On a property of the Fourier-cosine transform

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ON A PROPERTY
OF
THE FOURIER-COSINE TRANSFORM
by
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Reports on Applied and Numerical Analysis
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On a property of the Fourier-cosine transform

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SUMMARY

It is shown that the Fourier-cosine transform maps functions of the form

\[ t \mapsto \phi(1 - 2 \tanh^2 t) \cosh^{-1} t, \]

with \( \phi \) an entire analytic function and \( \Re \nu > 0 \), bijectively onto the functions

\[ x \mapsto \Gamma(\frac{1}{2} (\nu + ix)) \Gamma(\frac{1}{2} (\nu - ix)) \psi(x). \]

Here \( \psi \) is an even and entire analytic function of sub-exponential growth, i.e.

\[ \forall \nu > 0 : \sup_{z \in \mathbb{C}} e^{-\epsilon |z|} |\psi(z)| < \infty. \]
1. Introduction

The Fourier-Jacobi transform $f \mapsto g$ is defined by

$$ g(\lambda) = \int_0^\infty f(t) \phi^{(\alpha, \beta)}_h(t) \Delta_{\alpha, \beta}(t) \, dt, \quad \alpha > -1, \beta \in \mathbb{R} $$

with

$$ \phi^{(\alpha, \beta)}_h(t) = 2F_1 \left( \frac{1}{2} (\alpha + \beta + 1 + i\lambda), \frac{1}{2} (\alpha + \beta + 1 - i\lambda); \alpha + 1; -\sinh^2 t \right), \quad t > 0 $$

and

$$ \Delta_{\alpha, \beta}(t) = (2\sinh t)^{2\alpha+1} (2\cosh t)^{2\beta+1}. $$

If we take

$$ f(t) = (\cosh t)^{\delta+i\mu-2} P_{\alpha}^{(\alpha, \beta)} (1 - 2 \tanh^2 t), \quad \alpha, \delta > -1, \beta, \lambda, \mu \in \mathbb{R}, $$

then, following Koornwinder, [K],

$$ g(\lambda) = c_{\alpha, \beta, \mu} \left( \frac{1}{2} (\delta + i\mu + 1 - i\lambda)) \Gamma(\frac{1}{2} (\delta + i\mu + 1 - i\lambda)) \right). $$

Here the $P_{\alpha}^{(\alpha, \beta)}$ are Jacobi polynomials and the $W_\alpha$ are Wilson polynomials. For the constant $c_{\alpha, \beta, \mu}$ see [K]. If we abandon the factor between $\{ \}$ and if we keep the parameters $\alpha, \beta, \delta, \mu$ fixed, then it is clear that, via the Fourier-Jacobi transform, the space of polynomials is mapped linearly and bijectively on the space of even polynomials.

Let us denote this linear mapping by $F_{\alpha, \beta, \mu}$. Now we are in a position to put the following problems.

(i) Extend $F_{\alpha, \beta, \mu}$ bijectively to suitable spaces of analytic functions (entire functions, analytic functions on $[-1, 1]$, germs of analytic functions at 0, etc.).

(ii) Determine growth classes of analytic functions (as in [EG1], [EG2]) which are put in bijective correspondence by the extended $F_{\alpha, \beta, \mu}$.

In the present paper we work out a part of this program for the Fourier cosine transform, $\alpha = \beta = -\frac{1}{2}$. Even in this very special case the results seem to be new. The authors expect that the case with general parameters can be dealt with in the same spirit. We emphasize that our treatment is inspired by Koornwinders formula but does not use it.
2. A special infinite upper triangular matrix

In the sequel we take $v \in \mathbb{C}$, $\text{Re } v > 0$, fixed. We denote $\mathcal{N}_0 = \mathbb{N} \cup \{0\}$.

**Lemma 2.1.**

(i) For each $n \in \mathcal{N}_0$ there exist complex numbers $c_{j,n}, 0 \leq j \leq n$, such that

$$(2\tanh^2 t - 1)^n \cosh^{
u} t = \sum_{j=0}^{n} c_{j,n} \frac{d^{2j}}{dt^{2j}} \left[ \cosh^{
u} t \right], \quad t \in \mathbb{R}. $$

(ii) The numbers $c_{j,n}, 0 \leq j \leq n < \infty$, satisfy the recurrence relation

$$(n + \frac{1}{2}(1 + v)) (2n + v) c_{j,n+1} = c_{j-1,n} + (2n^2 + \frac{1}{2} v (1-v)) c_{j,n} +$$

$$+ n (2n-1+2v) c_{j,n-1} - 2n (n-1) c_{j,n-2}, \quad 0 \leq j \leq n + 1$$

with boundary conditions

$$c_{0,0} = 1, \quad c_{j,n} = 0 \quad \text{if } j < 0 \text{ or } j > n.$$

**Proof.**

It is obvious that (i) is true for $n = 0$, then $c_{0,0} = 1$. Now suppose, induitively, that (i) is true for $n = 0, 1, \cdots, N$.

Differentiating the expression in (i) twice according to $t$ and evaluating the derivatives using the induction hypothesis, leads to both assertions (i) and (ii) at once.

In the following theorem we gather some properties of the numbers $c_{j,n}, 0 \leq j \leq n < \infty$.

**Theorem 2.2.**

(i) $c_{j,j} = \frac{2^{j}}{\Gamma(v)} \left( \frac{\Gamma(v)}{\Gamma(v+2j)} \right)$, $j \in \mathcal{N}_0$.

(ii) $\left| \frac{c_{j,n}}{c_{j,j}} \right| \leq 4^n, \quad 0 \leq j \leq n < \infty$.

(iii) $\lim_{j \to \infty} c_{j,j} \left( \frac{e^{(2-\log 2)j}}{j^{2j} j^{-1}} \right)^{-1} = \frac{\sqrt{n} \ 2^v}{\Gamma(v)}$.

**Proof.**

(i) Take $n = j - 1$ in the recurrence relation, then $(v + 2j - 1)(v + 2j - 2) c_{j,j} = 2 c_{j-1,j-1}$, hence the result.
(ii) Put $d_{j,n} = \frac{c_{j,n}}{c_{i,j}}$, $0 \leq j \leq n < \infty$. Then from the recurrence relation, using $\text{Re}(\nu) > 0$,

$$\lim_{n \to \infty} \left| d_{j+1,n+1} \right| = \left| \frac{(2j+1+\nu)(2j+\nu)}{2(n+\frac{1}{2})(1+\nu)(2n+\nu)} d_{j,n} + \frac{2n^2 + \frac{1}{2} \nu(1-\nu)}{(n+\frac{1}{2})(1+\nu)(2n+\nu)} d_{j+1,n} + \right.$$  

$$+ \frac{n(2n-1+2\nu)}{(n+\frac{1}{2})(1+\nu)(2n+\nu)} d_{j+1,n-1} - \frac{2n(n-1)}{(n+\frac{1}{2})(1+\nu)(2n+\nu)} d_{j+1,n-2} \right| \leq$$  

$$\leq \left| d_{j,n} \right| + 2 \left| d_{j+1,n} \right| + 2 \left| d_{j+1,n-1} \right| \leq \left| d_{j,n} \right| + 2 \left| d_{j+1,n} \right| + 2 \left| d_{j+1,n-1} \right| + \left| d_{j+1,n-2} \right| .$$

Now apply induction.

(iii) Follows from Stirling's formula.

We gather the constants $c_{j,n}$ in an upper triangular matrix $C = [c_{j,n}]_{j=0}^{\infty}$. The next theorem gives some results on the inverse $C^{-1}$ of $C$ which is also an upper triangular matrix. The proof does not differ much from the preceding proofs.

**Theorem 2.3.**

(i) The elements $a_{k,j}$, $0 \leq k \leq j < \infty$ of $C^{-1}$ satisfy

$$\frac{d^j}{dt^j} \left[ \cosh^{-1} t \right] = (\cosh^{-1} t) \cdot \sum_{k=0}^{j} a_{k,j} (2 \tanh^2 t - 1)^k, \quad t \in \mathbb{R} .$$

(ii) The numbers $a_{1,m}$, $0 \leq l \leq m$, satisfy the recurrence relation

$$a_{k,j+1} = (k-\frac{1}{2})(1-\nu)(2k-2+\nu) a_{k-1,j} + (-2k^2 - \frac{1}{2} \nu(1-\nu)) a_{k,j} +$$  

$$- (k+1)(2k+1+2\nu) a_{k+1,j} + (2k+4)(k+1) a_{k+2,j} ,$$

with boundary conditions

$$a_{0,0} = 1, \quad a_{k,j} = 0 \text{ if } k < 0 \text{ or } k > j .$$

(iii) $|c_{j,k} \cdot a_{k,j}| \leq (11)^j$, $0 \leq k \leq j < \infty$.

3. The growth behaviour of the Fourier transform of a class of analytic functions.

For $\alpha = \beta = -\frac{1}{2}$ the Fourier-Jacobi transform reduces to the Fourier-cosine transform.
We take \( f(t) = \phi(1 - 2\tanh^2 t) \cdot \cosh^{-\nu} t \) with \( \phi(z) = \sum_{n=0}^{\infty} a_n z^n \) an arbitrary entire analytic function.

Consider the following formal computation

\[
g(u) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(ut) \, dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (\cosh^{-\nu} t) \left[ \sum_{n=0}^{\infty} a_n (1 - 2\tanh^2 t)^n \right] \cos(ut) \, dt = \\
= \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} a_n \int_0^{\infty} (\cosh^{-\nu} t) \left( 1 - 2\tanh^2 t \right)^n \cos(ut) \, dt \\
= \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} (-1)^n a_n \sum_{j=0}^{n} \left[ \frac{d^{2j}}{dt^{2j}} (\cosh^{-\nu} t) \right] \cos(ut) \, dt \\
= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} (-1)^{n+j} c_{j,n} a_n u^{2j} \int_0^{\infty} (\cosh^{-\nu} t) \cos(ut) \, dt \\
= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} (-1)^{n+j} c_{j,n} a_n u^{2j} \\
= \sqrt{\frac{2}{\pi}} \frac{2^{\nu-2}}{\Gamma(\nu)} \Gamma\left( \frac{1}{2} \right) \Gamma\left( \frac{1}{2} \right) \psi(u) \\
\psi(u) = \sum_{n=0}^{\infty} b_n u^{2n}, \quad b_n = \sum_{n=j}^{\infty} (-1)^{n+j} c_{j,n} a_n.
\]

For the Fourier integral in this calculation see e.g. [O], p. 35.

Note that, at \( \nu \), we used the results of lemma 2.1. In order to justify the remaining part of the calculation we proceed as follows.

Introduce the vectors

\[ a = \text{column } (a_0, a_1, a_2, ...) \]
\[ b = \text{column } (b_0, b_1, b_2, ...) \]

and the infinite diagonal matrix

\[ \tilde{I} = \text{diag } (1, -1, 1, -1, \cdots, (-1)^* \cdots). \]

Now the relation between the, supposed, Taylor coefficients of the functions \( \phi \) and \( \psi \) can be written as
The proof of the following characterization is elementary.

**Characterization 3.1.**

(i) Consider the Taylor series 
\[ \phi(z) = \sum_{n=0}^{\infty} a_n z^n. \]
\( \phi \) is an entire analytic function iff 
\[ \forall r > 0 : (a_n e^{nr})_{n=0} \in l_2. \]

(ii) Consider the Taylor series 
\[ \psi(z) = \sum_{n=0}^{\infty} b_n z^{2n}. \]
\( \psi \) is even entire and sub-exponential, i.e. 
\[ \forall r > 0 : \sup_{t \in \mathbb{R}} |\psi(t)| e^{-rt} < \infty, \]
iff 
\[ \forall r > 0 : (n^{2n} e^{nr} b_n)_{n=0} \in l_2. \]

In the next theorem we derive some fundamental estimates for the matrices \( C \) and \( C^{-1} \).

**Theorem 3.2.**

For each \( t > 0 \) there exists \( \tau > 0 \) such that the infinite matrices
\[ \Theta(t, \tau) := \text{diag}(n^{2n} e^{nr}) i C i \text{diag}(e^{-nr}) \]
\[ \Xi(t, \tau) := \text{diag}(e^{nr}) i C^{-1} i \text{diag}(e^{-nr} n^{-2n}) \]
are bounded as \( l_2 \)-operators.

**Proof.**

We have
\[ |\Theta_{j,n}(t, \tau)| = j^{2j} e^{\tau^2} |c_{j,j}| \left| \frac{c_{j,n}}{c_{j,j}} \right| e^{-\tau r} \]
\[ |\Xi_{k,j}(t, \tau)| = e^{\tau r} |c_{j,j} a_{k,j}| \frac{1}{|c_{j,j}|} j^{2j} e^{-\tau r}. \]

If we take \( \tau > t + 3 \), the wanted result follows with the aid of theorems 2.2, 2.3 and the estimate
\[ \|K\| \leq \sum_{k=1}^{\infty} \sup_{n \geq j} |K_{jk}| \]
for the \( l_2 \)-operator norm \( \|K\| \) of an infinite matrix \( K \).

Finally, our main result.
Theorem 3.3.
The mapping \( F_{-\frac{1}{2}, -\frac{1}{2}, A, \rho} \) which maps the space of polynomials bijectively on the space of even polynomials can be extended to a bijective continuous linear mapping between the space of entire functions and the space of even entire functions of subexponential growth.

Proof.
Let \( \tau > 0 \). Consider
\[
\text{diag}(n^2 e^{\tau t}) b = \text{diag}(n^2 e^{\tau t}) \tilde{C} \tilde{I} a = \\
= \{ \text{diag}(n^2 e^{\tau t}) \tilde{C} \tilde{I} \text{diag}(e^{-\tau t}) \} \text{diag}(e^{\tau t}) a.
\]
For \( \tau > \tau + 3 \) the operator between \{ \} is bounded in \( L_2 \) (Theorem 3.2), further \( \text{diag}(e^{\tau t}) a \in L_2 \) for all \( \tau > 0 \) (characterization 3.1 (i)). So \( \text{diag}(n^2 e^{\tau t}) b \in L_2 \).

Therefore \( \psi \) is entire and of subexponential growth (characterization 3.1 (ii)).

The inverse \( F_{-\frac{1}{2}, -\frac{1}{2}, A, \rho} \), which corresponds to the equality \( a = \tilde{C}^{-1} \tilde{I} b \), can be dealt with in a similar way.

Thus all formal calculations at the beginning of this section become justified.

Corollary 3.4.
The Fourier transform of \( \phi(1-2 \tanh^2 t) \cosh^{-1} t \), \( \phi \) entire, has the form
\[
\Gamma \left( \frac{1}{2} (v+iz) \right) \Gamma \left( \frac{1}{2} (v-iz) \right) \psi(z), \ \psi \text{ entire, even, of sub-exponential growth and vice versa.}
\]

Corollary 3.5.
Comparison with the general formula in section 1 shows
\[
W_N \left( \frac{1}{4}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, -i \mu \right) = \\
= \frac{N! (v)^{2N}}{(-4)^N} \sum_{j=0}^{N} \sum_{\alpha \in \mathbb{N}} (-1)^{\alpha j} c_{j, \alpha} a_{\alpha} \ x^{2j}
\]
with
\[
(v)^{2N} = (v+2N-1)(v+2N-2) \cdots (v) = \frac{\Gamma(v+2N)}{\Gamma(v)}
\]
and \( a_{\alpha} \) such that
\[
P_{N\frac{1}{4}, \frac{1}{2}}(z) = \sum_{k=0}^{N} a_{\alpha_k} z^k.
\]
References


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