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ON THE APPLICATION OF SEQUENTIAL QUADRATIC PROGRAMMING TO STATE-CONSTRAINED OPTIMAL CONTROL PROBLEMS.

by

J.L. de Jong
K.C.P. Machielsen

Paper prepared for presentation at the Fifth IFAC Workshop on Control Applications of Nonlinear Programming and Optimization, June 11-14 1985, Capri, Italy.

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ON THE APPLICATION OF SEQUENTIAL QUADRATIC PROGRAMMING TO STATE-CONSTRAINED OPTIMAL CONTROL PROBLEMS.

by

J.L. de Jong
Department of Mathematics
Eindhoven University of Technology

and

K.C.P. Machielsen
CAM-Centre CFT
Philips Eindhoven
The Netherlands

Abstract

In this paper a numerical method for the solution of state-constrained optimal control problems is presented. The method is derived from an infinite dimensional analogue of sequential quadratic programming. The main purpose of the paper is to present some theoretical aspects of the method. An experimental numerical implementation of the method is also discussed.

An analogue to finite dimensional sequential quadratic programming is developed in Banach spaces.

Application to state-constrained control problems follows similar lines as in case of deriving the minimum principle from the abstract necessary conditions for optimality.

In the setting of optimal control problems, the analogue to the inversion of the Hessian matrix of the Lagrangian is the solution of a linear multi point boundary value problem.

Each iteration of the method involves mainly the solution a linear multi point boundary value problem.

Numerically, a collocation method based on collocation with piecewise polynomial functions is proposed for the solution of the linear multi point boundary value problems.

The resulting set of linear equations is solved by Gauss elimination.

The method is derived considering only constraints of the equality type. Inequality constraints are transformed into equality constraints by means of an active set strategy or by using slack variables.
CHAPTER 1

INTRODUCTION

An important class of practical optimal control problems is the class of state-constrained problems. The optimality conditions for the solutions of these problems are pretty well understood at this moment thanks to Bryson et al. (1963), Jacobson et al. (1971), Norris (1973), Maurer (1979a), Kreindler (1982), etc. However general efficient numerical methods are still lacking.

A powerful tool for the numerical solution of state-constrained optimal control problems is the code developed by Bulirsch (1983), which is based on the ideas of multiple shooting (cf. Bock (1983), Well (1983)). An important drawback of this method for practical use however is that a priori knowledge about the structure of the solution, in particular about the sequence in which various constraints are active, c.q. passive, is required.

An other alternative is the Sequential Gradient-Restoration Algorithm (SGRA) of Miele (1980), which treats state-constraints by means of a slack-variable technique. A drawback for the practical use of this method is the low rate of convergence which is mainly due to the fact that SGRA is a first order gradient method. Consideration of possible alternative methods not having these drawbacks seems therefore useful.

At present one of the major topics in the area of numerical nonlinear programming is the development of sequential quadratic programming methods. Inspired by the many promising results in this area (cf. Tapia (1974a,b;1977;1978), Han (1976), Powell (1978), Gill et al. (1981), Schittkowski (1981), Stoer (1982), Bertsekas (1982)) a translation of these methods towards the application to state constrained optimal control problems was initiated. In this paper the first preliminary results of the translation effort is reported.

In section 2 the sequential quadratic programming method is discussed in the context of an abstract vector space. In section 3 the application of the method to state constrained optimal control problems is considered. An experimental implementation is discussed in section 4, whereas two worked out examples are given in section 5. Some final remarks are given in section 6.
CHAPTER 2

SEQUENTIAL QUADRATIC PROGRAMMING IN BANACH SPACES.

Consider the solution of the following abstract problem:

Problem (P):

Given the Banach spaces \( X, Y \) and \( Z \), the twice continuously
Frechet differentiable mappings \( f : X \rightarrow \mathbb{R}, g : X \rightarrow Y \) and
\( h : X \rightarrow Z \) and a closed convex cone \( B \) having nonempty
interior with vertex at zero, then find an \( x \in g^{-1}(B) \cap \mathcal{N}(h) \),
if such exist, such that

\[
\hat{f}(x) \leq f(x) \quad x \in g^{-1}(B) \cap \mathcal{N}(h) \tag{2.1}
\]

Sequential quadratic programming methods are methods based on the
observation that 'near' the solution \( \hat{x} \), the original problem may be
replaced by a suitable quadratic programming problem. They consist of
the sequential solution of quadratic subproblems which yield
directions of search. Along these directions better approximations to
the solution are generated.
Alternatively, they may be considered as methods which calculate
directions of search using Newton's method applied to the necessary
conditions for optimality.
In order to formally derive the subproblems, we require a brief review
of the optimality conditions for problem (P). These are given in the
following lemma, which is taken from Maurer et al. (1979b). A central
role in this lemma is played by the Lagrangian:

\[
\mathcal{L}(x, y^*, z^*) := f(x) - y^* \cdot g(x) - z^* \cdot h(x) \tag{2.2}
\]

(Variables with an asterisk are elements of the dual spaces cf.
Luenberger (1969)).

- 5 -
Lemma 2.1: If \( \hat{x} \in X \) is a solution to problem \( (P) \),
\[
\mathcal{Q}(h'(\hat{x})) = 2
\]  
(2.3)
and
\[
\exists d \in X : [h'(\hat{x})(d) = 0 \land g(\hat{x}) + g'(\hat{x})(d) \in \text{int}(B)]
\]  
(2.4)
then there exist multipliers \( \hat{y} \in Y \) and \( \hat{z} \in Z \) such that,
\[
L'(\hat{x}, \hat{y}, \hat{z})(d) = 0 \quad \forall d \in X
\]  
(2.5)
\[
\hat{y} \cdot g(\hat{x}) = 0
\]  
(2.6)
\[
\hat{y} \in B \quad \text{with} \quad B := \{ y \in Y : \langle y, y \rangle \geq 0, \forall y \in B \}
\]  
(2.7)
\[
L''(\hat{x}, \hat{y}, \hat{z})(d)(d) \geq 0
\]  
(2.8)

Conversely, if there exist multipliers \( \hat{y}^* \) and \( \hat{z}^* \) satisfying (2.5), (2.6) and (2.7) and numbers \( \delta > 0 \) and \( \beta > 0 \) such that
\[
L''(\hat{x}, \hat{y}^*, \hat{z}^*)(d)(d) \geq \delta \| d \|^2
\]  
(2.9)
\[
\forall d \in X : [\hat{y}^* \cdot (g(\hat{x}) + g'(\hat{x})(d)) \leq \beta \| d \|^2 \land h'(\hat{x})(d) = 0]
\]  
(2.10)

then \( \hat{x} \) is a local solution of problem \( (P) \).

We note that (2.4) is a Slater-type constraint qualification (cf. Maurer et al. (1979b)). In the following part of this paper, we will assume that this condition holds.

Of importance for the sequel is to note that the lemma states that the Lagrangian \( L(x, \hat{y}^*, \hat{z}^*) \) has a local minimum at \( x = \hat{x} \) in the subspace spanned by the linearized constraints. This fact gives the motivation for the idea to calculate a direction of search for the improvement of the current estimate of the solution by solving the linearly constrained subproblem:

\[
\begin{align*}
\text{minimize} & \quad \tilde{L}(x + \Delta x, y, z) \\
\Delta x & \quad i \ i \ i \ i \\
\text{subject to:} & \quad g(x) + g'(x)(\Delta x) \in B \\
& \quad h(x) + h'(x)(\Delta x) = 0
\end{align*}
\]  
(2.11)
In general this is a problem with a nonlinear objective function, which may be approximated by a second order expansion at $x = x^*$:

$$L(x + \Delta x, y, z) \approx L(x, y, z) + f'(x)(\Delta x) - y \cdot g'(x)(\Delta x)$$

$$- z \cdot h'(x)(\Delta x) + L''(x, y, z)(\Delta x)(\Delta x)/2$$

(2.13)

Based on this expression we may construct the following linearly constrained quadratic subproblem for the calculation of a direction of search $\Delta x$.

Problem (QPEI):

$$\text{minimize} \quad f'(x)(\Delta x) + L''(x, y, z)(\Delta x)(\Delta x)/2$$

$$\Delta x$$

(2.14)

subject to:

$$g(x) + g'(x)(\Delta x) \in B$$

(2.15)

$$h(x) + h'(x)(\Delta x) = 0$$

(2.16)

In this problem formulation, the term $(y', g'(x), + z, h'(x,))(\Delta x)$ is omitted. The reason for this is that we want to obtain a quadratic subproblem which has in the optimal point $x^*$, the same Lagrange multiplier $\bar{y}$ and $\bar{z}$ as the original problem. The Lagrange multipliers of the subproblem at $x = x^*$ would otherwise have been $\bar{y}' - y'$ and $\bar{z}' - z'$, which would have meant that the Lagrange multipliers of the subproblem would have converged to zero as $x, x^*$, $x^*$, $x$ would otherwise have been $\bar{y}' - y'$ and $\bar{z}' - z'$, which would have meant that the Lagrange multipliers of the subproblem would have converged to zero as $x, x^*$, $x$ would otherwise have been $\bar{y}' - y'$ and $\bar{z}' - z'$. Especially in the case of inequality constraints, this is an undesirable phenomenon. With this modification the Lagrange multipliers obtained via the solution of the problem (QPEI) may be used as new estimates of the Lagrange multipliers $\bar{y}$ and $\bar{z}$ of the original problem.

One of the difficulties of the solution, along these lines, of the original problem is the way in which the inequality constraints $g(x) \in B$ are handled. One way is to solve the problem (QPEI) as a quadratic programming problem with linear equality and inequality constraints. Another way is to first transform problem (P) into an equality constrained subproblem and then solve a quadratic subproblem with only equality constraints. In this paper we will follow the latter road and will almost exclusively deal with equality constrained subproblems.

The transformation of the constraint $g(x) \in B$ into an equality constraint can be done either by an active set strategy (this is called a preassigned active set strategy) or by means of slack variables. The details of the transformation are strongly dependent on the actual spaces $X$ and $Y$ involved. The transformation used by us will be considered in the next section.
Based on the sequential solution of equality constrained subproblems we are led to the following two, slightly different algorithms. In these algorithms use is made of a bounded, linear operator $G$, which may be interpreted as a mapping used to imitate an innerproduct in the Banach space $X$, which takes the form:

$$(x|y) = <Gx,y>$$

In Hilbert spaces the mapping $G$ becomes the identity operator.

Algorithm:

Given the invertible (normalisation) map $G \in B[X,X]$:

1. Given, $z = 0$, $i=0$
   $0$

2. Calculate a first order Lagrange multiplier estimate $\tilde{z}$ from
   $\begin{align*}
   Gd - h'(x).\tilde{z} &= -f'(x) \\
   h'(x).d &= -h(x)
   \end{align*}$
   (2.17)

3. Calculate an approximation to the Hessian of the Lagrangian at $x$:
   $\begin{align*}
   W(x,z) := f''(x) - \tilde{z}.h''(x) \\
   \end{align*}$
   (2.19)

4. Calculate a second order Lagrange multiplier estimate $\bar{z}$ and the Newton direction $d$ from
   $\begin{align*}
   W(x,\bar{z}).d - h'(x).\bar{z} &= -f'(x) \\
   h'(x).d &= -h(x)
   \end{align*}$
   (2.20)

5. If $\|d\| \leq \text{eps}$ then ready
   else goto (v)

(v) Calculate a step length $\alpha$ using bisection on the interval $(0,1]$, starting with $\alpha=1$, using the merit function:

$\begin{align*}
M(x,z) := f(x) - z.h(x) + \int_{x}^{\bar{z}} Q(L'(x,z)) / 2 \\
&+ \int_{2}^{\bar{z}} P(h(x))/2
\end{align*}$

(2.22)
Here $Q(.)$ denotes a mapping $X \rightarrow \mathbb{R}$ satisfying
\[ Q(x) = 0 \iff x = 0 \]
And $P(.)$ is a mapping $Z \rightarrow \mathbb{R}$ satisfying
\[ P(z) = 0 \iff z = 0 \]
\[
\int_1 \text{ and } \int_2 \text{ are penalty constants.}
\]

(vi) Set:
\[
x_{i+1} := x_i + \alpha_i d_i
\]
\[
z_{i+1} := z_i + \alpha_i (z_i - z_i)
\]

(vii) Either
   - (a) $z_{i+1} := z_i$ and goto (ii)
   - (b) goto (i)

The difference between the two algorithms is in step (vii), when (vii-a) is used, step (i) is only executed as part of the initialization, when (vii-b) is used step (i) is executed once in every iteration. In the sequel, we will refer to the different algorithms as algorithm a and algorithm b.

We note that steps (i) and (iii) are equivalent to the solution of the following quadratic subproblem:

Problem (QPE):
\[
\begin{align*}
\text{minimize} & \quad f'(x_i)(dx_i) + \bar{W}(dx_i)(dx_i)/2 \\
\text{subject to} & \quad h(x_i) + h'(x_i)(dx_i) = 0
\end{align*}
\]
where:
\[
\bar{W} = G \quad \text{in step (i)}
\]
\[
\bar{W} = L''(x,y,z) \quad \text{in step (iii)}
\]

A more detailed analysis of the algorithms presented will be a topic of our future research. We will end this section with the following remarks:

* Because algorithms a and b are Newton-like methods, each iteration involves calculations using the second derivatives of the problem functions. For many practical
problems, this may be a serious handicap. Therefore we intend also to consider the application of quasi-Newton and discrete Newton techniques to algorithms a and b in the future.

In algorithm b each iteration involves the solution of two sets of linear equations, whereas in algorithm a each iteration involves only the solution of one such set. For algorithm b a slightly stronger convergence result can be derived (cf. Tapia (1977)). However this difference may be regarded as insignificant for many practical problems. Hence algorithm a may be considered superior to algorithm b.

We note however to algorithm b that it provides a suitable initial estimate for the Lagrange multiplier $\overline{z}$ for algorithm a and we also expect that algorithm b will behave 'better' away from the solution. We intend to verify this in the future using numerical results.

The algorithms discussed are in fact infinite dimensional analogues to existing algorithms for nonlinear programming in $\mathbb{R}^n$.

As to the connection between algorithms a and b we refer to Bertsekas (1982), who shows that algorithm a is in fact a Newton-like method with variables in the space $\mathbb{R}^n$, whereas b is a Newton-like method in the space $\mathbb{R}^n$.

We expect that the use of inequality constrained subproblems (i.e. problem (QPEI)) may be more favourable than the use of equality constrained subproblems as presented in this section. This will be a topic of our future research.
CHAPTER 3

APPLICATION TO OPTIMAL CONTROL PROBLEMS.

We consider the application of the algorithm, given in the previous section to the following, state-constrained optimal control problem:

Problem (OCP):
Determine a control function \( \hat{u} \in L^\infty_{m}[0,T] \) and a state trajectory \( \hat{x} \in W_{m}^{\infty}[0,T] \) which minimize the functional:

\[
J = \int_{0}^{T} \left[ h(x(0)) + \int_{0}^{T} f(x,u) \, dt + g(x(T)) \right] \, \text{subject to the constraints:}
\]

\[
x = f(x,u) \quad \text{a.e. } 0 \leq t \leq T \quad (3.2)
\]
\[
D(x(0)) = 0 \quad (3.3)
\]
\[
E(x(T)) = 0 \quad (3.4)
\]
\[
S(x,u) \leq 0 \quad 0 \leq t \leq T \quad (3.5)
\]
\[
S(x) \leq 0 \quad 0 \leq t \leq T \quad (3.6)
\]

where:

\[
T \text{ is the fixed final time; } h : C(R \rightarrow R); f : C(R \times R \rightarrow R);
\]
\[
0 \quad 0
\]
\[
g : C(R \rightarrow R); f : C(R \times R \rightarrow R); D : C(R \rightarrow R); E : C(R \rightarrow R);
\]
\[
0 \quad 0
\]
\[
S : C(R \times R \rightarrow R); S : C(R \rightarrow R);
\]
\[
1 \quad 2
\]
\[
W_{m}^{\infty}[0,T] := \{ x : [0,T] \rightarrow R \, | \, \text{absolutely continuous,} \, x \in L_{m}^{\infty}[0,T] \}
\]

\( S_1 \) and \( S_2 \) represent the mixed and pure state constraints, respectively, with the properties,
The algorithm to be proposed for this problem can be derived in a rigorous fashion in a way closely related to the derivation of the optimality conditions for problem (OCP) from lemma 2.1 (cf. Jacobson et al. (1971); Morris (1973); Maurer (1981)). We will limit ourselves here to a formal treatment for the sake of brevity. We start by summarizing the necessary conditions for optimality for problem (OCP) in the following two lemmas, taken from Maurer (1979a):

Lemma 3.1: Let \((\hat{x}, \hat{u})\) be a solution of problem (OCP). Then there exist a number \(\lambda_0 \geq 0\), vectors \(\xi \in \mathbb{R}, \mu \in \mathbb{R}\); a vectorfunction \(\lambda : [0, T] \to \mathbb{R}\), a vectorfunction \(\eta : [0, T] \to \mathbb{R}\), and a vectorfunction \(\hat{s} : [0, T] \to \mathbb{R}\) of bounded variation, not all zero with the following properties:

\[ \lambda(0) = -\lambda_0 h(0) - \xi \cdot A x(0) - \mu \cdot E(0) \]  
\[ \lambda(T) = -\lambda_0 g(T) - \mu \cdot E(T) \]  
\[ \lambda(t) - \lambda(t^-) = -\int_0^t (H(t) + \hat{s} \cdot S(t)).dt - \int_0^T S(t) . \hat{N}(t) . dt \]  
\[ -\frac{T}{J}\]  

where the Hamiltonian \(H\) is defined as:

\[ H(x, u, \lambda, \mu) := \lambda \cdot f(x, u) + \mu \cdot f(x, u) \]  
and

\[ \hat{\eta}(t) \geq 0 \quad i=1, \ldots, k, \quad 0 \leq t \leq T \]  
\[ \hat{\eta}(t) \cdot S(t) = 0 \quad i=1, \ldots, k, \quad 0 \leq t \leq T \]
\( \dot{\lambda}_i \) is continuous from the right except possibly at \( t=0 \). (3.14)

\( \dot{\lambda}_i \) is nondecreasing on \([0,T]\), \( i=1,\ldots,k \). (3.15)

\( \dot{\lambda}_i \) is constant on intervals where \( S_n[t]<0 \) \( i=1,\ldots,k \). (3.16)

In lemma 3.1 we used the notation \([t]\) to show the implicit dependence of the relevant functions of \( t \), i.e. as \((\dot{x}(t),\dot{u}(t))\) or \((\dot{x}(t),\dot{u}(t),\dot{\lambda}(t))\).

A sufficient condition for \( \lambda_* > 0 \) and hence \( \lambda_* = 1 \) may be obtained by expressing the constraint qualification (2.3)-(2.4) in terms of problem (OCP) (cf. Maurer (1981)). As in the previous section we assume that there is at least one solution to problem (OCP) and that \( \lambda_* > 0 \) can be taken for all solutions.

The adjoint equation (3.9) is numerically not very tractable. It is possible, however, under additional assumptions, to transform this set of integral equations into a set of differential equations. Thereto we introduce the following terminology:

A subinterval \([t,t]\) \( C[0,T] \), \( t < t \), is called an interior arc of the constraint \( S \) \( (j=1,2; i=1,2 \ldots k \) ), if

\[ S_n[t]<0 \text{ for all } t \in [t,t] \]

Similarly, such a subinterval is called a boundary interval of the constraint \( S \) \( (j=1,2; i=1,2 \ldots k \) ), if

\[ S_n[t]=0 \text{ for all } t \in [t,t] \]

Entry- resp. exit-points (also referred to as junction points) and contact points are defined in an obvious way.

The order of the state constraint \( S_{2i} \) is defined as the integer \( p \) corresponding to the first time derivative of \( S \) which contains the control explicitly, i.e.

\[ \sum \frac{d}{dt} S_n(x(t)) = \begin{cases} 0 & \lambda = 0,1,\ldots,p-1 \chi_i \\ \neq 0 & \lambda = p \chi_i \end{cases} \]  

To this definition we note that we implicitly assumed that \( S_{2i} \) is sufficiently smooth and that the order of \( S \) is constant on \([0,T]\).

Using time derivatives of the Hamiltonian \( H \), it is possible to show that the function is differentiable on boundary arcs (cf. Jacobson et al. 1971; lemma 3.2 is taken from Maurer (1979a)). The lemma below states conditions under which the adjoint equation (3.9) may be transformed into a set of differential equations.
Lemma 3.2: Let \((\hat{x}, \hat{u})\) be a solution of problem (OCP) and let \(f, S\) be \(\mathbb{B}^{p+r}\) functions with \(\bar{p} = \max_{i \leq n} p + r \geq 0\).

Let \([t_i, t_{i+1}]\) be a boundary interval with \(t_i\) an entry and \(t_{i+1}\) an exit point, and let
\[
\begin{align*}
\text{rank}(S) &= k \quad \text{(3.20)} \\
u &= u^{\mathbb{B}^{p+r}}
\end{align*}
\]
Assume in addition that \(\hat{u}(t)\) is a \(C\) function for \(t \in (t_i, t_{i+1}).\)

Then the functions \(\hat{\lambda}\) and \(\hat{\xi}\) in the adjoint equation (3.9) are \(r+1\) \(C\) functions on \((t_i, t_{i+1}).\) In particular the adjoint equation
\[
\begin{align*}
\hat{\lambda} &= -\hat{\lambda}_T f [t] - \hat{\xi}_T S [t] - \hat{\eta}_T S [t] - f [t] \\
\text{X} &= \text{X}_2 \text{X}_1 \\
\text{Y} &= \text{Y}_2 \text{Y}_1 \text{Y}_3 \\
\text{Z} &= \text{Z}_2 \text{Z}_1 \text{Z}_4
\end{align*}
\]
holds on \((t_i, t_{i+1}).\)

where \(\hat{\gamma} := \hat{\xi} \geq 0\) is a \(C\) function, which satisfies an equation of the form
\[
\hat{\gamma} = \hat{\lambda}_T \hat{\xi}_T (\hat{x}, \hat{u}) \quad \text{for all } t \in (t_i, t_{i+1}) \quad \text{(3.22)}
\]
At junction and contact points the 'jump'-condition:
\[
\hat{\lambda}(t^+) = \hat{\lambda}(t^-) - \hat{\gamma} . S [t] \\
\text{X} &= \text{X}_2 \text{X}_1 \\
\text{Y} &= \text{Y}_2 \text{Y}_1 \text{Y}_3 \\
\text{Z} &= \text{Z}_2 \text{Z}_1 \text{Z}_4
\]
holds with \(\hat{\gamma} > 0.\)

\(*\): With \((S, u)\) we denote the \(k \times n\) matrix with elements \((S, u)\) where \(S\) has elements
\[
\begin{pmatrix}
S \\
u \\
1u \\
2u
\end{pmatrix}
\]
We note that equation (3.22) gives an explicit expression for the multiplier on boundary arcs (cf. appendix A).

In the sequel we will assume that the optimal trajectory has no contact points. The solution of problems with such solutions will be investigated in the future.

It may be noted that the necessary conditions for optimality presented in lemmas 3.1 and 3.2, are the conditions originally stated by Jacobson et al. (1971). It has been shown that they are equivalent to the conditions of Bryson et al. (1963) augmented with a number of conditions on the signs of various multipliers (Kreindler 1982). The reason for using the multipliers of Jacobson et al. (1971) is that
they can be used in a rather straightforward manner for the active set strategy to be discussed. It is of course also possible to calculate these multipliers from the multipliers of Bryson et al. (1963).

We next develop an analogue to problem (QPE). Essential in this approach is the way in which the inequality constraints $S$ are treated. We will consider two ways: using an active set strategy and using slack variables.

When an active set strategy is used, an estimate of the set of junction points is made. The constraint $S$ is supposed to be active at time $t$ if:

$$S[t] > -(\eta_j(t) + 2\frac{\eta_j(t)}{\rho_j})$$

where $\rho_j$ and $\rho_k$ are the penalty constants used for the merit function (cf. section 2) and where the notation $[t]$ is used to replace $(x'(t), u'(t))$, i.e. the current estimate of the solution of problem (OCP).

Using equation (3.24) an estimate of the set of junction points is made, taking care that the function $S$ is nondecreasing on boundary arcs.

Let:

- $t_{i=0, \ldots, i}$ be entry points
- $t_{i=0, \ldots, i}$ be exit points

With this estimate we obtain the following nonlinear equality constraints:

$$S[t] = 0 \text{ for all } (j,k) \in I(t) \text{ and } t_{i=0, \ldots, i} \leq t \leq t_{i=0, \ldots, i+1}$$

where $I(t)$ is the set of indices $(j,k)$ corresponding to the constraints which are active at time $t$.

Then the linearized constraints for the calculation of the direction of search $(Ax, Au)$ become:

$$S[t] + S[t].Ax(t) + S[t].Au(t) = 0 \text{ for all } (j,k) \in I(t) \text{ and } t_{i=0, \ldots, i} \leq t \leq t_{i=0, \ldots, i+1}$$

We next give the analogue to problem (QPE) for the case that an active set strategy is used.
Problem (QPE/OCP) : minimize $h \{0\} \Delta x(0) + \int_{0}^{T} \left( f \{t\} \Delta x + f \{t\} \Delta u \right) dt + g \{T\} \Delta x(T) + \Delta x, \Delta u$

subject to : 

\[ \Delta x = f \{t\} \Delta x + f \{t\} \Delta u + f \{t\} - x \quad 0 \leq t \leq T \quad \text{(3.28)} \]

\[ D[Q] \Delta x(0) = 0 \quad \text{(3.29)} \]

\[ E[T] \Delta x(T) = 0 \quad \text{(3.30)} \]

\[ S \{t\} + S \{t\} \Delta x(t) + S \{t\} \Delta u(t) = 0 \quad \text{for all } (j,k) \in I(t) \text{ and } t \leq t \leq T \quad i=0, \ldots, T \quad \text{(3.31)} \]

where $H$ is the augmented Hamiltonian, i.e.

\[ \bar{H} := f + \lambda f + \bar{\lambda} S \quad \text{(3.32)} \]

(the variables with a bar denote the corresponding variables from the previous step c.q. iteration).

Application of the lemmas 3.1 and 3.2 yields necessary conditions for optimality for the solution of problem (QPE/OCP). These conditions take the form of a linear multipoint boundary value problem and are counterpart to (2.20). Combination with the analogue of (2.21) yields the following linear multipoint boundary value problem, which is to be
solved at step (iii) of the algorithm:

\[ \Delta x = f(t).Ax + f(t).Au + f(t) - x \quad \text{a.e. } 0 \leq t \leq T \quad (3.33) \]

\[ \lambda = -\lambda.f(t) - \gamma'.s(t) - f(t) - \overline{H}(t).Ax \quad \text{a.e. } 0 \leq t \leq T \quad (3.34) \]

\[ D[0] + D[0].Ax(0) = 0 \quad (3.35) \]

\[ - (h[0] + \sigma.D[0]).Ax(0) = h[0] + \sigma.D[0] \quad (3.36) \]

\[ E[T] + E[T].Ax(T) = 0 \quad (3.37) \]

\[ (g[T] + \rho.E[T]).Ax(T) - \lambda(T) = -g[T] - \mu.E[T] \quad (3.38) \]

\[ f(t) + f(t).\lambda + S(t).\gamma + \overline{H}(t).Ax + \overline{H}(t).Au = 0 \quad 0 \leq t \leq T \quad (3.39) \]

\[ S(t) + S(t).Ax(t) + S(t).Au(t) = 0 \quad (3.40) \]

\[ S(t) + S(t).Ax(t) = 0 \quad (3.41) \]

\[ \lambda(t+) = \lambda(t-) - \gamma'.s(t) - \overline{S}(t).Ax(t) \quad i = 0, \ldots, 2i + 1 \quad (3.42) \]

\[ \gamma = 0 \quad \text{if } (2,j) \notin I(t) \quad (3.43) \]

\[ \gamma_{2i} = \phi[t].\lambda + \lambda.(\phi[t].Ax + \phi[t].Au) \quad t \leq t \leq T \quad (3.44) \]
We note that we used $S$ and $\gamma$ to replace

$$
S = \begin{bmatrix}
S_1 \\
1 \\
S_2
\end{bmatrix}, \quad \gamma = \begin{bmatrix}
\gamma_1 \\
\gamma_2
\end{bmatrix}
$$

(3.45)

The translation of (2.17) will follow straightforward following the previous approach, once the mapping $G$ is defined. As this definition is (within limits) arbitrary, we may take:

$$
\langle G.d , d \rangle := \int_1^T (\Delta x(t).\Delta x(t) + \Delta u(t).\Delta u(t)).dt
$$

(3.46)

This yields the following modification of (3.34), (3.36), (3.38), (3.39) and (3.41):

$$
\dot{\lambda} = - \lambda \cdot f [t] - \gamma \cdot S [t] - \dot{f} [t] - \Delta x \quad 0 \leq t \leq T
$$

(3.47)

$$
\lambda(0) = - h [0] - \Phi \cdot \delta [0]
$$

(3.48)

$$
\lambda(T) = g [T] + \mu \cdot E [T]
$$

(3.49)

$$
\dot{f} [t] + f [t] \cdot \lambda + S [t] \cdot \gamma + \Delta u = 0 \quad 0 \leq t \leq T
$$

(3.50)

$$
\lambda(t+) = \lambda(t-) - \gamma \cdot S [t] \quad i=0, \ldots, 2k+1
$$

(3.51)

When slack-variables are used, the constraints $S$ are transformed into equality constraints in the following way:

$$
S [t] + \text{diag}(y(t)).y(t)/2 = 0 \quad 0 \leq t \leq T
$$

(3.52)

$$
S [t] + \text{diag}(y(t)).y(t)/2 = 0 \quad 0 \leq t \leq T
$$

(3.53)

where: $y : [0,T] \rightarrow R$ and $\gamma : [0,T] \rightarrow R$

In this approach junction points are not treated explicitly. The modification of problem (QPE/OCP) for this case is straightforward. The boundary value problem to be solved in this case is given in appendix B.
To complete the translation of the algorithm, we consider the translation of the merit function in terms of the problem (OCP). The first step is to define the mappings \( P \) and \( Q \). We have chosen them similar to the criteria \( P \) and \( Q \) involved in the Sequential Gradient Restoration Algorithm of Miele (cf. Miele (1980)). \( P \) 'measures' the constraint violation, \( Q \) 'measures' the error in the optimality conditions.

For the sake of brevity we define:

\[
N(v) = v \cdot v = ||v||^2 \tag{3.54}
\]

\[
P := \int_0^T (N(x-f(x,u)) + N(S)).dt + N(D(x(0))) + N(E(x(T))) \tag{3.55}
\]

where:

\[
\bar{S} := \begin{bmatrix}
\max[S(x,u), -(\gamma + 2.\rho \text{diag}(\gamma).\gamma)/\rho] \\
\max[S(x), -(\gamma + 2.\rho \text{diag}(\gamma).\gamma)/\rho]
\end{bmatrix} \tag{3.56}
\]

in case that an active set strategy is used, (the max-operator is applied componentwise), otherwise

\[
\bar{S} := \begin{bmatrix}
S(x,u) + \text{diag}(\gamma).\gamma/2 \\
S(x) + \text{diag}(\gamma).\gamma/2
\end{bmatrix} \tag{3.57}
\]

in case that a slack-variable technique is used.

\[
Q := \int_0^T (N(\lambda + \lambda.f(x,u))^1 S(x,u) + f(x,u)) + \begin{bmatrix}
\lambda \\
\lambda
\end{bmatrix} (x(0)) + \sigma.D(x(0)) + \begin{bmatrix}
\lambda \\
\lambda
\end{bmatrix} (x(T)) + \begin{bmatrix}
\lambda \\
\lambda
\end{bmatrix} (\lambda(T) - g(x(T))) + \mu.E(x(T)) + \begin{bmatrix}
\lambda \\
\lambda
\end{bmatrix} (\lambda(x(T))) \tag{3.58}
\]

\[
\sum_{i=0}^{2T+1} N(\lambda_i(t) + \lambda_i(t) - \gamma S(x(t))) \tag{3.58}
\]
The merit function becomes:

\[
M = h(x(0)) + g(x(T)) + \mu . E(x(T)) +
\]

\[
\sum_{i=0}^{T-1} (f(x,u) - \lambda . (x-f(x,u)) + \gamma . S) . dt +
\]

\[
\sum_{i=0}^{2T+1} \gamma . S(x(t)) + (\int_0^T Q + P)/2
\]

We note that the norm on the update direction \(||d||\) is translated as:

\[
||d|| = \max_{0 \leq t \leq T} (||\Delta x||, ||\Delta x||, ||\Delta u||)
\]

where \(||.||\) denotes the euclidean norm on the space \(\mathbb{R}^n\) or \(\mathbb{R}^m\).
An experimental implementation of the method discussed in the previous sections has been developed. This implementation is based on the ideas of solving boundary value problems using collocation of piecewise polynomials (cf. de Boor et al. (1973) and Weiss (1974)). In an earlier stage of our investigations we developed a code based on collocation using piecewise cubic Hermite polynomials (cf. Dickmans et al. 1975). However, the nature of the solution (discontinuities in the control) yielded rather large errors. This approach has been abandoned.

One of the major parts of the method is the solution of the linear multipoint boundary value problem (3.33)-(3.44), which can be translated into the following problem:

\[ \begin{align*}
\dot{v} &= A(t) \cdot v + B(t) \cdot w + e(t) \quad \text{a.e. } 0 \leq t \leq T \\
0 &= C(t) \cdot v + D(t) \cdot w + g(t) \quad 0 \leq t \leq T \\
0 &= K_0 \cdot v(0) + L_0 \cdot s + I_0 \\
0 &= v(t^-) + F_i \cdot v(t^+) + G_i \cdot y_i \quad i=0, \ldots, 2^{i+1} \\
0 &= H_i \cdot v(t^+) + I_i \quad i=0, \ldots, 2^{i+1} \\
0 &= K_{-i} \cdot v(T) + L_{-i} \cdot y_i + I_i \quad \text{p} \\
\end{align*} \]

The relation between boundary value problem (3.33)-(3.44) and (4.1)-(4.6) is given on appendix C.
Following the concept of collocation with piecewise polynomials we introduce the following quantities:

* The breakpoint sequence:
\[
\Delta := \{ t_0, t_1, \ldots, t_p | 0 = t_0 < \ldots < t_p = T \} \tag{4.7}
\]

* The spaces consisting of piecewise polynomials:
\[
P_n^s, \Delta := \{ \text{all piecewise polynomials } v: [0, T] \rightarrow \mathbb{R} \text{ with breakpoint sequence } \Delta, \text{ where } v \text{ is an } s-\text{order polynomial on the open intervals } (t_i, t_{i+1}) \}
\]

* The numbers:
\[
-1 < \beta_1 < \beta_2 < \ldots < \beta_s < 1 \tag{4.8}
\]

and time differences
\[
h_j := t_{j+1} - t_j \quad j = 0, 1, \ldots, \delta - 1 \tag{4.9}
\]

and time instants
\[
\tau_{i+j.s} := \begin{cases} 
\frac{t_{i+j} + t_{i+j+1} + \beta_s h_j}{2} & \text{if } \beta_s = -1 \\
t + \frac{1}{2} h_j & \text{if } -1 < \beta_s < 1 \\
t_{i+j+1} & \text{if } \beta_s = +1 
\end{cases} \quad i = 0, 1, \ldots, \delta - 1 \tag{4.10}
\]

The collocation procedure is based on the requirement that \((4.1)\) and \((4.2)\) must hold at all timepoints \((j = 1, \ldots, s, \delta)\), which are referred to as collocation points. Furthermore the functions \(v\) and \(w\) are supposed to be piecewise polynomials. More specifically:
\[
v \in P^m+k_{s+1, \Delta} \quad \text{and} \quad w \in P^s_{s+1, \Delta} \tag{4.11}
\]

The functions \(w\) are allowed to be discontinuous at the breakpoints whereas the functions \(v\) must be continuous at these points, except in the case that such a point coincides with a junction point, in which case \((4.4)\) must hold.

In our procedure, we allow the junction points only to coincide with some of the breakpoints. In the sequel we shall treat every breakpoint as if it is a junction point. (When this is not the case, the column dimension of the matrix \(G\) and the row dimension of the matrix \(H\) will be zero. Taking \(F = -I\) (the \(2n\)-identity matrix), the function \(v\) will automatically be continuous in these points.)
de Boor et al. (1973) show that Gaussian points are a suitable choice for \( J \).

(Gaussian points are the points corresponding to the integration formulas of Gauss).

For \( v \) and \( w \) we select two different representations, which are based on the observation that for \( w \) we only need its values on the collocation points, whereas for \( v \) we also need the values on the breakpoints (cf. equations (4.1)–(4.6)).

This leads us to take the truncated power base for the representation of \( v \), i.e.

\[
v(t) = \sum_{j=0}^{s} c_{j} (t - t_{j})^{i} \quad t + \Delta t \leq t \leq t_{i+1}
\]

(4.12)

For \( w \) we use the Lagrange form of polynomial interpolation (cf. de Boor 1978), i.e.

\[
w(t) = \sum_{j=1}^{s} w_{j} z(t) \quad t + \Delta t \leq t \leq t_{i+1}
\]

(4.13)

with:

\[
z(t) = \frac{(t - t_{i})}{\prod_{k=1}^{s+i} \frac{(t_{k} - t_{i})}{(t_{k} + t_{i})}} \quad (4.14)
\]

Using these representations one can show that (4.1) is transformed into:

\[
\sum_{j=0}^{s} G_{ij} c_{rj} - B(t) w_{rj} - e(t) = 0 \quad i=1, \ldots, s
\]

(4.15)

\[
G_{ij} = (j, i - A(t)) \cdot (1 + \delta_{rj}) \cdot (he/2) \cdot (1 + \delta_{i}) \cdot (he/2)
\]

(4.16)

And (4.2) becomes:

\[
\sum_{j=0}^{s} C_{irj} c_{i+sr} + D(t) w_{i+sr} + g(t) = 0 \quad i=1, \ldots, s
\]

(4.17)

\[
C_{irj} = C(t) \cdot (1 + \delta_{i}) \cdot (he/2)
\]

(4.18)

Equations (4.1)–(4.6) are transformed into the following almost blockdiagonal system of linear equations:
This system is solved by means of Gaussian elimination, taking the sparsity pattern of the left hand side into account (subroutine CWIDTH from de Boor (1978)). Returning to the original algorithm of section 2, we see that steps (i) and (ii) are now clear, i.e. steps (i) and (ii) require the solution of system (4.20), whereas step (iii) is straightforward.
The last translation to be made is the calculation of the merit function $M$, whose calculation follows straightforward from (3.59) once a suitable quadrature formula for the calculation of the definite integrals is selected.

Referring to Weiss (1974), we note that there is a strong correspondence between this type of collocation and implicit Runge-Kutta methods based on interpolatory quadrature formulae. The definite integrals can be estimated on the basis of this correspondence, which yield the Gaussian quadrature formulae (cf. Bulirsh et al. 1976), i.e.

$$\int_{t_i}^{t_{i+1}} f(t) \, dt \approx h \sum_{j=1}^{s} w_j f(\tau_{j+\sigma})$$

(4.21)

As to the accuracy of the method discussed in this section we note that there are mainly three sources of error:

1. Discretization of the time-functions.

Convergence results from the theory of approximation indicate that this error is proportional to:

$$2s \left| \max_{i} h_{i} \right|$$

assuming that all functions involved are sufficiently smooth on the open intervals $(t_i, t_{i+1})$ for $i=0,...,s-1$.

2. Representation of the active set.

The representation of the active set is done by means of the junction points. These junction points are only allowed to coincide with the breakpoints. (The breakpoint sequence is kept fixed.) An error is due to the fact that in general the true junction points are interior to the intervals between successive breakpoints. In the future we will investigate variable stepsize strategies for improving a given breakpoint sequence, based on reducing this error.

3. Truncation of the iteration process.

Because we have local quadratic convergence of the iteration, this error can be made very small.
To perform some preliminary tests of the program we calculated the solution to the following two examples. The breakpoint sequence was taken to be uniform. For example 1 the algorithm was started with the slack variable technique, once the norm $\|d\|$ was below a certain limit (the value 1.0 was taken), a switch was made to the active set strategy. The calculations for example 2 were done using only the active set strategy.

Example 1 (cf. Miele 1980):

\begin{align*}
\text{minimize } & x(1) \\
\text{subject to : } & x = x_1 u_1 \\
& x = x_2 (u_2 - x - 0.5) \\
& x = 0 \\
& x(0) + x(0) = 0 \\
& x(1) = 1 \\
& x(1) = -\pi/4 \\
& x - u \leq 0
\end{align*}
Some convergence results are given in Table 5.1 below:

| I  | \( \alpha \)       | ||d||       | Merit function       |
|----|---------------------|------------|----------------------|
| 1  | 0.500000E+00        | 0.223730E+01| 0.238862E+01         |
| 2  | 0.250000E+00        | 0.258591E+01| 0.221859E+01         |
| 3  | 0.500000E+00        | 0.121017E+01| 0.221722E+01         |
| 4  | 0.100000E+01        | 0.369991E+00| 0.210637E+01         |

SWITCH OVER FROM SLACK-VARIABLE TECHNIQUE TO ACTIVE SET STRATEGY

| I  | \( \alpha \)       | ||d||       | Merit function       |
|----|---------------------|------------|----------------------|
| 5  | 0.100000E+01        | 0.844457E+00| 0.195140E+01         |
| 6  | 0.100000E+01        | 0.283447E+00| 0.182497E+01         |
| 7  | 0.100000E+01        | 0.220002E+00| 0.182240E+01         |
| 8  | 0.100000E+01        | 0.836837E-01| 0.182223E+01         |
| 9  | 0.100000E+01        | 0.191523E-02| 0.182223E+01         |
| 10 | 0.100000E+01        | 0.475043E-08| 0.182223E+01         |
| 11 | 0.195085E-15        | 0.195085E-15| 0.182223E+01         |

\( \mathcal{F} = 1.0E-01 \quad \mathcal{F} = 1.0E+01 \)

Example 2:

\[
\begin{aligned}
\text{minimize} & \quad \int_{0}^{1} \frac{1}{2} u \, dt \\
\text{subject to:} & \quad x_{1}(0) = 0 \quad x_{1}(1) = 1 \\
& \quad x_{2}(0) = 0 \quad x_{2}(1) = 0 \\
& \quad |u| \leq 5 \\
& \quad |x| \leq 1.4
\end{aligned}
\]
Some convergence results are given in table 5.2 below:

| I  | α         | ||d||   | Merit function |
|----|-----------|---------|---------------|
| 1  | 0.100000E+01 | 0.688235E+01 | 0.367874E+03 |
| 2  | 0.500000E+00 | 0.132081E+01 | 0.188972E+03 |
| 3  | 0.500000E+00 | 0.111806E+01 | 0.105495E+03 |
| 4  | 0.500000E+00 | 0.101895E+01 | 0.394670E+02 |
| 5  | 0.100000E+01 | 0.700380E+00 | 0.345637E+02 |
| 6  | 0.100000E+01 | 0.358886E+00 | 0.646068E+01 |
| 7  | 0.100000E+01 | 0.118366E-01 | 0.642857E+01 |
| 8  | 0.100000E+01 | 0.140097E-04 | 0.642857E+01 |
| 9  | 0.100000E+01 | 0.196261E-10 | 0.642857E+01 |
| 10 |           | 0.277556E-16 | 0.642857E+01 |

table 5.2

\[ p_1 = 1.0 \times 10^2 \quad p_2 = 1.0 \times 10^2 \]
CHAPTER 6

FINAL REMARKS.

Using the experimental implementation outlined in section 4, we were able to solve two problems, indicating that the method was feasible. It will be clear that there is still a lot of work to be done to obtain a fully tested operational code for solving state-constrained optimal control problems with the method presented here. We summarize the following topics:

* Solution of subproblems with inequality constraints.
* Solution of problems with contact points.
* More efficient implementation of the collocation method.
* Variable stepsize strategy (breakpoint sequence).
* Application of quasi-Newton and discrete Newton techniques.
* Choice of merit function, penalty constants, linesearch algorithm.
* Strategies in case of defective subproblems.

We hope to present results about this research in the future.
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APPENDIX A

EXPLICIT EXPRESSION FOR $\gamma_2$ ON BOUNDARY ARCS.

First we define the $nxm$ matrix functions $\phi : [0,T] \rightarrow \mathbb{R}$ (cf. Hamilton (1972)):

\begin{align*}
    \phi_0 & := f
    \\
    \phi_{i+1} & := \phi_i - \frac{\partial \phi_i}{\partial x_i}
\end{align*}

(A1)

(A2)

In the sequel we will assume that $k_1 = 0, k_2 = 1$ and $m = 1$ for simplicity. (i.e. we have only a single pure state constraint).

Using (A1) and (A2) it is possible to give an explicit expression for the time derivatives of $H_u$, i.e.

\begin{equation}
    H_u = \begin{cases}
        \lambda_i \phi_i & i = 0,1,\ldots,p-1 \\
        \lambda_i \phi_i - (-1)^{p-1} \gamma_2 S & i = p
    \end{cases}
\end{equation}

(A3)

where $p$ is the order of the state constraint $S$.

Because

\begin{equation}
    H = 0 \quad i = 0,1,\ldots,p \quad a.e. \ 0 \leq t \leq T
\end{equation}

(A3) implies:

\begin{equation}
    \gamma_2 = (-1)^{p-1} \frac{\lambda_i \phi_i}{S}
\end{equation}

(A4)

and hence:

\begin{equation}
    \phi(x,u) = (-1)^{p-1} \frac{\phi_i}{S}
\end{equation}

(A5)

in this case.

In the more general case of $k_1 \neq 0, k_2 > 1$ and $m > 1$, a similar relation
can be derived as the result of an elimination process, which is possible because of the regularity condition (3.20) (cf. Maurer 1979a).
APPENDIX B

BOUNDARY VALUE PROBLEM IN CASE OF SLACK VARIABLE TECHNIQUE.

The boundary value problem to be solved in case of the slack variable technique is:

\[
\begin{align*}
\dot{x} &= f(t) + \Delta x + f(t) - x, \\
\lambda &= -\lambda f(t) - S(t) \cdot \text{diag}(\gamma) \cdot \Theta - f(t) - H(t) \cdot \Delta x, \\
-x &= -H(t) \cdot \Delta u, \\
D(0) + D(0) \cdot \Delta x(0) &= 0, \\
-(h(0) + \Theta D(0)) \cdot \Delta x(0) - \lambda(0) &= h(0) + \Theta D(0), \\
E(T) + E(T) \cdot \Delta x(T) &= 0, \\
(g(T) + \Theta E(T)) \cdot \Delta x(T) - \lambda(T) &= -g(T) - \Theta E(T), \\
0 &= f(t) + \Delta x + S \cdot \text{diag}(\gamma) \cdot \Theta + H \cdot \Delta u, \\
S \cdot \Delta x + S \cdot \Delta u - 2 \cdot \text{diag}(\gamma) \cdot \Theta &= -S - \gamma, \\
\gamma &= \text{diag}(\gamma) \cdot \Theta.
\end{align*}
\]
Following the suggestions of Tapia (1977) the change in the slack variable in the $i$th iteration is:

'away from convergence' : $d\gamma(t) = |S(t)| - \bar{y}(t)$ \quad $j=1, \ldots, k$ \quad (B10)

\[ d\gamma(t) \quad \begin{cases} 
\text{for all } t \\
\text{where } j \\
\text{otherwise}
\end{cases} \quad (B11)

'near convergence' : $d\gamma(t) = \left\{ \begin{array}{ll}
\text{for all } t \\
\text{where } j \\
\text{otherwise}
\end{array} \right.$

$\theta(t) \neq 0$ \quad $j=1, \ldots, k$

Here $d\gamma(t)$ is defined as:

$\bar{d}\gamma(t) = \frac{2}{(I - \text{diag}(\theta(t))) - I}.\bar{y}(t)$ \quad (B12)

The condition 'near'/'away from' convergence is based on the value of the norm $|d\gamma|$. 

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APPENDIX C

RELATION BETWEEN BOUNDARY VALUE PROBLEMS.

\[
\begin{align*}
v &= \begin{bmatrix} x \\ \lambda \end{bmatrix}, \quad w = \begin{bmatrix} u \\ \gamma \end{bmatrix}, \quad e = \begin{bmatrix} f(t) - x \\ T & -f(t) \\ 0x & f(t) \end{bmatrix}, \quad g = \begin{bmatrix} S(t) \\ T & f(t) \end{bmatrix} \\
A(t) &= \begin{bmatrix} f(t) & 0 \\ x & T \\ -H(t) -f(t) & x \\ xx & x \end{bmatrix}, \quad B(t) = \begin{bmatrix} f(t) & 0 \\ u & -H(t) -S(t) \\ xu & x \end{bmatrix} \\
C(t) &= \begin{bmatrix} \tilde{S}(t) & 0 \\ T & -x \phi(t) - \tilde{\phi}(t) \\ \bar{H}(t) & T \\ u & f(t) \end{bmatrix}, \quad D(t) = \begin{bmatrix} \tilde{S}(t) & 0 \\ T & -x \phi(t) \\ \bar{H}(t) & T \\ uu & u \end{bmatrix} \\
F &= \begin{bmatrix} -I & 0 \\ \sqrt{x} S(t) & -I \\ i 2x & i \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ T & -S(t) \\ x & i \end{bmatrix} \\
H &= \begin{bmatrix} \tilde{S}(t) \\ 2x \end{bmatrix}, \quad \bar{I} = \begin{bmatrix} -\tilde{S}(t) \\ i \end{bmatrix}
\end{align*}
\]
\[ K = \begin{bmatrix} M & I \\ 0 & D[0] \end{bmatrix}, \quad L = \begin{bmatrix} T \\ D[0] \end{bmatrix}, \quad I_1 = \begin{bmatrix} T \\ 0 \end{bmatrix} \] (C6)

\[ K_2 = \begin{bmatrix} M & I \\ 0 & D[0] \end{bmatrix}, \quad L_2 = \begin{bmatrix} T \\ E[T] \end{bmatrix}, \quad I_2 = \begin{bmatrix} T \\ 0 \end{bmatrix} \] (C7)

where:

\[ M = h[0] + \bar{\sigma}D[0] \quad \text{and} \quad M = g[T] + \bar{\mu}E[T] \] (C8)

where:

\[ M = h[0] + \bar{\sigma}D[0] \quad \text{and} \quad M = g[T] + \bar{\mu}E[T] \] (C8)