2} \pi + \theta_2)) - \cos(m(\frac{3}{2} \pi + \theta_1)) \right)
\]

(C.42)

can be treated similarly:

\[
\cos(\frac{3}{2} m \pi - m \alpha) = \\
(-1)^m \cos(\frac{1}{3} m \pi \cos(m \alpha) - (-1)^m \sin(\frac{1}{3} m \pi \sin(m \alpha)
\]

(C.43)
leading to:
\[
\cos(m(\frac{3}{2} \pi + \theta_2)) - \cos(m(\frac{3}{2} \pi + \theta_1)) = \\
(-1)^m \cos(\frac{1}{m} \pi) (\cos(m \theta_2) - \cos(m \theta_1)) + \\
(-1)^m \sin(\frac{1}{m} \pi) (\sin(m \theta_2) - \sin(m \theta_1)) \]  
(C.44)

The Fourier integral (C.16) expands to:
\[
p_m = \frac{2(1 - (-1)^m) \sin(\frac{1}{m} \pi) p_{f}}{m \pi} (\sin(m \theta_2) - \sin(m \theta_1)) \]  
(C.45)

which is the same as the sine transform for the x-coil, except for the term \(-\sin(\frac{1}{m} \pi)\). For \(m\) is odd, this term is alternatingly unity and minus unity.

Summarising, the Fourier coefficients are:

for the \(x\)-coil:
\[
p^a_m = \frac{2(1 - (-1)^m) I}{m \pi} \left[ \frac{B_z (\sin(m \theta_2) - \sin(m \theta_1))}{B_r (\cos(m \theta_2) - \cos(m \theta_1))} \right], \quad p^b_m = 0 \]  
(C.46)

and for the \(y\)-coil:
\[
p^a_m = 0, \quad p^b_m = \sin(\frac{1}{m} \pi) \frac{2(1 - (-1)^m) I}{m \pi} \left[ \frac{-B_z (\cos(m \theta_2) - \cos(m \theta_1))}{B_r (\sin(m \theta_2) - \sin(m \theta_1))} \right] \left[ \frac{B_r (\cos(m \theta_2) - \cos(m \theta_1))}{B_z (\sin(m \theta_2) - \sin(m \theta_1))} \right] \]  
(C.47)
C. Fourier transform of the Lorentz excitation
Acknowledgements

A number of people contributed to my work, and helped me in writing this thesis, for which I am very grateful. I would like to thank Ard Kuipers, who is a great guy and who was my “partner in research” during the MRI-project. Bert Verbeek, thanks for being a friend and a very good coach, and for sharing The Funk. Jan Verheij, thanks for all your support, advice and patience. You are all great people to work together with, and I would like to continue to do so in the future. Dick van Campen, thanks for your help and confidence. I couldn’t forget to thank Rens Kodde for his guidance and assistance in setting up and performing the measurements. Your help was crucial. Also, thanks to Li He, Lars Paar and Jeroen Starreveld, who did part of my work as part of their study. Also thanks to the people at Philips Medical Systems for their assistance, especially Frank Kistemakers, Jan Konijn and Hans Tuijthof, and also to the people at Epicon. To all other participants of the STW meetings not mentioned before, thanks for your interest and useful comments. Pascal Etman, thanks for evaluating Chapter 6.

In the non-scientific department, I’d like to mention my room mates through the years: Francois-Xavier “Fix” DeBiesme, Leonie van den Heuvel, Ron Peerlings, Maykel Verschueren and Alex Zdravkov. Thanks for all the fun and support.