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Completion of the squares in the finite horizon $H^\infty$ control problem by measurement feedback

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ABSTRACT
In this paper we study the finite horizon version of the standard $H^\infty$ control problem by measurement feedback. Given a finite-dimensional linear, time-invariant system, together with a positive real number $\gamma$, we obtain necessary and sufficient conditions for the existence of a possibly time-varying dynamic compensator such that the $L_2[0,T]$-induced norm of the closed loop operator is smaller than $\gamma$. These conditions are expressed in terms of a pair of quadratic differential inequalities, generalizing the well-known Riccati differential equations that were introduced recently in the context of finite horizon $H^\infty$ control.
1. THE FINITE HORIZON $H^\infty$ CONTROL PROBLEM

Consider the linear time-invariant system

\[
\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \\
y(t) = C_1 x(t) + D_1 w(t), \\
z(t) = C_2 x(t) + D_2 u(t),
\]

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, $w \in \mathbb{R}^l$ an unknown disturbance input, $y \in \mathbb{R}^p$ the measured output and $z \in \mathbb{R}^q$ the output to be controlled. $A, B, E, C_1, D_1, C_2$ and $D_2$ are constant real matrices of appropriate dimensions. In addition, we assume that some fixed time interval $[0, T]$ is given. We shall be concerned with the existence and construction of dynamic compensators of the form

\[
\dot{p}(t) = K(t)p(t) + L(t)y(t), \\
u(t) = M(t)p(t) + N(t)y(t),
\]

where $K, L, M$ and $N$ are real, matrix valued, continuous functions on $[0, T]$. The feedback interconnection of $\Sigma$ and $\Sigma_F$ is the linear time-varying system $\Sigma_{cf}$ described by

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{p}(t)
\end{pmatrix} = \begin{pmatrix} A + BN(t)C_1 & BM(t) \\ L(t)C_1 & K(t) \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} + \begin{pmatrix} E + BN(t)D_1 \\ L(t)D_1 \end{pmatrix} w(t), \\
z(t) = (C_2 + D_2 N(t)C_1) \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} + D_2 N(t) D_1 w(t).
\]

Let us denote the matrices appearing in these equations by $A_\epsilon(t), E_\epsilon(t), C_\epsilon(t)$ and $D_\epsilon(t)$, respectively. Obviously, if we put $x(0) = 0, p(0) = 0$, then the closed loop system $\Sigma_{cf}$ defines a convolution operator $G_{cf}: L_2^1[0, T] \to L_2^1[0, T]$ given by

\[
(G_{cf}w)(t) = z(t) = \int_0^t C_\epsilon(t) \Phi_\epsilon(t, \tau) E_\epsilon(\tau) w(\tau) d\tau + D_\epsilon(t) w(t),
\]

where $\Phi_\epsilon(t, \tau)$ is the transition matrix of $A_\epsilon(t)$. Thus, the influence of disturbances $w \in L_2^1[0, T]$ on the output $z$ can be measured by the operator norm of $G_{cf}$, given in the usual way by

\[
\|G_{cf}\| := \sup \left\{ \frac{\|G_{cf}w\|_2}{\|w\|_2} \mid 0 \neq w \in L_2^1[0, T] \right\}.
\]

Here, $\|x\|_2$ denotes the $L_2[0, T]$ norm of the function $x$. The problem that we shall discuss in this paper is the following:

given $\gamma > 0$, find necessary and sufficient conditions for the existence of a dynamic compensator $\Sigma_F$ such that $\|G_{cf}\| < \gamma$. 
The problem as posed here will be referred to as the finite horizon $H^\infty$ control problem by measurement feedback. This problem was studied before in [6] and [2]. In the latter references it is however assumed that the following conditions hold: $D_1$ is surjective, $D_2$ is injective. In the present paper we shall extend the results obtained in [6] and [2] to the case that $D_1$ and $D_2$ are arbitrary.

2. QUADRATIC DIFFERENTIAL INEQUALITIES

A central role in our study of the problem posed is played by what we shall call the quadratic differential inequality. Let $\gamma > 0$ be given. For any differentiable matrix function $P : [0, T] \to \mathbb{R}^{n \times n}$, define $F_\gamma(P) : [0, T] \to \mathbb{R}^{(n+m) \times (n+m)}$ by

$$ F_\gamma(P) := \begin{bmatrix} \dot{P} + A'P + PA + C_2'C_2 + \frac{1}{\gamma} PEE'P & PB + C_2'D_2 \\ B'P + D_2'C_2 & D_2'D_2 \end{bmatrix} $$

If $F_\gamma(P)(t) \succeq 0$ for all $t \in [0, T]$, then we shall say that $P$ satisfies the quadratic differential inequality (at $\gamma$). Also a dual version of (2.1) will be important to us: for any differentiable $Q : [0, T] \to \mathbb{R}^{n \times n}$ define $G_\gamma(Q) : [0, T] \to \mathbb{R}^{(n+p) \times (n+p)}$ by:

$$ G_\gamma(Q) := \begin{bmatrix} -Q + AQ + QA' + EE' + \frac{1}{\gamma} QC_2'C_2Q & QC_1' + ED_1' \\ C_1Q + D_1E' & D_1D_1' \end{bmatrix} $$

If $G_\gamma(Q)(t) \succeq 0$ for all $t \in [0, T]$ then we shall say that $Q$ satisfies the dual quadratic differential inequality (at $\gamma$). In the sequel let

$$ G(s) := C_2(I - A)^{-1} B + D_2 , $$
$$ H(s) := C_1(I - A)^{-1} E + D_1 $$

denote the open loop transfer matrices from $u$ to $z$ and $w$ to $y$, respectively. Furthermore, denote by $\text{normrank}(G)$ and $\text{normrank}(H)$ the ranks of these transfer matrices considered as matrices with entries in the field of real rational functions. We are now ready to state our main result:

**THEOREM 2.1.** Let $\gamma > 0$. The following two statements are equivalent:

(i) There exists a time-varying dynamic compensator $\Sigma_F$ such that $\|G_{\Sigma_F}\| < \gamma$.

(ii) There exist differentiable matrix functions $P$ and $Q : [0, T] \to \mathbb{R}^{n \times n}$ such that

(a) $F_\gamma(P)(t) \succeq 0 \ \forall t \in [0, T]$ and $P(T) = 0$,
(b) \( \text{rank } F_y(P)(t) = \text{normrank } (G) \quad \forall t \in [0, T] \), \hspace{1cm} (2.4)

(c) \( G_y(Q)(t) \geq 0 \quad \forall t \in [0, T] \) and \( Q(0) = 0 \), \hspace{1cm} (2.5)

(d) \( \text{rank } G_y(Q)(t) = \text{normrank } (H) \quad \forall t \in [0, T] \), \hspace{1cm} (2.6)

(e) \( \gamma^2 I - Q(t)P(t) \) is invertible \( \forall t \in [0, T] \). \hspace{1cm} (2.7)

The aim of this paper is to outline the main steps and ideas involved in a proof of the latter theorem. For a more detailed discussion we would like to refer to [5].

It can be shown that, in general, if \( F_y(P) \geq 0 \) on \( [0, T] \) then

\[
\text{rank } F_y(P)(t) \geq \text{normrank } (G) \quad \forall t \in [0, T]
\]

and, likewise, if \( G_y(Q) \geq 0 \) on \( [0, T] \) then

\[
\text{rank } G_y(Q)(t) \geq \text{normrank } (H) \quad \forall t \in [0, T]
\]

This means that the pair of conditions (2.3), (2.4) can be reformulated as: \( P \) is a rank-minimizing solution of the quadratic differential inequality at \( y \), satisfying the end-condition \( P(T) = 0 \). A similar restatement is valid for the conditions (2.5), (2.6). It can also be shown that if \( P \) satisfies (2.3) and (2.4) then it is unique. Also, this unique solution turns out to be symmetric for all \( t \in [0, T] \). The same holds for \( Q \) satisfying (2.5) and (2.6).

We will now show that for the special case that \( D_1 \) and \( D_2 \) are assumed to be surjective and injective, respectively, our Theorem 2.1 specializes to the results obtained before in [6] and [2]. Indeed, if \( D_2 \) is injective then of course

\[
\text{normrank } (G) = \text{rank } D_2 = m
\]

Denote

\[
R_y(P) := \dot{P} + A'P + PA + C_2'C_2 + \frac{1}{\gamma} PEE'P - (PB + C_2'D_2)(D_2'D_2)^{-1}(B'P + D_2'C_2)
\]

Clearly, \( R_y(P) \) is the Schur-complement of \( D_2'D_2 \) in \( F_y(P) \). Therefore we have

\[
\text{rank } F_y(P)(t) = m + \text{rank } R_y(P)(t)
\]

for all \( t \in [0, T] \). This implies that the pair of conditions (2.3), (2.4) is equivalent to the condition: \( P \) is the solution of the Riccati differential equation \( R_y(P) = 0 \) with terminal condition \( P(T) = 0 \). A similar statement holds for \( Q \) satisfying (2.5) and (2.6). Thus we obtain

COROLLARY 2.2. Let \( \gamma > 0 \). Assume \( D_1 \) is surjective and \( D_2 \) is injective. Then the following statements are equivalent:
There exists a time-varying dynamic compensator \( \Sigma_F \) such that \( \|G_c\| < \gamma \).

(ii) There exist differentiable matrix-functions \( P \) and \( Q : [0, T] \rightarrow \mathbb{R}^{n \times n} \) such that for all \( t \in [0, T] \)

\[
\dot{P} = A'P + PA + C_2'C_2 + \frac{1}{\gamma^2} PEE'P - (PB + C_2'D_2)(D_2'D_2)^{-1}(B'P + D_2'C_2), \quad P(T) = 0
\]

and

\[
\dot{Q} = AQ + QA' + EE' + \frac{1}{\gamma^2} QC_2'C_2Q - (QC_1' + ED_1')(D_1'D_1)^{-1}(C_1Q + D_1E'), \quad Q(0) = 0.
\]

with, in addition,

\[
\gamma I - Q(t)P(t) \text{ invertible for all } t \in [0, T]. \tag{3.1}
\]

It can be shown that if the conditions in the statement of Theorem 2.1 (ii) indeed hold, then it is always possible to find a suitable compensator with dynamic order equal to \( n \), the dynamic order of the system to be controlled.

### 3. COMPLETION OF THE SQUARES

In this section we shall outline the proof of the implication (i) \( \Rightarrow \) (ii) of Theorem 2.1. Consider the system \( \Sigma \). For given \( u \) and \( w \), let \( z_{u,w} \) denote the output to be controlled, with \( x(0) = 0 \). Our starting point is the following lemma:

**LEMMA 3.1.** Let \( \gamma > 0 \). Assume that for all \( 0 \neq w \in L^2_2[0, T] \) we have

\[
\inf \{ \|z_{u,w}\|_2 - \gamma \|w\|_2 \mid u \in L^2_2[0, T] \} < 0. \tag{3.1}
\]

Then there exist a differentiable matrix function \( P : [0, T] \rightarrow \mathbb{R}^{n \times n} \) such that \( F_{\gamma}(P)(t) \geq 0 \) \( \forall t \in [0, T] \), \( P(T) = 0 \) and \( \text{rank} F_{\gamma}(P)(t) = \text{rank} (G) \) \( \forall t \in [0, T] \).

**PROOF.** A proof of this can be given by combining the result of [2, Theorem 2.3] with ideas used in the proof of [4, Theorem 5.4].

Now, assume that the condition (i) in the statement of Theorem 2.1 holds, i.e. assume there exists a dynamic compensator \( \Sigma_F \) such that \( \|G_c\| < \gamma \). Then condition (3.1) holds: let \( w \in L^2_2[0, T] \) and \( w \neq 0 \) and let \( z \) be the closed loop output with \( x(0) = 0 \) and \( p(0) = 0 \). Then \( z = z_{u,w} \), where \( u \) is the output of \( \Sigma_F \). Clearly
and hence \( \|z_u\|_2 - \gamma \|w\|_2 < 0 \). Then also the infimum in (3.1) is less than 0. We may then conclude that, indeed, a differentiable matrix function \( P \) exists that satisfies (2.3) and (2.4).

The fact that also (2.5) and (2.6) hold can be proven by the following dualization argument. Consider the dual system

\[
\begin{align*}
\dot{\xi} &= A_{11} \xi + C_{11} \psi + C_{21} \sigma, \\
\eta &= B_{11} \xi + D_{21} \sigma, \\
\zeta &= E_{11} \xi + D_{11} \psi,
\end{align*}
\]

and apply to \( \Sigma' \) the time-varying compensator

\[
\begin{align*}
\dot{q} &= K'(T-t) q + M'(T-t) \eta, \\
\dot{v} &= L'(T-t) q + N'(T-t) \eta.
\end{align*}
\]

It can be shown that if we denote by \( \tilde{G}_{el} \) the closed loop operator of \( \Sigma' \times \Sigma_p' \) (with \( \zeta(0)=0, q(0)=0 \)), and it \( G_{el}^* \) denotes the adjoint operator of \( G_{el} \) then the following equality holds:

\[
\tag{3.2}
\]

where \( R \) denotes the time-reversal operator \( (Rx)(t) := x(T-t) \). Now, if \( \|G_{el}\| < \gamma \) then also \( \|G_{el}^*\| < \gamma \) and therefore, by 3.2, \( \|G_{el}\| < \gamma \). We can therefore conclude that the quadratic differential inequality associated with \( \Sigma' \) has an appropriate solution, say \( \tilde{P}(t) \), on \([0,T]\). by defining \( Q(t) := \tilde{P}(T-t) \) we obtain a function \( Q \) that satisfies (2.5) and (2.6).

Finally, we have to show that condition (2.7) holds. We shall need the following lemma:

**LEMMA 3.2.** Assume that there exists \( P : [0, T] \to \mathbb{R}^{n \times n} \) such that \( F_\gamma(P)(t) \geq 0, \forall t \in [0, T] \), and rank \( F_\gamma(P)(t) = \operatorname{normrank}(G), \forall t \in [0, T] \). Then there exist continuous matrix functions \( C_{2,p} \) and \( D_p \) such that for all \( t \)

\[
F_\gamma(P)(t) = \begin{bmatrix} C_{2,p}(t) \\ D_p(t) \end{bmatrix} \begin{bmatrix} C_{2,p}(t) & D_p(t) \end{bmatrix}.
\]  

(3.3)

Assume \( F_\gamma(P) \) is factorized as in (3.3). Introduce a new system, say \( \Sigma_p \), by
\[ \dot{x}_p = (A + \frac{1}{\gamma} EEP)x_p + B u_p + E w_p , \]
\[
\Sigma_p \quad y_p = (C_1 + \frac{1}{\gamma} D_1 E' P)x_p + D_1 w_p , \tag{3.4}
\]
\[ z_p = C_{2,p} x_p + D_p u_p . \]

We stress that \( \Sigma_p \) is a time-varying system with continuous coefficient matrices. If \( \Sigma_F \) is a dynamic compensator of the form (1.2), let \( G_{P,cl} \) denote the operator from \( w_p \) to \( z_p \) obtained by interconnecting \( \Sigma_p \) and \( \Sigma_F \).

The crucial observation now is that \( \gamma \) if and only if \( \gamma \) that is, a compensator \( \Sigma_F \) "works" for \( \Sigma \) if and only if it "works" for \( \Sigma_p \). A proof of this can be based on the following "completion of the squares" argument:

**Lemma 3.3.** Assume that \( P \) satisfies (2.3) and (2.4). Assume \( x_p(0) = x(0) = 0, u_p(t) = u(t) \) for all \( t \in [0, T] \) and suppose that \( w_p \) and \( w \) are related by \( w_p(t) = w(t) - \gamma^2 E' P x(t) \) for all \( t \in [0, T] \). Then for all \( t \in [0, T] \) we have

\[ \|z(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2 = \frac{d}{dt}(x'(t) P(t) x(t)) + \|z_p(t)\|_2^2 - \gamma^2 \|w_p(t)\|_2^2 . \]

Consequently:

\[ \|z\|_2^2 - \gamma^2 \|w\|_2^2 = \|z_p\|_2^2 - \gamma^2 \|w_p\|_2^2 . \tag{3.5} \]

**Proof:** This can be proven by straightforward calculation, using the factorization (3.3).

**Theorem 3.4.** Let \( P \) satisfy (2.3) and (2.4). Let \( \Sigma_F \) be a dynamic compensator of the form (1.2). Then

\[ \|G_p\| < \gamma \Leftrightarrow \|G_{P,cl}\| < \gamma . \]

**Proof.** Assume \( \|G_{P,cl}\| < \gamma \) and consider the interconnection of \( \Sigma \) and \( \Sigma_F \).

Let \( 0 \neq w \in L_2^q [0, T] \), let \( x \) be the corresponding state trajectory of \( \Sigma \) and define \( w_p := w - \gamma^2 E' P x \). Then clearly \( y_p = y, x_p = x \) and therefore \( u_p = u \). This implies that the equality (3.4) holds. Also, we clearly have
Next, note that the mapping \( w_p \mapsto w_p + \gamma^2 E'P_x \) defines a bounded operator from \( L^2_{\mathcal{F}}[0, T] \) to \( L^2_{\mathcal{F}}[0, T] \). Hence there exists a constant \( \mu > 0 \) such that \( \|w_p\|_2^2 > \mu \|w\|_2^2 \). Define \( \delta > 0 \) by \( \delta^2 := \gamma^2 - \|G_{\mathcal{F},cl}\|^2 \). Combining (3.4) and (3.5) then yields
\[
\|z\|_2^2 - \gamma^2 \|w_p\|_2^2 \leq -\delta^2 \|w\|_2^2.
\]
Obviously, this implies that \( \|G_{\mathcal{F},cl}\|^2 \leq \gamma^2 - \delta^2 \mu < \gamma^2 \).

We will now prove that (2.7) holds. Again assume that \( \Sigma \) yields \( \|G_{\mathcal{F}}\| < \gamma \). By applying a version of Lemma 3.1 for time-varying systems it can then be proven that the dual quadratic differential inequality associated with \( \Sigma \):
\[
\bar G_{\gamma}(Y) := \begin{bmatrix} -\tilde Y + A_p Y + Y A_p' + EE' + \frac{1}{\gamma^2} Y C_{2,p} C_{2,p} Y & Y C_{1,p} + E D_1' \\ C_{1,p} Y + D_1 E' \\ D_1 D_1' \end{bmatrix} \geq 0
\]
has a solution \( Y(t) \) on \([0, T] \), satisfying \( Y(0) = 0 \) and
\[
\text{rank} \; \bar G_{\gamma}(Y)(t) = \text{normrank} \begin{bmatrix} t s - A_p(t) & -E \\ C_{1,p}(t) & D_1 \end{bmatrix} - n \quad (3.7)
\]
for all \( t \in [0, T] \). Here, we have denoted \( A_p = A + \gamma^2 EE'P \) and \( C_{1,p} = C_1 + \gamma^2 D_1 E'P \). Furthermore, it can be shown that \( Y \) is unique on each interval \([0, t_1] \) \((t_1 < T) \). On the other hand, it can be proven that on each interval \([0, t_1] \) on which \( I - QP \) is invertible, the function \( \tilde Y := (I - QP)^{-1} Q \) satisfies \( \bar G_{\gamma}(Y)(t) \geq 0 \), \( \tilde Y(0) = 0 \) and the rank condition (3.7). Thus on any such interval \([0, t_1] \) we must have \( Y(t) = \tilde Y(t) \). Clearly, since \( Q(0) = 0 \), there exists \( 0 < t_1 \leq T \) such that \( I - QP \) is invertible on \([0, t_1] \). Assume now that \( t_1 > 0 \) is the smallest real number such that \( I - Q(t_1)P(t_1) \) is not invertible. Then on \([0, t_1] \) we have
\[
Q(t) = (I - Q(t)P(t)) \tilde Y(t)
\]
and hence, by continuity
\[
Q(t_1) = (I - Q(t_1)P(t_1)) \tilde Y(t_1) . \quad (3.8)
\]
There exists \( x \neq 0 \) such that \( x'(I - Q(t_1)P(t_1)) = 0 \). By (3.8) this yields \( x'Q(t_1) = 0 \) whence \( x' = 0 \), which is a contradiction. We must conclude that \( I - Q(t)P(t) \) is invertible for all \( t \in [0, T] \).

This completes our proof of the implication (i) \( \Rightarrow \) (ii) of Theorem 2.1.
4. EXISTENCE OF COMPENSATORS

In the present section we will sketch the main ideas of our proof of the implication (ii) \( \Rightarrow \) (i) of Theorem 2.1. The main idea is as follows: starting from the original system \( \Sigma \) we shall define a new system, \( \Sigma_{p,Q} \), which has the following important properties:

1. Let \( \Sigma_{p} \) be any compensator. The closed loop operator \( G_{cl} \) of the interconnection \( \Sigma \times \Sigma_{p} \) satisfies \( \|G_{cl}\| < \gamma \) if and only if the closed loop operator of \( \Sigma_{p,Q} \times \Sigma_{p} \), say \( G_{p,Q,cl} \), satisfies \( \|G_{p,Q,cl}\| < \gamma \).

2. The system \( \Sigma_{p,Q} \) is almost disturbance decoupable by dynamic measurement feedback, i.e. for all \( \varepsilon > 0 \) there exists \( \Sigma_{p} \) such that \( \|G_{p,Q,cl}\| < \varepsilon \).

Property (1) states that a compensator \( \Sigma_{p} \) "works" for \( \Sigma \) if and only if it "works" for \( \Sigma_{p,Q} \). On the other hand, property (2) states that, indeed, there exists a compensator \( \Sigma_{p} \) that "works" for \( \Sigma_{p,Q} \); take any \( \varepsilon \leq \gamma \) and take a compensator \( \Sigma_{p} \) such that \( \|G_{p,Q,cl}\| < \varepsilon \). Then by, property (1), \( \|G_{cl}\| < \gamma \) so \( \Sigma_{p} \) works for \( \Sigma \). This would clearly establish a proof of the implication (ii) \( \Rightarrow \) (i) in Theorem 2.1.

We shall now describe how the new system \( \Sigma_{p,Q} \) is defined. Assume that there exist \( P \) and \( Q \) satisfying (2.3) to (2.7). Apply Lemma 3.2 to obtain a continuous factorization (3.3) of \( F \) and let the system \( \Sigma_{p} \) be defined by (3.4). Next, consider the dual quadratic differential inequality \( \bar{G}_{\gamma}(Y) \geq 0 \) associated with the system \( \Sigma_{p} \), together with the conditions \( Y(0) = 0 \) and the rank condition (3.7). As was already noted in the previous section, the conditions (2.5), (2.6) and (2.7) assure that there exists a unique solution \( Y \) on \( [0,T] \). (In fact, \( Y(t) = (\gamma^2 I - Q(t)P(t))^{-1}Q(t) \)).

Now, it can be shown that there exists a factorization

\[
\bar{G}_{\gamma}(Y)(t) = \begin{bmatrix} E_{p,Q}(t) & E_{p,Q}'(t) \\ D_{p,Q}(t) & D_{p,Q}'(t) \end{bmatrix},
\]

with \( E_{p,Q} \) and \( D_{p,Q} \) continuous on \( [0,T] \). Denote

\[
A_{p,Q}(t) := A_{p}(t) + Y(t) C_{2,p}'(t) C_{2,p}(t),
\]

\[
B_{p,Q}(t) := B + Y(t) C_{2,p}'(t) D_{p}(t).
\]

Then, introduce the new system \( \Sigma_{p,Q} \) by:

\[
\dot{x}_{p,Q} = A_{p,Q} x_{p,Q} + B_{p,Q} u_{p,Q} + E_{p,Q} w_{p,Q},
\]

\[
\Sigma \quad y_{p,Q} = C_{1,p} x_{p,Q} + D_{p,Q} w_{p,Q},
\]

\[
z_{p,Q} = C_{2,p} x_{p,Q} + D_{p} u_{p,Q}.
\]

Again, \( \Sigma_{p,Q} \) is a time-varying system with continuous coefficient matrices. We note that \( \Sigma_{p,Q} \) is in fact obtained by first transforming \( \Sigma \) into \( \Sigma_{p} \) and by subsequently applying the dual of this transformation to \( \Sigma_{p} \). We shall now first show that property (1) above holds. If \( \Sigma_{p} \) is a dynamic compensator, then let \( G_{p,Q,cl} \) be the closed loop operator from \( w_{p,Q} \) to \( z_{p,Q} \) in the interconnection of \( \Sigma_{p,Q} \) with \( \Sigma_{p} \).
Recall that $G_{cl}$ denotes the closed loop operator from $w$ to $z$ in the interconnection of $\Sigma$ and $\Sigma_F$.
We have the following:

**THEOREM 4.1.**

\[ \|G_{cl}\| < \gamma \iff \|G_{P,Q,cl}\| < \gamma. \]

**PROOF.** Assume $\Sigma_F$ yields $\|G_{cl}\| < \gamma$. By Theorem 3.4 then also $\|G_{p,d}\| < \gamma$, i.e., $\Sigma_F$ interconnected with $\Sigma_p$ (given by 3.4) also yields a closed loop operator with norm less than $\gamma$. It is easily seen that the dual compensator $\Sigma'_p$ (see section 3), interconnected with the dual of $\Sigma_p$:

\[
\begin{align*}
\xi &= A_p'(T-t)\xi + C_{1,p}'(T-t)z + C_{2,p}'(T-t)d \\
\eta &= B'_p\xi + D_p'(T-t)d \\
\zeta &= E'_p\zeta + D'_p z
\end{align*}
\]

yields a closed loop operator $\tilde{G}_{p,d}$ (from $d$ to $\zeta$) with $\|\tilde{G}_{p,d}\| < \gamma$. Now, the quadratic differential inequality associated with $\Sigma'_p$ is the transposed, time-reversed version of the inequality $\tilde{G}_p(Y) \geq 0$ and therefore has a unique solution $\tilde{Y}(t) = Y(T-t)$ such that $\tilde{Y}(T) = 0$ and the corresponding rank condition (3.7) holds. By applying Theorem 3.4 to the system $\Sigma'_p$ we may then conclude that the interconnection of $\Sigma'_p$ with the dual $\Sigma_{p,Q}'$ of $\Sigma_{p,Q}$ yields a closed loop operator with norm less than $\gamma$. Again by dualization we then conclude that $\|G_{P,Q,cl}\| < \gamma$. The converse implication is proven analogously.

Property (2) is stated formally in the following theorem:

**THEOREM 4.2.** For all $\epsilon > 0$ there exists a time-varying dynamic compensator $\Sigma_F$ such that $\|G_{P,Q,cl}\| < \gamma$.  

Due to space limitations, for a proof of the latter theorem we refer to [5]. By combining theorems 4.1 and 4.2 we immediately obtain a proof of the implication (ii) $\Rightarrow$ (i) in Theorem 4.2.
5. CONCLUDING REMARKS

In this paper we have studied the finite horizon $H^\infty$ control problem by dynamic measurement feedback. We have noted that the results obtained can be specialized to re-obtain results that were obtained before [6] and [2]. The development of our theory runs analogously to the theory developed in [4] and [3] around the standard $H^\infty$ control problem (the infinite horizon version of the problem studied in the present paper). In the latter references the main tools are the so-called quadratic matrix inequalities, the algebraic versions of the differential inequalities used in the present paper. For the special case that $D_1$ is surjective and $D_2$ is injective these quadratic matrix inequalities reduce to the algebraic Riccati equations that were also obtained in [6] and [1].

REFERENCES

List of COSOR-memoranda - 1989

<table>
<thead>
<tr>
<th>Number</th>
<th>Month</th>
<th>Author</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>M 89-01</td>
<td>January</td>
<td>D.A. Overdijk</td>
<td>Conjugate profiles on mating gear teeth</td>
</tr>
<tr>
<td>M 89-02</td>
<td>January</td>
<td>A.H.W. Geerts</td>
<td>A priori results in linear quadratic optimal control theory</td>
</tr>
<tr>
<td>M 89-03</td>
<td>February</td>
<td>A.A. Stoorvogel, H.L. Trentelman</td>
<td>The quadratic matrix inequality in singular $H_\infty$ control with state feedback</td>
</tr>
<tr>
<td>M 89-04</td>
<td>February</td>
<td>E. Willekens, N. Veraverbeke</td>
<td>Estimation of convolution tail behaviour</td>
</tr>
<tr>
<td>M 89-05</td>
<td>March</td>
<td>H.L. Trentelman</td>
<td>The totally singular linear quadratic problem with indefinite cost</td>
</tr>
<tr>
<td>M 89-06</td>
<td>April</td>
<td>B.G. Hansen</td>
<td>Self-decomposable distributions and branching processes</td>
</tr>
<tr>
<td>M 89-07</td>
<td>April</td>
<td>B.G. Hansen</td>
<td>Note on Urbanik's class $L_\alpha$</td>
</tr>
<tr>
<td>M 89-08</td>
<td>April</td>
<td>B.G. Hansen</td>
<td>Reversed self-decomposability</td>
</tr>
<tr>
<td>M 89-09</td>
<td>April</td>
<td>A.A. Stoorvogel</td>
<td>The singular zero-sum differential game with stability using $H_\infty$ control theory</td>
</tr>
<tr>
<td>M 89-10</td>
<td>April</td>
<td>L.J.G. Langenhoff, W.H.M. Zijm</td>
<td>An analytical theory of multi-echelon production/distribution systems</td>
</tr>
<tr>
<td>M 89-11</td>
<td>April</td>
<td>A.H.W. Geerts</td>
<td>The Algebraic Riccati Equation and Singular Optimal Control</td>
</tr>
<tr>
<td>Number</td>
<td>Month</td>
<td>Author</td>
<td>Title</td>
</tr>
<tr>
<td>--------</td>
<td>-----------</td>
<td>----------------------------------</td>
<td>-------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>M 89-12</td>
<td>May</td>
<td>D.A. Overdijk</td>
<td>De geometrie van de kroonwieloverbrenging</td>
</tr>
<tr>
<td>M 89-13</td>
<td>May</td>
<td>I.J.B.F. Adan, J. Wessels, W.H.M. Zijm</td>
<td>Analysis of the shortest queue problem</td>
</tr>
<tr>
<td>M 89-14</td>
<td>June</td>
<td>A.A. Stoorvogel</td>
<td>The singular $H_{\infty}$ control problem with dynamic measurement feedback</td>
</tr>
<tr>
<td>M 89-15</td>
<td>June</td>
<td>A.H.W. Geerts, M.L.J. Hautus</td>
<td>The output-stabilizable subspace and linear optimal control</td>
</tr>
<tr>
<td>M 89-16</td>
<td>June</td>
<td>P.C. Schuur</td>
<td>On the asymptotic convergence of the simulated annealing algorithm in the presence of a parameter dependent penalization</td>
</tr>
<tr>
<td>M 89-17</td>
<td>July</td>
<td>A.H.W. Geerts</td>
<td>A priori results in linear-quadratic optimal control theory (extended version)</td>
</tr>
<tr>
<td>M 89-18</td>
<td>July</td>
<td>D.A. Overdijk</td>
<td>The curvature of conjugate profiles in points of contact</td>
</tr>
<tr>
<td>M 89-19</td>
<td>August</td>
<td>A. Dekkers, J. van der Wal</td>
<td>An approximation for the response time of an open CP-disk system</td>
</tr>
<tr>
<td>M 89-20</td>
<td>August</td>
<td>W.F.J. Verhaeg</td>
<td>On randomness of random number generators</td>
</tr>
<tr>
<td>M 89-21</td>
<td>August</td>
<td>P. Zwietering, E. Aarts</td>
<td>Synchronously Parallel: Boltzmann Machines: a Mathematical Model</td>
</tr>
<tr>
<td>M 89-22</td>
<td>August</td>
<td>I.J.B.F. Adan, J. Wessels, W.H.M. Zijm</td>
<td>An asymmetric shortest queue problem</td>
</tr>
<tr>
<td>M 89-23</td>
<td>August</td>
<td>D.A. Overdijk</td>
<td>Skew-symmetric matrices in classical mechanics</td>
</tr>
<tr>
<td>M 89-24</td>
<td>September</td>
<td>F.W. Steutel, J.G.F. Thiemann</td>
<td>The gamma process and the Poisson distribution</td>
</tr>
<tr>
<td>M 89-25</td>
<td>September</td>
<td>A.A. Stoorvogel</td>
<td>The discrete time $H_{\infty}$ control problem: the full-information case</td>
</tr>
<tr>
<td>M 89-26</td>
<td>October</td>
<td>A.H.W. Geerts, M.L.J. Hautus</td>
<td>Linear-quadratic problems and the Riccati equation</td>
</tr>
<tr>
<td>M 89-27</td>
<td>October</td>
<td>H.L. Trentelman, A.A. Stoorvogel</td>
<td>Completion of the squares in the finite horizon $H_{\infty}$ control problem by measurement feedback</td>
</tr>
</tbody>
</table>