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Ray-optical analysis of fields on shadow boundaries of two parallel plates

S. W. Lee*

Department of Electrical Engineering, University of Illinois, Urbana, Illinois 61801

J. Boersma

Department of Mathematics, Technological University of Eindhoven, Eindhoven, Netherlands

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The electromagnetic diffraction by two parallel plates of semi-infinite length is treated by ray methods. Two special problems are considered: (i) calculation of the fields in the forward and backward directions due to diffraction of a normally incident plane wave by two nonstaggered parallel plates; (ii) calculation of the field due to a line source in the presence of two staggered parallel plates when the source, the two edges, and the observation point are on a straight line. The crucial step in the ray-optical analysis is the calculation of the interaction between the plates. This calculation is performed by two methods, namely, the uniform asymptotic theory of edge diffraction and the method of modified diffraction coefficient. The relative merits of the two methods are discussed. The ray-optical solution of problem (i) agrees with the asymptotic expansion (plate separation large compared to wavelength) of the exact solution.

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I. INTRODUCTION

This paper is concerned with the solution by ray methods, of some electromagnetic diffraction problems for a set of two perfectly conducting, parallel plates of semi-infinite length. More specifically, the paper consists of three parts dealing with:

(i) The calculation of the electromagnetic fields in the forward and backward directions in the case of diffraction of a normally incident plane wave by two nonstaggered parallel plates (Sec. II). This calculation is based on the uniform asymptotic theory of edge diffraction,¹⁻³ and its extension as utilized in Refs. 4, 5.

(ii) The study of the same problem as in (i) by the method of modified diffraction coefficient^{6,7} (Sec. III).

(iii) The calculation of the electromagnetic field due to a line source in the presence of two staggered parallel plates when the source, the two edges and the observation point are on a straight line (Sec. IV). The limiting case of plane wave excitation in a direction parallel to the line through the edges is discussed as well. The calculation is based on a combination of the uniform asymptotic theory and the method of modified diffraction coefficient.

The motivations and conclusions of our investigation are stated below.

First, the physical problems themselves are of interest as they relate to the wave propagation over sharp ridges; see the introduction of Ref. 8 and the literature quoted there.

Our second, and main, motivation is to show that ray methods provide an effective tool for the (high-frequency) asymptotic analysis of diffraction problems involving parallel-plate configurations. The analysis for such configurations is by no means trivial. In order to explain the difficulties encountered, we present a brief outline of the ray-optical approach to the diffraction problems stated above. In both problems, the incident

wave when hitting the first plate, generates a primary diffracted field. The latter field is a cylindrical wave centred at the diffracting edge and as such is determined by Keller's geometrical theory of diffraction.^{9,10} The primary diffracted field in turn acts as an incident wave on the second plate and gives rise to secondary diffraction. The secondary diffracted field will interact again with the first plate thus leading to higher-order diffractions. The actual calculation of the secondary diffracted field is complicated by the fact that the second edge lies on the geometrical-optics shadow boundary of the incident wave, due to the first plate. In the case of diffraction by two nonstaggered plates, an additional and similar difficulty comes up at the calculation of the higher-order interaction fields. In the case of multiple diffraction the backscattered direction coincides with the shadow boundary of the specularly reflected wave or, in other words, each edge lies on the ray-optical reflection boundary of the opposite plate. Now, as is well known, Keller's theory is not valid along shadow boundaries.

In order to overcome this difficulty, three different methods have been proposed in recent years, namely, the method of Yee, Felsen, and Keller (YFK),¹¹ the method of modified diffraction coefficient (MDC),^{6,7} and the uniform asymptotic theory of edge diffraction (UAT).¹⁻³ In the approach by YFK each interaction field is approximated by the field of an equivalent set of isotropic line sources, the source strengths being such as to provide the correct interaction field in the direction toward the opposite edge. Then the interaction fields are determined recursively by means of a special asymptotic formula for scattering of an isotropic cylindrical wave by a half-plane. Originally, YFK was devised in connection with a ray-optical treatment of reflection in an open-ended parallel-plate waveguide. In view of the approximate character of YFK, it is not surprising that the final ray-optical solution of the reflection problem fails to agree with the asymptotic expansion (width of waveguide large com-

pared to wavelength) of the exact solution. A corrected ray-optical solution, based on UAT and in complete agreement with the asymptotic form of the exact solution, was recently derived in Refs. 4, 5.

In the present paper, the successive diffracted fields are calculated by means of MDC and UAT. The first method, MDC, employs a modified diffraction coefficient for diffraction by a half-plane in the presence of a second parallel half-plane. This modified coefficient, which automatically includes the interaction between the diffracting edge and the second half-plane, is derived from the solution of a canonical problem. The second method, UAT, is applicable to diffraction of an arbitrary incident wave by a plane screen. UAT provides an asymptotic solution of the diffraction problem that is uniformly valid near the edge and the shadow boundaries. Away from these regions the solution reduces to an expansion for the diffracted field which contains Keller's result as its leading term. Higher-order terms are obtained as well whereas Keller's theory is incapable of determining these terms.

In the ray-optical analysis of the parallel-plate diffraction problems, both MDC and UAT turn out to be effective methods, although not to the same extent (see the discussion below). For the case of nonstaggered parallel plates, an exact solution to the diffraction problem is obtainable by the Wiener-Hopf technique^{12,13}; see Appendix A for a brief discussion of this exact solution. Our ray-optical solution given in (II.68), (II.70) and based on UAT, agrees exactly with the asymptotic expansion (plate separation large compared to wavelength) of the exact solution. A second ray-optical solution, given in (III.11), (III.12) and based on MDC, precisely recovers the exact far field solution. For the case of staggered parallel plates, a partial solution ignoring interaction between the plates was recently derived by Jones.⁸ Excluding interaction terms, our ray-optical solution (IV.27), (IV.30) is found to agree with Jones' rigorous asymptotic result.

The ray-optical analysis of this paper also provides a clear insight into the relative merits of MDC and UAT. Our conclusions are: (i) As Keller's theory, UAT describes a general method which in principle can be applied to all edge diffraction problems. On the other hand, MDC is designed to attack diffraction by special configurations involving two parallel plates, and those only. For example, in the diffraction problem for two staggered parallel plates (Sec. IV), the ray-optical solution can be obtained by UAT alone, but not by MDC alone. (ii) When both methods apply, MDC appears simpler than UAT, as demonstrated by the example in Secs. II and III.

Finally we list some conventions to be used throughout this paper: (i) The time factor is $\exp(-i\omega t)$ and is suppressed. (ii) All problems are two-dimensional (no z variation). Both the TM case (nonzero field components H_x, E_x, E_y) and the TE case (nonzero field components E_x, H_x, H_y) are treated simultaneously, with the help of two symbols u and τ such that

$$\begin{aligned} \text{for TM } u &= H_x, \quad \tau = +1, \\ \text{for TE } u &= E_x, \quad \tau = -1. \end{aligned}$$

It is convenient to associate τ with the reflection coefficient of the field u from a perfectly conducting plane. (iii) The total field u^t is the sum of the incident field u^i and the scattered field u . Additional subscripts in u^t and u (e.g., u_{cs}^t , u_3 , etc.) are employed to identify the sequence of fields arising in the multiple interaction between the parallel plates.

II. NONSTAGGERED PARALLEL PLATES: SOLUTION BY UNIFORM ASYMPTOTIC THEORY

A. Statement of problem and approach

The configuration of a pair of nonstaggered parallel plates and our choice of coordinates are sketched in Fig. 1. The polar coordinates $\{r_m, \phi_m\}$; $m=0, \pm 1, \pm 2, \dots$ have origins at $\{x=0, y=ma\}$. The angle ϕ_m is measured in a counterclockwise sense when m is positive, and clockwise when $m=0$ or m is negative; furthermore, $0 \leq \phi_m \leq 2\pi$. Let the incident plane wave propagate in the negative y direction and be given by

$$u^i(x, y) = \exp(-iky). \quad (\text{II.1})$$

The problem at hand is to derive a high-frequency approximation for the far field in the forward direction ($x=0, ky \rightarrow -\infty$) and the backward direction ($x=0, ky \rightarrow \infty$) of the incident plane wave.

Our approach is outlined below. The incident field (II.1) first reaches the upper plate $x \leq 0, y=a$, and scattering produces a total field $u_1^t(r_1, \phi_1)$ that is written as

$$u_1^t(r_1, \phi_1) = \exp(-iky) + u_1(r_1, \phi_1), \quad (\text{II.2})$$

where u_1 denotes the scattered field. The field u_1^t in turn acts as an incident field on the lower plate $x \leq 0, y=0$. Scattering of u_1^t at the lower plate gives rise to a scattered field $u_2(r_0, \phi_0)$, which will interact again with the upper plate and yield a scattered field $u_3(r_1, \phi_1)$. In this manner there results a sequence of scattered fields

$$u_1(r_1, \phi_1), u_2(r_0, \phi_0), u_3(r_1, \phi_1), u_4(r_0, \phi_0), \dots \quad (\text{II.3})$$

Note that $u_n(r_1, \phi_1)$ with n odd arises from a scattering at the upper plate; whereas $u_n(r_0, \phi_0)$ with n even arises

FIG. 1. Two nonstaggered parallel plates illuminated by a normally incident plane wave.

FIG. 2. A half-plane illuminated by a line source at $(x = -d \cos \Omega, y = d \sin \Omega)$.

from a scattering at the lower plate. A useful property of the scattered fields is

$$u_n(r_1, \phi_1) = -\pi u_n(r_1, 2\pi - \phi_1), \quad n \text{ odd}, \quad (\text{II. 4a})$$

$$u_n(r_0, \phi_0) = -\pi u_n(r_0, 2\pi - \phi_0), \quad n \text{ even}. \quad (\text{II. 4b})$$

This symmetry relation is a consequence of the fact that u_n is the scattered field from a single plate, as if the other plate were absent. For the sequence of scattered fields in (II. 3) we will determine them recursively instead of consecutively. A special form of u_n is assumed, and it is used to derive u_{n+1} by the uniform asymptotic theory, which is summarized in Sec. II B. Comparing the expression of u_{n+1} thus obtained with the assumed form of u_n after replacing n by $n+1$ in the latter, we obtain two recurrence relations in Sec. II C. Next we solve the recurrence relations in Sec. II D, and present the final results for the scattered fields on the shadow boundaries of the incident and reflected fields in Sec. II E.

B. Summary of uniform asymptotic theory

The uniform asymptotic theory of edge diffraction was developed in Refs. 1 and 3 for the scalar wave, and in Ref. 2 for the vectorial wave. Here we summarize its explicit formulas for a two-dimensional problem, and they constitute a theoretical basis for our analysis in Secs. II and IV of this paper.

Referring to Fig. 2, let the half-plane $x \leq 0, y = 0$ be illuminated by a cylindrical wave due to a line source located at $x = -d \cos \Omega, y = d \sin \Omega, 0 < \Omega < \pi$. Polar coordinates $\{r_1, \phi_1\}$ with origin at the source point, and $\{r_0, \phi_0\}$ with origin at the edge $\{x = 0, y = 0\}$ will be employed. We assume the incident cylindrical wave is given by the asymptotic representation:

$$u^i(r_1, \phi_1) \sim \exp(ikr_1) z^i(r_1, \phi_1), \quad k \rightarrow \infty, \quad (\text{II. 5a})$$

$$= \exp(ikr_1) \sum_{m=0}^{\infty} (ik)^{-m} z_m^i(r_1, \phi_1). \quad (\text{II. 5b})$$

Then the total field u^t is found to be

$$u^t(r_0, \phi_0) = U(r_0, \phi_0) + \tau U(r_0, 4\pi - \phi_0), \quad (\text{II. 6a})$$

where the double-valued function U is represented by a uniform asymptotic expansion:

$$U(r_0, \phi_0) \sim U^G(r_0, \phi_0) + U^d(r_0, \phi_0), \quad k \rightarrow \infty, \quad (\text{II. 6b})$$

where

$$U^G(r_0, \phi_0) \sim \exp[ik(r_0 + d)] [F(k^{1/2} \xi_0^*) - \hat{F}(k^{1/2} \xi_0^*)] z^i(r_1, \phi_1),$$

$$U^d(r_0, \phi_0) \sim \exp[ik(r_0 + d)] k^{-1/2} \sum_{m=0}^{\infty} (ik)^{-m} \hat{v}_m(r_0, \phi_0).$$

The various notations which appeared in (II. 6) are explained below. The Fresnel integral $F(x)$ is defined by

$$F(x) = \pi^{-1/2} \exp(-i\pi/4) \exp(-ix^2) \int_{-\infty}^x \exp(it^2) dt. \quad (\text{II. 7})$$

Its asymptotic expansion for large x is

$$F(x) \sim \exp(-ix^2) H(x) + \hat{F}(x), \quad x \rightarrow \pm \infty. \quad (\text{II. 8})$$

Here $H(x)$ is the unit step function, i. e., $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$, and

$$\hat{F}(x) = -\frac{\exp(i\pi/4)}{2\pi x} \sum_{m=0}^{\infty} \Gamma(m + \frac{1}{2}) (ix^2)^{-m}, \quad (\text{II. 9})$$

where the Gamma function $\Gamma(m + \frac{1}{2})$ is given by

$$\Gamma(m + \frac{1}{2}) = \sqrt{\pi} (\frac{1}{2})(\frac{3}{2}) \cdots ((2m-1)/2). \quad (\text{II. 10})$$

The Taylor expansion of $F(x)$ around $x = 0$ is

$$F(x) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{\exp(-iq\pi/4)}{\Gamma(q/2 + 1)} x^q, \quad (\text{II. 11})$$

which is convergent for each x . The function ξ_0^* in (II. 6) is defined by

$$\xi_0^* = (r_0 + d - r_1)^{1/2} \text{sgn}[\cos \frac{1}{2}(\phi_0 - \Omega)] \\ = [4dr_0/(r_0 + d + r_1)]^{1/2} \cos \frac{1}{2}(\phi_0 - \Omega). \quad (\text{II. 12})$$

Note that $\xi_0^* = 0$ along the shadow boundary $\phi_0 = \Omega + \pi$ of the incident wave. The sign of ξ_0^* is such that $\xi_0^* > 0$ ($\xi_0^* < 0$) when the observation point (r_0, ϕ_0) is in the illuminated region (shadow region) of the incident wave. Note that $(\xi_0^*)^2$ measures the excessive ray path from the source to the observation point via the edge of the half-plane. The two leading coefficients of the series in (II. 6) have been generally determined in Ref. 4 and in the present case are given by

$$\hat{v}_0(r_0, \phi_0) = -\frac{\exp(i\pi/4)}{2(2\pi)^{1/2}} z_0^i(r_1 = d, \phi_1 = \pi - \Omega) \\ \times r_0^{-1/2} \sec \frac{1}{2}(\phi_0 - \Omega), \quad (\text{II. 13})$$

$$\hat{v}_1(r_0, \phi_0) = -\frac{\exp(i\pi/4)}{2(2\pi)^{1/2}} \left[z_1^i(d, \pi - \Omega) r_0^{-1/2} \sec \frac{1}{2}(\phi_0 - \Omega) \right. \\ + \frac{1}{4} \left(\frac{3}{2d} z_0^i(d, \pi - \Omega) \sin^2 \frac{1}{2}(\phi_0 - \Omega) \right. \\ - \frac{\partial z_0^i(d, \pi - \Omega)}{\partial r_1} \cos(\phi_0 - \Omega) \\ + \frac{1}{d} \frac{\partial z_0^i(d, \pi - \Omega)}{\partial \phi_1} \sin(\phi_0 - \Omega) \Big) r_0^{-1/2} \sec^3 \frac{1}{2}(\phi_0 - \Omega) \\ \left. + \frac{1}{4} z_0^i(d, \pi - \Omega) r_0^{-3/2} \sec^3 \frac{1}{2}(\phi_0 - \Omega) \right]. \quad (\text{II. 14})$$

There exists a recursive formula for the determination of higher order $\hat{v}_m$.³ They are not needed here since throughout this paper we are only interested in terms up to the order of $k^{-3/2}$.

The expression in (II. 6) for the total field is uniformly valid for all $0 < r_0 < \infty$ and $0 \leq \phi_0 \leq 2\pi$. It is convenient to interpret the first term $U(r_0, \phi_0)$ in (II. 6a) as a contribution to the total field associated with the incident field, while the second term $U(r_0, 4\pi - \phi_0)$ as that associated with the reflected field. Let us concentrate on $U(r_0, \phi_0)$ given in (II. 6b), and consider the following two cases:

(i) Away from the shadow boundary and the edge $|k^{1/2} \xi_0^*| \gg 1$: The use of (II. 8) into (II. 6b) leads to the conclusion that U^G recovers the classical geometrical

optics field, and U^d gives the edge-diffracted field with its first term identical to Keller's results.^{9,10}

(ii) In the immediate neighborhood of the shadow boundary and/or edge $k^{1/2}\xi_0^* \rightarrow 0$: Both U^G and U^d become infinite, but the singularities in $\hat{F}(k^{1/2}\xi_0^*)$ cancel exactly those in $\{\hat{v}_m\}$. Consequently, U is continuous and well defined across $k^{1/2}\xi_0^* = 0$.

Between the above two extreme cases, $U(r_0, \phi_0)$ in (II. 6a) provides a smooth transition and, therefore, is called a uniform asymptotic expansion for large k . Similar comments apply to $U(r_0, 4\pi - \phi_0)$, the second term in (II. 6a).

C. Multiple scattering between plates

In this section we consider the multiple scattering of the incident field (II. 1) between the two parallel plates in Fig. 1 and derive recurrence relations for the multiply scattered fields.

First let us determine the total field $u_1^t(r_1, \phi_1)$ due to the scattering of the incident field (II. 1) at the upper plate. The solution of this Sommerfeld half-plane problem is well known, and can be written as (see Ref. 3)

$$u_1^t(r_1, \phi_1) = \exp[ik(r_1 - a)] \{ F[-\sqrt{2kr_1} \cos \frac{1}{2}(\phi_1 + \frac{1}{2}\pi)] + \tau F[-\sqrt{2kr_1} \cos \frac{1}{2}(\phi_1 - \frac{1}{2}\pi)] \}. \quad (\text{II. 15})$$

The latter result can be also derived by means of the uniform asymptotic theory. In the backward direction of the incident field $\phi_1 = 3\pi/2$, we may replace the first Fresnel integral in (II. 15) by its asymptotic expansion (II. 8) and the second Fresnel integral becomes equal to $F(0) = \frac{1}{2}$. Retaining only the leading terms we have

$$u_1^t(r_1, 3\pi/2) = u^i + \exp[ik(r_1 - a)] \times \left\{ \frac{1}{2}\tau - [\exp(i\pi/4)/2\sqrt{2\pi}] k^{-1/2} r_1^{-1/2} + O(k^{-3/2}) \right\}. \quad (\text{II. 16})$$

Furthermore, in the interior region $0 \leq \phi_1 \leq \pi$, the use of (II. 8) in the second Fresnel integral in (II. 15) leads to

$$u_1^t(r_1, \phi_1) = \exp[ik(r_1 - a)] \{ F[-\sqrt{2kr_1} \cos \frac{1}{2}(\phi_1 + \frac{1}{2}\pi)] + [\tau \exp(i\pi/4)/2\sqrt{2\pi}kr_1] \sec \frac{1}{2}(\phi_1 - \frac{1}{2}\pi) + O(k^{-3/2}) \}, \quad 0 \leq \phi_1 \leq \pi. \quad (\text{II. 17})$$

The field u_1^t acts as an incident field on the lower plate, and the resultant scattered field $u_2(r_0, \phi_0)$ is to be determined by means of the uniform asymptotic theory. However, the uniform theory cannot be immediately applied because of the fact that the incident field u_1^t in (II. 17) is not locally a cylindrical wave in the direction of $\phi_1 = \pi/2$. To circumvent this difficulty, we follow the method in Ref. 4: we replace the Fresnel integral $F(x)$ by its Taylor expansion in (II. 11), and (II. 17) becomes

$$u_1^t(r_1, \phi_1) = \exp[ik(r_1 - a)] \left\{ \sum_{q=0}^{\infty} \frac{\exp(-iq\pi/4)}{\Gamma(\frac{1}{2}q+1)} (-1)^q (2kr_1)^{q/2} \times \cos^{q\frac{1}{2}}(\phi_1 + \frac{1}{2}\pi) + \frac{\tau \exp(i\pi/4)}{2\sqrt{2\pi}kr_1} \sec \frac{1}{2}(\phi_1 - \frac{1}{2}\pi) + O(k^{-3/2}) \right\}, \quad 0 \leq \phi_1 \leq \pi. \quad (\text{II. 18})$$

The representation in (II. 18) comprises an infinite sum of cylindrical waves centered at the upper edge $r_1 = 0$, and is convergent throughout the interior region $0 \leq \phi_1 \leq \pi$. We now perform a term-by-term application of the uniform theory. To each cylindrical-wave term in (II. 18) the uniform theory is applied, and the corresponding scattered field constituent may be evaluated. Collecting the latter constituents, we obtain the scattered field $u_2(r_0, \phi_0)$. We do not perform this computation in detail, since later on we will derive a general result for the scattered field u_n which includes u_2 as a special case.

Consider now the scattered field $u_n(r_1, \phi_1)$, n odd, arising at the upper edge, and $u_n(r_0, \phi_0)$, n even, arising at the lower edge. Uniform expansions for these fields will be derived valid in the interior region $0 \leq \phi_1 \leq \pi$, $0 \leq \phi_0 \leq \pi$. Similar to the discussion in Sec. 7 of Ref. 4, we introduce the following ansatz for the uniform expansions:

$$u_n(r_1, \phi_1) = \frac{1}{2} \exp[ik[r_1 + (n-2)a]] \left\{ \sum_{q=0}^{\infty} \exp(-iq\pi/4) u_{n,q}(r_1, \phi_1) \times (k^{1/2}\xi_1)^q + \frac{\exp(i\pi/4)}{\sqrt{2\pi}} k^{-1/2} \sum_{q=0}^{\infty} \exp(-iq\pi/4) \times v_{n,q}(r_1, \phi_1) (k^{1/2}\xi_1)^q + O(k^{-1}) \right\} - \delta_{n1} u^i, \quad n \text{ odd}, \quad 0 \leq \phi_1 \leq \pi, \quad (\text{II. 19})$$

$$u_n(r_0, \phi_0) = \frac{\tau}{2} \exp[ik[r_0 + (n-2)a]] \left\{ \sum_{q=0}^{\infty} \exp(-iq\pi/4) u_{n,q}(r_0, \phi_0) \times (k^{1/2}\xi_0)^q + \frac{\exp(i\pi/4)}{\sqrt{2\pi}} k^{-1/2} \sum_{q=0}^{\infty} \exp(-iq\pi/4) \times v_{n,q}(r_0, \phi_0) (k^{1/2}\xi_0)^q + O(k^{-1}) \right\}, \quad n \text{ even}, \quad 0 \leq \phi_0 \leq \pi, \quad (\text{II. 20})$$

where $\delta_{n1} = 1$ if $n=1$ and $\delta_{n1} = 0$ if $n \neq 1$, and ξ_1 and ξ_0 are given by

$$\xi_1 = (r_1 + a - r_2)^{1/2} \text{sgn}[\cos \frac{1}{2}(\phi_1 + \frac{1}{2}\pi)] = \left(\frac{4ar_1}{r_1 + a + r_2} \right)^{1/2} \cos \frac{1}{2}(\phi_1 + \frac{1}{2}\pi), \quad (\text{II. 21})$$

$$\xi_0 = (r_0 + a - r_{-1})^{1/2} \text{sgn}[\cos \frac{1}{2}(\phi_0 + \frac{1}{2}\pi)] = \left(\frac{4ar_0}{r_0 + a + r_{-1}} \right)^{1/2} \cos \frac{1}{2}(\phi_0 + \frac{1}{2}\pi). \quad (\text{II. 22})$$

The ansatz in (II. 19) and (II. 20) describes the first and second terms of a high-frequency expansion in inverse powers of k . Each of these terms is represented by a convergent Taylor series with coefficients $\{u_{n,q}\}$ and $\{v_{n,q}\}$, respectively, which are to be determined. It should be emphasized that each of these Taylor series is to be considered in its entirety and should not be looked at as a series that can be truncated after several terms. Once the scattered fields $\{u_n\}$ are determined in the interior region from (II. 19) and (II. 20), those in the exterior region $\pi \leq \phi_1 \leq 2\pi$, $\pi \leq \phi_0 \leq 2\pi$ follow immediately from the symmetry relation in (II. 4).

For $n=1$, the expansion (II. 19) should agree with (II. 2) and (II. 18), thus yielding

$$u_{1,q}(r_1, \phi_1) = \frac{(-1)^q}{\Gamma(q/2+1)} \frac{(2r_1)^{q/2} \cos^{q/2}(\phi_1 + \frac{1}{2}\pi)}{\xi_1^q} \\ = \frac{(-1)^q}{\Gamma(\frac{1}{2}q+1)} \left(\frac{r_1+a+r_2}{2a} \right)^{q/2}, \quad (\text{II } 23)$$

$$v_{1,q}(r_1, \phi_1) = \delta_{q0} \tau r_1^{-1/2} \sec \frac{1}{2}(\phi_1 - \frac{1}{2}\pi). \quad (\text{II } 24)$$

Scattering of the incident field u_n at the upper or lower plates gives rise to the scattered field u_{n+1} . The field u_{n+1} can be determined by a term-by-term application of the uniform asymptotic theory as summarized in Sec. IIB. The result for u_{n+1} thus obtained is to be compared with the ansatz (II. 19) and (II. 20) with n replaced by $(n+1)$. By equating corresponding terms we are led to a set of recurrence relations for the coefficients $u_{n,q}$ and $v_{n,q}$. It is found that the recurrence relations are exactly the same as those given in Refs. 4 and 5. Upon specializing to $\phi_0 = \phi_1 = \pi/2$, the recurrence relations become

$$u_{n+1,m}(r_0, \pi/2) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{u_{n,q}(r_0+a, \pi/2)}{\Gamma(m/2-q/2+1)} \left(\frac{r_0}{r_0+2a} \right)^{q/2}, \quad (\text{II } 25)$$

$$v_{n+1,m}(r_0, \pi/2) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{v_{n,q}(r_0+a, \pi/2)}{\Gamma(m/2-q/2+1)} \left(\frac{r_0}{r_0+2a} \right)^{q/2} \\ - \frac{\tau}{2} \delta_{m0} u_{n,0}(a, \pi/2) r_0^{-1/2} - \delta_{m0} \pi^{1/2} \\ \times \frac{r_0^{1/2}}{a^{1/2}(r_0+a)^{1/2}} \sum_{q=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2}-q)} \\ \times \frac{\partial u_{n,2q}(r_0+a, \pi/2)}{\partial \phi_0} \left(\frac{r_0}{r_0+2a} \right)^q, \quad (\text{II } 26)$$

where $m=0, 1, 2, \dots$ and $n=1, 2, \dots$, provided that the following "finiteness condition" is satisfied:

$$u_{n,0}(a, \pi/2) = \pi^{1/2} \left(\frac{r_0+a}{a} \right)^{1/2} \sum_{q=0}^{\infty} \frac{u_{n,2q}(r_0+a, \pi/2)}{\Gamma(\frac{1}{2}-q)} \left(\frac{r_0}{r_0+2a} \right)^q. \quad (\text{II } 27)$$

In Sec. IID, it will be shown that coefficients $\{u_{n,q}\}$ do indeed satisfy (II. 27). The recurrence relations (II. 25) and (II. 26) are accompanied by the initial values:

$$u_{1,q}(r_0, \pi/2) = \frac{(-1)^q}{\Gamma(q/2+1)} \left(\frac{r_0+a}{a} \right)^{q/2}, \\ v_{1,q}(r_0, \pi/2) = \delta_{q0} \tau r_0^{-1/2}, \quad (\text{II } 28)$$

which are taken from (II. 23) and (II. 24). Furthermore, according to Ref. 4, the derivative $\partial u_{n,q}/\partial \phi_0$, which appeared in (II. 26), is determined by an additional recurrence relation

$$\frac{\partial u_{n+1,m}(r_0, \pi/2)}{\partial \phi_0} = \frac{1}{2} \frac{r_0}{r_0+a} \sum_{q=0}^{\infty} \frac{1}{\Gamma(m/2-q/2+1)} \\ \times \frac{\partial u_{n,q}(r_0+a, \pi/2)}{\partial \phi_0} \left(\frac{r_0}{r_0+2a} \right)^{q/2}, \quad (\text{II } 29)$$

subject to the initial conditions

$$\frac{\partial u_{1,q}(r_0, \pi/2)}{\partial \phi_0} = 0, \quad (\text{II } 30)$$

which is obtained by differentiation of (II. 23). Hence all derivatives $\{\partial u_{n,q}/\partial \phi_0\}$ vanish and (II. 26) simplifies to

$$v_{n+1,m}(r_0, \pi/2) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{v_{n,q}(r_0+a, \pi/2)}{\Gamma(m/2-q/2+1)} \left(\frac{r_0}{r_0+2a} \right)^{q/2} \\ - \frac{1}{2} \tau \delta_{m0} u_{n,0}(a, \pi/2) r_0^{-1/2}. \quad (\text{II } 31)$$

The latter recurrence relation holds for $n=1, 2, \dots$, $m=0, 1, \dots$. By defining

$$u_{0,q}(a, \pi/2) = -2, \quad v_{0,q}(r_0, \pi/2) = 0, \quad \text{for } q=0, 1, 2, \dots,$$

it is easily seen by comparing with (II. 28) that (II. 31) is also valid for $n=0$.

Let us summarize the results obtained so far. The coefficients $\{u_{n,q}(r_0, \pi/2)\}$ are determined by the recurrence relation and initial conditions

$$\left\{ \begin{aligned} u_{n+1,m}(r_0, \pi/2) &= \frac{1}{2} \sum_{q=0}^{\infty} \frac{u_{n,q}(r_0+a, \pi/2)}{\Gamma(m/2-q/2+1)} \left(\frac{r_0}{r_0+2a} \right)^{q/2}, \\ n &= 1, 2, \dots, m=0, 1, \dots, \\ u_{1,q}(r_0, \pi/2) &= \frac{(-1)^q}{\Gamma(q/2+1)} \left(\frac{r_0+a}{a} \right)^{q/2}, \quad q=0, 1, \dots \end{aligned} \right. \quad (\text{II } 32)$$

The coefficients $\{v_{n,q}(r_0, \pi/2)\}$ are determined by the recurrence relation and initial conditions:

$$\left\{ \begin{aligned} v_{n+1,m}(r_0, \pi/2) &= \frac{1}{2} \sum_{q=0}^{\infty} \frac{v_{n,q}(r_0+a, \pi/2)}{\Gamma(m/2-q/2+1)} \left(\frac{r_0}{r_0+2a} \right)^{q/2} \\ &\quad - \frac{1}{2} \tau \delta_{m0} u_{n,0}(a, \pi/2) r_0^{-1/2}, \\ n, m &= 0, 1, \dots, \\ v_{0,q}(r_0, \pi/2) &= 0, \quad q=0, 1, \dots, \quad u_{0,0}(a, \pi/2) = -2. \end{aligned} \right. \quad (\text{II } 33)$$

The solution of the recurrence relations (II. 32) and (II. 33) will be given in Sec. IID.

Once the recurrence relations are solved, we may calculate the desired field solutions as below. Setting $\phi_1 = \pi/2$ in (II. 19) and $\phi_0 = \pi/2$ in (II. 20), we have

$$u_n(r_1, \pi/2) = \frac{1}{2} \exp\{ik[r_1 + (n-2)a]\} \{u_{n,0}(r_1, \pi/2) \\ + [\exp(i\pi/4)/\sqrt{2\pi}] k^{-1/2} v_{n,0}(r_1, \pi/2) \\ + O(k^{-1})\} - \delta_{n1} \exp\{ik(r_1 - a)\}, \quad n \text{ odd}, \quad (\text{II } 34)$$

$$u_n(r_0, \pi/2) = \frac{1}{2} \tau \exp\{ik[r_0 + (n-2)a]\} \{u_{n,0}(r_0, \pi/2) \\ + [\exp(i\pi/4)/\sqrt{2\pi}] k^{-1/2} v_{n,0}(r_0, \pi/2) + O(k^{-1})\}, \\ n \text{ even}. \quad (\text{II } 35)$$

The total fields in the forward direction $\phi_0 = 3\pi/2$ and backward direction $\phi_1 = 3\pi/2$ of the incident field are given by

$$u^t(r_0, \phi_0 = 3\pi/2) = u_1^t(r_1 = r_0 + a, \phi_1 = \pi/2) + u_2(r_0, \phi_0 = 3\pi/2) \\ + \sum_{n=2}^{\infty} [u_{2n-1}(r_1 = r_0 + a, \phi_1 = \pi/2) + u_{2n}(r_0, \phi_0 = 3\pi/2)], \quad (\text{II } 36)$$

$$u^t(r_1, \phi_1 = 3\pi/2) = u_1^t(r_1, \phi_1 = 3\pi/2) + u_2(r_0 = r_1 + a, \phi_0 = \pi/2) \\ + \sum_{n=2}^{\infty} [u_{2n-1}(r_1, \phi_1 = 3\pi/2) \\ + u_{2n}(r_0 = r_1 + a, \phi_0 = \pi/2)]. \quad (\text{II } 37)$$

Let us consider the first terms in (II. 36) and (II. 37) in a little more detail. Since $u_1^t = u^i + u_1$, it follows from (II. 34) with $n=1$ and the symmetry relation in (II. 4) that

$$u_1^{\dagger}(r_1 = r_0 + a, \phi_1 = \pi/2) = \frac{1}{2} \exp(ikr_0) \{u_{1,0}(r_0 + a, \pi/2) + [\exp(i\pi/4)/\sqrt{2\pi}] k^{-1/2} \times v_{1,0}(r_0 + a, \pi/2) + O(k^{-1})\}, \quad (\text{II. 38})$$

$$u_1^{\dagger}(r_1, \phi_1 = 3\pi/2) = \exp[-ik(r_1 + a)] + u_1(r_1, \phi_1 = 3\pi/2) = \exp[-ik(r_1 + a)] - \tau u_1(r_1, \phi_1 = \pi/2) = \exp[-ik(r_1 + a)] + \tau \exp[ik(r_1 - a)] - \frac{1}{2} \tau \exp[ik(r_1 - a)] \times \{u_{1,0}(r_1, \pi/2) + [\exp(i\pi/4)/\sqrt{2\pi}] k^{-1/2} v_{1,0}(r_1, \pi/2) + O(k^{-1})\}. \quad (\text{II. 39})$$

When (II. 4), (II. 34), (II. 35), (II. 38), and (II. 39) are used in (II. 36) and (II. 37), we have the expressions for the total field in the forward and backward directions:

$$u^{\dagger}(r_0, \phi_0 = 3\pi/2) = \frac{1}{2} \exp(ikr_0) \sum_{n=0}^{\infty} \exp(i2nka) \{u_{2n+1,0}(r_0 + a, \pi/2) - u_{2n+2,0}(r_0, \pi/2) + [\exp(i\pi/4)/\sqrt{2\pi}] k^{-1/2} \times [v_{2n+1,0}(r_0 + a, \pi/2) - v_{2n+2,0}(r_0, \pi/2)] + O(k^{-1})\}, \quad (\text{II. 40})$$

$$u^{\dagger}(r_1, \phi_1 = 3\pi/2) = \exp[-ik(r_1 + a)] + \tau \exp[ik(r_1 - a)] - \frac{1}{2} \tau \exp[ik(r_1 - a)] \{u_{1,0}(r_1, \pi/2) + [\exp(i\pi/4)/\sqrt{2\pi}] k^{-1/2} \times v_{1,0}(r_1, \pi/2)\} + \frac{1}{2} \tau \exp[ik(r_1 - a)] \sum_{n=1}^{\infty} \exp(i2nka) \times \{[u_{2n,0}(r_1 + a, \pi/2) - u_{2n+1,0}(r_1, \pi/2)] + [\exp(i\pi/4)/\sqrt{2\pi}] k^{-1/2} [v_{2n,0}(r_1 + a, \pi/2) - v_{2n+1,0}(r_1, \pi/2)] + O(k^{-1})\}. \quad (\text{II. 41})$$

It is interesting to note that the total field in the forward and backward directions depends on $\{u_{n,0}\}$ and $\{v_{n,0}\}$ only.

D. Solution of recurrence relations

Consider first the recurrence relation in (II. 32). The same recurrence relation, subject to a different initial condition, was discussed in Ref. 4, Appendix C, where it was solved by a generating-function technique. Employing the same technique, we introduce the generating function

$$F_n(r_0; z) = \sum_{q=0}^{\infty} u_{n,q}(r_0, \pi/2) (iz)^q, \quad (\text{II. 42})$$

where z is a complex variable. Thus, it was shown in Ref. 4, Appendix C, that (II. 32) can be reduced to a recurrence relation for F_n expressed in terms of F_{n-1} , namely,

$$F_n(r_0; z) = \frac{1}{2\pi i} \int_{-\infty-i\alpha}^{\infty-i\alpha} \frac{\exp(-t^2)}{t-z} F_{n-1}\left(r_0 + a; t\left(\frac{r_0}{r_0 + 2a}\right)^{1/2}\right) dt \quad (\text{II. 43})$$

where $\alpha < \text{Im}z$. By repeated application of (II. 43), F_n can be expressed in terms of F_1 :

$$F_n(r_0; z) = \left(\frac{1}{2\pi i}\right)^{n-1} \int_{-\infty-i\alpha_1}^{\infty-i\alpha_1} \cdots \int_{-\infty-i\alpha_{n-1}}^{\infty-i\alpha_{n-1}} \exp\left(-\sum_{m=1}^{n-1} t_m^2\right) \times (t_1 - z)^{-1} \prod_{m=2}^{n-1} \left[t_m - t_{m-1} \left(\frac{r_0 + (m-2)a}{r_0 + ma}\right)^{1/2}\right]^{-1} \times F_1\left[r_0 + (n-1)a; t_{n-1} \left(\frac{r_0 + (n-2)a}{r_0 + na}\right)^{1/2}\right] \times dt_1 dt_2 \cdots dt_{n-1}, \quad (\text{II. 44})$$

where $\alpha_1 < \text{Im}z$, $\alpha_m < \alpha_{m-1} [r_0 + (m-2)a]^{1/2} [r_0 + ma]^{-1/2}$, $m = 2, 3, \dots, (n-1)$. From the initial condition in (II. 32) and the definition of F_1 in (II. 42), we find

$$F_1(r_0; z) = \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma(q/2 + 1)} \left(\frac{r_0 + a}{a}\right)^{q/2} (iz)^q = 2F\left[\exp(-i\pi/4)z\left(\frac{r_0 + a}{a}\right)^{1/2}\right]$$

according to (II. 11). Using a well-known integral representation for the Fresnel integral F , we have

$$F_1(r_0; z) = \frac{1}{\pi i} \int_{-\infty-i\alpha}^{\infty-i\alpha} \frac{\exp(-t^2)}{t + z[(r_0 + a)/a]^{1/2}} dt, \quad (\text{II. 45})$$

where $\alpha < -\text{Im}z\sqrt{(r_0 + a)/a}$, which is to be substituted in (II. 44). After simplification in a manner similar to that given in Ref. 4, Appendix C, we have the desired expression of F_n :

$$F_n(r_0; z) = 2\pi^{-n/2} \left(\frac{a}{r_0}\right)^{1/2} \int_0^{\infty} \cdots \int_0^{\infty} \exp\left[2ix_1z\left(\frac{r_0 + a}{r_0}\right)^{1/2} - \frac{r_0 + a}{r_0} x_1^2 - 2 \sum_{m=2}^{n-1} x_m^2 + 2 \sum_{m=1}^{n-2} x_m x_{m+1} - 2x_{n-1}x_n - x_n^2\right] \times dx_1 dx_2 \cdots dx_n. \quad (\text{II. 46})$$

The result in (II. 46) can easily be expanded in a power series of (iz) , comparable to (II. 42). Then it is found that the solution of $u_{n,q}(r_0, \pi/2)$ is given by

$$u_{n,q}(r_0, \pi/2) = \frac{2^{q+1}}{q!} \left(\frac{a}{r_0}\right)^{1/2} \left(\frac{r_0 + a}{r_0}\right)^{q/2} J_{n,q}(r_0), \quad n = 2, 3, \dots, q = 0, 1, 2, \dots, \quad (\text{II. 47})$$

where $J_{n,q}$ is an n -fold integral defined by

$$J_{n,q}(r_0) = \pi^{-n/2} \int_0^{\infty} \cdots \int_0^{\infty} x_1^q \exp\left(-\frac{r_0 + a}{r_0} x_1^2 - 2 \sum_{m=2}^{n-1} x_m^2 + 2 \sum_{m=1}^{n-2} x_m x_{m+1} - 2x_{n-1}x_n - x_n^2\right) \times dx_1 dx_2 \cdots dx_n. \quad (\text{II. 48})$$

The result in (II. 47) and (II. 48) together with the initial coefficient $u_{1,q}(r_0, \pi/2)$ in (II. 32) constitutes the solution for the recurrence relation in (II. 32). It can be shown that this solution satisfies the "finiteness condition" in (II. 27).

Next let us turn to the second recurrence relation in (II. 33). Except for the inhomogeneous term, this relation is identical to Eq. (C4) in Ref. 4. Hence, its solution can be derived in exactly the same manner with the result

$$v_{n,q}(r_0, \pi/2) = -\frac{\tau 2^q}{q! \sqrt{r_0}} \left(\frac{r_0 + a}{r_0}\right)^{q/2} \sum_{m=1}^n J_{m-1,0}(a) I_{n-m,q}(r_0), \quad (\text{II. 49})$$

where

$$J_{0,0}(a) = -1, \quad J_{1,0}(a) = \frac{1}{2}, \quad I_{0,q}(r_0) = \delta_{q0}, \quad (\text{II. 50})$$

$$I_{n,q}(r_0) = \pi^{-n/2} \int_0^\infty \cdots \int_0^\infty x_1^n \exp\left(-\frac{r_0+a}{r_0} x_1^2\right) \\ - 2 \sum_{m=2}^n x_m^2 + 2 \sum_{m=1}^{n-1} x_m x_{m+1} \Big) dx_1 dx_2 \cdots dx_n, \\ n=1, 2, \dots, q=0, 1, 2, \dots \quad (\text{II. 51})$$

According to (II. 40) and (II. 41), the total field in the directions $\phi_0 = 3\pi/2$ and $\phi_1 = 3\pi/2$ only depends on the coefficients $\{u_{n,0}(r_0, \pi/2)\}$ and $\{v_{n,0}(r_0, \pi/2)\}$. Therefore, we present the special results of (II. 47) and (II. 49):

$$u_{1,0}(r_0, \pi/2) = 1; \quad u_{n,0}(r_0, \pi/2) = 2(a/r_0)^{1/2} J_{n,0}(r_0), \\ n=2, 3, \dots, \quad (\text{II. 52})$$

$$v_{n,0}(r_0, \pi/2) = -\tau(1/r_0)^{1/2} \sum_{m=1}^n J_{m-1,0}(a) I_{n-m,0}(r_0), \\ n=1, 2, \dots \quad (\text{II. 53})$$

For later use, we derive simple closed-form results for $u_{2,0}$ and $v_{2,0}$. We evaluate the double integral

$$J_{2,0}(r_0) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \exp\left(-\frac{r_0+a}{r_0} x_1^2 - 2x_1 x_2 - x_2^2\right) dx_1 dx_2 \\ (\text{II. 54})$$

by introducing the new variables $y_1 = (a/r_0)^{1/2} x_1$, $y_2 = x_1 + x_2$; then, (II. 54) passes into

$$J_{2,0}(r_0) = (1/\pi)(r_0/a)^{1/2} \int \int_S \exp(-y_1^2 - y_2^2) dy_1 dy_2, \\ (\text{II. 55})$$

where S is a sector described by $y_1 \geq 0$, $y_2 \geq (r_0/a)^{1/2} y_1$. The sector S has an interior angle $(\pi/2) - \tan^{-1}(r_0/a)^{1/2}$. Thus, we find easily

$$J_{2,0}(r_0) = \left(\frac{r_0}{a}\right)^{1/2} \frac{(\pi/2) - \tan^{-1}(r_0/a)^{1/2}}{2\pi} \quad (\text{II. 56})$$

and, consequently,

$$u_{2,0}(r_0, \pi/2) = \frac{1}{2} - (1/\pi) \tan^{-1}(r_0/a)^{1/2}. \quad (\text{II. 57})$$

The latter result has been checked by a direct computation based on (II. 32). The coefficient $v_{2,0}$ in (II. 49) becomes

$$v_{2,0}(r_0, \pi/2) = -\tau r_0^{-1/2} [J_{0,0}(a) I_{1,0}(r_0) + J_{1,0}(a) I_{0,0}(r_0)] \\ = \frac{1}{2} \tau (r_0 + a)^{-1/2} - \frac{1}{2} \tau r_0^{-1/2}, \quad (\text{II. 58})$$

which was also checked by a direct computation based on (II. 33).

Furthermore, we need the values of $J_{n,0}(r_0)$ and $I_{n,0}(r_0)$ as $r_0 \rightarrow \infty$, and $J_{n,0}(a)$. Their evaluations are given in Ref. 14. Here we list the final results:

$$J_{n,0}(\infty) = \frac{1}{2\pi\sqrt{n-1}}, \quad n=2, 3, \dots, \quad (\text{II. 59})$$

$$I_{n,0}(\infty) = \frac{1}{n!} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}, \quad n=0, 1, 2, \dots, \quad (\text{II. 60})$$

$$J_{n,0}(a) = \frac{-1}{n!} \frac{\Gamma(n-\frac{1}{2})}{\Gamma(-\frac{1}{2})}, \quad n=0, 1, 2, \dots \quad (\text{II. 61})$$

A comparison of (II. 59), (II. 61) with (II. 56) shows that they agree for $n=2$.

In summary, the solutions of the recurrence relations in (II. 32) and (II. 33) are given, respectively, in (II. 47) and (II. 49). The explicit solutions for $\{u_{n,0}(r_0, \pi/2)\}$ and $\{v_{n,0}(r_0, \pi/2)\}$ as $r_0 \rightarrow \infty$ are given in (II. 50), (II. 52), (II. 53), and (II. 59)–(II. 61).

E. Far fields in the forward and backward directions

Consider first the total field $u^t(r_0, 3\pi/2)$ in the forward direction $\phi_0 = 3\pi/2$, as given in (II. 40). On substitution of results in (II. 52), (II. 53), (II. 57), and (II. 58) for $\{u_{n,0}\}$ and $\{v_{n,0}\}$, we obtain

$$u^t\left(r_0, \frac{3\pi}{2}\right) = \exp(ikr_0) \left\{ \left[\frac{1}{4} + \frac{1}{2\pi} \tan^{-1}\left(\frac{r_0}{a}\right)^{1/2} \right] \right. \\ + a^{1/2} \sum_{n=1}^{\infty} \exp(i2nka) \left(\frac{J_{2n+1,0}(r_0+a)}{(r_0+a)^{1/2}} - \frac{J_{2n+2,0}(r_0)}{r_0^{1/2}} \right) \\ + k^{-1/2} \frac{\tau \exp(i\pi/4)}{2\sqrt{2}\pi} \left[\frac{1}{2\sqrt{r_0+a}} + \frac{1}{2\sqrt{r_0}} \right. \\ + \sum_{n=1}^{\infty} \exp(i2nka) \left(\frac{-1}{\sqrt{r_0+a}} \sum_{m=1}^{2n+1} J_{m-1,0}(a) \right. \\ \times I_{2n+1-m,0}(r_0+a) + \frac{1}{\sqrt{r_0}} \sum_{m=1}^{2n+2} J_{m-1,0}(a) \\ \times I_{2n+2-m,0}(r_0) \Big) \Big] + O(k^{-1}) \Big\}. \quad (\text{II. 62})$$

It has been verified that the first term in (II. 62), i. e.,

$$u^t(r_0, 3\pi/2) \Big|_{\text{first term}} = \exp(ikr_0) \left[\frac{1}{4} + (1/2\pi) \tan^{-1}(r_0/a)^{1/2} \right] \quad (\text{II. 63})$$

agrees with the result that is obtained by specialization of a rigorous asymptotic expansion for the field due to Jones.⁸ Jones did not take into account the interaction between the edges of the two plates, and hence did not obtain the other terms in (II. 62). For large values of r_0 , (II. 62) can be simplified, and we obtain the total far field in the forward direction

$$u^t\left(r_0, \frac{3\pi}{2}\right) = \exp(ikr_0) \left\{ \frac{1}{2} - \frac{1}{2\pi} \left(\frac{a}{r_0}\right)^{1/2} + \left(\frac{a}{r_0}\right)^{1/2} \right. \\ \times \sum_{n=1}^{\infty} \exp(i2nka) [J_{2n+1,0}(\infty) - J_{2n+2,0}(\infty)] \\ + \frac{\tau \exp(i\pi/4)}{2\sqrt{2}\pi k r_0} \left[1 + \sum_{n=1}^{\infty} \exp(i2nka) \left(- \sum_{m=1}^{2n+1} J_{m-1,0}(a) I_{2n+1-m,0}(\infty) \right. \right. \\ \left. \left. + \sum_{m=1}^{2n+2} J_{m-1,0}(a) I_{2n+2-m,0}(\infty) \right) \right] + O(k^{-1}) + O(r_0^{-3/2}) \Big\}. \quad (\text{II. 64})$$

The term of order $k^{-1/2}$ in (II. 64) can be considerably simplified. From (II. 60) and (II. 61) it follows that

$$- \sum_{m=1}^{p+1} J_{m-1,0}(a) I_{p+1-m,0}(\infty) = - \sum_{m=0}^p J_{m,0}(a) I_{p-m,0}(\infty) \\ = \sum_{m=0}^p \frac{\Gamma(m-\frac{1}{2})\Gamma(p-m+\frac{1}{2})}{m! \Gamma(-\frac{1}{2})(p-m)! \Gamma(\frac{1}{2})}, \quad (\text{II. 65})$$

where $p=2n$ or $p=2n+1$. The latter sum in (II. 65) is just the coefficient of t^p in the power-series expansion of the product

$$\left(\sum_{q=0}^{\infty} \frac{\Gamma(q-\frac{1}{2})}{q! \Gamma(-\frac{1}{2})} t^q\right) \left(\sum_{q=0}^{\infty} \frac{\Gamma(q+\frac{1}{2})}{q! \Gamma(\frac{1}{2})} t^q\right) = (1-t)^{1/2} (1-t)^{-1/2} = 1. \quad (\text{II. 66})$$

Note that both series in (II. 66) are binomial series which have been explicitly summed. Since the coefficient of t^0 in (II. 66) is equal to $\delta_{p,0}$, it follows immediately that

$$-\sum_{m=1}^{p+1} J_{m-1,0}(a) I_{p+1-m,0}(\infty) = \delta_{p,0}. \quad (\text{II. 67})$$

The use of (II. 67) and (II. 59) in (II. 64) leads to the final expression for the total far field in the forward direction:

$$u^{\dagger}\left(r_0, \phi_0 = \frac{3\pi}{2}\right) = \exp(ikr_0) \left[\frac{1}{2} - \frac{1}{2\pi} \left(\frac{a}{r_0}\right)^{1/2} + \frac{1}{2\pi} \left(\frac{a}{r_0}\right)^{1/2} \times \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \right) \exp(i2nka) + \frac{\tau e^{i\pi/4}}{2\sqrt{2\pi k r_0}} + O(k^{-1}) + O(r_0^{-3/2}) \right], \quad (\text{II. 68})$$

which agrees exactly with (A6) in Appendix A, which is an asymptotic expansion of the exact solution derived by the Wiener-Hopf technique.

Next consider the total field $u^{\dagger}$ in the backward direction $\phi_1 = 3\pi/2$, as given by (II. 41). On substitution of the results of (II. 52) and (II. 53) for the coefficients $\{u_{n,0}\}$, $\{v_{n,0}\}$, we obtain

$$u^{\dagger}\left(r_1, \frac{3\pi}{2}\right) = \exp[-ik(r_1+a)] + \exp[ik(r_1-a)] \times \left[\frac{\tau}{2} + \tau a^{1/2} \sum_{n=1}^{\infty} \exp(i2nka) \left(\frac{J_{2n,0}(r_1+a)}{\sqrt{r_1+a}} - \frac{J_{2n+1,0}(r_1)}{\sqrt{r_1}} \right) \right] + \frac{\exp(i\pi/4)}{2\sqrt{2\pi}} k^{-1/2} \exp[ik(r_1-a)] \times \left[-r_1^{-1/2} + \sum_{n=1}^{\infty} \exp(i2nka) \times \left(-\frac{1}{\sqrt{r_1+a}} \sum_{m=1}^{2n} J_{m-1,0}(a) I_{2n-m,0}(r_1+a) + \frac{1}{\sqrt{r_1}} \sum_{m=1}^{2n+1} J_{m-1,0}(a) I_{2n+1-m,0}(r_1) \right) \right] + O(k^{-1}). \quad (\text{II. 69})$$

As $r_1 \rightarrow \infty$, (II. 69) can be simplified in a similar manner as the reduction of (II. 62). The final expression for the total far field in the backward direction is given by

$$u^{\dagger}\left(r_1, \phi_1 = \frac{3\pi}{2}\right) = \exp[-ik(r_1+a)] + \exp[ik(r_1-a)] \times \left[\frac{\tau}{2} + \frac{\tau}{2\pi} \left(\frac{a}{r_1}\right)^{1/2} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n}} \right) \times \exp(i2nka) - \frac{\exp(i\pi/4)}{2\sqrt{2\pi k r_1}} + O(k^{-1}) + O(r_1^{-3/2}) \right] \quad (\text{II. 70})$$

which again agrees with the asymptotic expansion of the exact solution given in (A3), Appendix A.

Let us now comment on several key steps in the derivation of the final solution in (II. 68) and (II. 70):

(i) In the calculation of multiple scattering between

edges, the term-by-term application of the uniform asymptotic theory to the incident field in (II. 19) or (II. 20) is a formal procedure. As other formal procedures in ray-optical methods, its "justification" is its correct final result.

(ii) The derivation of the recurrence relations in (II. 25) and (II. 26) depends critically on the fact that the q th constituent of the incident field in (II. 19) [or (II. 20)] is proportional to ξ_1^q (or ξ_0^q), and ξ_1 is identically zero at the observation point, the location of the lower edge. Had the two plates been slightly staggered, simple recurrence relations as those in (II. 25) and (II. 26) could not have been derived.

(iii) The evaluations of the integral $J_{n,q}$ in (II. 48) and $I_{n,q}$ in (II. 51) are themselves interesting mathematical problems. In Ref. 14, two methods are used for their evaluations: one is elementary and involves transformation of variables in n -dimensional space and generating-function techniques, while the other uses integral equations, Fourier transforms, and Wiener-Hopf technique.

(iv) In two occasions in our derivation, the argument of analytical continuation was resorted to for extending the domain of convergence of the series involved. One occurs in the derivation of (II. 57) by a direct computation from (II. 32):

$$u_{2,0}\left(r_0, \frac{\pi}{2}\right) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{u_{1,q}(r_0+a, \pi/2)}{\Gamma(1-\frac{1}{2}q)} \left(\frac{r_0}{r_0+2a}\right)^{q/2} = \frac{1}{2} \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma(1-\frac{1}{2}q)\Gamma(1+\frac{1}{2}q)} \left(\frac{r_0+2a}{a}\right)^{q/2} \left(\frac{r_0}{r_0+2a}\right)^{q/2} = \frac{1}{2} - \frac{1}{\pi} \sum_{q=0}^{\infty} \frac{(-1)^q}{2q+1} \left(\frac{r_0}{a}\right)^{q+(1/2)}. \quad (\text{II. 71})$$

Note that this series converges to the right-hand side of (II. 57) only in the range $0 \leq r_0 \leq a$. To show that (II. 57) also holds for $r_0 > a$, one may invoke some analytical continuation argument. The other similar situation arises in the verifications of (II. 25) and (II. 27) by a direct substitution from (II. 47). [Yet another occurs later in the derivation of (IV. 19) in Sec. IV, where three series converge only when $|\eta| < 1$.]

Some numerical results calculated from (II. 68) will be presented in Sec. III C.

III. NONSTAGGERED PARALLEL PLATES: SOLUTION BY MODIFIED DIFFRACTION COEFFICIENT

A. Outline of approach

In this part of the paper, the same problem sketched in Fig. 1, namely, the diffraction of a normally incident plane wave by two nonstaggered parallel plates is attacked by a different ray method—the method of modified diffraction coefficient described in Refs. 6 and 7. The solution so obtained turns out to be in complete agreement with the exact far field solution given in Appendix A.

First let us outline the general approach. From the symmetry of the problem it follows (see pp. 137–138 of Ref. 13) that the original problem sketched in Fig. 1 can be replaced by two auxiliary ones: (i) a problem with a perfect electric wall (where the tangential electric field is zero) at $y = a/2$ (Fig. 3a), and (ii) a problem with a

FIG. 3. Two auxiliary problems for the problem sketched in Fig. 1.

magnetic wall (where the tangential magnetic field is zero) at $y=a/2$ (Fig. 3b). Once these two auxiliary problems are solved their solutions will be properly superimposed to yield the solution of the original problem. For the convenience of applying the method of modified diffraction coefficient, we generalize the problem by letting the incident field u^i come from the direction ϕ_1^i , where $(3\pi/2) < \phi_1^i \leq 2\pi$:

$$u^i(r_1, \phi_1) = \exp[-ikr_1 \cos(\phi_1 - \phi_1^i)] \exp(-ika), \quad (\text{III. 1})$$

where $\{r_1, \phi_1\}$ are polar coordinates with origin at $\{x=0, y=a\}$ (Fig. 3a). The problem is to determine the total field at an observation point (r_1, ϕ_1) , where $kr_1 \rightarrow \infty$ and $\pi \leq \phi_1 \leq 2\pi$. After the field is derived, we will set $\phi_1 = 3\pi - \phi_1^i$, and let ϕ_1^i go to $3\pi/2$ in order to obtain the desired field solutions in the forward and backward directions.

Let us concentrate on the problem with an electric wall (Fig. 3a). The incident field u^i in (III. 1) reaches the upper edge $x=0, y=a$, and diffraction there produces a scattered field $u_1(r_1, \phi_1)$. The field u_1 propagates along diffracted rays emanating from the edge. Then the field along the diffracted ray in the direction $\phi_1 = \pi/2$ is specularly reflected from the electric wall at $y=a/2$. The reflected field u_2 strikes the upper edge again, and diffraction there produces a scattered field $u_3(r_1, \phi_1)$. Such a diffraction and reflection sequence continues. The total field at (r_1, ϕ_1) is then given by

$$u^t(r_1, \phi_1) = u^i(r_1, \phi_1) + [u_1(r_1, \phi_1) + u_{\text{int}}(r_1, \phi_1)] \times [1 - H(x) \exp(ika |\sin \phi_1|)]. \quad (\text{III. 2})$$

Here u_{int} is the contribution from the interaction between the upper edge and the electric wall

$$u_{\text{int}}(r_1, \phi_1) = \sum_{n=1}^{\infty} u_{2n+1}(r_1, \phi_1). \quad (\text{III. 3})$$

The term with the unit step function $H(x)$ in (III. 2) is to account for the possible specular reflection at the electric wall of the outgoing diffracted rays emanating from the upper edge. When the observation point (r_1, ϕ_1) has a negative x coordinate, i. e., $3\pi/2 < \phi_1 \leq 2\pi$, $H(x) = 0$ in agreement with the fact that there is no such a specular reflection. When $x > 0$, the same factor $\exp(ika |\sin \phi_1|)$ accounts for the contribution of the specular reflection for both TM case ($\tau = +1$) and TE case ($\tau = -1$). This independence of τ is due to the combination of the facts that (i) the scattered field u_n satisfies the symmetry relation in (II. 4) and (ii) the reflection coefficient of u_n from the electric wall is τ .

The central step is to determine u_{int} . In the present approach, instead of determining $u_2, u_3, \dots$ successively, we will introduce a diffraction coefficient for the upper edge, a modified version of Keller's diffraction coefficient, and write down u_{int} in a single step.

B. Far fields in the forward and backward directions

Let us consider $u_1(r_1, \phi_1)$, the scattered field from the upper plate $x < 0, y = a$ due to an incident field (III. 1) (as if the electric wall at $y=a/2$ and lower plate $x < 0, y=0$ were absent). Following Keller's geometrical theory of diffraction,^{9,10} the far field solution of u_1 is the sum of the usual geometrical optics field and a diffracted field u_1^d . The latter is

$$u_1^d(r_1, \phi_1) \sim \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} D(\phi_1, \phi_1^i) u^i(r_1=0), \quad kr_1 \rightarrow \infty, \quad (\text{III. 4})$$

where $D(\phi_1, \phi_1^i)$ is known as Keller's diffraction coefficient

$$D(\phi_1, \phi_1^i) = - \left(\sec \frac{\phi_1 - \phi_1^i}{2} + \tau \sec \frac{\phi_1 + \phi_1^i}{2} \right). \quad (\text{III. 5})$$

The result in (III. 4) and (III. 5) is not valid in the neighborhood of shadow boundary of the incident field $\phi_1 = \phi_1^i - \pi$, or that of reflected field $\phi_1 = 3\pi - \phi_1^i$. In those neighborhoods we may use the exact Sommerfeld half-plane solution for the scattered field:

$$u_1(r_1, \phi_1) = \exp(ikr_1) \left[-F \left(-\sqrt{2kr_1} \cos \frac{\phi_1 - \phi_1^i}{2} \right) + \tau F \left(\sqrt{2kr_1} \cos \frac{\phi_1 + \phi_1^i}{2} \right) \right] u^i(r_1=0), \quad (\text{III. 6})$$

where F is the Fresnel integral defined in (II. 7). The result in (III. 6) can be also derived, of course, by the uniform asymptotic theory described in Sec. II B. When $\phi_1 \neq \phi_1^i - \pi$ and $\phi_1 \neq 3\pi - \phi_1^i$, the Fresnel integral in (III. 6) can be replaced by its asymptotic expansion according to (II. 8) and (II. 9). Retaining only the leading term, we recover (III. 4) and (III. 5), plus the usual geometrical optics field.

According to Ref. 6 and 7, the interaction term u_{int} in (III. 2) can be written in a similar form as (III. 4), and is given by

$$u_{\text{int}}(r_1, \phi_1) \sim \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} [\bar{D}(\phi_1, \phi_1^i) - D(\phi_1, \phi_1^i)] \quad (III. 7a)$$

where $\bar{D}(\phi_1, \phi_1^i)$ is a modified diffraction coefficient and is related to Keller's diffraction coefficient in (III. 5) by

$$\bar{D}(\phi_1, \phi_1^i) = D(\phi_1, \phi_1^i) f(\phi_1) f(\phi_1^i), \quad (III. 7b)$$

$$f(\phi_1) = \begin{cases} 1/G_+(k|\cos\phi_1|), & \pi/2 < \phi_1 < 3\pi/2, \\ G_+(k|\cos\phi_1|), & 0 \leq \phi_1 < \pi/2, \text{ or } 3\pi/2 < \phi_1 \leq 2\pi. \end{cases} \quad (III. 7c)$$

The function $G_+(\alpha)$ is described in Appendix B. Several remarks about the formula in (III. 7) are in order:

(i) $\bar{D}$ is the exact diffraction coefficient for the edge diffraction by a perfectly conducting half-plane in the presence of a parallel, infinite electric wall at distance $a/2$. It was derived from the rigorous solution of a canonical problem.

(ii) In case that the infinite electric wall (Fig. 3a) is replaced by an infinite magnetic wall (Fig. 3b), (III. 7) remains valid after replacing $G_+(\alpha)$ by $\tilde{G}_+(\alpha)$. The function $\tilde{G}_+(\alpha)$ is also described in Appendix B.

(iii) The formula (III. 7) is valid for both TM and TE cases. The difference in these two cases enters through D in (III. 5).

(iv) Apparently, $f(\phi_1)$ and hence $\bar{D}(\phi_1, \phi_1^i)$ are not continuous across $\phi_1 = 3\pi/2$, since $G_+(0) = [1 - \exp(ika)]^{1/2} \neq 1$. However, in (III. 2) this discontinuity is compensated by the term with unit step function $H(x)$, and as a result the total field u^i is continuous across $\phi_1 = 3\pi/2$.

(v) In Refs. 6 and 7, u_1^i in (III. 4) and u_{int} in (III. 7) are combined in a single term. For the present application it is more convenient to separate out u_1^i , which is the component that becomes infinite on shadow boundaries and should be replaced by u_i in (III. 6).

Concerning the result in (III. 7), we are particularly interested in the field exactly on the reflected shadow boundary. Letting $\phi_1 \rightarrow (3\pi - \phi_1^i)$ in (III. 7), we obtain in the limit

$$u_{\text{int}}(r_1, \phi_1 = 3\pi - \phi_1^i) \sim \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} \left(2\tau \sin\phi_1^i \frac{kG'_+(k|\cos\phi_1^i|)}{G_+(k|\cos\phi_1^i|)} \right) u^i(r_1 = 0), \quad (III. 8)$$

where $G'_+(\alpha)$ means the derivative of $G_+(\alpha)$ with respect to α . $G'_+(\alpha)$ is also discussed in Appendix B.

In summary, for the problem sketched in Fig. 3a with an incident field in (II. 1), the total far field solution ($kr_1 \rightarrow \infty$) is given by (III. 2), (III. 6), and (III. 7) when $3\pi/2 < \phi_1 \leq 2\pi$, $\pi \leq \phi_1 \leq 3\pi/2$. For the special case $\phi_1 = 3\pi - \phi_1^i$ and $\phi_1^i \rightarrow 3\pi/2$, we obtain the total far field on shadow boundary of the reflected field from (III. 2), (III. 6), and (III. 8), namely,

electric wall: $u^i(r_1, \phi_1 = 3\pi/2)$

$$\sim \exp[-ik(r_1 + a)] + \exp[ik(r_1 - a)] G(0) \times \left[\frac{\tau}{2} - \frac{\exp(i\pi/4)}{2\sqrt{2\pi kr_1}} \left(1 + 2\tau \frac{kG'_+(0)}{G_+(0)} \right) \right], \quad kr_1 \rightarrow \infty, \quad (III. 9)$$

where $G(\alpha) = G_+(\alpha)G_+(-\alpha)$ is defined in Appendix B and we have written the factor $[1 - \exp(ika)]$ as $G(0)$. In the above derivation the case $\phi_1^i = 3\pi/2$ is obtained as a limit $\phi_1^i = 3\pi/2 + \delta$, $\delta \rightarrow 0+$. It can be shown that the identical result is obtained when the limit is approached from the other side $\phi_1^i = 3\pi/2 - \delta$, $\delta \rightarrow 0+$.

Following exactly the same procedure we can solve the problem sketched in Fig. 3b. For the special case $\phi_1 = 3\pi - \phi_1^i$ and $\phi_1^i \rightarrow 3\pi/2$, the total far field is found to be

$$\text{magnetic wall: } u^i(r_1, \phi_1 = 3\pi/2) \sim \exp[-ik(r_1 + a)] + \exp[ik(r_1 - a)] \tilde{G}(0) \times \left[\frac{\tau}{2} - \frac{\exp(i\pi/4)}{2\sqrt{2\pi kr_1}} \left(1 + 2\tau \frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} \right) \right], \quad kr_1 \rightarrow \infty. \quad (III. 10)$$

Note that (III. 10) is identical to (III. 9) except for the replacement of $[G(0), G_+(0), G'_+(0)]$ by $[\tilde{G}(0), \tilde{G}_+(0), \tilde{G}'_+(0)]$, as discussed in (ii) following (III. 7).

Now let us return to the original problem sketched in Fig. 1, with incident field given in (II. 1). The scattered far field in the forward direction $\phi_0 = 3\pi/2$ is simply $(\tau/2)$ times the difference of (III. 9) and (III. 10) after replacing (r_1, ϕ_1) by (r_0, ϕ_0) . This is evident from the sketch in Fig. 3. Including the incident field (II. 1), we have the total far field in the forward direction:

$$u^i(r_0, \phi_0 = \frac{3\pi}{2}) \sim \frac{1}{2} \exp(ikr_0) + \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi kr_0}} \left[\tau + [1 - \exp(-ika)] \frac{kG'_+(0)}{G_+(0)} + [1 + \exp(-ika)] \frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} \right], \quad kr_0 \rightarrow \infty. \quad (III. 11)$$

The total far field in the backward direction $\phi_1 = 3\pi/2$ is simply $\frac{1}{2}$ times of the sum of (III. 10) and (III. 11), and the result is

$$u^i(r_1, \phi_1 = \frac{3\pi}{2}) \sim \exp[-ik(r_1 + a)] + \frac{1}{2} \tau \exp[ik(r_1 - a)] - \frac{\exp[ik(r_1 - a) + i(\pi/4)]}{2\sqrt{2\pi kr_1}} \times \left[1 + \tau [1 - \exp(ika)] \frac{kG'_+(0)}{G_+(0)} + \tau [1 + \exp(ika)] \frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} \right], \quad kr_1 \rightarrow \infty. \quad (III. 12)$$

The results in (III. 11) and (III. 12) are in complete agreement with the rigorous far field solutions given by (A5) and (A2) in Appendix A. We emphasize that (III. 11) and (III. 12) are valid for arbitrary values of ka . When ka is large, we may use the asymptotic formulas for $G_+(\alpha)$, $\tilde{G}_+(\alpha)$, etc., in (III. 11) and (III. 12). Retaining the leading terms up to $O(k^{-1}a^{-1})$, we recover (II. 68) and (II. 70) exactly.

Compared with the derivation given in Sec. II, we arrive at the solution in (III. 11) and (III. 12) in fewer steps. The key to this simplification is that the interaction field u_{int} is calculated from (III. 7), instead of from (III. 3). Looking from a different viewpoint, it is rather satisfactory that the use of the uniform asymptotic theory in Sec. II also recovered the exact asymptotic solution. This was done without introducing a new canonical problem, with the interaction between two edges being "built up" from the local consideration of a single edge. In more general edge diffraction problems, formula (III. 7) may not be applicable, while the uniform asymptotic theory can always be employed. One such example is given in Sec. IV.

C. Numerical results and discussion

For the problem sketched in Fig. 1 with incident field given in (II. 1), the solutions for the total far field ($kr_0 \rightarrow \infty, k\gamma_1 \rightarrow \infty$) in the forward and backward directions are given in (III. 11) and (III. 12), respectively. When ka is large, the solutions reduce to those in (II. 68) and (II. 70). Some remarks concerning the numerical evaluations of those results are in order.

First let us concentrate on (II. 68), and normalize it with respect to the incident field:

$$\frac{u^t}{u^i} \Big|_{\phi_0=3\pi/2} \approx \frac{1}{2} + \tau \frac{\exp(i\pi/4)}{2\sqrt{2\pi k r_0}} - \frac{1}{2\pi} \left(\frac{a}{r_0}\right)^{1/2} \left[1 - S\left(\frac{ka}{\pi}\right)\right], \quad (\text{III. 13})$$

where $S(x)$ is a short notation for the infinite series

$$S(x) = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \right) \exp(i2n\pi x). \quad (\text{III. 14})$$

The latter series is slowly convergent. It is advantageous to transform it into an integral:

$$\begin{aligned} S(x) &= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \exp(i2n\pi x) \left(\int_0^{\infty} \exp(-2nt) t^{-1/2} dt \right. \\ &\quad \left. - \int_0^{\infty} \exp[-(2n+1)t] t^{-1/2} dt \right) \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{\exp(i2\pi x - 2t^2) [1 - \exp(-t^2)]}{1 - \exp(i2\pi x - 2t^2)} dt, \end{aligned} \quad (\text{III. 15})$$

which is rapidly convergent and can be easily evaluated by numerical integration. The series $S(x)$ is periodic with period 1, and in fact a Fourier series. For later use we examine the behavior of $S(x)$ in the vicinity of $x=0$. Referring to Section 1.11 in Ref. 15, $S(x)$ can be expressed in terms of Lerch's transcendent $\Phi(z, s, v)$, viz.,

$$\begin{aligned} S(x) &= 2^{-1/2} \exp(i2\pi x) \{ \Phi[\exp(i2\pi x), \frac{1}{2}, 1] \\ &\quad - \Phi[\exp(i2\pi x), \frac{1}{2}, \frac{3}{2}] \}. \end{aligned} \quad (\text{III. 16})$$

By means of formula 1.11(8) in Ref. 15, we obtain the Taylor expansion of $S(x)$, and its leading terms are

$$S(x) = S(0) - \frac{1}{2\pi} \exp(-i\pi/4) x^{1/2} + O(x), \quad (\text{III. 17})$$

valid around $x=0$, where $x^{1/2} = i|x|^{1/2}$ when $x < 0$. The initial constant term in (III. 17) is equal to

$$S(0) = (1/\sqrt{2}) \left[\zeta\left(\frac{1}{2}\right) - \zeta\left(\frac{1}{2}, \frac{3}{2}\right) \right] = 0.3951013566 \dots, \quad (\text{III. 18})$$

where $\zeta(s)$ and $\zeta(s, v)$ are, respectively, ordinary, and generalized zeta functions, and the numerical values were taken from a table in Ref. 16. Since $S(x)$ is periodic with period 1, the expansion in (III. 17) is also valid after replacing x by $(x - m)$, where m is an arbitrary integer. When this result is used in (III. 13), we have the normalized total field in the vicinity of $(ka/\pi) = m$, $m=1, 2, 3, \dots$ (i. e., the width a between the plates being a multiple of half wavelength):

$$\begin{aligned} \frac{u^t}{u^i} \Big|_{\phi_0=3\pi/2} &\approx \frac{1}{2} + \frac{1}{2\sqrt{2\pi k r_0}} \left[\exp(i\pi/4) \tau - 0.605\sqrt{2m} \right. \\ &\quad \left. - \exp(-i\pi/4) \frac{m\pi}{\sqrt{2}} \left(\frac{ka}{m\pi} - 1 \right)^{1/2} + O\left(\frac{ka}{m\pi} - 1 \right) \right]. \end{aligned} \quad (\text{III. 19})$$

From (III. 13), (III. 15), and (III. 19), it follows that u^t/u^i is a smooth function of ka , except at $ka = m\pi$. At the latter locations, the amplitude and phase plots of u^t/u^i vs ka exhibit vertical tangents.

FIG. 4. Normalized total field on the incident shadow boundary of two nonstaggered parallel plates (Fig. 1) for TM case. The solid curves are calculated from (III. 20), and the dashed curves from (III. 13).

FIG. 5. Normalized total field on the incident shadow boundary of two nonstaggered parallel plates (Fig. 1) for TE case. The solid curves are calculated from (III. 20), and the dashed curves from (III. 13).

Next consider (III. 11), which after normalization becomes

$$\frac{u^t}{u^i} \Big|_{\phi_0=3\pi/2} \approx \frac{1}{2} + \frac{\exp(i\pi/4)}{2\sqrt{2}\pi k r_0} \left(\tau + [1 - \exp(-ika)] \frac{kG_*(0)}{G_*(0)} + [1 + \exp(-ika)] \frac{k\tilde{G}_*(0)}{\tilde{G}_*(0)} \right). \quad (\text{III. 20})$$

The evaluation of the last two terms in (III. 20) is most easily done by numerical integration of the following representations, cf. Appendix B,

$$\frac{kG_*(0)}{G_*(0)} = \frac{\exp(-i\pi/4)}{\pi\sqrt{2}} ka \int_{-\infty}^{\infty} \frac{\exp(ika - kat^2)}{1 - \exp(ika - kat^2)} \times \frac{dt}{(1 + \frac{1}{2}it^2)^{1/2}}, \quad (\text{III. 21})$$

$$\frac{k\tilde{G}_*(0)}{\tilde{G}_*(0)} = -\frac{\exp(-i\pi/4)}{\pi\sqrt{2}} ka \int_{-\infty}^{\infty} \frac{\exp(ika - kat^2)}{1 + \exp(ika - kat^2)} \times \frac{dt}{(1 + \frac{1}{2}it^2)^{1/2}}. \quad (\text{III. 22})$$

Some caution should be taken when $ka = m\pi$, $m = 1, 2, 3, \dots$. When m is even, the integral (III. 21) is divergent; when m is odd, the integral in (III. 22) is divergent. However, these divergent integrals are compensated by the factors $[1 \mp \exp(-ika)]$ in (III. 20) such that their combined values become zero in the respective cases. It can be shown that the amplitude and phase plots of u^t/u^i vs ka , based on (III. 20), exhibit the same behavior at "resonance values" $ka = m\pi$ as the previous curves based on (III. 13).

In Figs. 4 and 5, numerical results for the total far field in the forward direction are presented as a function of the plate separation-to-wavelength ratio a/λ , with the observation point at a fixed distance from the lower edge $r_0 = 2\lambda$. The solid curves are calculated from (III. 20)–(III. 22), while the dashed curves stem from (III. 13) and (III. 15). Note that these two sets of curves are in good agreement even for a is about half wavelength.

IV. STAGGERED PARALLEL PLATES

A. Statement of problem and approach

In this part of the paper, we consider the diffraction by two perfectly conducting, parallel plates staggered a length l . We assume l to be positive, finite, and not close to zero. The separation of the plates is $a/2$, which is written as b hereafter (Fig. 6). The incident field is that from an isotropic line source:

$$u^i(r_2, \phi_2) = (i/4)H_0^{(1)}(kr_2) = \frac{\exp[i(kr_2 + \pi/4)]}{2\sqrt{2}\pi k r_2} [1 + (1/8ikr_2) + O(k^{-2})]. \quad (\text{IV. 1})$$

The polar coordinates $\{r_0, \phi_0\}$, $\{r_1, \phi_1\}$, and $\{r_2, \phi_2\}$ have origins at the lower edge, the upper edge, and the source point, respectively. We are interested in the case when the line source, the two edges, and the observation point are exactly on a straight line (Fig. 6), i. e.,

$$\text{line source: } r_0 = c + d, \quad \phi_0 = \Omega, \quad (\text{IV. 2a})$$

$$\text{observation point: } r_0 = r_0, \quad \phi_0 = \pi + \Omega. \quad (\text{IV. 2b})$$

Except for the special situations $l = 0$ or $l = \infty$, rigorous analytical solution to this problem is not known. In Ref. 6, two coupled Wiener–Hopf equations were formulated and an approximate method for solving them valid for large kl was presented. However, for the case described in (IV. 2) (the most difficult one), no explicit result was obtained. Recently Jones⁸ studied the same problem with a plane wave incidence (instead of incidence from a line source). He first considered the

FIG. 6. Two staggered parallel plates illuminated by an incident cylindrical wave from a line source at $r_2 = 0$.

scattering of the incident plane wave at the upper plate and obtained an exact result for the scattered field in terms of Fresnel integrals. Then the diffraction of this scattered field at the second (lower) plate is treated by the conventional Wiener-Hopf technique. The final result thus obtained may be considered as the field scattered by two parallel plates when the interaction between the plates is ignored. Jones' analysis includes the special case when the incident plane wave propagates in a line through the edges of the two plates. It is this special case that is comparable to our result to be derived next. Excluding interaction terms, Jones' result and our result are found in agreement.

To attack the problem sketched in Fig. 6 with incident field in (IV. 1), we will use a combination of the uniform asymptotic theory (cf. Sec. II) and the method of modified diffraction coefficient (cf. Sec. III). Again our solution is asymptotic for large k , and contains terms up to and including the order of $k^{-3/2}$. The steps of solution are described below. The incident field u^i in (IV. 1) reaches the upper plate $x < -l$, $y = b$, and diffraction there produces a scattered field $u_1(r_1, \phi_1)$. The field u_1 on the diffracted ray traveling in the direction $\phi_1 = \pi/2$ is bounced back and forth between the lower plate and the upper edge. This multiple interaction is accounted for by a single scattered field $u_{\text{int}}(r_1, \phi_1)$ emanating from the upper edge. The calculation of u_1 and u_{int} follows a procedure similar to that used in Sec. III. For the diffraction at the lower edge $x = 0$, $y = 0$, the incident field is taken to be

$$\tilde{u}^i(r_1, \phi_1) = u^i + u_1 + u_{\text{int}} = \tilde{u}_{\text{cy}}^i + \tilde{u}_{\text{no}}^i. \quad (\text{IV. 3})$$

In the neighborhood of the lower edge, $\tilde{u}^i$ is further divided into two components: cylindrical wave component $\tilde{u}_{\text{cy}}^i$ and noncylindrical component $\tilde{u}_{\text{no}}^i$. Their respective diffractions give rise to u_{cy}^t and u_{no}^t , which are calculated by the uniform asymptotic theory described in Sec. II. The further successive diffraction of u_{no}^t by the upper edge results in u_{up}^t . Successive diffraction of u_{cy}^t by the upper edge gives rise to a field of order k^{-2} , and hence this contribution is ignored. The total field solution in the direction $\phi_0 = \pi + \Omega$, correct to the order $k^{-3/2}$, is then given by the sum of u_{cy}^t , u_{no}^t , and u_{up}^t .

B. Far field solution in forward direction

The scattering of u^i in (IV. 1) at the upper plate gives rise to a scattered field $u_1(r_1, \phi_1)$. To derive an asymptotic expression of u_1 valid in the region $\pi/2 \leq \phi_1 \leq \pi$, we may use the uniform asymptotic theory summarized in Sec. II B. The result is

$$\begin{aligned} u_1(r_1, \phi_1) = & \frac{-\exp[i(kr_1 + kc + \pi/4)]}{2\sqrt{2\pi kr_2}} \left(1 + \frac{1}{8ikr_2}\right) F(-k^{1/2}\xi_1^*) \\ & + \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} \frac{\exp[i(kc + \pi/4)]}{2\sqrt{2\pi kc}} \\ & \times \left[\left(\frac{2r_1c}{r_2}\right)^{1/2} (\xi_1^*)^{-1} - \sec \frac{\phi_1 - (2\pi - \Omega)}{2} \right. \\ & \left. - \tau \sec \frac{\phi_1 + (2\pi - \Omega)}{2} \right] + O(k^{-2}), \end{aligned} \quad (\text{IV. 4})$$

where F is the Fresnel integral defined in (II. 7) and

$$\begin{aligned} \xi_1^* = & (r_1 + c - r_2)^{1/2} \operatorname{sgn} \left(\cos \frac{\phi_1 - (2\pi - \Omega)}{2} \right) \\ = & \left(\frac{4r_1c}{r_1 + c + r_2} \right)^{1/2} \cos \frac{\phi_1 - (2\pi - \Omega)}{2}. \end{aligned} \quad (\text{IV. 5})$$

For $\pi/2 \leq \phi_1 \leq \pi$ and $r_1 \neq 0$, $u_1(r_1, \phi_1)$ given in (IV. 4) is finite and continuous everywhere.

The field u_1 on the diffracted ray traveling in the direction $\phi_1 = \pi/2$ is bounced back and forth between the upper edge and the lower plate, resulting in a scattered field u_{int} . Since l is assumed to be positive and not close to zero, this interaction is locally the same as that discussed in Sec. III. Thus, using (III. 7), one obtains

$$\begin{aligned} u_{\text{int}}(r_1, \phi_1) = & \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} \frac{\exp[i(kc + \pi/4)]}{2\sqrt{2\pi kc}} \\ & \times [\bar{D}(\phi_1, 2\pi - \Omega) - D(\phi_1, 2\pi - \Omega)] + O(k^{-2}). \end{aligned} \quad (\text{IV. 6})$$

In the region $\pi/2 \leq \phi_1 \leq \pi$, u_{int} can be written more explicitly

$$\begin{aligned} u_{\text{int}}(r_1, \phi_1) = & \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} \frac{\exp[i(kc + \pi/4)]}{2\sqrt{2\pi kc}} \\ & \times \left(1 - \frac{G_+(k \cos(2\pi - \Omega))}{G_+(k |\cos \phi_1|)} \right) \\ & \times \left(\sec \frac{\phi_1 - (2\pi - \Omega)}{2} + \tau \sec \frac{\phi_1 + (2\pi - \Omega)}{2} \right) \\ & + O(k^{-2}). \end{aligned} \quad (\text{IV. 7})$$

Assuming that Ω or ϕ_1 is not close to $\pi/2$, we may use the asymptotic expansion for $G_+(\alpha)$ given in (B11), Appendix B (remembering $a = 2b$), and (IV. 7) passes into

$$\begin{aligned} u_{\text{int}}(r_1, \phi_1) = & \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} \frac{\exp[i(kc + \pi/4)]}{2\sqrt{2\pi kc}} \\ & \times \left(\frac{\exp(i\pi/4)}{2\sqrt{\pi kb}} \sum_{n=1}^{\infty} \frac{\exp(i2nkb)}{n^{3/2}} \right) \\ & \times [\sec(2\pi - \Omega) + \sec \phi_1] \left(\sec \frac{\phi_1 - (2\pi - \Omega)}{2} \right. \\ & \left. + \tau \sec \frac{\phi_1 + (2\pi - \Omega)}{2} \right) + O(k^{-2}), \end{aligned} \quad (\text{IV. 8})$$

valid for $\pi/2 < \phi_1 \leq \pi$, away from $r_1 = 0$. It should be remarked that the result in (IV. 8) can be also derived by using the uniform asymptotic theory described in Sec. II. Such a derivation, however, is quite involved, whereas the use of (III. 7) enables us to write down (IV. 8) readily as we did above.

Next consider the diffraction at the lower edge $x = 0$, $y = 0$. The solutions of u_1 and u_{int} having been found, the incident field $\tilde{u}^i$ defined in (IV. 3) now can be written explicitly as $\tilde{u}^i = \tilde{u}_{\text{cy}}^i + \tilde{u}_{\text{no}}^i$, which consists of a cylindrical wave component $\tilde{u}_{\text{cy}}^i$ and a noncylindrical wave component $\tilde{u}_{\text{no}}^i$. From (IV. 3), (IV. 4), and (IV. 8) we find the cylindrical wave component to be

$$\begin{aligned} \tilde{u}_{\text{cy}}^i(r_1, \phi_1) = & \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} C(r_1, \phi_1) + O(k^{-2}), \\ & \pi/2 < \phi_1 \leq \pi, \end{aligned} \quad (\text{IV. 9})$$

where

$$C(r_1, \phi_1) = \frac{\exp[i(kc + \pi/4)]}{2\sqrt{2\pi kb}} \left[\left(\frac{2r_1 c}{r_2} \right)^{1/2} (\xi_1^*)^{-1} - \sec \frac{\phi_1 - (2\pi - \Omega)}{2} \right. \\ \left. - \tau \sec \frac{\phi_1 + (2\pi - \Omega)}{2} + \left(\frac{\exp(i\pi/4)}{2\sqrt{\pi kb}} \sum_{n=1}^{\infty} \frac{\exp(i2nkb)}{n^{3/2}} \right) \right. \\ \left. \times [\sec(2\pi - \Omega) + \sec \phi_1] \right. \\ \left. \times \left(\sec \frac{\phi_1 - (2\pi - \Omega)}{2} + \tau \sec \frac{\phi_1 + (2\pi - \Omega)}{2} \right) \right]. \quad (\text{IV. 10})$$

The noncylindrical wave component in (IV. 9) is found to be

$$\tilde{u}_{\text{no}}^i(r_1, \phi_1) = \frac{\exp[i(kr_1 + kc + \pi/4)]}{2\sqrt{2\pi kr_2}} \left(1 + \frac{1}{8ikr_2} \right) F(k^{1/2} \xi_1^*). \quad (\text{IV. 11})$$

The diffraction of $\tilde{u}_{\text{cy}}^i, \tilde{u}_{\text{no}}^i$ at the lower edge gives rise to total field components $u_{\text{cy}}^t, u_{\text{no}}^t$, respectively. Their calculations are considered below.

For an incident cylindrical wave $\tilde{u}_{\text{cy}}^i$, we can apply formula (II. 6) directly. Retaining only the leading terms, we have

$$u_{\text{cy}}^t(r_0, \phi_0) = \frac{\exp[i(kr_0 + kd + \pi/4)]}{2\sqrt{2\pi kr_1}} F(k^{1/2} \xi_0^*) C(r_1, \phi_1) \\ + \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi kr_0}} \frac{\exp[i(kd + \pi/4)]}{2\sqrt{2\pi kd}} \\ \times \left[\left(\frac{2r_0 d}{r_1} \right)^{1/2} (\xi_0^*)^{-1} C(r_1, \phi_1) - \left(\sec \frac{\phi_0 - \Omega}{2} \right. \right. \\ \left. \left. + \tau \sec \frac{\phi_0 + \Omega}{2} \right) C(d, \pi - \Omega) \right] + O(k^{-2}), \quad (\text{IV. 12})$$

valid for $\pi \leq \phi_0 \leq 2\pi$, away from the lower edge $r_0 = 0$. The function ξ_0^* was defined in (II. 12). Of particular interest is the field in the forward direction $\phi_0 = \pi + \Omega$. In this direction, $\xi_0^* = 0$ and $\sec(\phi_0 - \Omega)/2$ becomes infinite. However, the resultant singularities do cancel and u_{cy}^t remains finite as shown below. Let us assume that ϕ_0 deviates from $(\pi + \Omega)$ by a small number δ :

$$\phi_0 = (\pi + \Omega) - \delta. \quad (\text{IV. 13})$$

Then, it follows from simple geometry in Fig. 7 and the definitions in (II. 12) and (IV. 5) that

FIG. 7. Limiting case when the observation point falls on the line through the source point and two edges: r_0 is fixed and $\phi_0 \rightarrow \pi + \Omega$ (or equivalently $\delta \rightarrow 0$).

$$\phi_1 = (\pi - \Omega) + \frac{r_0}{r_0 + d} \delta [1 + A_0 \delta^2 + O(\delta^4)], \quad A_0 = \frac{d(r_0 - d)}{6(r_0 + d)^2}, \\ r_1 = (r_0 + d)[1 + A_1 \delta^2 + O(\delta^4)], \quad A_1 = -\frac{r_0 d}{2(r_0 + d)^2}, \\ r_2 = (r_0 + d + c)[1 + A_2 \delta^2 + O(\delta^4)], \quad A_2 = -\frac{r_0(d + c)}{2(r_0 + d + c)^2}, \\ \xi_0^* = \delta \left(\frac{r_0 d}{2(r_0 + d)} \right)^{1/2} [1 + A_3 \delta^2 + O(\delta^4)], \quad A_3 = \frac{1}{8} \left(\frac{r_0 d}{(r_0 + d)^2} - \frac{1}{3} \right), \\ \xi_1^* = \delta \frac{r_0}{r_0 + d} \left(\frac{(r_0 + d)c}{2(r_0 + d + c)} \right)^{1/2} [1 + A_4 \delta^2 + O(\delta^4)], \\ A_4 = \frac{1}{8} \left[\frac{r_0}{r_0 + d + c} \left(\frac{d}{r_0 + d} + \frac{d + c}{r_0 + d + c} \right) - \frac{r_0^2 + 2r_0 d + 4d^2}{3(r_0 + d)^2} \right]. \quad (\text{IV. 14})$$

Substituting (IV. 13) and (IV. 14) into (IV. 10) and (IV. 12) and letting $\delta \rightarrow 0$, we obtain

$$u_{\text{cy}}^t(r_0, \phi_0 = \pi + \Omega) \\ = \frac{\exp[i(kr_0 + kd + \pi/4)]}{2\sqrt{2\pi k(r_0 + d)}} \frac{\exp[i(kc + \pi/4)]}{2\sqrt{2\pi kb}} \\ \times \left(\frac{\tau}{2 \sin \Omega} + \frac{\exp(i\pi/4) \sin \Omega}{2\sqrt{\pi kb} \cos^2 \Omega} \sum_{n=1}^{\infty} \frac{\exp(i2nkb)}{n^{3/2}} \right) \\ + \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi kr_0}} \frac{\exp[i(kd + \pi/4)]}{2\sqrt{2\pi kd}} \frac{\exp[i(kc + \pi/4)]}{2\sqrt{2\pi kb}} \\ \times \left(\frac{cr_0}{2(r_0 + d + c)^2} + \frac{1}{\sin^2 \Omega} + \frac{\tau r_0}{r_0 + d} \frac{\cos \Omega}{\sin^2 \Omega} \right) + O(k^{-2}). \quad (\text{IV. 15})$$

We note also that the successive diffraction of u_{cy}^t by the upper edge (including interaction) leads to terms of $O(k^{-2})$ for the field in the direction $\phi_0 = \pi + \Omega$. Hence they are ignored.

It remains to calculate u_{no}^t , the total field component due to the incidence of $\tilde{u}_{\text{no}}^i$. Because of the rapid variation of the Fresnel function across $\xi_1^* = 0$, $\tilde{u}_{\text{no}}^i$ cannot be regarded as a cylindrical wave, and the uniform asymptotic theory cannot be directly applied to calculate its diffraction at the lower edge. Following the method in Ref. 4, we expand the Fresnel integral in a Taylor series around $\xi_1^* = 0$, viz.,

$$\tilde{u}_{\text{no}}^i(r_1, \phi_1) = \exp(ikr_1) \left(\sum_{q=0}^{\infty} z^{(q)}(r_1, \phi_1) k^{(q-1)/2} \right. \\ \left. + \frac{1}{8i} \sum_{q=0}^{\infty} r_2^{-1} z^{(q)}(r_1, \phi_1) k^{(q-3)/2} \right), \quad (\text{IV. 16})$$

where $z^{(q)}$ is determined from (II. 11) and is given by

$$z^{(q)}(r_1, \phi_1) = \frac{\exp[i(kc + \pi/4)]}{4\sqrt{2\pi r_2}} \frac{\exp(-iq\pi/4)}{\Gamma(q/2 + 1)} (\xi_1^*)^q, \\ q = 0, 1, 2, \dots \quad (\text{IV. 17})$$

Each term in (IV. 16) is now considered as a cylindrical wave constituent. We apply the formula of uniform asymptotic theory in (II. 6) to each constituent separately and then sum up the resultant fields to obtain u_{no}^t , namely,

$$u_{\text{no}}^t(r_0, \phi_0) \\ = \exp[ik(r_0 + d)] \left[\sum_{q=0}^{\infty} \left(F(k^{1/2} \xi_0^*) + \frac{\exp(i\pi/4)}{2\pi} k^{-1/2} (\xi_0^*)^{-1} \right) \right]$$

$$\begin{aligned}
& \times \sum_{m=0}^{[(q+1)/2]} \Gamma(m + \frac{1}{2})(ik)^{-m}(\xi_0^*)^{-2m} z^{(q)}(r_1, \phi_1) k^{(q-1)/2} \\
& + \frac{1}{8i} \sum_{q=0}^{\infty} \left(F(k^{1/2} \xi_0^*) + \frac{\exp(i\pi/4)}{2\pi} k^{-1/2} (\xi_0^*)^{-1} \right. \\
& \times \sum_{m=0}^{[(q-1)/2]} \Gamma(m + \frac{1}{2})(ik)^{-m}(\xi_0^*)^{-2m} r_2^{-1} z^{(q)}(r_1, \phi_1) k^{(q-3)/2} \Big] \\
& - \frac{\exp[i(kr_0 + kd + \pi/4)]}{2\sqrt{2\pi}r_0} \left[k^{-1} z^{(0)}(d, \pi - \Omega) \left(\sec \frac{\phi_0 - \Omega}{2} \right. \right. \\
& + \tau \sec \frac{\phi_0 + \Omega}{2} \Big) - ik^{-3/2} \frac{1}{4d} \frac{\partial z^{(1)}(d, \pi - \Omega)}{\partial \phi_1} \\
& \times \left(\sin(\phi_0 - \Omega) \sec^3 \frac{\phi_0 - \Omega}{2} - \tau \sin(\phi_0 + \Omega) \sec^3 \frac{\phi_0 + \Omega}{2} \right) \Big] \\
& + O(k^{-2}), \tag{IV. 18a}
\end{aligned}$$

where ξ_0^* was defined in (II. 12), $[(q \pm 1)/2]$ is the largest integer $\leq (q \pm 1)/2$, and

$$z^{(0)}(d, \pi - \Omega) = \frac{\exp[i(kc + \pi/4)]}{4\sqrt{2\pi}(c+d)}, \tag{IV. 18b}$$

$$\frac{\partial z^{(1)}(d, \pi - \Omega)}{\partial \phi_1} = \frac{\exp(ikc)}{4\pi} \frac{\sqrt{cd}}{c+d}. \tag{IV. 18c}$$

In deriving (IV. 18a), we have made use of the fact that, due to the factor $(\xi_1^*)^q$, the incident field amplitude $z^{(q)}(r_1, \phi_1)$ and its first $(q-1)$ derivatives vanish at the lower edge $r_1 = d$, $\phi_1 = \pi - \Omega$. The result in (IV. 18) can be simplified considerably. The steps follow exactly those in Ref. 4, Sec. 6, except that terms of $O(k^{-3/2})$ in (IV. 18a) were not present in Ref. 4. After simplification, (IV. 18a) becomes

$$\begin{aligned}
u_{no}^t(r_0, \phi_0) &= \frac{\exp[i(kr_0 + kc + kd + \pi/4)]}{2\sqrt{2\pi}kr_2} \left\{ F(k^{1/2} \xi_0^*) F(k^{1/2} \xi_1^*) \right. \\
&+ \frac{1}{2} F(k^{1/2} \xi_0^* (1 + \eta^2)^{1/2}) - \frac{1}{2} \exp(-ik \xi_1^{*2}) F(k^{1/2} \xi_0^*) \\
&+ G(\eta, k^{1/2} \xi_0^*) \left(1 + \frac{1}{8ikr_2} \right) + \frac{\exp(i\pi/4)}{4\sqrt{\pi}k} (\xi_0^*)^{-1} \\
&\times \frac{1}{(1 + \eta^2)^{1/2}} - \frac{i}{4\pi k} (\xi_0^*)^{-2} \frac{\eta}{1 + \eta^2} \Big\} \\
&- \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi}kr_0} \frac{\exp[i(kc + kd + \pi/4)]}{2\sqrt{2\pi}k(c+d)} \\
&\times \frac{1}{2} \left(\sec \frac{\phi_0 - \Omega}{2} + \tau \sec \frac{\phi_0 + \Omega}{2} \right) + \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi}kr_0} \\
&\times \frac{i \exp[ik(c+d)]}{16\pi k(c+d)} \left(\frac{c}{d} \right)^{1/2} \left(\sin(\phi_0 - \Omega) \sec^3 \frac{\phi_0 - \Omega}{2} \right. \\
&- \tau \sin(\phi_0 + \Omega) \sec^3 \frac{\phi_0 + \Omega}{2} \Big) \\
&+ O(k^{-2}), \tag{IV. 19}
\end{aligned}$$

valid for $\pi \leq \phi_0 \leq 2\pi$, away from the edge $r_0 = 0$. In (IV. 19), the following notations were used:

$$\eta = \frac{\xi_1^*}{\xi_0^*}, \tag{IV. 20}$$

$$G(\eta, t) = \frac{\exp(-i\eta^2 t^2)}{2\pi} \int_0^\pi \frac{\exp(i\sigma^2 t^2)}{1 + \sigma^2} d\sigma. \tag{IV. 21}$$

On the shadow boundary $\phi_0 = \pi + \Omega$, $\xi_0^* = \xi_1^* = 0$, and $\sec(\phi_0 - \Omega)/2$ becomes infinity. As before, the resultant singularities do cancel, and a finite u_{no}^t is obtained, namely,

$$\begin{aligned}
u_{no}^t(r_0, \phi_0 = \pi + \Omega) &= \frac{\exp[i(kr_0 + kd + kc + \pi/4)]}{2\sqrt{2\pi}k(r_0 + d + c)} \left[\frac{1}{4} + \frac{1}{2\pi} \tan^{-1} \left(\frac{r_0 c}{(r_0 + d + c)d} \right)^{1/2} \right] \\
&\times \left(1 + \frac{1}{8ik} \frac{1}{r_0 + d + c} \right) + \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi}kr_0} \frac{\exp[i(kc + kd + \pi/4)]}{2\sqrt{2\pi}k(c+d)} \\
&\times \frac{\tau}{2 \sin \Omega} + \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi}kr_0} \frac{\exp[i(kc + kd + \pi/2)]}{8\pi k(c+d)} \left(\frac{c}{d} \right)^{1/2} \\
&\times \left(-\tau \frac{\cos \Omega}{\sin^2 \Omega} + A_5 \right) + O(k^{-2}), \tag{IV. 22}
\end{aligned}$$

where

$$\begin{aligned}
A_5 &= 4 \left(3A_3 - A_4 + \frac{1}{2}A_2 + \frac{2r_0 c(A_4 - A_3)}{(r_0 + d)(c+d)} \right) - \frac{1}{6} \\
&= -\frac{1}{2} - \frac{(c+d)(r_0^2 - cd - d^2)}{2(r_0 + d)(r_0 + c + d)^2}. \tag{IV. 23}
\end{aligned}$$

The previous expansions (IV. 14) were intermediate in deriving these results. It remains to determine the diffraction of u_{no}^t by the upper edge. The field on the diffracted ray of u_{no}^t traveling in the direction $\phi_0 = \Omega$ is

$$\begin{aligned}
&\frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi}kr_0} \frac{\exp[i(kc + kd + \pi/4)]}{2\sqrt{2\pi}k(c+d)} \left(\frac{1}{2} \right) \\
&\times (-1 - \tau \sec \Omega) + O(k^{-3/2}), \tag{IV. 24}
\end{aligned}$$

which was calculated from (III. 4) with $u^t(r_1 = 0)$ replaced by $u_{no}^t(d, \pi - \Omega)$ as given by (IV. 11). After diffraction at the upper edge, it gives rise to a field in the direction $\phi_1 = \pi - \Omega$:

$$\begin{aligned}
&\frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi}kr_1} \frac{\exp[i(kd + \pi/4)]}{2\sqrt{2\pi}kd} \frac{\exp[i(kc + kd + \pi/4)]}{2\sqrt{2\pi}k(c+d)} \\
&\times \left(\frac{1}{2} \right) (-1 - \tau \sec \Omega) [-1 - \tau \sec(\pi - \Omega)] + O(k^{-2}), \tag{IV. 25}
\end{aligned}$$

which was calculated from (III. 4) with u^t equal to the quantity in (IV. 24). Diffraction of the field in (IV. 25) at lower edge simply reduces the dominant term by a half. Thus, the successive diffractions of u_{no}^t at the upper and lower edges result in a contribution to the total field in the direction $\phi_0 = \pi + \Omega$:

$$\begin{aligned}
u_{up}^t(r_0, \phi_0 = \pi + \Omega) &= \frac{\exp[i(kr_0 + kd + \pi/4)]}{2\sqrt{2\pi}k(r_0 + d)} \frac{\exp[i(kd + \pi/4)]}{2\sqrt{2\pi}kd} \\
&\times \frac{\exp[i(kc + kd + \pi/4)]}{2\sqrt{2\pi}k(c+d)} \left(-\frac{\tan^2 \Omega}{4} \right) + O(k^{-2}). \tag{IV. 26}
\end{aligned}$$

Further interaction of u_{up}^t with the upper edge is of no interest because it produces fields of $O(k^{-5/2})$ in the direction $\phi_0 = \pi + \Omega$.

Summing up (IV. 15), (IV. 22), and (IV. 26), we obtain the desired total field in the direction $\phi_0 = \pi + \Omega$ due to an incidence given in (IV. 1), viz.,

$$u^t(r_0, \phi_0 = \pi + \Omega) = k^{-1/2} P + k^{-1} Q + k^{-3/2} R + O(k^{-2}), \tag{IV. 27a}$$

where

$$P = \frac{\exp[i(kr_0 + kd + kc + \pi/4)]}{2\sqrt{2\pi}(r_0 + d + c)} \left[\frac{1}{4} + \frac{1}{2\pi} \tan^{-1} \left(\frac{r_0 c}{(r_0 + d + c)d} \right)^{1/2} \right], \tag{IV. 27b}$$

FIG. 8. Normalized total field on the incident shadow boundary of two staggered parallel plates (Fig. 6) for TM case. u^t is calculated from Eq. (IV.27) with one, two, and three terms (indicated by 1, 2, and 3, respectively, next to the curves); $u^t = \exp[ikr_2 + i\pi/4]/2\sqrt{2\pi kr_2}$ where $r_2 = c + d + r_0$.

$$Q = \frac{\exp[i(kr_0 + kd + kc + \pi/2)]}{8\pi} \left(\frac{1}{\sqrt{r_0(c+d)}} + \frac{1}{\sqrt{(r_0+d)c}} \right) \frac{\tau}{2\sin\Omega} \quad (\text{IV. 27c})$$

$$R = \frac{\exp[i(kr_0 + kd + kc + 3\pi/4)]}{8(2\pi)^{3/2}} \left\{ \frac{\sqrt{2}}{\sqrt{(r_0+d)bc}} \frac{\sin\Omega}{\cos^2\Omega} \times \sum_{n=1}^{\infty} \frac{\exp(i2nkb)}{n^{3/2}} - \frac{\pi}{(r_0+d+c)^{3/2}} \left[\frac{1}{4} + \frac{1}{2\pi} \times \tan^{-1} \left(\frac{r_0 c}{(r_0+d+c)d} \right)^{1/2} \right] + \frac{1}{\sqrt{r_0 d c}} \left(-\frac{c}{2(c+d)} + \frac{cd}{2(r_0+d)(r_0+d+c)} + \frac{1}{\sin^2\Omega} + \frac{d(r_0-c)}{(r_0+d)(c+d)} \frac{\tau \cos\Omega}{\sin^2\Omega} \right) - \frac{\exp(i2kd)}{\sqrt{(r_0+d)(d+c)d}} \frac{\tan^2\Omega}{4} \right\}, \quad (\text{IV. 27d})$$

which is valid for Ω not close to $\pi/2$ (or $l \neq 0$). Some numerical results calculated from (IV.27) are presented in Figs. 8 and 9 pertaining to a configuration with $\Omega = \pi/4$, $c = d = 2\lambda$.

From the result in (IV.27) we can also obtain the total field $u^t(r_0, \phi_0 = \pi + \Omega)$ when the incident field is a

plane wave coming from the direction $\phi_0 = \Omega$. To this end, let us multiply (IV.1) by the factor

$$2\sqrt{2\pi kc} \exp[-ik(c+d) - i(\pi/4)]. \quad (\text{IV. 28})$$

In the limit $c \rightarrow \infty$, the incident field in (IV.1) then becomes a plane wave given by

$$u^i = \exp[-ikr_0 \cos(\phi_0 - \Omega)]. \quad (\text{IV. 29})$$

Multiplying the final result (IV.27) by the same factor (IV.28) and letting $c \rightarrow \infty$, we obtain the total field on the incident shadow boundary when the incident field is given by (IV.29), namely,

$$u^t(r_0, \phi_0 = \pi + \Omega) = \bar{P} + k^{-1/2} \bar{Q} + k^{-1} \bar{R} + O(k^{-3/2}), \quad (\text{IV. 30a})$$

where

$$\bar{P} = \exp(ikr_0) \left[\frac{1}{4} + \frac{1}{2\pi} \tan^{-1} \left(\frac{r_0}{d} \right)^{1/2} \right], \quad (\text{IV. 30b})$$

$$\bar{Q} = \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi}} \left(\frac{1}{\sqrt{r_0}} + \frac{1}{\sqrt{r_0+d}} \right) \frac{\tau}{2\sin\Omega}, \quad (\text{IV. 30c})$$

$$\bar{R} = \frac{i \exp(ikr_0)}{8\pi \sqrt{r_0 d}} \left(-\frac{r_0}{2(r_0+d)} + \frac{1}{\sin^2\Omega} - \frac{d}{r_0+d} \frac{\tau \cos\Omega}{\sin^2\Omega} \right) + \bar{R}_{\text{int}}, \quad (\text{IV. 30d})$$

FIG. 9. Normalized total field on the incident shadow boundary of two staggered parallel plates (Fig. 6) for TE case. u^t is calculated from Eq. (IV.27) with one, two, and three terms (indicated by 1, 2, and 3, respectively, next to the curves); $u^t = \exp[ikr_2 + i\pi/4]/2\sqrt{2\pi kr_2}$, where $r_2 = c + d + r_0$.

$$\bar{R}_{\text{int}} = \frac{i \exp(ikr_0)}{8\pi\sqrt{r_0+d}} \left[\left(\frac{2}{b} \right)^{1/2} \frac{\sin\Omega}{\cos^2\Omega} \sum_{n=1}^{\infty} \frac{\exp(i2nkb)}{n^{3/2}} - \exp(i2kd) \right] \times \frac{\tan^2\Omega}{4\sqrt{d}}. \quad (\text{IV. 30e})$$

The final result in (IV. 30) may be compared with the rigorous solution derived by Jones in Ref. 8. Jones' solution is a uniform asymptotic expansion for the total field $u^t(r_0, \phi_0)$ in the forward region $\pi \leq \phi_0 \leq 2\pi$; this expansion contains terms up to $O(k^{-1})$ [term of $O(k^{-3/2})$ is not included]. In Ref. 17, Jones' solution is specialized to the case $\phi_0 = \pi + \Omega$ (a step that involves some tedious limit computations). It is found that the result agrees precisely with (IV. 30), except that $\bar{R}_{\text{int}}$ in (IV. 30e) does not appear in Jones' solution. The term $\bar{R}_{\text{int}}$ describes the interactions between the parallel plates: (i) the first term in (IV. 30e) accounts for the multiple reflections and diffractions along the shadow boundary $x = -l$, $0 \leq y \leq b$ (Fig. 6), and (ii) the second term results from the interaction between the edges along their connecting line. Jones did not consider these two types of interactions which explains the absence of $\bar{R}_{\text{int}}$ in his solution.

It is also interesting to observe that the interaction contribution described by $\bar{R}_{\text{int}}$ is of order $O(k^{-1})$ provided that $\Omega \neq \pi/2$ and Ω is not close to $\pi/2$. When $\Omega = \pi/2$ precisely, the interaction contribution is of order $O(k^0)$, as can be seen by comparison of the results in Sec. II and that of Jones. Thus, as $\Omega \rightarrow \pi/2$, the interaction contribution increases from $O(k^{-1})$ to $O(k^0)$. It would be desirable to derive a result which is uniform in Ω ; however, such a derivation seems beyond our means for the moment.

APPENDIX A: EXACT SOLUTION OF DIFFRACTION BY TWO NONSTAGGERED PARALLEL PLATES

For diffraction of an incident plane wave by two non-staggered parallel plates, the problem has been solved exactly by the Wiener-Hopf technique (see Ref. 12, Sec. 3.2, or Ref. 13, Sec. 3-12). The solution is given in terms of an inverse Fourier transform, and it is exact. We have evaluated the inverse Fourier transform and obtained the far field solution. When the incident field is given by (II. 1) (normal incidence), the total field u^t in the upper half-space ($y \geq a$), far away from the edges, is found to be (Fig. 1)

$$u^t(r_1, \phi_1) \sim u_u^t(r_1, \phi_1) + \exp(-ika) \frac{\exp[i(kr_1 + \pi/4)]}{2\sqrt{2\pi kr_1}} D(\phi_1, 3\pi/2) \times \left(\frac{G_+(k \cos \phi_1) G_+(k \cos 3\pi/2) + \tilde{G}_+(k \cos \phi_1) \tilde{G}_+(k \cos 3\pi/2)}{2} - 1 \right), \quad kr_1 \rightarrow \infty, \pi \leq \phi_1 \leq 2\pi, \quad (\text{A1a})$$

where u_u^t is the (exact) total field due to the upper plate when the lower plate is removed, and is given by

$$u_u^t = \exp[ik(r_1 - a)] \{ F[\sqrt{2kr_1} \cos \frac{1}{2}(\phi_1 - 3\pi/2)] + \tau F[\sqrt{2kr_1} \cos \frac{1}{2}(\phi_1 + 3\pi/2)] \}. \quad (\text{A1b})$$

Here the Fresnel integral F is defined in (II. 7), Keller's diffraction coefficient D in (III. 5), and $G_+(\alpha)$ and $\tilde{G}_+(\alpha)$ in Appendix B. The solution in (A1) is valid uniformly for all ϕ_1 , between π and 2π , and for an arbitrary ka .

In deriving (A1) from the said inverse Fourier transform, we have used the following procedure: First, the integrand of the inverse Fourier transform is decomposed into a term which exhibits a pole singularity and a second term which has no such a pole singularity but contains a branch singularity. Evaluation of the first constituent yields the field u_u^t in (A1b). Saddle point integration of the second constituent yields the remainder of (A1a).

We are interested particularly in the total field exactly on the reflected shadow boundary $\phi_1 = 3\pi/2$. Setting $\phi_1 = (3\pi/2) - \delta$, where $\delta \rightarrow 0$ and making use of the relations

$$\begin{aligned} D(\phi_1, 3\pi/2) &= (2\tau/\delta)[1 + O(\delta)], \\ G_+(k \cos \phi_1) &= G_+(0) - \delta k G'_+(0) + O(\delta^2), \\ G_+(0) &= [G(0)]^{1/2}, \end{aligned}$$

we obtain

$$\begin{aligned} u^t(r_1, \phi_1 = 3\pi/2) &\sim \exp[-ik(r_1 + a)] + \frac{1}{2} \tau \exp[ik(r_1 - a)] \\ &- \frac{\exp[ik(r_1 - a) + i(\pi/4)]}{2\sqrt{2\pi kr_1}} \left(1 + \tau[1 - \exp(ika)] \frac{kG'_+(0)}{G_+(0)} \right. \\ &\left. + \tau[1 + \exp(ika)] \frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} \right), \quad kr_1 \rightarrow \infty. \quad (\text{A2}) \end{aligned}$$

For large ka , formulas (B20) and (B21), Appendix B, may be used in (A2); then u^t becomes

$$\begin{aligned} u^t(r_1, \phi_1 = \frac{3\pi}{2}) &\sim \exp[-ik(r_1 + a)] + \exp[ik(r_1 - a)] \\ &\times \left[\frac{\tau}{2} + \frac{\tau}{2\pi} \left(\frac{a}{r_1} \right)^{1/2} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n}} \right) \right. \\ &\left. \times \exp(i2nka) - \frac{\exp(i\pi/4)}{2\sqrt{2\pi kr_1}} \right], \quad kr_1 \rightarrow \infty \text{ and } ka \rightarrow \infty. \quad (\text{A3}) \end{aligned}$$

Corresponding to (A1), the solution of u^t in the lower half-space ($y \leq 0$), far away from the edges, is found to be (Fig. 1)

$$\begin{aligned} u^t(r_0, \phi_0) &\sim u_l^t(r_0, \phi_0) - \exp(-ika) \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi kr_0}} D(\phi_0, \pi/2) \\ &\times \left(\frac{G_+(k \cos \phi_0) G_+(k \cos \pi/2) - \tilde{G}_+(k \cos \phi_0) \tilde{G}_+(k \cos \pi/2)}{2} \right. \\ &\left. + \exp(ika) \right), \quad kr_0 \rightarrow \infty, \pi \leq \phi_0 \leq 2\pi, \quad (\text{A4a}) \end{aligned}$$

where u_l^t is the (exact) total field due to the lower plate when the upper plate is removed, and is given by

$$\begin{aligned} u_l^t &= \exp(ikr_0) \{ F[\sqrt{2kr_0} \cos \frac{1}{2}(\phi_0 - \pi/2)] \\ &+ \tau F[\sqrt{2kr_0} \cos \frac{1}{2}(\phi_0 + \pi/2)] \}. \quad (\text{A4b}) \end{aligned}$$

Exactly on the incident shadow boundary $\phi_0 = 3\pi/2$, we obtain the total field u^t from (A4), namely,

$$\begin{aligned} u^t(r_0, \phi_0 = \frac{3\pi}{2}) &\sim \frac{1}{2} \exp(ikr_0) + \frac{\exp[i(kr_0 + \pi/4)]}{2\sqrt{2\pi kr_0}} \\ &\times \left(\tau + [1 - \exp(-ika)] \frac{kG'_+(0)}{G_+(0)} + [1 + \exp(-ika)] \right. \\ &\left. \times \frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} \right), \quad kr_0 \rightarrow \infty. \quad (\text{A5}) \end{aligned}$$

For large ka , the use of (B20) and (B21) in (A5) leads to

$$u^*(r_0, \phi_0 = \frac{3\pi}{2}) \sim \exp(ikr_0) \left[\frac{1}{2} - \frac{1}{2\pi} \left(\frac{a}{r_0} \right)^{1/2} + \frac{1}{2\pi} \left(\frac{a}{r_0} \right)^{1/2} \times \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n+1}} \right) \exp(i2nka) + \tau \frac{\exp(i\pi/4)}{2\sqrt{2\pi k r_0}} \right], \quad kr_0 \rightarrow \infty \text{ and } ka \rightarrow \infty. \quad (\text{A6})$$

APPENDIX B: FUNCTIONS $G_+(\alpha)$ AND $\tilde{G}_+(\alpha)$

We are interested here in the factorization of two functions

$$G(\alpha) = 1 - \exp[-a(\alpha^2 - k^2)^{1/2}], \quad (\text{B1})$$

$$\tilde{G}(\alpha) = 1 + \exp[-a(\alpha^2 - k^2)^{1/2}], \quad (\text{B2})$$

in the manner

$$G(\alpha) = G_+(\alpha)G_+(-\alpha), \quad (\text{B3})$$

$$\tilde{G}(\alpha) = \tilde{G}_+(\alpha)\tilde{G}_+(-\alpha), \quad (\text{B4})$$

where $G_+(\alpha)$ and $\tilde{G}_+(\alpha)$ are regular in the upper half complex α -plane ($\text{Im}\alpha > 0$) and have algebraic behavior at infinity. These two factorizations have been studied extensively in the literature. Here we simply list several useful final results.

(i) Infinite product forms^{6, 7, 12, 13}

$$G_+(\alpha) = [2(1 + \alpha/k) \sin(ka/2)]^{1/2} \exp(-i\pi/4) \times \exp[(i\alpha a/2\pi)(1 - C + \ln(4\pi/ka) + \frac{1}{2}\pi i)] \times \exp[(i\alpha\gamma/2\pi) \ln[(\alpha - \gamma)/k]] \prod_{n=1}^{\infty} (1 + \alpha/i\gamma_n) \times \exp(i\alpha a/2n\pi), \quad (\text{B5})$$

$$\tilde{G}_+(\alpha) = [2 \cos(ka/2)]^{1/2} \exp[(i\alpha a/2\pi)(1 - C + \ln(\pi/ka) + \frac{1}{2}\pi i)] \times \exp[(i\alpha\gamma/2\pi) \ln[(\alpha - \gamma)/k]] \prod_{n=1}^{\infty} (1 + \alpha/i\gamma_{n-1/2}) \times \exp[i\alpha a/(2n-1)\pi], \quad (\text{B6})$$

where $C = 0.57721 \dots$ is Euler's constant, $\gamma = (\alpha^2 - k^2)^{1/2} = -i(k^2 - \alpha^2)^{1/2}$, and $\gamma_m = [(2m\pi/a)^2 - k^2]^{1/2} = -i[k^2 - (2m\pi/a)^2]^{1/2}$ with $m = n$ or $n - \frac{1}{2}$.

(ii) Relation to Weinstein's functions^{18, 19}:

$$G_+(\alpha) = \exp[U(\alpha/k, ka)], \quad (\text{B7})$$

$$\tilde{G}_+(\alpha) = \exp[\tilde{U}(\alpha/k, ka)]. \quad (\text{B8})$$

Here $U(s, ka)$ denotes the exact Weinstein function

$$U(s, ka) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln[1 - \exp(ika - kat^2)] \times \frac{(1 + it^2)(1 + \frac{1}{2}it^2)^{-1/2}}{t(1 + \frac{1}{2}it^2)^{1/2} - 2^{-1/2}s \exp(i\pi/4)} dt, \quad (\text{B9})$$

and $\tilde{U}(s, ka)$ is again given by (B9) after replacing $\ln[1 - \exp(ika - kat^2)]$ by $\ln[1 + \exp(ika - kat^2)]$. In Ref. 18, an approximation for $U(s, ka)$ was introduced,

$$U_{\text{app}}(s, ka) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln[1 - \exp(ika - kat^2)] \times \frac{1}{t - 2^{-1/2}s \exp(i\pi/4)} dt, \quad (\text{B10})$$

and various properties of $U_{\text{app}}(s, ka)$ were discussed. It has been shown¹⁹ that the error of this approximation is

$O(k^{-1}a^{-1})$, uniformly in s .

(iii) Asymptotic expansion for large ka when α/k is not close to zero^{18, 19}:

$$G_+(\alpha) = 1 - \frac{\exp(i\pi/4)}{(2\pi ka)^{1/2}} \frac{k}{\alpha} \sum_{n=1}^{\infty} \frac{\exp(inka)}{n^{3/2}} + O(k^{-1}a^{-1}), \quad (\text{B11})$$

$$\tilde{G}_+(\alpha) = 1 - \frac{\exp(i\pi/4)}{(2\pi ka)^{1/2}} \frac{k}{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n \exp(inka)}{n^{3/2}} + O(k^{-1}a^{-1}), \quad (\text{B12})$$

which can be derived starting from either (B9) or (B10).

(iv) Logarithmic derivative at $\alpha = 0$:

$$\frac{kG'_+(0)}{G_+(0)} = \frac{1}{2} + \frac{ika}{2\pi} \left[-C + \ln\left(\frac{4\pi}{ka}\right) + i\frac{\pi}{2} \right] + \frac{ika}{2\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n^2 - (ka/2\pi)^2}} \right), \quad (\text{B13})$$

$$\frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} = \frac{ika}{2\pi} \left[-C + \ln\left(\frac{\pi}{ka}\right) + i\frac{\pi}{2} \right] + \frac{ika}{2\pi} \sum_{n=1}^{\infty} \left(\frac{1}{(n - \frac{1}{2})} - \frac{1}{[(n - \frac{1}{2})^2 - (ka/2\pi)^2]^{1/2}} \right), \quad (\text{B14})$$

where $[m^2 - (ka/2\pi)^2]^{1/2} = -i[(ka/2\pi)^2 - m^2]^{1/2}$ with $m = n$ or $n - \frac{1}{2}$. The derivations of (B13) and (B14) follow from (B5) and (B6), respectively. An alternative representation of the derivatives reads

$$\frac{kG'_+(0)}{G_+(0)} = \frac{1}{2}ka \sum_{n=1}^{\infty} H_0^{(1)}(nka), \quad (\text{B15})$$

$$\frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} = \frac{1}{2}ka \sum_{n=1}^{\infty} (-1)^n H_0^{(1)}(nka). \quad (\text{B16})$$

To establish (B15), let us start with the logarithmic differentiation of (B7), viz.,

$$\frac{kG'_+(0)}{G_+(0)} = \frac{\partial}{\partial s} U(s, ka) \Big|_{s=0} = \frac{2^{-1/2} \exp(i\pi/4)}{2\pi i} \int_{-\infty}^{\infty} \ln[1 - \exp(ika - kat^2)] \times \frac{(1 + it^2)(1 + \frac{1}{2}it^2)^{-1/2}}{[t(1 + \frac{1}{2}it^2)^{1/2} - 2^{-1/2}s \exp(i\pi/4)]^2} dt \Big|_{s=0}.$$

Integrating by parts, we find

$$\frac{kG'_+(0)}{G_+(0)} = \left(\frac{2^{-1/2} \exp(i\pi/4)}{2\pi i} \ln[1 - \exp(ika - kat^2)] \times \frac{(-1)}{t(1 + \frac{1}{2}it^2)^{1/2} - 2^{-1/2}s \exp(i\pi/4)} \Big|_{t=-\infty}^{\infty} + \frac{2^{-1/2} \exp(i\pi/4)}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp(ika - kat^2)}{1 - \exp(ika - kat^2)} \times \frac{2kat}{t(1 + \frac{1}{2}it^2)^{1/2} - 2^{-1/2}s \exp(i\pi/4)} dt \Big|_{s=0} \right) = \frac{\exp(-i\pi/4)ka}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp(ika - kat^2)}{1 - \exp(ika - kat^2)} \times \frac{dt}{(1 + \frac{1}{2}it^2)^{1/2}}. \quad (\text{B17})$$

The integrand can be expanded in a geometric series yielding

$$\frac{kG'_+(0)}{G_+(0)} = \frac{\exp(-i\pi/4)ka}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \exp(inka) \int_{-\infty}^{\infty} \frac{\exp(-nkat^2)}{(1 + \frac{1}{2}it^2)^{1/2}} dt. \quad (\text{B18})$$

By applying the substitution $t^2 = 2s$, the integral in (B18) becomes

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{\exp(-nkat^2)}{(1 + \frac{1}{2}it^2)^{1/2}} dt \\ &= \exp(-i\pi/4)\sqrt{2} \int_0^{\infty} \frac{\exp(-2nkas)}{(s^2 - is)^{1/2}} ds \\ &= \frac{\pi}{\sqrt{2}} \exp(i\pi/4) \exp(-inka) H_0^{(1)}(nka), \end{aligned} \quad (\text{B19})$$

according to Equation 4.3 (16) in Ref. 20. On substitution of (B19) into (B18) we obtain (B15). A similar proof holds for (B16).

(v) Logarithmic derivative at $\alpha=0$ for large ka :

$$\frac{kG'_+(0)}{G_+(0)} = \frac{\exp(-i\pi/4)}{\sqrt{2\pi}} (ka)^{1/2} \sum_{n=1}^{\infty} \frac{\exp(inka)}{n^{1/2}} + O(k^{-1/2}a^{-1/2}), \quad (\text{B20})$$

$$\frac{k\tilde{G}'_+(0)}{\tilde{G}_+(0)} = \frac{\exp(-i\pi/4)}{\sqrt{2\pi}} (ka)^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n \exp(inka)}{n^{1/2}} + O(k^{-1/2}a^{-1/2}) \quad (\text{B21})$$

which are obtained from (B15) and (B16) after replacing the Hankel functions by their asymptotic expansions.

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