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THE TURNING POINT PROBLEM: AN EXAMPLE OF THE OPTIMIZATION OF QUASI-STATIONARY SAILPLANE TRAJECTORIES BY MEANS OF CONVEX COMBINATIONS

by

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Summary

Quasi-stationary sailplane trajectory problems are relatively simple, practical nonlinear optimization problems which for educational reasons will be of interest to a much larger audience than the sailplane pilot community alone. One interesting problem is the turning point problem which concerns the determination of the optimal velocities with which the sailplane pilot should fly when he wants to optimally round a turning point on a return or on a triangle flight in the presence of wind. This problem lends itself very well for solution by the convex-combinations approach discussed by the author in a number of recent papers dealing with other sailplane trajectory problems. This convex-combinations approach is a purely geometric approach that is based on an interesting relation between the average velocity over a broken trajectory and a certain convex combination of the vectors representing the velocities over the different legs of the trajectory. In the paper first the basic ideas governing quasi-stationary sailplane trajectory optimization problems as well as the convex-combinations approach are reviewed. Then the solution is derived of the turning point problem by means of this convex-combinations approach. With the help of a special graph, the turning point graph, introduced in the paper, this solution may be easily implemented in flight. This turning point graph may be constructed by graphical means by any sailplane pilot without too much trouble.

Keywords
Aerospace trajectories, nonlinear optimization
1. Introduction

Quasi-stationary sailplane trajectory problems are relatively simple, practical nonlinear optimization problems of which there exist a number of different versions with varying degrees of difficulty. Several approaches may be used - and have been used in the past with success - towards their solution. A number of them have to be solved on every flight. Therefore, they are of much interest to sailplane pilots (cf. Reichmann 1975, Weinholtz 1975). For educational purpose, it is believed, they will also be of interest to a much larger audience than the sailplane pilot community alone. In fact, some of the problems may serve very well as classroom examples of practical nonlinear optimization problems.

One of the more complicated problems in sailplane flight trajectory optimization is the turning point problem. This is the problem of the determination of the optimal velocities for the rounding of the turning point(s) on a return or a triangle flight in the presence of wind. In practice, this problem has to be solved once or twice on every successful flight. Yet, as far as this author knows, the problem has not received much attention in the soaring literature. The reason for that might well have been the relatively complicated problem formulation combined with an analytical solution that cannot easily be interpreted. Also, not too many sailplane pilots might have been aware that there is really an optimization problem here.

An exception in this respect was the former Dutch national soaring champion Dick Teuling, who drew the author's attention to the problem and told him of his analytical solution. He also suggested to try to solve the problem by means of the convex-combinations approach discussed by the author in a number of recent papers (cf. de Jong 1977, 1978a, 1979, 1980a). This paper is the result of this suggestion.

The paper starts off, in Section 2, with a review of the classical MacCready problem, which is the basic problem in sailplane flight trajectory optimization. Its classical, graphical solution, together with its practical implementation is outlined in Section 3. Then, in Section 4, the convex-combinations approach is introduced and applied towards the solution of the MacCready problem. In Section 5, this solution procedure is formally repeated for the MacCready problem in the presence of wind. This discussion sets the stage for the solution of the turning point problem in case of a return flight with wind.
in the plane of flight. This is the topic of Section 6 and the main topic of the paper. The generalization of the problem to a turning point on a triangle flight with wind from an arbitrary direction is thereafter treated in Section 7. After some concluding remarks in Section 8, the paper ends with two appendices which present an analytical solution of the turning point problem and some numerical data to allow realistic computations.

2. The MacCready problem and related definitions

The basic problem in quasi-stationary sailplane trajectory optimization is the MacCready problem, named after the 1956 world champion Paul B. MacCready, who invented a device to implement the optimal solution. The MacCready problem is the problem of the determination (cf. figure 1) of the optimal cruise velocity in between thermals. The name thermals thereby stands for the columns of rising air which the sailplane pilots use to regain their lost height. In summer thermals are often found under cumulus clouds. To be optimized is the time to cover a given distance or, equivalently, the average or travel velocity of the sailplane.

The reason that the cruise velocity in between the thermals is in practice the only variable in the optimization problem is that for any sailplane in equilibrium flight there exists a fixed relationship between its cruise or horizontal velocity and its vertical velocity. This relationship (cf. figure 2), that is different for different sailplanes and that varies with the weight of the sailplane is called the velocity polar. In mathematical terms this may be denoted by the expression (cf. Appendix B)

\[ w = w_p(v) \]

where \( w \) stands for the vertical velocity of the sailplane and \( v \) for its horizontal velocity. If the air in which the sailplane flies is not at rest but instead moves with a constant vertical velocity \( u_a \), then the vertical velocity of the sailplane relative to the earth is given by

\[ w = w_p(v) + u_a \]

The graph (see figure 2) which represents this relationship is an example of an absolute velocity polar (velocities relative to the earth are called absolute velocities).
When it is assumed (cf. figure 1) that the horizontal distance between two subsequent thermals is equal to \( L \), that the absolute rate of climb is equal to \( z \) and that the atmosphere in between the two thermals has a constant vertical velocity \( u_a \), then in mathematical terms the MacCready problem may be formulated as

\[
\min_{v, \Delta h} \left\{ \frac{L}{v} + \frac{\Delta h}{z} \bigg| \Delta h + \frac{L}{v} \left[ w_p(v) + u_a \right] = 0, \quad \Delta h \geq 0, \quad v_{\text{min}} \leq v \leq v_{\text{max}} \right\}.
\]

If the MacCready problem is restricted to one single thermal (as in figure 1), then in most practical cases the inequalities for \( \Delta h \) and \( v \) are satisfied as strict inequalities. In the usual formulation these restrictions are therefore left out in which case the problem formulation reduces to

\[
\min_v \left\{ \frac{L}{v} - \frac{L}{v} \frac{w_p(v) + u_a}{z} \right\}.
\]

Differentiation of this expression with respect to \( v \) and setting the derivative equal to zero yields as necessary condition for an extremum the equation

\[
-w \frac{P(v)}{v^2} + w_p(v) + u_a = z.
\]

This equation is often referred to as the MacCready equation. It plays a central role in the theory and implementation of optimal quasi-stationary sailplane flight trajectories. For reasons to be explained below the right hand side of the MacCready equation is called the MacCready-ring-setting, or short, the MacCready value.

For the case that the vertical velocity of the atmosphere \( u_a \) in between the thermals is equal to zero the optimal cruise velocity is a well defined function of the MacCready value \( z \). This velocity is called the MacCready cruise velocity, which as function of \( z \), is denoted by \( \Phi_{\text{cr}}(z) \). It should be remarked that the solution of the MacCready problem in case of a constant vertical velocity of the atmosphere in between the two thermals in terms of the MacCready cruise velocity may be represented as \( \Phi_{\text{cr}}(z - u_a) \). (cf. figure 2)

The inequality constraints for the velocity \( v \) are almost always satisfied in practice as strict inequalities. The same is not the case with the inequality constraint for \( \Delta h \). This constraint turns out to play an important role in case of cloudstreet flying (cf. de Jong, 1977) as well as in case that a trajectory with more than one thermal is to be optimized. In the latter
situation the MacCready problem may be stated as

$$\min_{\Delta h_i, v_i} \left\{ \sum_{i=1}^{m} \left( \frac{L_i}{v_i} + \frac{\Delta h_i}{z_i} \right) \mid \sum_{i=1}^{m} \left( \frac{\Delta h_i}{v_i} + \frac{L_i}{z_i} \left[ w_p(v_i) + u_{a,i} \right] \right) = 0 \right\}$$

$$\Delta h_i \geq 0, \ i = 1, \ldots, m \}$$.

The optimality conditions for this case may be found through the use of the Lagrange-multiplier technique for solving nonlinear constrained optimization problems (cf. Dixon, 1972). If one defines a Lagrangean function by

$$\mathcal{L}(\Delta h, v, \lambda, u) = \sum_{i=1}^{m} \left( \frac{L_i}{v_i} + \frac{\Delta h_i}{z_i} \right) - \lambda \sum_{i=1}^{m} \left( \frac{\Delta h_i}{v_i} + \frac{L_i}{z_i} \left[ w_p(v_i) + u_{a,i} \right] \right) -$$

$$- \sum_{i=1}^{m} \mu_i \Delta h_i$$

where \( \lambda \) and \( \mu_1, \ldots, \mu_m \) represent Lagrange multipliers, then necessary conditions for optimality for each \( i, i = 1, \ldots, m \), are given by

$$\frac{\partial \mathcal{L}}{\partial v_i} = \frac{L_i}{v_i} \left[ -1 + \lambda \left( -\frac{1}{v_i} \frac{dp}{dv_i} (v_i) + w_p(v_i) + u_{a,i} \right) \right] = 0 \ ,$$

$$\frac{\partial \mathcal{L}}{\partial \Delta h_i} = \frac{1}{z_i} - \lambda - \mu_i = 0 \ ,$$

$$\mu_i \geq 0 \ \ \ \Delta h_i \geq 0 \ \ \ \mu_i \Delta h_i = 0 \ .$$

From this it follows that for \( i \)-th part of the trajectory the cruise velocity \( v_i \) should be chosen to satisfy

$$-v_i \frac{dp}{dv_i} (v_i) + w_p(v_i) + u_{a,i} = 1/\lambda \ ,$$

where \( 1/\lambda \) is a constant MacCready value which should satisfy

$$1/\lambda \geq z_i = \max_i z_i \ .$$
The optimal value of the height $\Delta h_i$ over which the sailplane should climb in the $i$-th thermal at the end of the $i$-th part of the trajectory depends on the value of the absolute rate of climb $z_i$ in that thermal. The optimality conditions stipulate that

$$
\Delta h_i = 0 \quad \text{if} \quad z_i \neq 1/\lambda,
$$

$$\Delta h_i \geq 0 \quad \text{if} \quad z_i = 1/\lambda .$$

In practical terms the latter conditions imply the almost obvious rule that for optimality height should only be gained in the strongest thermal.

Proper application of nonlinear optimization theory (cf. Dixon, 1972) to the problem of the optimization of a complete cross-country flight with a number of different thermals leads to the theoretical optimality rule that the cruise velocities in between the thermals should be determined from the MacCready equation with one and the same MacCready value for all subsequent elementary trajectories. This MacCready value is equal to the inverse of the Lagrange multiplier corresponding to the restriction requiring that the overall height gain or loss should be zero. For almost all practical situations this MacCready value equals the value of the absolute rate of climb in the strongest thermal on the trajectory (cf. de Jong, 1977).

The theoretical rule that gaining height should be limited to the strongest thermal somewhere on the trajectory is in practice, of course, impossible as, next to the constraints already mentioned, there are also limitations to the height bands over which the thermals may be used. The practical optimal strategy therefore reduces to the rule to realize the maximal height gain in the strongest thermal on each part of the trajectory (cf. de Jong, 1978b, Litt and Sander, 1978).

The MacCready equation with the proper MacCready value also represents a necessary condition for optimality when the vertical velocity of the atmosphere in between the thermals is not constant but instead piecewise constant. A regular limiting argument may be used to make it plausible that the MacCready relation even continues to hold as optimality condition in case the vertical velocity of the atmosphere is an arbitrary piecewise continuous
function of the distance coordinate $x$ with $0 \leq x \leq L$. The optimal velocity history of the sailplane $v(x)$ will then satisfy the appropriately adapted general MacCready equation

$$-v(x) \frac{d}{dv}(v(x)) + w_p(v(x)) + u_a(x) = z.$$ 

A rigorous derivation of this result is a simple exercise in the calculus of variations (cf. de Jong, 1977).

The general MacCready equation will continue to be the optimality condition of the sailplane as long as the trajectory may be considered to be quasi-stationary, i.e. as long as it may be assumed that at any moment the sailplane is in equilibrium flight. As soon as the dynamics of the sailplane come into the picture, a different optimization strategy may be optimal (cf. de Jong, 1980b).

3. The solution of the MacCready problem and its practical implementation

For the solution of the MacCready equation two almost identical graphical procedures are in common use (cf. figure 2). The first of these consists of the graphical construction of the point on the regular velocity polar where the line through the point $(0, z-u_a)$ on the vertical axis is tangent to the polar (figure 2: line I). The second consists of the similar construction of the point on the absolute velocity polar where the line through the point $(0,z)$ on the vertical axis is tangent to the absolute polar (figure 2: line II). Which of these procedures is to be preferred depends on the particular case at hand.

An interesting graphical result that becomes directly available when the second of the two graphical procedures is used is the fact that the resulting horizontal velocity from the initial point to the final point at the same height can be read off directly from the graph. Indeed, this velocity, which is given by

$$v_r = \frac{L}{t_{tot}} = \frac{z - (w_p(\vartheta_c r(z - u_a)) + u_a) \vartheta_c r(z - u_a)}{z - (w(\vartheta_c r(z - u_a)) + u_a) \vartheta_c r(z - u_a)},$$

is precisely represented (cf. figure 2) by the piece of the horizontal velocity axis that is cut off by the line through $(0,z)$ and tangent to the absolute velocity polar.
It may be noted that the optimal average or resulting horizontal velocity which is the solution of the MacCready problem for the case, that the vertical velocity $u_a$ of the atmosphere in between the thermals is zero, is a well-defined function of the absolute rate of climb in the next thermal. As such this velocity is given by the expression

$$\hat{v}_r(z) = \frac{z}{z - w_p(\hat{v}_{cr}(z))} \hat{v}_{cr}(z).$$

In line with the previous definitions, this velocity is called the MacCready travel velocity.

The second graphical solution procedure (cf. figure 2: line II) may also be considered to be the basis for the operation of two special devices, the MacCready ring and its modern successor, the Sollfahrtgeber or speeddirector (cf. Reichmann, 1975). Each of these presents the pilot a means to continuously check in flight whether the MacCready equation is satisfied. The main idea behind their operation is that a signal proportional to $-\frac{dv_p}{dv}(v)$ is added to the signal produced by a regular rate of climb indicator, which measures the absolute rate of climb $w_p(v) + u_a$. The pilot has only to continuously adjust his velocity so that the combined signal equals the MacCready value he desires. Proper knowledge of the correct MacCready value combined with the use of a MacCready ring or a Sollfahrtgeber or speeddirector thus allows any sailplane pilot today to fly at the optimal cruise velocity at any moment.

4. Convex combinations and the MacCready problem

An interesting different approach to the solution of quasi-stationary sailplane trajectory problems is the purely geometric approach based on the use of convex combinations of velocity vectors. The basis of this convex-combinations approach (cf. de Jong, 1980a) is the observation that is very simple to construct geometrically the resulting velocity vector over a broken trajectory (cf. figure 3) once the velocity vectors on the legs of the broken trajectory are given. This resulting velocity ($\hat{v}_{ABC}$ in figure 3), namely, may be constructed in a velocity diagram by the determination of the intersection of a line in the desired direction and the connection line of the endpoints of the velocity vectors on the different legs ($\hat{v}_{AB}$ an $\hat{v}_{BC}$ in figure 3). That this yields the correct result follows immediately from the
definition of the resulting velocity $\vec{v}_{ABC}$ from A via B to C

$$\vec{v}_{ABC} = \frac{\vec{AC}}{t_{AB} + t_{BC}}.$$

With the vector equality

$$\vec{AC} = \vec{AB} + \vec{BC} = \vec{v}_{AB}t_{AB} + \vec{v}_{BC}t_{BC},$$

this may be rewritten as

$$\vec{v}_{ABC} = \frac{t_{AB}}{t_{AB} + t_{BC}} \vec{v}_{AB} + \frac{t_{BC}}{t_{AB} + t_{BC}} \vec{v}_{BC}$$

or as

$$\vec{v}_{ABC} = \vec{v}_{AB} - \frac{t_{BC}}{t_{AB} + t_{BC}} \left( \vec{v}_{BC} - \vec{v}_{AB} \right).$$

The observation that the difference vector $(\vec{v}_{BC} - \vec{v}_{AB})$ is just a vector that is equal in direction and size to the connection line of the endpoints of the vectors $\vec{v}_{AB}$ and $\vec{v}_{BC}$ completes the argument. Linear combinations of vectors with coefficients that add up to 1.0 are known as **convex combinations**.

Another point of interest in the present context is the observation that the intersection point divides the connection line into pieces with a length ratio, that is inversely proportional to the ratio of the times spent on the legs. To wit (cf. figure 3)

$$\frac{|\vec{v}_{BC} - \vec{v}_{ABC}|}{|\vec{v}_{AB} - \vec{v}_{ABC}|} = \frac{t_{AB}}{t_{BC}}.$$

The basis for the application of the convex-combinations approach to the solution of the MacCready problem is the observation that the MacCready problem of the determination of the largest resulting velocity over a broken trajectory that consists of a cruise from the initial point A (cf. figure 4) to the point B, where the thermal is reached, followed by a climb in vertical direction from B to C. With the absolute velocity vector $\vec{v}_{BC}$ in the thermal given, the problem reduces to the problem of determining that vector $\vec{v}_{AB}$ (cf. figure 4) with its end point on the absolute velocity polar, which,
in combination with the absolute vertical velocity \( \vec{v}_{BC} \) yields the largest resulting horizontal velocity. This is the same problem as finding the line which connects the endpoint of the vector \( \vec{v}_{BC} \) and which cuts off the largest piece from the horizontal axis. The solution of that problem is that \( \vec{v}_{AB} \) with its endpoint on the absolute vertical velocity vector \( \vec{v}_{BC} \) is tangent to the absolute velocity polar. This result is of course the same vector as constructed as solution to the MacCready equation by the second graphical procedure discussed in the preceding section.

An interesting observation that may be made at this point is that the convex-combinations approach also immediately yields the solution of the little more general version of the MacCready problem, which arises when a height difference between the initial and final point in the problem formulation is specified (cf. figure 4, trajectory ABCD). The solution to this problem, which is known as the generalized MacCready problem, is immediate as soon as one realizes that the problem is no more than the problem of the determination of the largest resulting velocity in the direction of the line AD instead of in horizontal direction. The optimal strategy is to fly as before on the leg AB with the same absolute velocity \( \vec{v}_{AB} \) and then climb in the thermal with the same absolute vertical velocity \( \vec{v}_{BC} \) until point D is reached. The difference between the solution of the generalized MacCready problem and the solution of the classical MacCready problem evidently only lies in the difference in the times spent in climbing in the thermal. The cruise parts of both solutions, i.e. the legs AB, should both be flown with the same velocity \( \vec{v}_{AB} \) that is uniquely determined by the same MacCready value \( z \).

5. The MacCready problem in case of wind

The most striking difference between the solution of the MacCready problem by means of the convex combinations approach and the same solution following the usual analytical approach, lies in the explicit use that is made in the convex-combinations approach of the absolute velocity vectors as vectors. This use has the advantage that the same approach may be followed for the solution of the MacCready problem in case the air mass in which the sailplane flies has a motion of itself. The latter situation occurs as well in case of a constant up- or down wind, as discussed in the preceding section, as in case of some horizontal wind. Necessary for a solution by means of the convex-combinations
approach is only that use is made of the absolute velocity vectors, i.e. the vectors that represent the velocity of the sailplane relative to the earth. The MacCready problem in case of a horizontal wind (cf. figure 5) may then be interpreted as the problem of the determination of that absolute velocity vector $\vec{v}_{AB}$ with its endpoint on the absolute velocity polar which in combination with the absolute velocity vector $\vec{v}_{BC}$, representing the sailplane in climb, yields the largest resulting absolute velocity vector $\vec{v}_{ABC}$ in horizontal direction. As follows immediately from figure 5, this wanted absolute velocity vector $\vec{v}_{AB}$ will have its endpoint on the absolute velocity polar in that point where the tangent line is such that it goes through the endpoint of the absolute velocity vector $\vec{v}_{BC}$. In this way, again, the largest piece is cut off from the horizontal (absolute velocity) axis, which implies the largest resulting absolute velocity in horizontal direction.

A closer look at the geometrical solution of the MacCready problem with wind presented in figure 5 immediately reveals that the solution in terms of the absolute velocity vectors could also have been obtained by adding the absolute wind velocity vector to all velocity vectors involved in the solution of the corresponding MacCready problem without wind. Said differently, the solution of the MacCready problem with wind may be obtained by just considering the problem relative to the moving air mass in which the flight takes place. This equivalence which implies that the optimal strategy for the MacCready problem is not influenced by wind, has been well-known to sailplane pilots for years.

Another look at the solution of the MacCready problem with wind presented in figure 5 may lead to one more, different conclusion, namely that the solution of the MacCready problem with wind is also equivalent to the solution of a hypothetical MacCready problem without wind involving a sailplane having a hypothetical velocity polar equal to absolute velocity polar (as drawn in figure 5) and a hypothetical vertical velocity in the point where tangent to the absolute velocity polar in the point $\vec{v}_{AB}$ intersects the vertical axis of the absolute velocity axis system. This value, which would have been the MacCready value for the pilot of the hypothetical sailplane, is called the equivalent MacCready value. For the standard MacCready problem situation, where the vertical velocity of the atmosphere between the thermals is zero, an expression for the equivalent MacCready value immediately follows from geometry (cf. figure 5)
\[ z_{eq} = \frac{\Phi_r(z) + v_w}{\Phi_r(z)} z \]

where \( \Phi_r(z) \) is the MacCready travel velocity, defined in the preceding section, and \( v_w \) is the (tail)wind velocity on the trajectory (taken to be positive in case of tailwind and negative in case of headwind).

6. The turning point problem on a return flight with wind in the plane of the flight

From an optimization point of view a return flight with wind in the plane of flight may be considered as a series of subsequent MacCready problems with the wind direction changing at the moment of the rounding of the turning point. As long as cruising and climbing take place with the same wind direction, a regular MacCready problem with wind results. The optimal strategy in that situation is not influenced by the wind and the solution of the MacCready problem is the same as in the case of no wind: One should fly according to the MacCready ring or speed director fed with a MacCready value equal to the expected absolute rate of climb in the next thermal. The more interesting problem arises if one considers that MacCready problem which involves the actual rounding of the turning point. In that case the first part of the cruise is flown with a wind direction which is 180° different from the direction during the second part of the cruise and during the climb thereafter. It is this problem that is meant by the name turning point problem.

The general formulation of the turning point problem requires an arbitrary fixation of the location at the initial time of the thermal that should be flown to after the rounding of the turning point. That arbitrariness is due to the fact that after the rounding of the turning point an ordinary generalized MacCready problem (cf. Section 4) remains in which the distance to the next thermal is of no importance. Without losing any generality it may therefore be assumed that the initial location of the next thermal at the initial time is just above the turning point itself. The problem that results with that choice is sketched in figure 6a.

The general turning point problem thus formulated is also equivalent to another hypothetical problem for which it is assumed that the next thermal is just above the turning point at the moment that the sailplane arrives there.
This situation is sketched in figure 6b. It is clear from this figure that when the sailplane would regain all its lost height in this thermal then just a standard MacCready problem would remain which would bring the sailplane to the initial height again in the thermal assumed above for the general problem formulation.

With this second equivalent problem formulation in mind, it will be clear that the general turning point problem can also be interpreted as succession of two equivalent generalized MacCready problems (in the sense as discussed in Section 4): the first one, as sketched in figure 6c, with an absolute velocity polar that is a translation of the original velocity polar to the left (under the assumption of a head wind), the second one, also sketched in figure 6c, with an absolute velocity polar that is a reflection of the translation of the original velocity polar to the right (under the assumption of a tailwind). For a succession of two equivalent generalized MacCready problems like this, it may be easily shown, that, in analogy to the regular MacCready theory for trajectories with more than one thermal, a necessary condition for optimality is that the equivalent MacCready values that specify the optimal solutions for both subproblems should be identical. The second subproblem being a standard generalized MacCready problem with a tailwind \( v_{w,2} \) and a MacCready value \( z_2 \) being equal to the (expected) rate of climb in the next thermal, it follows immediately that this equivalent MacCready value is given by

\[
\zeta_{\text{eq}} = \frac{\vartheta_r(z_2) + v_{w,2}}{\vartheta_r(z_2)} z_2.
\]

Analogously, for the first equivalent generalized MacCready problem, the similar relation

\[
\zeta_{\text{eq}} = \frac{\vartheta_r(z_1) + v_{w,1}}{\vartheta_r(z_1)} z_1
\]

should hold. Equating these relations gives an equation from which the MacCready value for the first subproblem may determined

\[
\frac{\vartheta_r(z_1) + v_{w,1}}{\vartheta_r(z_1)} z_1 = \frac{\vartheta_r(z_2) + v_{w,2}}{\vartheta_r(z_2)} z_2.
\]
A special form of this equation, which suggests a simple graphical procedure for the solution of $z_1$ when $z_2$ is given, is

$$z_1 = \frac{\vartheta_r(z_1)}{\vartheta_r(z) + v_w,1} \cdot \frac{\vartheta_r(z_2) + v_w,2}{\vartheta_r(z_1)} z_2.$$  

The graphical procedure is sketched in figure 6c: Given $z_2$, one determines $z_{eq}$ by extending the tangent line through $z_2$ to the vertical axis of the absolute velocity system. Thereafter $z_1$ may be determined by extending the line through $z_{eq}$ and tangent to the right absolute velocity polar in figure 4c to the vertical axis of the relative coordinate system.

Although already very simple the graphical procedure does not lend itself very well for actual use in practice in flight. A better practical method is to make of use of a specially constructed graph, called the turning point graph (cf. figure 7) in which the equivalent MacCready values, $z_{eq}$ determined graphically or computed with the formulas given above, are presented as a function of the wind velocity $v_w$ and the MacCready value $z$. For practical reasons $z_{eq}$ is depicted horizontally.

The determination of the MacCready value $z_1$ with the turning point graph follows in two steps (cf. figure 7): First, the value of $z_{eq}$ is determined which corresponds to the value of $v_w,2$ and $z_2$ on the second part of the trajectory. Thereafter, a vertical line is drawn and the intersection determined with the line in the graph that corresponds to the wind velocity $v_w,1$ on the first part of the trajectory. The value of $z_1$ that corresponds to that point of intersection is read off by interpolation. (In figure 7 the determination of $z_1$ in a situation with $v_{w1} = -v_{w2} = -10$ m/s and $z_2 = 2.5$ m/s is illustrated.) It may be noted that a turning point graph may be constructed by purely graphical means by any sailplane pilot.

Application of the procedures sketched above for different combinations of expected climb rates $z$ and wind velocities $v_w$ for an LS-3-sailplane results in the MacCready values presented in Table 1. From this table the influence of the wind on the MacCready values to be used for the cruise flight towards the turning point may be observed. Especially in case of turning points against the wind the influence of the wind turns out to be important.
As a final remark it may be noted that the knowledge of the MacCready value for the cruise flight to the turning point is not only of importance for the optimal execution of this part of the flight. As observed by Dick Teuling (cf. Teuling, 1981), even more important for the pilot is to know which thermals he should skip on this part of the flight. This he should do with all thermals he meets before the turning point and which offer an absolute rate of climb less than the MacCready value $z_1$ for the cruise flight towards the turning point.

7. The turning point problem on a triangle flight with an arbitrary wind direction

In case of return flights with wind from a direction that does not coincide with the desired course direction as well as in case of triangle flights with wind from an arbitrary direction, use may be made of the same equation for the determination of the MacCready value for the cruise flight towards the turning point as derived in the preceding section

$$z_1 = \frac{\vartheta_r(z_1)}{\vartheta_r(z_1) + v_{w,1}} \cdot \frac{\vartheta_r(z_2) + v_{w,2}}{\vartheta_r(z_2)} \cdot z_2.$$

For the wind velocities $v_{w,1}$ and $v_{w,2}$ in this equation, one should now substitute, instead of the total wind velocity, the magnitudes of those components of the total wind velocity which are of influence for the wind correction on the particular legs of the trajectory. These components turn out to be the components of the wind velocity in the direction of the heading $\varphi_s$ (cf. figure 8), which is the course direction which the pilot should steer relative to the moving air mass in which the sailplane flies. This heading is such that the motion that results as the vector sum of the motion relative to the moving air mass and the motion of the moving air mass itself is precisely in the direction $\varphi_t$ of the desired track on the ground (cf. figure 8). That the components of the wind velocity in the direction of the heading are indeed the only ones to take into account follows from the important observation that the motion of the sailplane relative to the moving air mass is restricted to the vertical plane in the direction of the heading. The component of the motion of the air mass perpendicular to that plane only moves that plane sideways and is of no importance for the motion in the moving plane itself.
The procedure to determine the heading in the presence of wind is a standard navigation procedure (see figure 8). If $\varphi_w$ is the course direction of the wind (i.e. the direction into which the wind blows, which is the usual wind direction minus 180$^\circ$), $\varphi_t$ the course direction of the desired track over the ground, then the heading $\varphi_s$ is given by

$$\varphi_s = \varphi_t + \Delta\varphi_w$$

where the wind correction $\Delta\varphi_w$ satisfies the relation (cf. figure 8)

$$v_r \sin(-\Delta\varphi_w) = v_w \sin(\varphi_w - \varphi_t)$$

which leads to the expression

$$\Delta\varphi_w = -\arcsin\frac{v_w}{v_r} \sin(\varphi_w - \varphi_t)$$

In this expression $v_r$ represents the resulting or average velocity relative to the air in the plane of the heading. On the trajectory part towards the turning point, this resulting velocity $v_r$ is equal to the cruise velocity $v_{cr}(z_1)$, on the trajectory part after the rounding of the turning point the optimal resulting velocity $v_r$ is the resulting velocity that corresponds to the solution of the generalized MacCready problem. This velocity depends on the height at which the turning point is rounded and the distance at the moment of rounding from the turning point to the next thermal. Thus, the exact determination of the optimal resulting velocity of both parts of the turning point problem trajectory in an arbitrary situation will be very complicated. Therefore one will in general have to resort to approximations for the evaluation of the wind correction.

Given the wind correction $\Delta\varphi_w$ and therewith given the heading $\varphi_s$, the wind-component $v_{w,\text{eff}}$ in the vertical plane of the heading is easily found from geometry (cf. figure 8)

$$v_{w,\text{eff}} = v_w \cos(\varphi_w - \varphi_s)$$

$$= v_w \cos(\varphi_w - \varphi_t) \cos \Delta\varphi_w + v_w \sin(\varphi_w - \varphi_t) \sin \Delta\varphi_w$$

$$= v_w \cos(\varphi_w - \varphi_t) \cos \Delta\varphi_w - v_r \sin^2 \Delta\varphi_w$$

If, as is often the case, $\Delta\varphi_w$ is small, then the wind component in the plane of the heading, $v_{w,\text{eff}}$ may be approximated by
\[ v_{w,\text{eff}} \approx v_w \cos(\varphi_w - \varphi_c) . \]

With either of these expressions for the wind component in the plane of the heading the approximate evaluation of the MacCready value for the cruise flight to the turning point may be worked out very easily with the turning point graph presented in figure 7.

8. Concluding remarks

A simple, practicable solution has been derived for the turning point problem in soaring using the convex-combinations approach. Noteworthy is that this approach makes the rather complicated solution to this problem easy to understand. With the help of the turning point graph, introduced here, the solution itself may be easily implemented in flight. This turning point graph, from which the pilot may read off the data for the optimal strategy may be constructed by graphical means by any sailplane pilot without too much trouble.

It should be noted, that the theory presented here has its main value in the insight it presents in the relative importance of the parameters governing the optimization problem. As in actual practice the circumstances will not always be so neat as hypothesized for the model, the presented numerical data should be taken as guide numbers. The numerical MacCready values presented only exactly hold for the situation where the air through which the sailplane flies during the rounding maneuver has a vertical velocity equal to zero. If this assumption is not true, then the presented MacCready values \( z_1 \) for the cruise flight towards the turning point are no longer the optimal MacCready values. In fact, if the air goes up, the presented MacCready value \( z_1 \) will be larger than optimal, if the air goes down, the presented value for \( z_1 \) will be smaller than optimal. In theory these optimal values could be determined by solving the problem once more for every value of the vertical velocity of the atmosphere. For practical applications this does not seem to be worthwhile however.
9. References


Figure 1: The classical MacCready problem

Figure 2: The standard and the absolute (regular and extended) velocity polar and the graphical construction of the solution of the Mac Cready problem sketched in figure 1
Figure 3: The resulting velocity over a broken trajectory

Figure 4: The solution of the MacCready problem by means of the convex-combinations approach
Figure 5: The solution of the MacCready problem with wind

Figure 6a: The turning point problem on a return flight in the presence of wind
Figure 6b: An equivalent formulation of the turning point problem in figure 6a.

Figure 6c: The velocities involved in the turning point problem of figure 6a.
Figure 7: The turning point graph for the determination of the MacCready values for the cruise flight towards the turning point for an LS-3-sailplane (cf. App. B)

Figure 8: Angles involved in the determination of the windcorrection
Table 1: Optimal MacCready-ring settings (= MacCready values) for the cruise flight towards the turning point in the presence of wind \( v_{w,1} = v_w, v_{w,2} = -v_w \) for an LS-3-sailplane (cf. App. B)
Appendix A: The analytical solution of the turning point problem

Let the situation for the turning point maneuver be as discussed in Section 6 and sketched in figure 6a. Let L be the distance from the initial point A to the turning point B, let \( v_1 \) and \( w_p(v_1) \) be the horizontal and vertical velocity of the sailplane relative to the air on the trajectory to the turning point, let \( v_2 \) and \( w_p(v_2) \) the corresponding velocities on the trajectory from the turning point to the thermal after the rounding, let \( v_{w1} \) and \( v_{w2} \) with \( v_{w1} = -v_w \) and \( v_{w2} = v_w \) be the wind velocities on these trajectories and assume that the vertical velocity of the atmosphere during the rounding maneuver is zero. If \( z_2 \) is the absolute vertical climb rate in the thermal, which at the initial time is just above the turning point then the total time to fly from the initial point to the point at the same height in the moving thermal is equal to the sum of the time to fly to the turning point, the time to fly from the turning point to the thermal and the time to climb in the thermal.

In formula form this yields

\[
\begin{align*}
    t_{\text{tot}} &= \frac{L}{v_1 + v_{w1}} + \frac{L}{v_1 + v_{w1}} \cdot \frac{v_{w2}}{v_2} - \frac{L}{v_1 + v_{w1}} \cdot \frac{w_p(v_1)}{z_2} \\
    &= \frac{L}{v_1 + v_{w1}} \cdot \frac{v_{w2}}{v_2} - \frac{L}{v_1 + v_{w1}} \cdot \frac{w_p(v_2)}{z_2} \\
    &= \frac{L}{v_1 + v_{w1}} \left( \frac{z_2 - w_p(v_1)}{z_2} + \frac{v_{w2}}{v_2} \left( \frac{z_2 - w_p(v_2)}{z_2} \right) \right) .
\end{align*}
\]

(A.1)

Necessary for this total time being minimal is that the derivatives with respect to \( v_1 \) and \( v_2 \) equal zero

\[
\begin{align*}
    \frac{\partial t_{\text{tot}}}{\partial v_1} &= 0 \quad \frac{\partial t_{\text{tot}}}{\partial v_2} = 0
\end{align*}
\]

(A.2)

For \( v_2 \) this leads to the condition

\[
0 = -\frac{L}{v_1 + v_{w1}} \cdot \frac{v_{w2}}{z_2} \left( -v_2 \frac{d w_p(v_2)}{d v_2} - z_2 + w_p(v_2) - \frac{v_2^2}{v_2^2} \right)
\]

which yields the MacCready equation

\[
(A.3) \quad -v_2 \frac{d w_p}{d v} (v_2) + w_p(v_2) = z_2 .
\]
For \( v_1 \) one obtains

\[
0 = -\frac{L}{(v_1 + v_{\text{w},1})^2} \left( \frac{z_2 - w_p(v_1)}{z_2} + \frac{v_w,2}{v_2} \left( \frac{z_2 - w_p(v_2)}{z_2} \right) \right) + \frac{1}{v_1 + v_{\text{w},1}} \left( \frac{dP(v_1)}{dv} \right)
\]

which yields the equation

\[
(A.4) \quad -(v_1 + v_{\text{w},1}) \frac{dP}{dv} (v_1) + w_p(v_1) = z_2 + \frac{v_w,2}{v_2} (z_2 - w_p(v_2))
\]

which with the definition of the equivalent MacCready value

\[
(A.5) \quad z_{\text{eq}} = z_2 + \frac{v_w,2}{v_2} (z_2 - w_p(v_2))
\]

\[
= z_2 \left( 1 + \frac{v_w,2}{v_2} \left( \frac{z_2 - w_p(v_2)}{z_2} \right) \right)
\]

can be written as

\[
(A.6) \quad -(v_1 + v_{\text{w},1}) \frac{dP}{dv} (v_1) + w_p(v_1) = z_{\text{eq}}
\]

or as

\[
(A.7) \quad -v_1 \frac{dP}{dv} (v_1) + w_p(v_1) = z_{\text{eq}} - \frac{v_{\text{w},1}}{v_1} (-v_1 \frac{dP}{dv}) (v_1).
\]

The cruise velocity \( v_1 \) towards the turning point thus should satisfy a MacCready equation (cf. (A.3))

\[
(A.8) \quad -v_1 \frac{dP}{dv} (v_1) + w_p(v_1) = z_1
\]

with a MacCready value \( z_1 \) given by

\[
(A.9) \quad z_1 = z_{\text{eq}} - \frac{v_{\text{w},1}}{v_1} (-v_1 \frac{dP}{dv})(v_1)
\]

or, equivalently, satisfying

\[
(A.10) \quad z_1 = z_{\text{eq}} - \frac{v_{\text{w},1}}{v_1} (z_1 - w_p(v_1)).
\]
Combination of this equation with the definition (A.5) of the equivalent MacCready value \( z_{eq} \) yields as condition for \( z_1 \) a symmetric relation

\[
(A.11) \quad z_1 + \frac{v_{w,1}}{v_1} (z_1 - w_1(v_1)) = z_{eq} = z_2 + \frac{v_{w,2}}{v_2} (z_2 - w_2(v_2))
\]

which with the definition of the MacCready travel velocity (for \( i = 1,2 \))

\[
(A.12) \quad \phi_r(z_i) = \frac{z_i}{z_i - w_i(v_i)} v_i
\]

can be rewritten as

\[
(A.13) \quad \frac{\phi_r(z_1) + v_{w,1}}{\phi_r(z_1)} z_1 = \frac{\phi_r(z_2) + v_{w,2}}{\phi_r(z_2)} z_2.
\]

The graphical representation of this equation is (with \( v_{w,1} = -v_{w,2} = -v_w \)) precisely as sketched in figure 6c.
Appendix B: Numerical data for the velocity polar of an LS-3 sailplane

For the numerical evaluation of the turning point data (turning point graph (figure 7) and turning point table (table I)) use was made of a fourth order polynomial approximation of the velocity polar measurements of Stich (1978) of an LS-3 sailplane of the form

\[ v_p(v) = \sum_{i=-1}^{3} a_i v^i. \]

For \( v \) expressed in m/s the coefficients used were

\[
\begin{align*}
    a_{-1} &= 103.553713 \\
    a_0 &= -12.350022 \\
    a_1 &= 0.595341 \\
    a_2 &= 0.012336 \\
    a_3 &= 0.000107
\end{align*}
\]

The measurements of Stich related to an LS-3 sailplane with a weight of 373 kgf and a wing area of 10.5 m\(^2\). A less good but still acceptable approximation of the same data is given by the quadratic approximation

\[ v_p(v) = \sum_{i=0}^{2} c_i v^i \]

with

\[
\begin{align*}
    c_0 &= +1.748 \\
    c_1 &= +0.002 \\
    c_2 &= -0.094
\end{align*}
\]

Reference: