Analysis of eddy currents in a gradient coil

Citation for published version (APA):

Kroot, J. M. B. (2005). *Analysis of eddy currents in a gradient coil*. [Phd Thesis 1 (Research TU/e / Graduation TU/e), Mathematics and Computer Science]. Technische Universiteit Eindhoven.
<https://doi.org/10.6100/IR591714>

DOI:

[10.6100/IR591714](https://doi.org/10.6100/IR591714)

Document status and date:

Published: 01/01/2005

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Analysis of Eddy Currents in a Gradient Coil

J.M.B. Kroot

Copyright ©2005 by J.M.B. Kroot, Eindhoven, The Netherlands.

All rights are reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the author.

This research was supported by Philips Medical Systems and Stan Ackermans Institute.

Printed by Ponsen & Looijen, Wageningen

Cover design: Paul Verspaget

CIP-DATA LIBRARY TECHNISCHE UNIVERSITEIT EINDHOVEN

Kroot, J.M.B.

Analysis of eddy currents in a gradient coil /

by Johannes Marius Bartholomeus Kroot -

Eindhoven : Technische Universiteit Eindhoven, 2005. Proefontwerp. -

ISBN 90-386-0604-4

NUR 919

Subject headings: eddy currents / skin effect / Legendre polynomials / magnetic resonance / electric coils / integral equations / magnetic fields / Galerkin method / numerical simulation

2000 Mathematics Subject Classification: 45E99, 78A25, 78A55, 45C10, 65Z05, 49M15

Analysis of Eddy Currents in a Gradient Coil

PROEFONTWERP

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van de
Rector Magnificus, prof.dr.ir. C.J. van Duijn, voor een
commissie aangewezen door het College voor
Promoties in het openbaar te verdedigen
op donderdag 23 juni 2005 om 16.00 uur

door

Johannes Marius Bartholomeus Kroot

geboren te Loon op Zand

Dit proefontwerp is goedgekeurd door de promotor:

prof.dr. W.H.A. Schilders

Copromotoren:

dr.ir. A.A.F. van de Ven

en

dr.ir. S.J.L. van Eijndhoven

Acknowledgements

During the time that I was a PhD student, I have had support from many people. Therefore, I would like to express my sincere gratitude towards them. First of all, I would like to thank prof.dr. W.H.A. Schilders for the coordination of the project. He always came up with new interesting ideas that turned out to be useful in the realization of this thesis. In particular, he brought me into contact with people from different institutes, from which I got a clear overview of the research previously conducted in this topic.

Next, I am greatly indebted to my supervisors dr.ir. S.J.L. van Eijndhoven and dr.ir. A.A.F. van de Ven. They have provided me with a lot of mathematical input, in both the modeling part and the analytical computations. I also want to thank them for the time they have devoted to me and the discussions we have had on the Thursday afternoons, which were stimulating and pleasant.

Apart from the innovative mathematics in this thesis, the essence for me was to come to a serviceable application in the industry. I could not have been more fortunate than to carry out the project in the MRI department of Philips Medical Systems. Therefore, I would like to thank my supervisors dr.ir. H. Boschman and dr. J. Konijn for the collaboration and in particular for the practical input. Special gratitude is due to dr.ir. G. Peeren, who has supported me greatly in the realization of the design tool Eddy.

I am sincerely grateful to prof.dr. U. van Rienen from the Universität Rostock, prof.dr. P.W. Hemker from the Centrum voor Wiskunde en Informatica, Amsterdam, and prof.dr. A.G. Tijhuis from the Eindhoven University of Technology for participation in my doctoral committee and for carefully reading this thesis.

From the numerous people I worked with, I would like to thank three people in particular. First of all I thank dr.ing. H. de Gersem from the Technische Universität Darmstadt for his support in the numerical computations that I used for the validation of my results. His explanations gave me the inspiration for the approach used in Chapter 6 of this thesis. Second, I thank dr.ir. M.C. van Beurden from the electromagnetics department of the Eindhoven University of Technology for his help in obtaining the correct description of the current source. Third, I thank ir. J.M.C. Tas, who just graduated at the Eindhoven University of Technology.

Although I was his supervisor in his final project, he has helped me a lot in the analysis of the magnetic fields.

Naturally, I would like to thank all colleagues of the CASA group of the Eindhoven University of Technology and the Magnets and Mechanics group of Philips Medical Systems for the pleasant work atmosphere during the last three years. Besides, I want to thank the group of mechanical developers of Philips Medical Systems for the nice lunches and coffee breaks.

Last, but not least, I would like to thank my parents Mari and Lia and my sister Suzanne for their continuing support and patience, and André Snelders and Maarten van der Kamp for their assistance during the thesis defense.

Jan M.B. Kroot
Eindhoven, June 2005

Nomenclature

General

$\mathbf{a}$	vector
$\mathbf{A}$	matrix
$\mathbf{a} \cdot \mathbf{b}$	inner product of vectors
$\mathbf{a} \times \mathbf{b}$	cross product of vectors
$ \mathbf{a} $	length of vector
$\llbracket \mathbf{a} \rrbracket$	jump of vector
$\mathbf{a}^*$	complex conjugate
$\Re\{\}$	real part
r, φ, z	cylindrical coordinates
x, y, z	Cartesian coordinates
$\tilde{x}$	dimensionless x
$\bar{x}$	time-averaged x
$\hat{x}$	Fourier transform of x
$(\cdot, \cdot)$	inner product of functions
$\mathbf{1}_{[a,b]}$	characteristic function on the interval $[a, b]$

Operators

∇	gradient operator
$\nabla \cdot$	divergence operator
$\nabla \times$	curl operator
∇^2, Δ	Laplace operator
$\frac{\partial}{\partial a}$	partial derivative to a
$\mathcal{F}_z$	Fourier transform operator to z
$\mathbf{K}$	integral operator
$\mathbf{K}^{(N)}$	finite rank integral operator
Π	projection operator

Subscripts

i	island
r	ring
s	surface
x, y, z	x -, y -, z -component
0	begin position
1	end position

Superscripts

c	characteristic scale value
e	eddy
imp	imposed
ind	induced
$prim$	primary field
s	source
sec	secondary field
(c)	cosine mode
(L)	per unit of length
(s)	sine mode
—	inner region
+	outer region

Greek symbols

α_n	coefficient of n -th basis function
γ	Euler's constant
δ	penetration depth
ϵ_0	electric permittivity
κ	system parameter
$\hat{\kappa}$	characteristic system parameter
λ_n	n -th eigenvalue
μ_0	magnetic permeability
μ_n	n -th eigenvalue
ρ	electric charge
ρ_s	surface charge
σ	electric conductivity
ϕ	electric scalar potential
ϕ_n	n -th basis function
$\Delta\phi$	phase-lag
χ_e	electric susceptibility
χ_m	magnetic susceptibility
$\Psi^{(0)}$	polygamma function
ψ_l	characteristic function of group l
ω	angular frequency

Roman symbols

A	vector potential
B	magnetic induction
c	central position of a strip
c_0	speed of light
d	half the width of a strip
D	width of a strip
D	dielectric displacement
da	surface element
ds	line element
dv	volume element
∂S	curve enclosing S
∂V	closed boundary surface of V
e	exponential function
e	unity vector
E	elliptic integral of the second kind
E	electric field
${}_2F_1$	hypergeometric function
f	frequency
f_{char}	characteristic frequency
$\mathbf{F}_e$	Coulomb force
$\mathbf{F}_m$	Lorentz force
G	gradient
$\mathbf{G}_f$	momentum of the field
$\mathbf{G}_{mech}$	momentum of moving charges
h	thickness of strips
H	magnetic field
I	total current
I_e	effective current
I_n	modified Bessel function of order n of the first kind
$\mathbf{j}$	surface current density
J	current density
J_k	Bessel function of order k of the first kind
K	elliptic integral of the first kind
$\mathcal{K}$	kernel function
K_n	modified Bessel function of order n of the second kind
L	number of groups
$\bar{L}$	self-inductance
L_2	space of functions of integrable square
M	total number of basis functions
M	magnetization field
n	normal vector
N	number of strips
P	polarization field
P_{diss}	dissipated power
P_{ind}	power caused by inductive effects

P_k	Legendre polynomial of order k
Q_k	Legendre function of the second kind of order k
R	radius
$\bar{R}$	resistance
$\mathcal{R}$	distance function
R_s	radius of plane circular strip
$\mathbf{S}$	Poynting vector
$\mathcal{S}$	surface
S_c	surface of the cylinder
$S_{\cup}$	unified surface of the strips
$\mathbf{t}$	tangent vector
t	time
T_1	longitudinal relaxation time
T_2	transverse relaxation time
u	length coordinate of central line
U_m	magnetic energy
u_n	n -th eigenfunction
v	width coordinate of strip
$\mathcal{V}$	volume
$\bar{Z}$	impedance
Z_k	integrated Legendre polynomial of order k

Abbreviations

BEM	Boundary Element Method
CRT	Cathode-Ray Tube
CT	Computed Tomography
DC	Direct Current
FDM	Finite Difference Method
FDTD	Finite Difference Time Domain
FEM	Finite Element Method
FIT	Finite Integration Technique
FVM	Finite Volume Method
MR	Magnetic Resonance
MRI	Magnetic Resonance Imaging
PMS	Philips Medical Systems
PNS	Peripheral Nerve Stimulation
RF	Radio Frequency
SAR	Specific Absorption Rate

Contents

Acknowledgements	v
Nomenclature	vii
1 Introduction	1
1.1 Magnetic Resonance Imaging	1
1.2 Gradient coils	6
1.2.1 Design of gradient coils	6
1.2.2 Eddy currents	8
1.3 Objective and outline of thesis	9
1.3.1 Objective	9
1.3.2 Literature	10
1.3.3 Outline	12
1.4 Concluding remarks	14
2 Model formulation	17
2.1 Electromagnetic field theory	17
2.1.1 Maxwell's equations	17
2.1.2 Boundary conditions	19
2.1.3 Constitutive equations and driving source	20
2.1.4 General solution of Maxwell's equations in free space	21
2.1.5 Energy flow and electromagnetic forces	23

2.2	Geometry of the model	28
2.3	Mathematical model	30
2.3.1	Basic assumptions and reduced model	30
2.3.2	Dimensionless formulation	36
2.3.3	Integral formulation	38
2.3.4	Coordinate-free problem setting	43
2.4	Summary	44
3	Mathematical Analysis	45
3.1	Leading integral equation, with logarithmic kernel	46
3.1.1	Type one: The ring	46
3.1.2	Type two: The plane circular strip	49
3.1.3	Type three: The plane rectangular patch	51
3.2	Solution procedure	53
3.2.1	The Galerkin method	54
3.2.2	Choice of basis functions	55
3.3	A purely logarithmic kernel function	57
3.4	Special case: Plane rectangular strips	60
3.4.1	Model formulation	61
3.4.2	Display of results	66
3.5	Summary	71
4	Circular loops of strips	73
4.1	Model formulation	73
4.1.1	Configuration	73
4.1.2	Adjustment of the integral equation	75
4.2	Composition of the matrices	78
4.3	Numerical results	83
4.3.1	One ring	84

4.3.2	The Maxwell pair	92
4.3.3	A realistic z-coil	98
4.3.4	Validation	102
4.4	Summary	104
5	Rings and islands	107
5.1	Model formulation	108
5.2	Solution procedure	110
5.2.1	Rewriting the kernel function	110
5.2.2	Basis functions	112
5.2.3	Calculation of the matrix elements	116
5.3	Numerical results	118
5.3.1	One ring and one island	119
5.3.2	Two rings and one island	120
5.3.3	Two rings and four islands	122
5.4	Summary	126
6	Plane circular strips	129
6.1	Problem formulation	129
6.2	Solution method	133
6.2.1	Construction of a linear set of equations	133
6.2.2	Solving the linear set of equations	139
6.3	Numerical results	140
6.3.1	One plane circular strip	141
6.3.2	Ten plane circular strips	142
6.4	Summary	144
7	Aspects and conclusions of the design	145
7.1	Aspects of the design	145
7.2	Achievements	149

7.3	Conclusions	150
A	Derivation of (3.61)	153
	Bibliography	154
	Index	161
	Summary	163
	Samenvatting	165
	Curriculum vitae	167

Introduction

Since 1973, when the first images of an inhomogeneous object were provided by the Magnetic Resonance Imaging (MRI) technique, the development of clinical MRI has been rapid. Its greatest impact is in the field of neurology. Apart from (in-vivo) diagnosis of the brain, central nervous system, and spinal cord, where it has largely become the superior method, MRI has made major contributions in oncology, cardiovascular or abdominal imaging, and the investigation of musculoskeletal problems. Challenges in MRI research are the improvement of image acquisition times and the reduction of loss of image quality due to organ movements and due to disturbing electromagnetic phenomena such as eddy currents. In the gradient coil, which is part of the scanner, eddy currents occur due to acquisition sequences. The research presented in this thesis focusses on the analysis and simulation of these eddy currents.

This research was carried out for Philips Medical Systems (PMS), in the group Magnets and Mechanics. PMS designs, develops, and manufactures MRI-scanners, see Figure 1.1, and wishes to gain insight in factors that determine the quality of the scanner.

Before we discuss the objectives and results of the research reported in this thesis, an introduction to magnetic resonance imaging is given; see Section 1.1. In Section 1.2, we describe the design of gradient coils and the problems of eddy currents occurring in these coils. The objectives of this research are to analyze the eddy currents in the gradient coils and to design a software tool that simulates the eddy currents. A more detailed description of the objectives, together with an outline of the thesis, are presented in Section 1.3. In Section 1.4, we present concluding remarks.

1.1 Magnetic Resonance Imaging

MRI is a revolutionary imaging technique that plays an important role in the medical community. It provides images of cross-sections of the body without ionizing radiation. Detailed information is obtained because MR signals are sensitive for several tissue parameters. Its

Figure 1.1: *MRI-Scanner (Philips Intera 3.0T)*

excellent soft-tissue contrast makes MRI the ideal technique for viewing the brain, the heart, and cardiac functions, as well as the muscular, skeletal and abdominal systems. Unlike CT-scanning (Computed Tomography), MRI can take scans from any angle. It is noninvasive, involves no radiation, and no pain. Also, risks for patients are not reported. We explain briefly the principles of magnetic resonance and MRI systems. For an extensive overview, see for example [11], [79].

Roughly, the principle of magnetic resonance imaging comes down to the following: hydrogen protons are stimulated by a strong external magnetic field and additional radio pulses, resulting in small electromagnetic signals emitted by the protons. The emitted signals are received by an acquisition system and processed to become an image with contrast differences.

Specifically, the core of an atom, the nucleus, is a collection of positively charged and neutral particles that account for the bulk of the atomic mass. Certain nuclei possess an inherent angular momentum, or spin, which induces a magnetic field with magnitude and direction represented by the magnetic moment (μ) of the spin; see Figure 1.2 (a). In the absence of an applied external magnetic field, the individual magnetic moments are randomly oriented due to the kinetics produced by thermal energy. Thus, the macroscopic magnetic moment (M) is zero. When an external magnetic field is applied to a collection of protons, the magnetic dipoles tend to align with the magnetic field, either with (parallel to) or against (anti-parallel to) the direction of the applied field.

However, the alignment of the magnetic moment with the applied magnetic field is not perfect. In the presence of an applied magnetic field, the spin vectors of the nuclei experience a torque, which causes them to rotate about the axis of the applied field with a specific frequency; see Figure 1.2 (b). This cone-shaped rotation is called Larmor precession. The rate

Figure 1.2: (a) Each charged nucleus induces a magnetic moment μ due to spinning; (b) In the presence of an applied magnetic field B_0 , the spinning nuclei experience a torque; (c) A rotating B_1 -field causes an RF flip angle θ of the macroscopic magnetization M .

of precession is dependent on the specific physical characteristics of the isotope involved and on the strength of the applied magnetic field. This relationship is expressed as $\omega = \gamma B_0$, where ω is the Larmor frequency, γ is a constant of proportionality (gyromagnetic ratio), which is specific for the nuclei involved, and B_0 is the magnetic field strength. Larmor precession is a resonance phenomenon. If a system has a natural frequency of oscillation, the energy can be most efficiently transferred to the system at this frequency.

When looking at the sum of the individual magnetic moments, which is called the macroscopic magnetic moment, we observe a small excess favoring the low energy (parallel) orientation. The strength of the macroscopic magnetization M is a function both of the temperature and of the strength of the applied magnetic field.

In order to obtain imaging information from the spins, they have to induce a current in the receiver coil. In its equilibrium state, according to Faraday's law of induction, the macroscopic magnetization is static and does not induce a current. Thus, the spins must be perturbed or excited. This can be achieved by irradiating the spin system with an RF-pulse, a short burst of radio frequency, matching the Larmor frequency of the nuclei of interest.

The RF excitation pulse can be represented by an additional magnetic field, B_1 , which is perpendicular to B_0 and rotates about B_0 with the Larmor frequency for a short period of time. The effect of applying the B_1 -field is that the magnetization precesses about this second field, and hence spirals away from the longitudinal direction, towards the transverse plane. This is represented in Figure 1.2 (c).

Each time, changing of direction of the vector induces an electrical current in a receiver coil. The received current is the MRI-signal used for creating the images. The transverse vector moves towards and away from the receiver coil with the precession frequency. Therefore, also the resulting MRI-signal has this frequency. The amplitude of the signal depends on the magnitude of the transverse vector.

When the radio pulse stops, all nuclei return to their original position. The magnetization returns to its original value. This is called relaxation, where longitudinal relaxation refers to longitudinal magnetization and transverse relaxation refers to transverse magnetization. The time required for longitudinal magnetization to return to 63% of its original value is the relaxation time T_1 . After complete relaxation, the transverse magnetization is zero. The relaxation time T_2 is the time it takes for the transverse magnetization to decrease to about 37%. The relaxation times T_1 and T_2 are important characteristics for tissues. For example, water has a high T_1 and a high T_2 . This property of water is used as follows:

- if the receiving time of the signal is long, i.e. a long echo-time, then the water particles are emitting stronger signals than other particles;
- if the time between two pulses is short, i.e. a short repetition time, then the water particles are not completely relaxed before they flip again.

Figure 1.3: *Schematic overview of an MRI-scanner.*

In the MRI-scanner, the external magnetic field is realized by the so-called main magnet. This magnet produces a strong static homogeneous field. RF-coils are installed to excite the nuclei and to detect the re-emitted signals from the nuclei. For the spatial differentiation of the signals emitted from parts of the human body to be diagnosed, MRI utilizes magnetic field gradients. A magnetic field gradient is a weak magnetic field, superimposed on the main magnetic field, which changes linearly with position. The magnetic field gradients in the x , y , and z directions required for an imaging study are produced by three sets of orthogonally

positioned coils. A schematic picture of an MRI scanner is presented in Figure 1.3. In addition to producing gradients oriented along the x , y , or z axes, by powering the gradient coils in combination, it is possible to generate magnetic field gradients in any direction.

The gradients generated by the gradient coils should be linear over the imaging volume, and should be stable for the duration of the applied gradient. Together with the change in time of the gradient of the field, the precession frequency is affected. For the reconstruction of images, the gradient coils have three important functions:

- selection of a slice;
- location by use of frequency encoding;
- location by use of phase encoding.

Figure 1.4: *Magnetic resonance imaging of slices of the head in three directions, accomplished by the gradient coils.*

First, the slice selection gradient, G_z , is switched on, and together with the RF pulse it defines a slice. To determine the position within the slice, the other gradients are used. If, subsequently, the phase encoding gradient G_y is switched on, the spins will have their resonant frequencies, and hence their rate of precession, altered according to their position along the y -axis. When the phase encoding gradient is switched off, spins at points along the y -axis again begin to precess at the same frequency. However, their phases will no longer be the same. The frequency encoding gradient G_x , when switched on, causes the excited nuclei to precess at increasing frequency along the x -axis. The frequency distribution spectrum of the response signal provides a one-dimensional profile of signal intensity in the subject. The intensity of the received magnetic resonance signals is a function of several parameters, including proton density and relaxation times T_1 and T_2 . The relative contribution of each parameter is controlled by adjusting the RF pulses, the gradients applied, and the timing of the data acquisition. These parameters are set in the pulse sequence. The slice selection in the three directions is represented by Figure 1.4.

1.2 Gradient coils

The main magnet causes a strong static and substantially uniform magnetic field with a strength of a few Tesla (T). The gradient system incorporates three spatially independent and time controllable gradient fields, which are significantly smaller than the main field. The gradient fields are normally expressed in millitesla per meter (mT/m). The amplitude of the total field determines the Larmor frequency. The z -component of the magnetic field is highly dominant. Therefore, we are only interested in this component. Hence, if we talk about the magnetic field of the gradient system, we refer to the z -component only, unless explicitly stated otherwise.

The linearity of the gradient fields is measured by the variation of the gradients G_x , G_y and G_z . The transverse gradients, G_x and G_y , are defined as the gradient of the z -component of the magnetic field restricted to the variables x and y , respectively. The derivative of B_z to z is called the longitudinal gradient, G_z . For imaging purposes, for ideal gradient fields, G_x , G_y , and G_z are uniform within the region of interest. In this section, we discuss some practical problems that are encountered in the design of gradient coils.

1.2.1 Design of gradient coils

Linearity, self-inductance, resistance, stored energy, dissipated power, external fields, and region of interest are the multiple aspects we have to take into consideration in the design of a gradient coil. They are related to functionality, image processing, image quality, and costs. Issues of the performance of a gradient coil are the following:

- Spatial non-linearity of the gradient results in image distortions. Nuclei are excited that do not belong to the desired slice. The system detects resonance signals from different positions and processes them as if they come from the slice. The image appears blurred and for large non-linearities ghosts appear, which can even result in complete organs at a wrong position.
- The rate of change of the gradient field is proportional to the rate of change of the current. The voltage is related to the current by means of the self-inductance $\bar{L}$ and the Ohmic resistance $\bar{R}$ of the coil. The ratio of $\bar{L}$ and $\bar{R}$ defines the relaxation time τ . Assuming a voltage turned on instantaneously at the moment $t = 0$, with no current present in the electric circuit before this moment, then the current will rise exponentially. The relaxation time τ is the time it takes for the current to rise 63% of its full value. In order to reduce the voltages and to have shorter switching times, the self-inductance of the coil has to be decreased.
- The stored energy in a gradient coil is defined as the squared amplitude of the magnetic field, integrated over the volume of the coil. This means that the stored energy increases

rapidly if the magnetic fields and the volume of the coil are increased. Moreover, the self-inductance is linearly related to the stored energy.

- For a given current, the dissipated power in the coil is proportional to the Ohmic resistance. In order to reduce the energy costs, the dissipated power has to be minimized.
- A coil creates internal as well as external fields. The external fields induce eddy currents in other conductors, which affect the total field. Therefore, a shielding coil is included, concentric with the first coil, but on a larger radius, and with a current in opposite direction. The total field outside the shielding coil can thus be made very small. The disadvantages are that more energy is needed to maintain the same gradient field inside the inner coil and that an additional coil has to be installed.
- The region of interest cannot merely be enlarged, because that would affect the feasibility of the requirements on the previous points.

Figure 1.5: Selection of a thinner slice by using a higher gradient.

The quality of a gradient coil is measured primarily by three factors. The first factor is the *coil constant*, which is a measure of the field gradient at the center of the coil produced by one unit of current. A steep gradient is needed to obtain high resolution images; by a steeper gradient, while using the same bandwidth of the radio frequency pulse a thinner slice can be selected; see Figure 1.5. For example, specialized gradient coils are developed to determine blood flow shear stress very near a vessel wall, particularly in the vicinity of carotid bifurcation; see e.g. [12]. The second factor is the *self-inductance*, which should be as small as possible to enable low switching rates. The third factor is *gradient homogeneity*, which is measured by the difference between the desired field and the field actually achieved over the region of interest. To properly measure the quality of a gradient coil, Turner [74], [75] defined a figure

of merit as a relation of the mentioned three factors. To design a coil of high quality is not a straightforward thing to do. For example, the gradient efficiency and switching speed can be improved by reducing the physical size of the coil. However, to access the region of uniform gradient is typically more difficult in smaller coils than in larger ones.

To create G_z , the principle of a Maxwell pair is used. A Maxwell pair, see e.g. [28, Sect. 2.4], consists of two circular loops of wire with a radius of r meters, separated by a distance of $r\sqrt{3}$ meters, and carrying the same current in opposite direction. In the vicinity of their midpoint $z = 0$ the z -component of the magnetic field vanishes, as well as its third and all even derivatives with respect to z . Therefore, along the z -axis in the neighborhood of $z = 0$, B_z is linear up to z^4 . A standard example of a coil that creates a magnetic field whose axial component is linearly increasing in the transversal direction is the Golay coil. The Golay coil consists of four sets of symmetrically placed saddle coils. The optimal dimensions are derived using the zonal spherical harmonics. For more details on the Golay coil, see [28, Sect. 3.4].

In practise, gradient coils are manufactured using copper strips instead of wires. The design can be formulated as an optimization problem: find the most efficient conductor shape, subject to geometrical, electrical or electromagnetic constraints. This means that the total delivered energy should be minimized, such that

- the conductor shape is within a given volume;
- the current density is as high as possible;
- the magnetic field complies with a predetermined shape.

A proposition to solve such a problem is described by Peeren [46],[47]. Peeren makes use of the calculation of stream functions. The stream functions determine streamlines, which are the corresponding equipotential lines. In the practical design of a gradient coil thin strips of copper are placed along the streamlines. The magnetic field induced by the currents in the conductor pattern is computed, assuming that the current density is constant and uniform in the strips. In Chapter 2, Figure 2.1 shows possible designs of an x-coil and a z-coil.

1.2.2 Eddy currents

In MRI scanning coils, the effects of eddy currents are playing an increasingly important role. The design and operation of electrical devices require detailed knowledge of electrical phenomena. Some devices are based on the development of eddy currents such as generators of high magnetic fields, flaw detectors, devices for magnetic forming, material separators, and so forth. In these applications, eddy currents are desirable and exploited. In other applications, eddy currents are undesirable as they cause energy losses and should therefore be minimized. In both cases, the eddy currents need to be accurately described and analyzed.

One of the major problems in the use of switched gradient coils is the interaction of the rapidly changing fields with other conducting structures in the MRI-scanner including the gradient coils themselves, resulting in eddy currents on the surfaces of these conductors. The eddy currents cause forces on the conductors that result in noise and reduction of the lifetime. Moreover, the eddy currents cause perturbations on the expected gradient field, leading to MR pictures with blurring and ghosting; signals are detected from points outside the field of view, because they have the same magnetic field value. So, eddy currents should be controlled as much as possible. In general, the appearance of eddy currents is important in the design of gradient coils. The way to reduce eddy currents is by active shielding, applying pre-emphasis currents and cutting slits in the conductors.

Especially inside the gradient coils, it is difficult to predict the eddy currents, because of the complex geometries of the coils and the mutual inductances between the different parts. For example, the z -coil is influenced by the x -coil and the y -coil, but also by the shielding coils and the cryostat. The cryostat has the function to cool down the superconductors in the main magnet.

As we will discuss, eddy current response is greatly affected by the frequencies used. The electrical conductivity is another direct effect on the eddy current flow; the greater the conductivity, the greater the flow of eddy currents on the surface. Other effects are geometrical features such as curvature, edges, grooves, and the distances between the conductors.

1.3 Objective and outline of thesis

1.3.1 Objective

The gradient coils are designed to create a uniform gradient in the magnetic field. For that, a pattern of streamlines is computed that results in the desired magnetic field, subject to design requirements, such as low energy dissipation, low self-inductance and use of a small amount of conducting material. On the pattern of streamlines, thin strips of copper are installed.

The currents in the strips are assumed static, and uniformly distributed. However, even in the static case, a uniform distribution is not always a valid assumption, especially not for curved strips. In the non-static case, when gradient coils are switched on and off subsequently, the current is not distributed uniformly, because eddy currents are induced in the strips. Eddy currents cause undesirable disturbances in the gradient field. Moreover, the presence of eddy currents will increase the resistance of the coil, resulting in a higher energy dissipation. Meanwhile, the self-inductance will decrease, resulting in the need for a smaller voltage supply.

The main goal of this thesis is to give a detailed analysis and a simulation of the eddy currents that are present in gradient coils. The analysis will be of support to the overall design of such coils at PMS. To get to such an analysis, a mathematical model must be derived for

the current distribution in a gradient coil of a given configuration. The model should incorporate changing fields and mutual electromagnetic field coupling. As far as the geometry is concerned, the model should describe conductors on a cylindrical surface.

In particular, the analysis should lead to the design of a software tool that simulates the current distribution and the electromagnetic behavior quantitatively. In the simulation, special attention should be devoted to time effects (different frequencies) and spatial effects (space dependent magnetic fields), and in particular to the derivation of characteristics that determine the system. Time dependence of the current distribution is an important aspect in the relationship to the question how problem areas can be identified regarding image quality. Localizing the problem areas is a necessity for the development of a method to eliminate unwanted perturbations in the gradient field.

The analysis in this thesis provides insight in the behavior of eddy currents in a gradient coil. Characteristic quantities of a system are derived, in particular self-inductance, resistance, characteristic frequency, phase-lag with respect to the source, dissipated energy, stored energy, variation of the magnetic field, the linearity of the gradient field, and the error of the achieved field in comparison with the desired field. For all these characteristic quantities, we investigate their dependence on the frequency of the applied source, the shape of the conductors, the distance between the conductors, and the conductivity.

1.3.2 Literature

From the numerous books on electromagnetic fields, three books have to be mentioned that deal with eddy currents. The book by Tegopoulos and Kriezis [68] is concerned with analytical methods applied mainly to two-dimensional configurations. The book by Stoll [63] is concerned with both analytical and numerical methods, giving a wide spectrum of eddy current considerations; especially the finite difference method has been widely documented. The book by Lammeraner and Staff [35] is devoted to the analytical approach only and covers one-dimensional configurations.

Research on eddy currents is mostly devoted to localizing the areas where they occur. From the devices in which eddy currents are desirable and therefore exploited, we mention the eddy current waste separation belt. Rem describes in [51] how eddy current separation is used in the recycling of non-ferrous metals from waste. Due to a changing magnetic field around a moving belt, eddy currents are induced in the metal particles, which cause these particles to repel from the belt. Other areas in which eddy currents are desirable are brake systems [58] and induction heaters [56]. In all these examples, the electromagnetic energy from the eddy currents is converted to work done by forces or heat dissipation.

In most electromagnetic machines, eddy currents are undesirable. An example of undesired presence of eddy currents is in Cathode-Ray Tube (CRT) deflection coils; see [24]. The CRT is the predominant display device for televisions and computers. Eddy currents in the

deflection coils contribute to heat dissipation and ringing (annoying patterns on the screen). Another unfavorable aspect of eddy currents is generation of destructive forces [69]. In MRI applications, eddy currents not only occur in the scanner itself, but they can also be induced in the patient. The patient might be heated locally due to energy dissipation in tissue. This phenomenon is measured by the Specific Absorption Rate (SAR); see e.g. [25], [45]. Also, nerves might be stimulated by the induced currents. This phenomenon is referred to as Peripheral Nerve Stimulation (PNS); see e.g. [40], [83].

The design of gradient coils is an important subject in the development of MRI-scanners. As already mentioned in Section 1.2, the principle of a Maxwell pair is used to design a z-coil. The Maxwell pair consists of two circular loops of wire carrying a current in opposite direction. A straightforward approach to obtain a better gradient field is to introduce more loops or a coil with multiple turns. In [53], [65], methods are described to determine the positions of the loops and the intensities of the currents. Because of the anti-symmetry of the configuration, the field and all its even derivatives vanish at the origin. By properly selecting the positions and the currents, we can systematically remove the third, fifth and higher order odd derivatives, thus achieving a better gradient field.

In other methods, the values of the currents are fixed and the corresponding positions are computed. This leads to optimization methods, such as the conjugate gradient method [84] and the simulated annealing method [15]. Some basic methods for the design of x- and y-coils are described by [17], [57]. Another approach is the target field method [73]. In this approach the desired field is specified and the corresponding current distribution on the cylindrical surface is computed. Approaches using stream functions to determine optimal surface structures are reported among others by Peeren [46], [47], and Tomasi [71]. In [46], [47], the optimal structure is related to a minimization of the magnetic energy. In this approach, currents are assumed static. Possible displacements of the currents due to inductive effects are not taken into account. In [71], the current distribution is discretized by use of one-dimensional wires.

As a result of eddy currents, the gradient homogeneity can be degraded and the rise and decay times of the switched field can be increased. A solution to this problem is to place a shield between gradient coils and their surrounding structures, so that the gradient field is reduced to zero outside the shield. This technique is often referred to as passive shielding [72]. An even better solution is to design gradient coils that produce a low intensity magnetic field outside the coils and thus hardly induce eddy currents in other conducting structures. Such coils must consist of at least two coils of different sizes. The outer coil, i.e. the shield coil, produces a field that cancels the one of the primary coil outside the shield coil. This technique is referred to as active shielding [8], [41]. Another issue in gradient coil design is the reduction of disturbing sound. Some recent publications on acoustic control in gradient coils are e.g. by Mansfield et al. [42] and Li et al. [37].

In the mathematical description of dynamic current distributions through strip-like structures, aspects as skin depth and edge-effects play an important role; see e.g. [36]. In a paper by

Genenko et al. [18], the reduction of edge currents due to magnetic shielding is simulated. Besides in gradient coils, strip-like structures also appear as striplines in electromagnetic transmission lines used for the excitation of antennas, see e.g. Collin [14]. However, in contrast to the low (quasi-static) frequency range in which gradient coils act (less than 10^4 Hz), the frequencies for antenna systems are very high ($\approx 10^9$ Hz), and in the latter case the strips can be modeled as perfectly conducting. Whether a conducting strip, dependent on the range of applied frequencies, can be considered as infinitely thin and/or perfectly conducting, is indicated by Bekers et al. [6].

It is a trend nowadays to do research on high-frequency applications, such as antennas [5], circuit simulators and interconnects [54], [55]. Oddly enough, low-frequency applications seem to be outdated. The author could not find attempts in literature that analytically treat the rather simple configurations considered in this research. Apart from numerous numerical packages that use the Finite Difference Method (FDM), the Finite Element Method (FEM), the Finite Integration Technique (FIT), the Finite Volume Method (FVM), or the Boundary Element Method (BEM), no calculations have been performed that make use of the properties of the governing equations, in order to derive a solution by analytical means. In previous research for Philips Medical Systems, Ulicevic [76] modeled the edge effect in a gradient coil by use of infinitely long plane rectangular strips. This model was extended in [77], and forms the basis of the present work. The eddy currents in a set of rings have been determined by the author [31], [34] and the characteristics of the magnetic field induced by the resulting currents are investigated by Tas [67].

The analytical approach in this thesis leads to an integral equation of the second kind for the current distribution. Atkinson [3], [4] proposed methods, such as the Nyström method and the collocation method to solve such an integral equation. The asymptotic behavior of the integral kernel is logarithmic and therefore, inspired by the paper of Reade [50], we use the Galerkin method with Legendre polynomials as the basis functions. Abdou [1] also presents a method to solve Fredholm-Volterra integral equations with logarithmic singular kernels by means of Legendre polynomials.

1.3.3 Outline

The modeling starts with the derivation of the governing equations, describing the current distribution in the conductors of a cylindrical gradient coil; see Chapter 2. These equations are based on Maxwell's equations, the constitutive equations for conductors of copper, boundary conditions restricting the current and a given supplied total current. The resulting equation for the current distribution is an integral equation of the second kind, valid on the cylindrical surface of the gradient coil. The configuration of copper strips on this coil is arbitrary.

Before we consider a specific configuration, we first investigate the integral equation. In Chapter 3, we show that the kernel contains a logarithmic singularity. Induced currents have a preferable direction and the singularity is logarithmic in the coordinate perpendicular to this

direction. To determine an approximation of the current distribution, we apply the Galerkin method. The basis functions that we use are defined globally, i.e. on the whole domain of a strip, without having limited support. We show that the most appropriate choice of the basis functions is Fourier modes in the coordinate representing the preferable direction and Legendre polynomials in the coordinate perpendicular to it. A software tool has been designed, called *Eddy*, which simulates the current distribution by use of this method. In Section 3.4, we present the special case of infinitely long plane rectangular strips. Although this example is physically not feasible, it contains the characteristics that appear in the realistic geometries of the following chapters. The integral equation for this special case has a purely logarithmic kernel. Exploiting the relations of Special Functions, we are able to calculate the inner products for the Galerkin method analytically.

In Chapter 4, we consider the configuration consisting of circular loops of strips, called rings. This ring model is a good approximation of the z-coil. The general integral equation for the current distribution of an arbitrary configuration is adjusted to one for an axi-symmetric current. The kernel can be expressed in terms of elliptic integrals and from their asymptotic expansion the logarithmic function is established. Thus, Legendre polynomials are appropriate basis functions for the current distribution and the analytical result from Chapter 3 can be used. We compute the amplitude of the currents in the rings and their phase-lags with respect to the applied source. Both effects cause a distortion in the magnetic field. The magnitude of the field changes and the time behavior does not correspond to the one of the desired field. We want to get more insight in these effects, because the quality of the images in MRI is highly susceptible to them. In this chapter, we also compute the magnetic field induced by the currents in the rings as well as the resulting resistance and self-inductance of such a system of rings.

As an extension of the ring model, which is axi-symmetric, we introduce pieces of copper that are placed between the rings. We call these pieces of copper islands. They are used in gradient coils to fill up empty spaces for a better heat transfer and a higher stability of the coil. Another case in which islands can be used in the model is when slits are cut into the strips to force the current to flow in a certain direction. Even though the islands are not connected to a voltage source, eddy currents occur due to inductive effects. In Chapter 5, the current distribution is determined in a set of rings and islands. Here, the Fourier mode decomposition is useful and the Legendre polynomials are again appropriate to describe the dependence of the axial coordinate. Furthermore, we show how the islands affect the resistance and the self-inductance.

In Chapter 6, we consider a set of plane circular loops of strips. This configuration is used to model the x- and y-coil. Although these coils are installed on a cylindrical surface, the current distribution is approximately the same as in plane circular strips. The reason for that is that the currents are only affected by local influences. We determine the current distribution in these plane circular strips by use of Legendre polynomials and compute the resulting magnetic

field, resistance and self-inductance.

The conclusions are summarized in Chapter 7. We give an overview of the design aspects that are used in the development of the software tool Eddy. This chapter is concluded with a summary of the most important results of this thesis.

1.4 Concluding remarks

As mentioned, part of the objective is to develop a software tool that simulates the current distribution in a gradient coil of an MRI-scanner. This tool should be able to predict tendencies with respect to specific behavior of time-varying currents and the corresponding magnetic field, such as edge-effects and field distortions. Due to induced eddy currents, the magnitude of the field changes and the time behavior does not correspond to the one of the desired field.

In MRI, the occurrence of eddy currents is an undesirable phenomenon. The quality of the images is very much susceptible to these effects. Due to distortions in the magnetic field, the images become blurred or show wrong information. Apart from high quality of the MRI-scans, another design aspect of a scanner is economical energy consumption. High switching rates of a gradient coil require a lot of power to bring the field to a desired strength. Moreover, energy is dissipated due to heated copper. Quantitative values that represent these energy aspects are the resistance and the self-inductance of a gradient coil. With our software tool, eddy currents in a coil can be determined and thus the resistance and the self-inductance of the coil can be computed.

Figure 1.6: *Examples of MRI scans.*

Model formulation

In this chapter, we present a model of a gradient coil. It is the basis of the analysis in this thesis. We start in Section 2.1 with an overview of the fundamentals of electromagnetic theory, as introduced in the book by Stratton [64]. The geometry of our model, as specified in Section 2.2, is a cylinder covered with thin conducting strips. Since we wish to model different types of gradient coils, we start from a general distribution of strips on the cylinder. In Section 2.3, the electromagnetic equations are applied to a particular conducting geometry, leading to an integral equation for the current distribution. The summary of this chapter is presented in Section 2.4.

2.1 Electromagnetic field theory

2.1.1 Maxwell's equations

Electromagnetic phenomena are governed by Maxwell's equations and the continuity equation (conservation of charge); see [64, Ch.1]. The equations are based on four basic global conservation laws: Gauss' law, Ampère's law, Faraday's law, and the law of conservation of magnetic induction. In this subsection, we present the mathematical formulation of these laws. The electromagnetic quantities used are summarized in Table 2.1.

1. In global form, Gauss' law is given by

$$\int_{\partial\mathcal{V}} \mathbf{D} \cdot \mathbf{n} \, da = \int_{\mathcal{V}} \rho \, dv, \quad (2.1)$$

or, in words, the total flux of the dielectric displacement through any closed surface, $\partial\mathcal{V}$, is equal to the total electric charge within the volume, $\mathcal{V}$, enclosed by the surface.

2. In global form, Ampère's law is given by

$$\int_{\partial\mathcal{S}} \mathbf{H} \cdot \mathbf{t} \, ds = \int_{\mathcal{S}} \mathbf{J} \cdot \mathbf{n} \, da + \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{D} \cdot \mathbf{n} \, da. \quad (2.2)$$

Table 2.1: *Electromagnetic field quantities*

<i>Symbol</i>	<i>Name</i>	<i>SI Units</i>
E	Electric field	Volt/meter [V/m]
B	Magnetic induction	Tesla [T]
D	Dielectric displacement	Coulomb/square meter [C/m ²]
H	Magnetic field	Ampère/meter [A/m]
J	Current density	Ampère/square meter [A/m ²]
ρ	Charge density	Coulomb/cubic meter [C/m ³]

In words, this relation states that the work done by the magnetic field along any closed curve, ∂S , is equal to the flux of the electric current through the surface, S , enclosed by the curve, plus the time rate of change of the total flux of the dielectric displacement through the same surface.

3. In global form, Faraday's law is given by

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{t} \, ds = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n} \, da. \quad (2.3)$$

In words, this relation states that the work done by the electric field along any closed curve, ∂S , is equal to the time rate of change of the total magnetic flux through the surface, S , enclosed by the curve.

4. In global form, conservation of magnetic induction is represented by the relation

$$\int_{\partial \mathcal{V}} \mathbf{B} \cdot \mathbf{n} \, da = 0. \quad (2.4)$$

In words, the total flux of the magnetic induction through any closed surface, $\partial \mathcal{V}$, is zero.

The local form of (2.1) - (2.4) is presented by the set of differential equations ($\mathbf{D} = \mathbf{D}(\mathbf{x}, t)$, etc.)

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (2.5)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.6)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (2.7)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (2.8)$$

From the differential equations (2.5) and (2.7), we deduce the equation of continuity of charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (2.9)$$

The global form of this equation is given by

$$\int_{\partial\mathcal{V}} \mathbf{J} \cdot \mathbf{n} \, da = -\frac{\partial}{\partial t} \int_{\mathcal{V}} \rho \, dv. \quad (2.10)$$

In words, conservation of electric charge reveals that the flux of the current density, $\mathbf{J}$, through any closed surface, $\partial\mathcal{V}$, is equal to the time rate of change of the free electric charge density contained in $\mathcal{V}$.

2.1.2 Boundary conditions

The global balance laws formulated in the preceding subsection are also used in the derivation of jump conditions over a surface of discontinuity. For the normal field components, these laws are applied to thin volumes (pill boxes), which enclose a part of the surface. Thereafter, the thickness of each volume tends to zero; see [64, Sect.1.13]. For the tangential field components, contours are used that cross the surface. Let $\mathcal{S}$ be a fixed surface that separates medium G^- and medium G^+ , where medium G^- and medium G^+ have different electromagnetic characteristics. The unit normal on $\mathcal{S}$, pointing from medium G^- into medium G^+ , is denoted by $\mathbf{n}$. With a superindex $-$, we denote the electromagnetic quantities at the side of medium G^- and by the superindex $+$ the electromagnetic quantities of medium G^+ . We introduce the jump of a vector function $\mathbf{G}$ across $\mathcal{S}$ by $[\![\mathbf{G}]\!]$, according to

$$[\![\mathbf{G}]\!] = (\mathbf{G}^+ - \mathbf{G}^-)|_{\mathcal{S}}. \quad (2.11)$$

Further, we define $[\![\mathbf{G} \cdot \mathbf{n}]\!] = [\![\mathbf{G}]\!] \cdot \mathbf{n}$ and $[\![\mathbf{G} \times \mathbf{n}]\!] = [\![\mathbf{G}]\!] \times \mathbf{n}$. Discontinuities in the electromagnetic fields can occur when material parameters, characterizing the media, have different values in medium G^- and medium G^+ . From (2.1) and (2.4) we derive the jump conditions

$$[\![\mathbf{B} \cdot \mathbf{n}]\!] = 0, \quad [\![\mathbf{D} \cdot \mathbf{n}]\!] = \rho_s, \quad (2.12)$$

across $\mathcal{S}$, respectively. Here, ρ_s is the surface charge density, the charge per unit of area at $\mathcal{S}$. From (2.2) and (2.3) we derive

$$[\![\mathbf{E} \times \mathbf{n}]\!] = \mathbf{0}, \quad [\![\mathbf{H} \times \mathbf{n}]\!] = -\mathbf{j}_s, \quad (2.13)$$

across $\mathcal{S}$, respectively. Here, $\mathbf{j}_s$ is the surface current flowing along $\mathcal{S}$, the current per unit of length in $\mathcal{S}$. Finally, from (2.10) we derive

$$[\![\mathbf{J} \cdot \mathbf{n}]\!] = 0, \quad (2.14)$$

at $\mathcal{S}$.

To conclude this subsection, we mention the classical conditions at infinity. We demand that $|\mathbf{H}| \sim r^{-2}$, for $r \rightarrow \infty$, where $r = |\mathbf{x}|$, when currents are only located within a finite distance to the origin. From Faraday's law, it then follows that also the magnitude of $\nabla \times \mathbf{E}$ is proportional to r^{-2} , for $r \rightarrow \infty$.

2.1.3 Constitutive equations and driving source

The Maxwell system (2.5) - (2.9) consists of 7 independent equations for the 16 unknowns: $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$, $\mathbf{J}$, and ρ . Therefore, we need 9 constitutive relations to make the system complete. These relations express the influence of matter on the electromagnetic fields and vice versa. In free space, we have

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad (2.15)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad (2.16)$$

$$\mathbf{J} = \mathbf{0}, \quad (2.17)$$

where $\mu_0 = 4\pi \cdot 10^{-7}$ H/m and $\epsilon_0 = 1/(\mu_0 c_0^2) = 1/(36\pi) \cdot 10^{-9}$ F/m are the magnetic permeability and the electric permittivity in free space, respectively, and c_0 is the speed of light in free space.

In the presence of matter, we have

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad (2.18)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (2.19)$$

where we have introduced the polarization field $\mathbf{P}$ and the magnetization field $\mathbf{M}$. We now distinguish primary fields and secondary fields; see [7, Sect.2.2]. A primary field is used to represent the physical sources of the electromagnetic fields. Primary fields are enforced externally and will be used as the driving sources in our problems. A secondary field is the result of the interaction between $\mathbf{E}$ and $\mathbf{H}$ and the charged particles in the medium. We use superscripts prim and sec to indicate the difference. Thus, we arrive at the following representations:

$$\rho = \rho^{prim} + \rho^{sec}, \quad (2.20)$$

$$\mathbf{J} = \mathbf{J}^{prim} + \mathbf{J}^{sec}, \quad (2.21)$$

$$\mathbf{P} = \mathbf{P}^{prim} + \mathbf{P}^{sec}, \quad (2.22)$$

$$\mathbf{M} = \mathbf{M}^{prim} + \mathbf{M}^{sec}. \quad (2.23)$$

For linear, homogeneous, isotropic, and stationary media, we use the constitutive relations

$$\mathbf{J}^{sec}(\mathbf{x}, t) = \sigma \mathbf{E}(\mathbf{x}, t), \quad (2.24)$$

$$\mathbf{P}^{sec}(\mathbf{x}, t) = \epsilon_0 \chi_e \mathbf{E}(\mathbf{x}, t), \quad (2.25)$$

$$\mathbf{M}^{sec}(\mathbf{x}, t) = \chi_m \mathbf{H}(\mathbf{x}, t). \quad (2.26)$$

The material constants σ , χ_e and χ_m are known as conductivity, electric susceptibility, and magnetic susceptibility, respectively. For the secondary charge ρ^{sec} a separate relation is not needed; for given $\mathbf{J}^{prim}$, $\mathbf{J}^{sec}$ and ρ^{prim} , the secondary charge can directly be determined

from the equation of continuity (2.9). We now arrive at

$$\mathbf{B} = \mu \mathbf{H} + \mu_0 \mathbf{M}^{prim}, \quad (2.27)$$

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P}^{prim}, \quad (2.28)$$

$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}^{prim}, \quad (2.29)$$

where $\mu = \mu_0(1 + \chi_m)$ is the permeability and $\epsilon = \epsilon_0(1 + \chi_e)$ is the permittivity of the medium. Equation (2.29) is also referred to as Ohm's law.

Table 2.2: *Values of electromagnetic constants for copper.*

Symbol	Name	Value
σ	Electric conductivity	$5.88 \cdot 10^7 \quad \Omega^{-1}\text{m}^{-1}$
μ	Magnetic permeability	$1.26 \cdot 10^{-6} \quad \text{Hm}^{-1}$
ϵ	Electric permittivity	$8.85 \cdot 10^{-12} \quad \text{Cm}^{-1}$

In Table 2.2, the numerical values of σ , μ , and ϵ are listed, as they are used in this thesis. Here, σ , $\mu(= \mu_0)$ and $\epsilon(= \epsilon_0)$ are the electric conductivity, the magnetic permeability, and the electric permittivity of copper, respectively.

2.1.4 General solution of Maxwell's equations in free space

The media used in this thesis are copper and air. The material parameters of air are equal to the ones of free space and the copper is assumed to be non-polarizable and non-magnetizable ($\epsilon = \epsilon_0$, $\mu = \mu_0$, everywhere). We consider the electromagnetic system without having any interaction with other objects, i.e. we assume that the system is in free space. Since the divergence of $\mathbf{B}$ is zero, we introduce the vector potential $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$, according to

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (2.30)$$

With (2.30), using Stokes' theorem, we can write Faraday's law (2.3) as

$$\begin{aligned} \int_{\partial S} \mathbf{E} \cdot \mathbf{t} \, ds &= -\frac{\partial}{\partial t} \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da = -\int_S (\nabla \times \frac{\partial \mathbf{A}}{\partial t}) \cdot \mathbf{n} \, da \\ &= -\int_{\partial S} \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{t} \, ds. \end{aligned} \quad (2.31)$$

Thus, for any ∂S , the vector fields $\mathbf{E}$ and $\mathbf{A}$ satisfy

$$\int_{\partial S} (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) \cdot \mathbf{t} \, ds = 0. \quad (2.32)$$

This is the necessary and sufficient condition for the existence of a scalar potential $\phi = \phi(\mathbf{x}, t)$, for the field $\mathbf{E} + \partial \mathbf{A} / \partial t$, such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi. \quad (2.33)$$

Since we are dealing with a physical system, the current density $\mathbf{J}(\mathbf{x})$ and the charge density $\rho(\mathbf{x})$ are finite and confined to a finite volume $\mathcal{V}$. Substituting (2.33) into (2.7), we obtain the relation

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \frac{\rho}{\epsilon_0}, \quad (2.34)$$

while (2.5) becomes

$$-\nabla^2\mathbf{A} + \epsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla\left[(\nabla \cdot \mathbf{A}) + \epsilon_0\mu_0\frac{\partial\phi}{\partial t}\right] = \mu_0\mathbf{J}. \quad (2.35)$$

Thus, both the electric and the magnetic field are put in terms of the vector potential $\mathbf{A}$ and the scalar potential ϕ . Note that there is no unique solution for $\mathbf{A}$ and ϕ . Namely,

$$\mathbf{A}' = \mathbf{A} + \nabla\psi, \quad \phi' = \phi - \frac{\partial\psi}{\partial t}, \quad (2.36)$$

is also a solution. We restrict the number of solutions, $\mathbf{A}$ and ϕ , by imposing a gauge. One option is the Lorentz gauge, defined by

$$\nabla \cdot \mathbf{A} = -\epsilon_0\mu_0\frac{\partial\phi}{\partial t}, \quad (2.37)$$

which decouples (2.34) and (2.35) for $\mathbf{A}$ and ϕ resulting in the inhomogeneous wave equations

$$\nabla^2\phi - \epsilon_0\mu_0\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad (2.38)$$

$$\nabla^2\mathbf{A} - \epsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \quad (2.39)$$

A favorable aspect here is that $\mathbf{A}$ and ϕ satisfy similar equations.

Another option is the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0, \quad (2.40)$$

which yields the Poisson equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}, \quad (2.41)$$

and the wave equation

$$\nabla^2\mathbf{A} - \epsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \nabla\left[\epsilon_0\mu_0\frac{\partial\phi}{\partial t}\right]. \quad (2.42)$$

Provided that there are no electric charges outside $\mathcal{V}$, the solution of the Poisson equation (2.41), satisfying the condition that $\phi = \mathcal{O}(r^{-1})$, $r \rightarrow \infty$, is given by (see [64, Sect.3.4])

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\boldsymbol{\xi}, t)}{|\mathbf{x} - \boldsymbol{\xi}|} dv(\boldsymbol{\xi}). \quad (2.43)$$

The vector potential $\mathbf{A}$ can be solved from (2.42), when ϕ is meanwhile a known function.

In our problem, a quasi-static approach may be applied (see (2.99) further on, for explanation), implying that we may replace (2.42) by

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \quad (2.44)$$

Similar to (2.43), we have (see [64, Sect.4.15] and note that there are no electric currents outside $\mathcal{V}$)

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J}(\boldsymbol{\xi}, t)}{|\mathbf{x} - \boldsymbol{\xi}|} dv(\boldsymbol{\xi}). \quad (2.45)$$

From the definition of $\mathbf{A}$ in (2.30), the solution for $\mathbf{B}$ can be derived as

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \mathbf{J}(\boldsymbol{\xi}, t) \times \frac{\mathbf{x} - \boldsymbol{\xi}}{|\mathbf{x} - \boldsymbol{\xi}|^3} dv(\boldsymbol{\xi}), \quad (2.46)$$

which is also known as the law of Biot-Savart.

In case we are dealing with a surface current density $\mathbf{j}_s(\mathbf{x}, t)$ or a surface charge density $\rho_s(\mathbf{x}, t)$, confined to a finite surface $\mathcal{S}$ ($\rho = 0, \mathbf{J} = \mathbf{0}$ in $\mathcal{V}$) the equations (2.41) and (2.44) are replaced by

$$\nabla^2 \phi = 0, \quad \nabla^2 \mathbf{A} = \mathbf{0}, \quad (2.47)$$

and they should hold everywhere in $\mathbb{R}^3$ except on $\mathcal{S}$. Additionally, we have to impose jump conditions at $\mathcal{S}$. For instance $[\partial\phi/\partial n]|_{\mathcal{S}} = -\rho_s/\epsilon_0$ and $[\mathbf{B} \times \mathbf{n}]|_{\mathcal{S}} = -\mu_0 \mathbf{j}_s$. The solutions of ϕ and $\mathbf{A}$ in the quasi-static approach are then (again see [64, Sect.3.4] and [64, Sect.4.15])

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{S}} \frac{\rho_s(\boldsymbol{\xi}, t)}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}), \quad (2.48)$$

and

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{S}} \frac{\mathbf{j}(\boldsymbol{\xi}, t)}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}). \quad (2.49)$$

For a volume current density $\mathbf{J}$ and a surface current density $\mathbf{j}$, the normal components at the boundary of the conductors are equal to zero.

2.1.5 Energy flow and electromagnetic forces

From conservation of energy, it is apparent that a change in time of the field intensity and the energy density must be associated with a flow of energy from or towards the source. In global form, the balance of energy is given by; see [64, Sect.2.19],

$$\int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J} dv + \int_{\partial\mathcal{V}} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} da = - \int_{\mathcal{V}} \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) dv, \quad (2.50)$$

for any volume $\mathcal{V}$. In differential form, (2.50) reads

$$\mathbf{E} \cdot \mathbf{J} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}, \quad (2.51)$$

which is also obtained when pre-multiplying (2.5) by $\mathbf{E}$, pre-multiplying (2.6) by $\mathbf{H}$, and subtracting the results. The interpretation of (2.50) is as follows: the first term on the left-hand side is the total dissipated power, $P_{diss}(t)$, which is partly accounted for by the heat loss and partly compensated by energy introduced through impressed forces. The second term on the left-hand side stands for the flow of electromagnetic energy across the surface $\partial\mathcal{V}$. It can also be expressed as

$$\int_{\partial\mathcal{V}} \mathbf{S} \cdot \mathbf{n} \, da, \quad \text{with} \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}. \quad (2.52)$$

Vector $\mathbf{S}$ is called the Poynting vector, which can be interpreted as the density of the energy flow in the field. Together, the two parts of the left-hand side of (2.50) form the total loss of available stored energy. The integral in the right-hand side represents the rate of decrease of electric and magnetic energy stored within the volume $\mathcal{V}$.

In case of a quasi-static approach with time-harmonic currents and frequency ω , we can write (here, $\mathbf{J}^*$ is the complex conjugate of $\mathbf{J}$)

$$\mathbf{J}(\mathbf{x}, t) = \Re\{\mathbf{J}(\mathbf{x})e^{-i\omega t}\} = \frac{1}{2}(\mathbf{J}(\mathbf{x})e^{-i\omega t} + \mathbf{J}^*(\mathbf{x})e^{i\omega t}), \quad (2.53)$$

and with this

$$\begin{aligned} P_{diss}(t) &= \int_{\mathcal{V}} \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{J}(\mathbf{x}, t) \, dv(\mathbf{x}) \\ &= \frac{1}{2\sigma} \int_{\mathcal{V}} \Re\{\mathbf{J}(\mathbf{x}) \cdot \mathbf{J}^*(\mathbf{x}) + \mathbf{J}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x})e^{-2i\omega t}\} \, dv(\mathbf{x}). \end{aligned} \quad (2.54)$$

In the right-hand side of (2.50), the rate of increase of electric energy can be set equal to zero, because in a quasi-static approach the change in time of the dielectric displacement is negligible. The rate of increase of magnetic energy is the power caused by inductive effects, and therefore denoted by P_{ind} . If we substitute

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{B} \cdot \mathbf{H}), \quad (2.55)$$

and use that

$$(\nabla \times \mathbf{A}) \cdot \mathbf{H} = \mathbf{A} \cdot (\nabla \times \mathbf{H}) + \nabla \cdot (\mathbf{A} \times \mathbf{H}) = \mathbf{A} \cdot \mathbf{J} + \nabla \cdot (\mathbf{A} \times \mathbf{H}), \quad (2.56)$$

then P_{ind} can be written as

$$\begin{aligned} P_{ind} &= \int_{\mathcal{V}} \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \, dv \\ &= \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{J} \, dv + \frac{1}{2} \frac{\partial}{\partial t} \int_{\partial\mathcal{V}} \mathbf{A} \times \mathbf{H} \, da. \end{aligned} \quad (2.57)$$

Volume $\mathcal{V}$ can be chosen arbitrarily, and so, also the enclosing surface $\partial\mathcal{V}$ can. We choose $\mathcal{V}$ to be a sphere with radius r . If we let r tend to infinity, then the contribution of the

surface integral disappears, because $\mathbf{A}$ and $\mathbf{H}$ satisfy the far field conditions, where they are proportional to $1/r$ and $1/r^2$, respectively. Moreover, as in (2.53),

$$\mathbf{A}(\mathbf{x}, t) = \Re\{\mathbf{A}(\mathbf{x})e^{-i\omega t}\} = \frac{1}{2}(\mathbf{A}(\mathbf{x})e^{-i\omega t} + \mathbf{A}^*(\mathbf{x})e^{i\omega t}). \quad (2.58)$$

Therefore,

$$\begin{aligned} P_{ind}(t) &= \frac{1}{2} \frac{\partial}{\partial t} \int_V \mathbf{A}(\mathbf{x}, t) \cdot \mathbf{J}(\mathbf{x}, t) dv(\mathbf{x}) \\ &= -\frac{\omega}{2} \int_V \Re\{i\mathbf{A}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x})e^{-2i\omega t}\} dv(\mathbf{x}). \end{aligned} \quad (2.59)$$

The second term on the left-hand side of (2.50), will not be considered any longer, because it is simply the sum of P_{diss} and P_{ind} as described above. Often, only the time-averaged values are taken into account,

$$\bar{P}_{diss} = \frac{1}{2\sigma} \int_V \mathbf{J} \cdot \mathbf{J}^* dv, \quad (2.60)$$

$$\bar{P}_{ind} = 0. \quad (2.61)$$

The result $\bar{P}_{ind} = 0$ is physically predictable, because the energy needed for inductive effects is maintained in the system, such that in average no energy is lost.

For the computation of the resistance, the self-inductance and the impedance, we first need to define the time-averaged values of the magnetic energy and the squared total current. Since the magnetic energy $U_m(t)$ satisfies $\partial U_m / \partial t = P_{ind}$, it can be written as

$$\begin{aligned} U_m(t) &= \frac{1}{2} \int_V \mathbf{H}(\mathbf{x}, t) \cdot \mathbf{B}(\mathbf{x}, t) dv(\mathbf{x}) \\ &= \frac{1}{4} \int_V \Re\{\mathbf{A}(\mathbf{x}) \cdot \mathbf{J}^*(\mathbf{x}) + \mathbf{A}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x})e^{-2i\omega t}\} dv(\mathbf{x}). \end{aligned} \quad (2.62)$$

Moreover, the total current is defined as the normal component of $\mathbf{J}(\mathbf{x}, t)$ integrated over a cross-section of the conductors present. Introducing $I(t) = Ie^{-i\omega t}$, we obtain

$$I^2(t) = \frac{1}{2} \Re\{I^2 e^{-2i\omega t} + II^*\}. \quad (2.63)$$

The time-averaged values of the magnetic energy $\bar{U}_m$ and the squared total current I_e^2 then become

$$\bar{U}_m = \frac{1}{4} \int_V \mathbf{A} \cdot \mathbf{J}^* dv, \quad (2.64)$$

$$I_e^2 = \frac{1}{2} II^*. \quad (2.65)$$

Note that (2.45) can be used to see that $\mathbf{A} \cdot \mathbf{J}^*$ is real-valued. The term I_e is called the effective current. We now come to the definitions of the resistance $\bar{R}$ and the self-inductance $\bar{L}$,

$$\bar{R} = \frac{\bar{P}_{diss}}{I_e^2} = \frac{\int_V \mathbf{J} \cdot \mathbf{J}^* dv}{\sigma II^*}, \quad (2.66)$$

$$\bar{L} = \frac{\bar{U}_m}{\frac{1}{2} I_e^2} = \frac{\int_V \mathbf{A} \cdot \mathbf{J}^* dv}{II^*}. \quad (2.67)$$

The impedance is a complex value defined as $\bar{Z} = \bar{R} - i\omega\bar{L}$. Therefore, $\bar{Z}$ follows directly from (2.66) and (2.67). The imaginary part of the impedance is commonly referred to as the inductive reactance.

For a finite volume $\mathcal{V}$ in equilibrium, the resultant force exerted on the matter within $\mathcal{V}$ must be zero. Contributing to this resultant force are body and surface forces. We define the net mechanical force acting on a charged volume, $\mathbf{F}_e$, and the net force exerted on a volume with current, $\mathbf{F}_m$, by

$$\mathbf{F}_e = \int_{\mathcal{V}} \rho \mathbf{E} dv, \quad (2.68)$$

$$\mathbf{F}_m = \int_{\mathcal{V}} \mathbf{J} \times \mathbf{B} dv. \quad (2.69)$$

We refer to $\mathbf{F}_e$ as the Coulomb force and to $\mathbf{F}_m$ as the Lorentz force. Together, $\mathbf{F}_e$ and $\mathbf{F}_m$ represent the change in time of the total linear momentum of the moving charges, also denoted as $\partial \mathbf{G}_{mech} / \partial t$. In the dynamic case, an extra body force arises from the variation in time of the linear momentum of the field $\mathbf{G}_f$ in the volume $\mathcal{V}$, defined by; see [64, Sect.2.29]

$$\frac{\partial \mathbf{G}_f}{\partial t} = \int_{\mathcal{V}} \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}) dv. \quad (2.70)$$

The total surface forces are equal to the total outward flux of momentum through the surface $\mathcal{S}$, which can be expressed as

$$\int_{\mathcal{S}} \boldsymbol{\tau} da = \int_{\mathcal{S}} \mathbf{T} \cdot \mathbf{n} da. \quad (2.71)$$

Here, $\boldsymbol{\tau}$ represents the traction exerted by matter outside $\mathcal{S}$ on a unit area of $\mathcal{S}$, and $\mathbf{T}$ is the corresponding stress tensor. Equilibrium of forces is now formulated in global form as

$$\int_{\mathcal{V}} \left[\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} + \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}) \right] dv + \int_{\mathcal{S}} \mathbf{T} \cdot \mathbf{n} da = 0, \quad (2.72)$$

or in local form (see [27, Eq.2.80] or [64, Sect.2.29]), resulting in the differential equation

$$\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} + \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}) + \nabla \cdot \mathbf{T} = 0. \quad (2.73)$$

For completeness, we define the divergence operator applied on a tensor

$$\nabla \cdot \mathbf{T} = \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial T_{kj}}{\partial x_k} \mathbf{e}_j. \quad (2.74)$$

Equation (2.73) can also be obtained by vector multiplication of (2.5) by $\mathbf{B}/\mu_0$, vector multiplication of (2.6) by $\epsilon_0 \mathbf{E}$, and by addition of the results.

(a) x-coil

(b) z-coil

Figure 2.1: (a) Sketch of the x-coil. The y-coil has an identical shape and is placed in the scanner with a rotation of 90 degrees about the x-coil; (b) Sketch of the z-coil.

2.2 Geometry of the model

A gradient coil consists of a long strip of copper arranged on a cylinder. Figure 2.1 shows design configurations of gradient coils. The width of the strip is a few centimeters, its thickness a few millimeters. The mutual distance between separate windings of the coil varies from just a few millimeters to several centimeters.

To model a gradient coil, we consider a geometry of strips on a cylinder. We use cylindrical coordinates (r, φ, z) , where the z -axis coincides with the central axis of the cylinder. The radius of the cylinder is R , such that the surface S_c of the cylinder is defined as

$$S_c = \{(r, \varphi, z) \mid r = R, -\pi \leq \varphi \leq \pi, -\infty < z < \infty\}. \quad (2.75)$$

The cylinder is considered as infinitely long, whereas the strips occupy the finite surface S_U on the cylinder S_c . All strips have equal thickness h . Not all strips are necessarily of the same width, and the mutual distances may be different.

We distinguish three types of strips. The first type is the patch, a simply connected piece of copper placed on the cylinder. The second type is the strip that forms a closed loop around the cylinder, encircling the cylinder once. The third type also forms a closed loop, but closed on top of the surface of the cylinder, instead of encircling; see Figure 2.2.

Figure 2.2: Three types of strips: 1. Patch; 2. Closed loop encircling the cylinder; 3. Closed loop on top of the cylinder.

We assume that there are N distinct strips and that the width of each strip is uniform. In each of the N strips, we have to determine the current distribution, driven by prescribed sources. The currents in different strips can be driven by the same source. Therefore, we introduce L groups of strips, $L \leq N$, such that the current through all strips of one group l is driven by one source. The total current through each group is prescribed, having the value $I_l(t)$,

$l = 1, \dots, L$. Each group consists of N_l strips, such that $\sum_{l=1}^L N_l = N$. We define $S_n^{(l)}$ as the surface occupied by the n -th strip in group l , $n \in \{1, \dots, N_l\}$, $l \in \{1, \dots, L\}$, and $S_\cup^{(l)}$ as the surface occupied by the strips in the l -th group. So,

$$S_\cup^{(l)} = \sum_{n=1}^{N_l} S_n^{(l)}, \quad \text{and} \quad S_\cup = \sum_{l=1}^L S_\cup^{(l)}. \quad (2.76)$$

Here, $\sum$ denotes the disjoint union of sets. Note that $S_n^{(l)} \subset S_\cup^{(l)} \subset S_\cup \subset S_c$. By G^- , we denote the inner region of the cylinder S_c , and by G^+ the outer region.

For the description of the geometry used in the model, we consider the central line $\mathbf{c} = \mathbf{c}(u)$ of a strip S , where u represents the path length along the line. The central line $\mathbf{c}(u)$ is extended geometrically to a strip with uniform width. For that, we define in every point of $\mathbf{c}(u)$ a curve $\mathbf{d}(v; u)$, which crosses $\mathbf{c}(u)$ perpendicularly and is restricted to the cylindrical surface. A way to achieve this, is by considering the plane through the point of $\mathbf{c}(u)$ with its normal tangent to the curve $\mathbf{c}(u)$. The plane intersects with the cylinder and forms a curve $\mathbf{D}(v; u)$, which is an ellipse, or a straight line. In every point of $\mathbf{c}(u)$ we choose a part of the curve $\mathbf{D}(v; u)$, referred to as $\mathbf{d}(v; u)$, with length equal to the width of the strip, such that on both sides of $\mathbf{c}(u)$ we have half of the width of the strip. We assume that the radius of curvature of the central line is larger than half the width of the strip throughout the whole configuration. For strips with widths that are much smaller than the radius of the cylinder, it is plausible to give a local description of the surface of the strip. Therefore, we define the local coordinate system for the central line, $\{\mathbf{e}_r, \mathbf{e}_u, \mathbf{e}_v\}$, where $\mathbf{e}_u$ is the normalized tangent along the curve

$$\mathbf{e}_u(u) = \frac{\mathbf{c}'(u)}{|\mathbf{c}'(u)|}. \quad (2.77)$$

The coordinate vector $\mathbf{e}_r = \mathbf{e}_r(u) = \mathbf{e}_r|_{\mathbf{c}(u)}$ is the outward normal on the cylinder surface along the curve, and $\mathbf{e}_v = \mathbf{e}_r \times \mathbf{e}_u$ tangent to the cylinder and normal to the central line.

The simplest example of a closed loop is a ring of width D , with surface

$$\left\{ (r \cos u, r \sin u, z) \mid r = R, -\pi \leq u \leq \pi, -\frac{D}{2} \leq z \leq \frac{D}{2} \right\}, \quad (2.78)$$

and central line

$$\mathbf{c}(u) = (R \cos u, R \sin u, 0), \quad -\pi \leq u \leq \pi. \quad (2.79)$$

Locally, we approximate the coordinate system by

$$\mathbf{e}_u(u) = \mathbf{e}_\varphi, \quad (2.80)$$

$$\mathbf{e}_v(u) = \mathbf{e}_z, \quad (2.81)$$

$$\mathbf{e}_r(u) = \mathbf{e}_r. \quad (2.82)$$

Another example of a closed loop is the circular strip, with radius R_s , on the cylindrical surface, with radius R ($R_s < R$). A Cartesian basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is chosen such that the

y -axis passes through the center of the circular strip on the cylinder. The central line is presented in Cartesian coordinates by

$$\{(x, y, z) \mid x^2 + y^2 = R^2, x^2 + z^2 = R_s^2, y > 0\}. \quad (2.83)$$

In the $\{\mathbf{e}_r, \mathbf{e}_u, \mathbf{e}_v\}$ system, we obtain

$$\mathbf{c}(u) = (R_s \cos u, \sqrt{R^2 - R_s^2 \cos^2 u}, R_s \sin u), \quad -\pi \leq u \leq \pi, \quad (2.84)$$

and the local coordinate vectors

$$\mathbf{e}_r(u) = \frac{1}{R}(R_s \cos u, \sqrt{R^2 - R_s^2 \cos^2 u}, 0), \quad (2.85)$$

$$\mathbf{e}_u(u) = \frac{\sqrt{R^2 - R_s^2 \cos^2 u}}{R_s \sqrt{R^2 - R_s^2 \cos^4 u}} \left(-R_s \sin u, \frac{R_s^2 \cos u \sin u}{\sqrt{R^2 - R_s^2 \cos^2 u}}, R_s \cos u \right), \quad (2.86)$$

$$\mathbf{e}_v(u) = \mathbf{e}_u(u) \times \mathbf{e}_r(u). \quad (2.87)$$

2.3 Mathematical model

In this section, we present the boundary value problem for the specific geometry as described in Section 2.2. To obtain the mathematical description of our problem, we start from Maxwell's theory as described in Section 2.1 and we reduce the equations by means of some appropriate assumptions. The equations are written in dimensionless form. An integral equation is derived for the current distribution on the surface of the cylinder.

2.3.1 Basic assumptions and reduced model

To reduce the set of equations exposed in Section 2.1, we use the following assumptions:

1. The current distribution is time harmonic with angular frequency ω and with that also the magnetic field and the electric field are.
2. The only driving source is a primary current source.
3. The strips are isotropic, homogeneous, non-polarizable, and non-magnetizable conductors (copper). The space outside the strips is air.
4. The frequency ω is sufficiently small, such that we can neglect the displacement current in Ampère's law, i.e. we may use a quasi-static approach.
5. The thickness of the strips is neglected.
6. The conductors are rigid; magneto-mechanical effects (vibrations) are not considered.

Some comments on the assumptions 1,2,4,5 are given below.

1. The time dependent current in a gradient coil can be represented by a sum of time-harmonic currents, i.e.

$$\mathbf{J}(\mathbf{x}, t) = \Re\left\{\sum_{n=1}^N \mathbf{J}_n(\mathbf{x})e^{-i\omega_n t}\right\}. \quad (2.88)$$

A typical acquisition sequence for a gradient coil is the trapezoidal sequence; see [79]. In [67], Tas derives time series expansions as in (2.88) for several trapezoidal sequences, yielding the values of ω_n . We investigate the solution for each frequency, $\omega = \omega_n$, separately. For one frequency component, the electric field, the magnetic field, and the current density satisfy

$$\mathbf{E}(\mathbf{x}, t) = \Re\{\mathbf{E}(\mathbf{x})e^{-i\omega t}\}, \quad (2.89)$$

$$\mathbf{H}(\mathbf{x}, t) = \Re\{\mathbf{H}(\mathbf{x})e^{-i\omega t}\}, \quad (2.90)$$

$$\mathbf{J}(\mathbf{x}, t) = \Re\{\mathbf{J}(\mathbf{x})e^{-i\omega t}\}, \quad (2.91)$$

respectively. Without further comments, the terms $\Re\{\}$ and $e^{-i\omega t}$ are omitted in the equations: the remaining (complex) amplitudes of the fields are only space dependent and all time derivatives are replaced by the factor $-i\omega$. The Maxwell equations, valid in both G^- and G^+ , become (expressed in $\mathbf{E} = \mathbf{E}(\mathbf{x})$ and $\mathbf{H} = \mathbf{H}(\mathbf{x})$ only)

$$\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}, \quad \nabla \cdot \mathbf{E} = 0, \quad (2.92)$$

$$\nabla \times \mathbf{H} = \mathbf{0}, \quad \nabla \cdot \mathbf{H} = 0, \quad (2.93)$$

where absence of currents and free charges in G^- and G^+ is used.

2. In most practical problems, currents are caused by a voltage source. In modeling such problems, one usually prescribes the voltage source at the boundaries of the conductors, with that defining the begin and end of the system. See for example [26, p.128], [78, Sect.2.1.1]. In all models we use throughout this thesis, only conductors are excited that form closed loops of strips. For such geometries, without begin and end, a voltage source is not easily applied. Therefore, in our approach, we assume a time-harmonic primary current $\mathbf{J}^{prim}$. We shall further on assign $\mathbf{J}^{prim}$ either to one strip or to a group of strips. The primary current $\mathbf{J}^{prim}$ is enforced locally. The conductivity of the copper takes care of the transportation of the current, yielding a distribution $\mathbf{J}^s(\mathbf{x}, t)$, which is commonly referred to as the *source current*; see [29, Eq.(2.1.4)], [80, Sect.2.1].

Due to the time-harmonic source current, $\mathbf{J}^s(\mathbf{x}, t) = \Re\{\mathbf{J}^s(\mathbf{x})e^{-i\omega t}\}$, the electromagnetic fields vary simultaneously, causing induced currents. These induced currents are called *eddy currents*, $\mathbf{J}^e(\mathbf{x}, t) = \Re\{\mathbf{J}^e(\mathbf{x})e^{-i\omega t}\}$. The total current $\mathbf{J}(\mathbf{x})$ is the sum of the source current $\mathbf{J}^s(\mathbf{x})$ and the eddy current $\mathbf{J}^e(\mathbf{x})$,

$$\mathbf{J}(\mathbf{x}) = \mathbf{J}^s(\mathbf{x}) + \mathbf{J}^e(\mathbf{x}). \quad (2.94)$$

Note that $\mathbf{J}^s$ partly contains the primary current $\mathbf{J}^{prim}$ and that the remaining part, together with $\mathbf{J}^e$, forms the secondary current $\mathbf{J}^{sec}$; see Section 2.1.3. However, because it is more common to talk about eddy currents for induced currents, we use from now on the terminology $\mathbf{J}^s$ and $\mathbf{J}^e$. Eddy currents can be caused by currents in other conductors as well as by a current in the conductor itself, i.e. self-induction. In the latter case, the induced currents are often referred to as *self-eddy currents* or *self-eddies*.

4. The source current $\mathbf{J}^s$ causes an electric field $\mathbf{E}^{imp}$, which is called the imposed electric field and is related to the term $-\nabla\phi$ in (2.33). The induced eddy current $\mathbf{J}^e$ causes an induced electric field, $\mathbf{E}^{ind}$, which is related to the term $\partial\mathbf{A}/\partial t$ in (2.33). The only driving source is $\mathbf{J}^{prim}$, so no primary fields $\mathbf{M}^{prim}$ and $\mathbf{P}^{prim}$ are imposed; see (2.22) and (2.23). Therefore, (2.27) - (2.29) can be written as

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad (2.95)$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad (2.96)$$

$$\mathbf{J} = \mathbf{J}^s + i\omega\sigma\mathbf{A}, \quad (2.97)$$

where (2.97) holds on the conducting strips. Substituting these constitutive equations and (2.89) - (2.91) into Maxwell's equation (2.5), we obtain for $\mathbf{H}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ on the conductors that

$$\nabla \times \mathbf{H} = \left(1 - \frac{i\omega\epsilon_0}{\sigma}\right)\mathbf{J}. \quad (2.98)$$

According to Tas [67], the frequency spectrum mainly consists of low frequencies, where low means in the order of kHz. If we use frequencies $f = \omega/2\pi < 10^4$ Hz, then we have enough information to reconstruct the signal. Using the values from Table 2.2, we obtain

$$\frac{\epsilon_0\omega}{\sigma} < 10^{-14} \ll 1. \quad (2.99)$$

This justifies the neglect of the displacement current, i.e. the contribution of $\partial\mathbf{D}/\partial t$ in Ampère's law, which yields the *quasi-static* version of (2.5)

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad (2.100)$$

on the conductors. Within the same order of approximation, the equation of conservation of charge (2.9) becomes

$$\nabla \cdot \mathbf{J} = 0. \quad (2.101)$$

Hence, the current in the conductor is divergence free.

5. The cross-section of a ring is a narrow rectangle, meaning that the thickness h is small compared to its width. More relevant for us is the ratio of half the thickness of the ring, $h/2$, with respect to the penetration depth δ . The penetration depth is a characteristic length for the decline of the current density in the thickness direction, and is defined as $\delta = \sqrt{2/\mu_0\sigma\omega}$; see e.g. [30, Eq.(35.7)]. The values for the electric conductivity σ and the magnetic permeability

μ_0 are given in Table 2.2. A typical measure for the thickness is $h = 2.5 \cdot 10^{-3}$ m. For frequencies $f < 10^4$ Hz, we find that $\delta > h/2$, implying that the current density is approximately uniform in the thickness direction. Consequently, we can assume that the rings are infinitely thin and that the radial component of the current density on the strip is zero. We replace the current density $\mathbf{J}$ (in A/m²) by the surface current per unit of length $\mathbf{j}$ (in A/m), defined as

$$\mathbf{j}(\varphi, z) = \int_R^{R+h} \mathbf{J}(r, \varphi, z) dr \approx h\mathbf{J}(R, \varphi, z). \quad (2.102)$$

We recall that Maxwell's equations (2.92) - (2.93) are valid for ordinary points, i.e. for points in whose neighborhood the physical properties of the medium vary continuously. The points on the cylinder S_c between the strips are ordinary points, whereas points on the strips, i.e. on $S_\cup$, are not. Since the thickness of the strips is negligibly small, the regions G^- and G^+ are both bounded by $r = R$. We describe the field in G^- and G^+ in terms of cylindrical coordinates: in G^- we have the field components $\{E_r^-, E_\varphi^-, E_z^-, H_r^-, H_\varphi^-, H_z^-\}$ and in G^+ we have $\{E_r^+, E_\varphi^+, E_z^+, H_r^+, H_\varphi^+, H_z^+\}$.

For the set of strips, $S_\cup$, arbitrarily distributed on the cylinder, the surface current $\mathbf{j}$ satisfies

$$\mathbf{j} = j_\varphi(\varphi, z)\mathbf{e}_\varphi + j_z(\varphi, z)\mathbf{e}_z. \quad (2.103)$$

The current distribution has to be free of divergence, which is for this surface current represented by

$$\frac{1}{R} \frac{\partial j_\varphi}{\partial \varphi} + \frac{\partial j_z}{\partial z} = 0. \quad (2.104)$$

We apply the boundary conditions (2.12) - (2.13) to the fields in G^- and G^+ . The first boundary condition of (2.12) gives $[\mu_0 H_r] = 0$ across both sides of each strip. Because the strips are infinitely thin, we can replace the two jump conditions from inner region to strip and from strip to outer region by one condition as

$$H_r^-(R, \varphi, z) = H_r^+(R, \varphi, z), \quad (2.105)$$

for $\mathbf{x} \in S_\cup$. Let $\rho_s^{(1)}$ be the surface charge and $E_r^{(1)}$ the radial component of the electric field, at the outer surface of the strip. Then the jump condition from G^+ to the strip is $[\mathbf{n} \cdot \mathbf{D}] = \epsilon_0 E_r^+ - \epsilon_0 E_r^{(1)} = \rho_s^{(1)}$. Analogously, let $\rho_s^{(2)}$ be the surface charge and $E_r^{(2)}$ the radial component of the electric field, at the inner surface. Then $[\mathbf{n} \cdot \mathbf{D}] = \epsilon_0 E_r^- - \epsilon_0 E_r^{(2)} = \rho_s^{(2)}$. Now, when we let the thickness of the strip tend to zero, $\rho_s^{(1)}$ and $\rho_s^{(2)}$ become one and the same surface charge, which leaves us with

$$\rho_s^{(1)} - \rho_s^{(2)} = \epsilon_0 (E_r^+(R, \varphi, z) - E_r^-(R, \varphi, z)) = 0. \quad (2.106)$$

Hence, the second condition of (2.12) can be written as

$$E_r^-(R, \varphi, z) = E_r^+(R, \varphi, z). \quad (2.107)$$

Thus, the radial components of $\mathbf{H}$ and $\mathbf{E}$ are continuous across $S_\cup$.

Similarly, from (2.13) we obtain

$$E_\varphi^-(R, \varphi, z) = E_\varphi^+(R, \varphi, z), \quad (2.108)$$

$$E_z^-(R, \varphi, z) = E_z^+(R, \varphi, z), \quad (2.109)$$

and

$$H_\varphi^+(R, \varphi, z) - H_\varphi^-(R, \varphi, z) = j_z(\varphi, z), \quad (2.110)$$

$$H_z^+(R, \varphi, z) - H_z^-(R, \varphi, z) = -j_\varphi(\varphi, z), \quad (2.111)$$

valid on $S_\cup$. Note that $\mathbf{j}_s$ has been replaced by $\mathbf{j}$ as the surface current, and the selected orientation determines the minus sign.

The points $\mathbf{x}$ on the cylinder S_c that do not belong to the strips, are ordinary points, implying that at these points of the cylinder the components of $\mathbf{E}$ and $\mathbf{H}$ are continuous. Putting $j_\varphi(\varphi, z)$ and $j_z(\varphi, z)$ to zero at these points, the boundary conditions are satisfied at all points of the cylinder.

The Maxwell equations are a set of coupled first-order partial differential equations. To solve these equations, we introduce potentials according to Section 2.1.4, with the purpose of transforming the Maxwell equations into a reduced number of second-order differential equations. The vector potential $\mathbf{A}$ satisfies $\mathbf{B} = \nabla \times \mathbf{A}$ and the scalar potential ϕ satisfies

$$\mathbf{E} = i\omega\mathbf{A} - \nabla\phi, \quad (2.112)$$

where the time-harmonic behavior has been included. Using the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, and the quasi-static approximation, we obtain, cf. (2.47),

$$\nabla^2 \mathbf{A} = \mathbf{0}, \quad (2.113)$$

$$\nabla^2 \phi = 0. \quad (2.114)$$

Hence, the scalar potential ϕ is harmonic in the region $G^- \cup G^+$, is continuous everywhere, and vanishes at infinity. Also $\partial\phi/\partial n$ is continuous everywhere, because there is no surface charge. So, $\phi \equiv 0$ and $\mathbf{E} = i\omega\mathbf{A}$. Solving the problem for $\mathbf{A}$ solves the problem for $\mathbf{E}$ and we continue with $\mathbf{A}$.

The Maxwell equations in terms of $\mathbf{A}$ and $\mathbf{H}$ become

$$\nabla \times \mathbf{A} = \mu_0 \mathbf{H}, \quad \nabla \cdot \mathbf{A} = 0, \quad (2.115)$$

$$\nabla \times \mathbf{H} = \mathbf{0}, \quad \nabla \cdot \mathbf{H} = 0, \quad (2.116)$$

for $\mathbf{x} \in G^- \cup G^+$, together with the boundary conditions

$$\mathbf{A}^-(R, \varphi, z) = \mathbf{A}^+(R, \varphi, z), \quad (2.117)$$

and

$$H_r^-(R, \varphi, z) = H_r^+(R, \varphi, z), \quad (2.118)$$

$$H_\varphi^+(R, \varphi, z) - H_\varphi^-(R, \varphi, z) = j_z(\varphi, z), \quad (2.119)$$

$$H_z^+(R, \varphi, z) - H_z^-(R, \varphi, z) = -j_\varphi(\varphi, z), \quad (2.120)$$

for all $\mathbf{x} \in S_c$, with $\mathbf{x}$ represented by the coordinates (R, φ, z) . The conditions at infinity require

$$\mathbf{H} \rightarrow \mathbf{0}, \quad \nabla \times \mathbf{A} \rightarrow \mathbf{0}, \quad \text{for } |\mathbf{x}| \rightarrow \infty. \quad (2.121)$$

Using Ohm's law (2.97), we obtain for the total current

$$j_\varphi(\varphi, z) = j_\varphi^s(\varphi, z) + i\omega h\sigma A_\varphi(R, \varphi, z), \quad (2.122)$$

$$j_z(\varphi, z) = j_z^s(\varphi, z) + i\omega h\sigma A_z(R, \varphi, z). \quad (2.123)$$

The description of the source current $\mathbf{j}^s(\varphi, z)$ depends on the geometry under consideration, and is therefore discussed in more detail in the specific examples presented in this thesis; see Chapters 4 - 6. The parts of (2.122) and (2.123) with the vector potential components A_φ and A_z correspond with the eddy current parts, j_φ^e and j_z^e , of the total current components j_φ and j_z .

The only condition we have not used yet, is (2.14), which states that the normal component of the current at an edge of a strip must be zero. In the r -direction, this condition is automatically satisfied, because the current $\mathbf{j}$ has no r -component. For a general description of the problem, it is necessary to write the problem in terms of the local coordinate systems $\{\mathbf{e}_r, \mathbf{e}_u, \mathbf{e}_v\}$, as introduced in Section 2.2. For this, we write

$$\mathbf{j}(u, v) = j_u(u, v)\mathbf{e}_u(u) + j_v(u, v)\mathbf{e}_v(u). \quad (2.124)$$

Then (2.118) - (2.120) become

$$([\mathbf{H}]|_{S_\cup}, \mathbf{e}_r) = 0, \quad (2.125)$$

$$([\mathbf{H}]|_{S_\cup}, \mathbf{e}_u) = j_v, \quad (2.126)$$

$$([\mathbf{H}]|_{S_\cup}, \mathbf{e}_v) = -j_u. \quad (2.127)$$

The condition that the normal component of the current $\mathbf{j}$ at the edge of a strip is zero becomes $j_u = 0$ or $j_v = 0$ at the edge. For a loop, we have a periodicity condition on j_u .

In a gradient coil, distinct strips may be present, connected to distinct voltage sources. To each voltage source one strip is connected, or a group of strips, connected in parallel. In our model we avoid the use of voltage sources by introducing imposed currents. In order to describe the quantity of current imposed on the strips, we prescribe the total current that flows through the cross-sections of the strips. For strips or groups of strips, which are connected in parallel, one total source current is required.

If we assume a system consisting of L groups, then we need to add L additional conditions to obtain a well-defined problem description. For that, we prescribe the complex amplitudes I_l of the total currents, $I_l(t) = \Re\{I_l e^{-i\omega t}\}$, $l = 1, \dots, L$. The complex amplitudes I_l introduce phase and intensity differences between the sources.

2.3.2 Dimensionless formulation

The free variables in our mathematical model are r, φ, z , the dependent variables are $\mathbf{A}, \mathbf{H}, \mathbf{j}$, and the parameters are $R, h, \mu_0, \sigma, \omega, I_l$. The Maxwell equations (2.115) - (2.116) for G^- and G^+ and the boundary conditions (2.117) - (2.120) present the relations among the dependent variables. For scaling, we introduce

$$\tilde{r} = \frac{r}{D^c}, \quad \tilde{z} = \frac{z}{D^c}, \quad \tilde{\mathbf{A}} = \frac{\mathbf{A}}{A^c}, \quad \tilde{\mathbf{H}} = \frac{\mathbf{H}}{H^c}, \quad \tilde{\mathbf{j}} = \frac{\mathbf{j}}{j^c}, \quad (2.128)$$

where the scaling factors D^c, A^c, H^c and j^c are characteristic values, to be chosen. All distances are scaled by the characteristic distance D^c . Here, it is convenient to choose the radius of the cylinder as the characteristic distance, i.e. $D^c = R$. Using the scaling in (2.115), we obtain

$$\tilde{\nabla} \times \tilde{\mathbf{A}} = \frac{\mu_0 R H^c}{A^c} \tilde{\mathbf{H}}, \quad (2.129)$$

where

$$\tilde{\nabla} = \left(\frac{\partial}{\partial \tilde{r}}, \frac{1}{\tilde{r}} \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \tilde{z}} \right). \quad (2.130)$$

We obtain the simplest formulation by imposing the relation

$$A^c = \mu_0 R H^c. \quad (2.131)$$

The equations (2.115) - (2.116) then become

$$\tilde{\nabla} \times \tilde{\mathbf{A}} = \tilde{\mathbf{H}}, \quad \tilde{\nabla} \times \tilde{\mathbf{H}} = \mathbf{0}, \quad \tilde{\nabla} \cdot \tilde{\mathbf{A}} = 0, \quad \tilde{\nabla} \cdot \tilde{\mathbf{H}} = 0. \quad (2.132)$$

Normalization of the jump conditions (2.119) and (2.120) yields

$$\tilde{H}_\varphi^+(1, \varphi, \tilde{z}) - \tilde{H}_\varphi^-(1, \varphi, \tilde{z}) = \frac{j^c}{H^c} \tilde{\mathbf{j}}_z(\varphi, \tilde{z}), \quad (2.133)$$

$$\tilde{H}_z^+(1, \varphi, \tilde{z}) - \tilde{H}_z^-(1, \varphi, \tilde{z}) = -\frac{j^c}{H^c} \tilde{\mathbf{j}}_\varphi(\varphi, \tilde{z}), \quad (2.134)$$

showing that the optimal choice for H^c is

$$H^c = j^c, \quad \Rightarrow A^c = \mu_0 R j^c. \quad (2.135)$$

Consequently, the boundary conditions can be written as

$$\tilde{H}_r^-(1, \varphi, \tilde{z}) = \tilde{H}_r^+(1, \varphi, \tilde{z}), \quad (2.136)$$

$$\tilde{H}_\varphi^+(1, \varphi, \tilde{z}) - \tilde{H}_\varphi^-(1, \varphi, \tilde{z}) = \tilde{\mathbf{j}}_z(\varphi, \tilde{z}), \quad (2.137)$$

$$\tilde{H}_z^+(1, \varphi, \tilde{z}) - \tilde{H}_z^-(1, \varphi, \tilde{z}) = -\tilde{\mathbf{j}}_\varphi(\varphi, \tilde{z}), \quad (2.138)$$

$$\tilde{\mathbf{A}}^-(1, \varphi, \tilde{z}) = \tilde{\mathbf{A}}^+(1, \varphi, \tilde{z}). \quad (2.139)$$

The condition at infinity (2.121) remains unchanged. The relations (2.122) and (2.123) in dimensionless form are transformed into

$$i\kappa\tilde{A}_\varphi^-(1, \varphi, \tilde{z}) = i\kappa\tilde{A}_\varphi^+(1, \varphi, \tilde{z}) = \tilde{j}_\varphi(\varphi, \tilde{z}) - \tilde{j}_\varphi^s(\varphi, \tilde{z}), \quad (2.140)$$

$$i\kappa\tilde{A}_z^-(1, \varphi, \tilde{z}) = i\kappa\tilde{A}_z^+(1, \varphi, \tilde{z}) = \tilde{j}_z(\varphi, \tilde{z}) - \tilde{j}_z^s(\varphi, \tilde{z}), \quad (2.141)$$

where $\kappa = \hbar\sigma\mu_0\omega R$ is a system parameter. Moreover, the condition that states that the normal components of the currents at the edges of the strips must be zero remains unchanged and (2.104) becomes

$$\frac{\partial}{\partial\varphi}\tilde{j}_\varphi + \frac{\partial}{\partial\tilde{z}}\tilde{j}_z = 0. \quad (2.142)$$

We are still free to choose j^c . It seems convenient to choose j^c as the average current over all strips

$$j^c = \frac{\sum_{l=1}^L I_l}{\sum_{l=1}^L D_l}, \quad (2.143)$$

writing D_l for the total width of the strips of group l . Note that j^c may have a complex value.

According to (2.49), the solution of the vector potential caused by a surface current is in dimensional form given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{S_U} \frac{\mathbf{j}(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}). \quad (2.144)$$

Using (2.144) in Ohm's law (2.97), with $\mathbf{J}$ replaced by $\mathbf{j}/h$, we obtain the integral equation for $\mathbf{j}(\mathbf{x})$:

$$\mathbf{j}(\mathbf{x}) - \frac{i\hbar\sigma\mu_0\omega}{4\pi} \int_{S_U} \frac{\mathbf{j}(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}) = \mathbf{j}^s(\mathbf{x}). \quad (2.145)$$

In the dimensionless form, the vector potential $\tilde{\mathbf{A}}(\tilde{\mathbf{x}})$ is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 R j^c}{4\pi} \int_{S_U} \frac{\tilde{\mathbf{j}}(\tilde{\boldsymbol{\xi}})}{|\tilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}|} da(\tilde{\boldsymbol{\xi}}) = \mu_0 R j^c \tilde{\mathbf{A}}(\tilde{\mathbf{x}}), \quad (2.146)$$

such that the integral equation from (2.145) is transformed into the dimensionless one, according to

$$\tilde{\mathbf{j}}(\tilde{\mathbf{x}}) - \frac{i\kappa}{4\pi} \int_{S_U} \frac{\tilde{\mathbf{j}}(\tilde{\boldsymbol{\xi}})}{|\tilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}|} da(\tilde{\boldsymbol{\xi}}) = \tilde{\mathbf{j}}^s(\tilde{\mathbf{x}}). \quad (2.147)$$

This is the leading integral equation of this thesis. We show in the next section that (2.147) in cylindrical coordinates is consistent with (2.140) and (2.141).

Similarly, the magnetic induction $\mathbf{B}(\mathbf{x})$, see (2.46), can be expressed in $\tilde{\mathbf{B}}(\tilde{\mathbf{x}})$,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 j^c}{4\pi} \int_{S_U} \tilde{\mathbf{j}}(\tilde{\boldsymbol{\xi}}) \times \frac{\tilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}}{|\tilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}|^3} da(\tilde{\boldsymbol{\xi}}) = \mu_0 j^c \tilde{\mathbf{B}}(\tilde{\mathbf{x}}). \quad (2.148)$$

For $P_{diss}(t)$ and $P_{ind}(t)$, we use a dimensionless time variable $\tilde{t} = \omega t$ to obtain

$$\begin{aligned} P_{diss}(t) &= \frac{R^2 |j^c|^2}{2h\sigma} \int_{S_U} \Re \{ \tilde{\mathbf{J}}(\tilde{\mathbf{x}}) \cdot \tilde{\mathbf{J}}^*(\tilde{\mathbf{x}}) + \left(\frac{j^c}{|j^c|} \right)^2 \tilde{\mathbf{J}}(\tilde{\mathbf{x}}) \cdot \tilde{\mathbf{J}}(\tilde{\mathbf{x}}) e^{-2i\tilde{t}} \} da(\tilde{\mathbf{x}}) \\ &= \frac{R^2 |j^c|^2}{h\sigma} \tilde{P}_{diss}(\tilde{t}), \end{aligned} \quad (2.149)$$

and

$$\begin{aligned} P_{ind}(t) &= -\frac{\omega \mu_0 R^3 |j^c|^2}{8\pi} \int_{S_U} \int_{S_U} \Re \left\{ i \left(\frac{j^c}{|j^c|} \right)^2 \frac{\tilde{\mathbf{J}}(\tilde{\boldsymbol{\xi}}) \cdot \tilde{\mathbf{J}}^*(\tilde{\mathbf{x}})}{|\tilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}|} e^{-2i\tilde{t}} \right\} da(\tilde{\boldsymbol{\xi}}) da(\tilde{\mathbf{x}}) \\ &= -\omega \mu_0 R^3 |j^c|^2 \tilde{P}_{ind}(\tilde{t}). \end{aligned} \quad (2.150)$$

Furthermore,

$$\begin{aligned} \bar{P}_{diss} &= \frac{R^2 |j^c|^2}{2h\sigma} \int_{S_U} \tilde{\mathbf{J}}(\tilde{\mathbf{x}}) \cdot \tilde{\mathbf{J}}^*(\tilde{\mathbf{x}}) da(\tilde{\mathbf{x}}) \\ &= \frac{R^2 |j^c|^2}{h\sigma} \tilde{P}_{diss}, \end{aligned} \quad (2.151)$$

$$\begin{aligned} \bar{U}_m &= \frac{\mu_0 R^3 |j^c|^2}{16\pi} \int_{S_U} \int_{S_U} \frac{\tilde{\mathbf{J}}(\tilde{\boldsymbol{\xi}}) \cdot \tilde{\mathbf{J}}^*(\tilde{\mathbf{x}})}{|\tilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}|} da(\tilde{\boldsymbol{\xi}}) da(\tilde{\mathbf{x}}), \\ &= \mu_0 R^3 |j^c|^2 \tilde{U}_m. \end{aligned} \quad (2.152)$$

The characteristic quantities $\bar{R}$ and $\bar{L}$ are determined from the formulae (2.66) and (2.67), respectively, and we express them in dimensionless variables, such that all computations can immediately be performed using the known dimensionless current density. Using $I(t) = R j^c \tilde{I}(t)$, we obtain

$$\bar{R} = \frac{1}{h\sigma} \frac{\tilde{P}_{diss}}{\tilde{I} \tilde{I}^*} = \frac{1}{h\sigma} \tilde{R}, \quad (2.153)$$

$$\bar{L} = \mu_0 R \frac{\tilde{U}_m}{\frac{1}{2} \tilde{I} \tilde{I}^*} = \mu_0 R \tilde{L}. \quad (2.154)$$

2.3.3 Integral formulation

In this section, we derive from the set of dimensionless differential equations for our model a formulation expressed in terms of the current distribution only. We obtain a set of integral equations for the current distribution $\mathbf{j} = j_\varphi(\varphi, z) \mathbf{e}_\varphi + j_z(\varphi, z) \mathbf{e}_z$. The tildes are from now on omitted for the sake of readability.

We start from $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{0}$, which is a direct consequence of $\nabla \times \mathbf{A} = \mathbf{H}$ and $\nabla \times \mathbf{H} = \mathbf{0}$ in (2.132). This relation is valid for both the inner region G^- and the outer region G^+ . We introduce the complex Fourier series expansions of the 2π -periodic vector potential $\mathbf{A}$, the

magnetic field $\mathbf{H}$ and the current density $\mathbf{j}$, according to

$$A_q(r, \varphi, z) = \sum_{n=-\infty}^{\infty} A_q^{(n)}(r, z) e^{in\varphi}, \quad (\text{for } q = r, \varphi, z), \quad (2.155)$$

$$H_q(r, \varphi, z) = \sum_{n=-\infty}^{\infty} H_q^{(n)}(r, z) e^{in\varphi}, \quad (\text{for } q = r, \varphi, z), \quad (2.156)$$

$$j_q(\varphi, z) = \sum_{n=-\infty}^{\infty} j_q^{(n)}(z) e^{in\varphi}, \quad (\text{for } q = \varphi, z). \quad (2.157)$$

Using the gauge $\nabla \cdot \mathbf{A} = 0$, we obtain $\Delta \mathbf{A} = \mathbf{0}$. To the equations obtained by substituting (2.155) into $\Delta \mathbf{A} = \mathbf{0}$, we apply Fourier integral transformation (see e.g. [60, Sect.2.3]) with respect to the variable z . A function $g(r, z)$ is Fourier transformed to $\hat{g}(r, p)$, according to

$$\hat{g}(r, p) = \mathcal{F}_z\{g(r, \cdot)\}(p) = \int_{-\infty}^{\infty} g(r, z) e^{-ipz} dz. \quad (2.158)$$

Introducing the Fourier integral transforms $\hat{A}_q^{(n)}(r, p) = \mathcal{F}_z\{A_q^{(n)}(r, \cdot)\}(p)$, for both the inner and outer region, we obtain from $\Delta \mathbf{A} = \mathbf{0}$,

$$-\left(\frac{n^2}{r^2} + p^2\right) \hat{A}_z^{(n)} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{A}_z^{(n)}}{\partial r} \right) = 0, \quad (2.159)$$

$$-\left(\frac{n^2 + 1}{r^2} + p^2\right) \hat{A}_\varphi^{(n)} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{A}_\varphi^{(n)}}{\partial r} \right) + \frac{2ni}{r^2} \hat{A}_r^{(n)} = 0, \quad (2.160)$$

$$-\left(\frac{n^2 + 1}{r^2} + p^2\right) \hat{A}_r^{(n)} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{A}_r^{(n)}}{\partial r} \right) - \frac{2ni}{r^2} \hat{A}_\varphi^{(n)} = 0. \quad (2.161)$$

In the derivation of (2.159) - (2.161), the radiation conditions at infinity are used. The general solution can be expressed in terms of I_n and K_n , the modified Bessel functions of order n of the first and second kind, respectively (see e.g. [38], [82]). This yields

$$\hat{A}_r^{(n)-}(r, p) = -iC_1^{(n)}(p)I_{n-1}(|p|r) + iC_2^{(n)}(p)I_{n+1}(|p|r), \quad (2.162)$$

$$\hat{A}_\varphi^{(n)-}(r, p) = C_1^{(n)}(p)I_{n-1}(|p|r) + C_2^{(n)}(p)I_{n+1}(|p|r), \quad (2.163)$$

$$\hat{A}_z^{(n)-}(r, p) = C_3^{(n)}(p)I_n(|p|r), \quad (2.164)$$

for the inner region (denoted by the superindex $-$), with still arbitrary functions $C_1^{(n)}(p)$, $C_2^{(n)}(p)$ and $C_3^{(n)}(p)$. For the outer region (denoted by the superindex $+$), we obtain

$$\hat{A}_r^{(n)+}(r, p) = -iC_4^{(n)}(p)K_{n-1}(|p|r) + iC_5^{(n)}(p)K_{n+1}(|p|r), \quad (2.165)$$

$$\hat{A}_\varphi^{(n)+}(r, p) = C_4^{(n)}(p)K_{n-1}(|p|r) + C_5^{(n)}(p)K_{n+1}(|p|r), \quad (2.166)$$

$$\hat{A}_z^{(n)+}(r, p) = C_6^{(n)}(p)K_n(|p|r), \quad (2.167)$$

with still arbitrary functions $C_4^{(n)}(p)$, $C_5^{(n)}(p)$ and $C_6^{(n)}(p)$. Using $\nabla \times \mathbf{A} = \mathbf{H}$, the boundary

conditions (2.136) - (2.139), and the divergence-free condition (2.142), we obtain

$$C_1^{(n)}(p) = \frac{1}{2}K_{n-1}(|p|)\hat{j}_\varphi^{(n)}(p), \quad C_4^{(n)}(p) = \frac{1}{2}I_{n-1}(|p|)\hat{j}_\varphi^{(n)}(p), \quad (2.168)$$

$$C_2^{(n)}(p) = \frac{1}{2}K_{n+1}(|p|)\hat{j}_\varphi^{(n)}(p), \quad C_5^{(n)}(p) = \frac{1}{2}I_{n+1}(|p|)\hat{j}_\varphi^{(n)}(p), \quad (2.169)$$

$$C_3^{(n)}(p) = \frac{1}{2}K_n(|p|)\hat{j}_z^{(n)}(p), \quad C_6^{(n)}(p) = \frac{1}{2}I_n(|p|)\hat{j}_z^{(n)}(p). \quad (2.170)$$

For the inverse Fourier transformation, we use the convolution principle. Both the limits $r \uparrow 1$ for the inner region and $r \downarrow 1$ in the outer region give the same results. We obtain

$$A_r(1, \varphi, z) = \int_{S_c} \mathcal{K}_r(\varphi - \theta, z - \zeta) j_\varphi(\theta, \zeta) \, da(\theta, \zeta), \quad (2.171)$$

$$A_\varphi(1, \varphi, z) = \int_{S_c} \mathcal{K}_\varphi(\varphi - \theta, z - \zeta) j_\varphi(\theta, \zeta) \, da(\theta, \zeta), \quad (2.172)$$

$$A_z(1, \varphi, z) = \int_{S_c} \mathcal{K}_z(\varphi - \theta, z - \zeta) j_z(\theta, \zeta) \, da(\theta, \zeta), \quad (2.173)$$

valid on the whole cylinder S_c , with kernel functions

$$\mathcal{K}_r(\varphi, z) = \frac{1}{8\pi^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} [I_{n-1}(|p|)K_{n-1}(|p|) - I_{n+1}(|p|)K_{n+1}(|p|)] e^{ipz} e^{in\varphi} \, dp, \quad (2.174)$$

$$\mathcal{K}_\varphi(\varphi, z) = \frac{1}{8\pi^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} [I_{n-1}(|p|)K_{n-1}(|p|) + I_{n+1}(|p|)K_{n+1}(|p|)] e^{ipz} e^{in\varphi} \, dp, \quad (2.175)$$

and

$$\mathcal{K}_z(\varphi, z) = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} I_n(|p|)K_n(|p|) e^{ipz} e^{in\varphi} \, dp. \quad (2.176)$$

Finally, the complete integral equations for the current density $\mathbf{j} = j_\varphi(\varphi, z)\mathbf{e}_\varphi + j_z(\varphi, z)\mathbf{e}_z$ are obtained from (2.140) and (2.141), in which the currents are restricted to the surface $S_\cup$, yielding

$$i\kappa \int_{S_\cup} \mathcal{K}_\varphi(\varphi - \theta, z - \zeta) j_\varphi(\theta, \zeta) \, da(\theta, \zeta) = j_\varphi(\varphi, z) - j_\varphi^s(\varphi, z), \quad (2.177)$$

$$i\kappa \int_{S_\cup} \mathcal{K}_z(\varphi - \theta, z - \zeta) j_z(\theta, \zeta) \, da(\theta, \zeta) = j_z(\varphi, z) - j_z^s(\varphi, z). \quad (2.178)$$

Note that, by restricting the currents to $S_\cup$, the boundary conditions, which state that the normal component of the current is zero at the edges of the strips, are implicitly included. This is enforced by the electric conductivity σ , which is set equal to zero on the cylindrical surface between the strips. The linear integral equations (2.177) and (2.178) are integral equations of the second kind.

The important advantage of the integral equation in the form of (2.177) is that it is valid for an arbitrary set of conductors $S_\cup$, as long as they are infinitely thin and fixed on the surface of

the cylinder. For computations, they are applicable to a ring, a rectangular patch and a plane circular strip; see the following chapters.

Next, we prove that the resulting integral equations (2.177) and (2.178) for the surface current density $\mathbf{j}$ in cylindrical coordinates correspond with the coordinate-free equations, given by (2.145). The derivation is shown for (2.177), i.e. for the φ -component of the current. We need to prove that the kernel function $\mathcal{K}_\varphi(\varphi, z)$, given by (2.175), can be expressed as

$$\mathcal{K}_\varphi(\varphi, z) = \frac{1}{4\pi} \frac{\cos(\varphi)}{\sqrt{z^2 + 4 \sin^2(\varphi/2)}}. \quad (2.179)$$

The derivation uses the following three formulas (see [38, p.315], [82, p.358], [82, p.384]):

$$\frac{\pi}{2} \int_0^\infty J_\mu(at) J_\nu(bt) e^{-|c|t} t^\lambda dt = \int_0^\infty K_\mu(at) I_\nu(bt) \cos(|c|t + \frac{1}{2}(\mu - \nu + \lambda)\pi) t^\lambda dt, \quad (2.180)$$

$$\sum_{k=-\infty}^{\infty} J_k^2(s) e^{ik\varphi} = J_0(2s \sin \frac{\varphi}{2}), \quad (2.181)$$

$$\int_0^\infty J_0(bt) e^{-|a|t} dt = \frac{1}{\sqrt{a^2 + b^2}}, \quad (2.182)$$

where the functions J_k , I_k and K_k are the k^{th} -order Bessel function of the first kind, the modified k^{th} -order Bessel function of the first kind, and the modified k^{th} -order Bessel function of the second kind, respectively. Thus, we derive

$$\begin{aligned} & \int_{-\infty}^{\infty} [I_{k-1}(|p|)K_{k-1}(|p|) + I_{k+1}(|p|)K_{k+1}(|p|)] e^{ipz} dp \\ &= 2 \int_0^\infty [I_{k-1}(s)K_{k-1}(s) + I_{k+1}(s)K_{k+1}(s)] \cos(sz) ds \\ &= \pi \int_0^\infty [J_{k-1}^2(s) + J_{k+1}^2(s)] e^{-sz} ds, \end{aligned} \quad (2.183)$$

and

$$\begin{aligned} & \pi \sum_{k=-\infty}^{\infty} \int_0^\infty [J_{k-1}^2(s) + J_{k+1}^2(s)] e^{-sz} e^{ik\varphi} ds \\ &= \pi \sum_{k=-\infty}^{\infty} \int_0^\infty [J_k^2(s) e^{i\varphi} + J_k^2(s) e^{-i\varphi}] e^{-sz} e^{ik\varphi} ds \\ &= 2\pi \sum_{k=-\infty}^{\infty} \int_0^\infty J_k^2(s) \cos(\varphi) e^{-sz} e^{ik\varphi} ds, \end{aligned} \quad (2.184)$$

to obtain

$$\begin{aligned}
 \mathcal{K}_\varphi(\varphi, z) &= \frac{1}{4\pi} \sum_{k=-\infty}^{\infty} \int_0^\infty J_k^2(s) \cos(\varphi) e^{-sz} e^{ik\varphi} ds \\
 &= \frac{\cos(\varphi)}{4\pi} \int_0^\infty J_0(2s \sin \frac{\varphi}{2}) e^{-sz} ds \\
 &= \frac{1}{4\pi} \frac{\cos(\varphi)}{\sqrt{z^2 + 4 \sin^2(\varphi/2)}}. \tag{2.185}
 \end{aligned}$$

Likewise, the kernel functions $\mathcal{K}_r(\varphi, z)$ and $\mathcal{K}_z(\varphi, z)$, introduced in (2.174) and (2.176), and appearing in the integral equations for the vector field components A_r and A_z , respectively, can be written as

$$\mathcal{K}_r(\varphi, z) = \frac{1}{4\pi} \frac{\sin \varphi}{\sqrt{z^2 + 4 \sin^2(\varphi/2)}}, \tag{2.186}$$

$$\mathcal{K}_z(\varphi, z) = \frac{1}{4\pi} \frac{1}{\sqrt{z^2 + 4 \sin^2(\varphi/2)}}. \tag{2.187}$$

In a general setting, the integral operation in (2.144) can be applied to any surface current. Let the observation point $\mathcal{P}$ be described by coordinates $\mathbf{p}$ with respect to a coordinate system attached to the origin $\mathcal{O}_1$ and consider in $\mathcal{P}$ the local orthonormal system $\{\mathbf{e}_k, \mathbf{e}_l, \mathbf{e}_m\}$. Then any vector $\mathbf{v}|_{\mathcal{P}}$ attached to $\mathcal{P}$ can be written as $\mathbf{v}|_{\mathcal{P}} = v_k \mathbf{e}_k + v_l \mathbf{e}_l + v_m \mathbf{e}_m$.

Moreover, let the reference point $\mathcal{Q}$ be described by coordinates $\mathbf{q}$ with respect to a coordinate system attached to the origin $\mathcal{O}_2$ and consider in $\mathcal{Q}$ the local orthonormal system $\{\mathbf{e}_\kappa, \mathbf{e}_\lambda, \mathbf{e}_\mu\}$. Then any vector $\mathbf{v}|_{\mathcal{Q}}$ attached to $\mathcal{Q}$ can be written as $\mathbf{v}|_{\mathcal{Q}} = v_\kappa \mathbf{e}_\kappa + v_\lambda \mathbf{e}_\lambda + v_\mu \mathbf{e}_\mu$.

In the integral for the vector potential (2.146), we need to compute the distance between observation point $\mathcal{P}$ and reference point $\mathcal{Q}$. It is helpful to perform this computation in a common coordinate system, such as the Cartesian system. The variables in the integral can all be transformed to these coordinates (we can talk about Cartesian coordinates without loss of generality). The coordinate transformation can be substituted into the integral. Moreover, the change of the vector $\mathbf{j}$ to the new coordinate system implies a vector transformation. Define T_p as the transformation operator from the $\{\mathbf{e}_k, \mathbf{e}_l, \mathbf{e}_m\}$ system to the Cartesian system and T_q as the transformation operator from the $\{\mathbf{e}_\kappa, \mathbf{e}_\lambda, \mathbf{e}_\mu\}$ system to the Cartesian system. Let the value of a vector function, e.g. the current density $\mathbf{j}$, in a point $\mathcal{Q}$ be expressed by $\mathbf{j}|_{\mathcal{Q}}$, which implicitly means that the vector $\mathbf{j}$ is then also defined on the same reference frame as $\mathcal{Q}$. The vector potential in observation point $\mathcal{P}$ is then determined from

$$\mathbf{A}|_{\mathcal{P}} = T_p^{-1} \frac{1}{4\pi} \int_{S_U} \frac{1}{\mathcal{R}(\mathcal{P}, \mathcal{Q})} T_q \mathbf{j}|_{\mathcal{Q}} da(\mathcal{Q}), \tag{2.188}$$

where $\mathcal{R}(\mathcal{P}, \mathcal{Q})$ is the distance between $\mathcal{P}$ and $\mathcal{Q}$.

Consider as an example that $\mathcal{P}$ and $\mathcal{Q}$ are both described by cylindrical coordinates. Then, the local coordinate systems are defined as

$$\mathbf{e}_k = (\cos \varphi, \sin \varphi, 0), \quad \mathbf{e}_l = (-\sin \varphi, \cos \varphi, 0), \quad \mathbf{e}_m = (0, 0, 1), \quad (2.189)$$

$$\mathbf{e}_\kappa = (\cos \theta, \sin \theta, 0), \quad \mathbf{e}_\lambda = (-\sin \theta, \cos \theta, 0), \quad \mathbf{e}_\mu = (0, 0, 1), \quad (2.190)$$

and T_p^{-1} and T_q become

$$T_p^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.191)$$

$$T_q = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.192)$$

Moreover, T_p^{-1} is independent of the integration variables, so T_p^{-1} and T_q can be computed successively

$$T_p^{-1}T_q = \begin{pmatrix} \cos(\varphi - \theta) & \sin(\varphi - \theta) & 0 \\ -\sin(\varphi - \theta) & \cos(\varphi - \theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.193)$$

The distance between $\mathcal{P}$ and $\mathcal{Q}$, which is in the Cartesian coordinates equal to $\mathcal{R}(\mathcal{P}, \mathcal{Q}) = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$, yields after coordinate transformation

$$\mathcal{R}(\mathcal{P}, \mathcal{Q}) = \sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}. \quad (2.194)$$

Thus, the expressions (2.185) - (2.187) indeed match with the kernel functions in the integral formulations of the vector field components A_φ , A_r , and A_z , derived from (2.144). We remark that in the derivations of (2.185) - (2.187), we came across some expressions that are going to be very useful in the analysis coming up.

2.3.4 Coordinate-free problem setting

The preceding sections have shown how a gradient coil can be modeled as a set of infinitely thin strips on a cylindrical surface. We have presented how to derive from the Maxwell equations, the boundary conditions, the constitutive equations and the prescription of the total current through the strips, an integral equation for the dimensionless vector function $\mathbf{j}(\mathbf{x})$:

$$\mathbf{j}(\mathbf{x}) - \frac{i\kappa}{4\pi} \int_{S_\cup} \frac{\mathbf{j}(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}) = \mathbf{j}^s(\mathbf{x}). \quad (2.195)$$

In operator form, the integral equation (2.195) can be written as

$$(\mathbf{I} - i\kappa\mathbf{K})\mathbf{j} = \mathbf{j}^s. \quad (2.196)$$

The currents have no r -component, such that $\mathbf{j}(\varphi, z) = j_\varphi(\varphi, z)\mathbf{e}_\varphi + j_z(\varphi, z)\mathbf{e}_z$. The operator K is then expressed by a diagonal operator matrix with diagonal elements K_φ and K_z , defined as

$$(K_\varphi j_\varphi)(\varphi, z) = \frac{1}{4\pi} \int_{S_U} \frac{\cos(\varphi - \theta) j_\varphi(\theta, \zeta)}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} da(\theta, \zeta), \quad (2.197)$$

$$(K_z j_z)(\varphi, z) = \frac{1}{4\pi} \int_{S_U} \frac{j_z(\theta, \zeta)}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} da(\theta, \zeta). \quad (2.198)$$

The resulting integral equations are given by (2.177) and (2.178).

From a physical point of view, we know that both $\mathbf{j}^s(\varphi, z)$ and $\mathbf{j}(\varphi, z)$ are finite. Moreover, the currents are defined on the surface S_U , which has a finite size. Therefore,

$$\int_{S_U} |\mathbf{j}^s(\varphi, z)|^2 d\varphi dz < \infty, \quad \int_{S_U} |\mathbf{j}(\varphi, z)|^2 d\varphi dz < \infty, \quad (2.199)$$

such that the currents belong to L_2 , the functions of integrable square. This means that all possible currents form a linear vector space that is bounded; a Banach space. More specifically, the solution $\mathbf{j}$ must be a differentiable vector function that satisfies the conditions at the edges, $(\mathbf{j} \cdot \mathbf{n})|_{S_U} = 0$, and is free of divergence, $\nabla \cdot \mathbf{j} = 0$.

The operator K is a linear operator, the kernel functions of which are even in φ and z . Therefore, K is a self-adjoint (Hermitian) operator:

$$\begin{aligned} (K\mathbf{f}, \mathbf{g}) &= \frac{1}{4\pi} \int_{S_U} \int_{S_U} \frac{\mathbf{f}(\boldsymbol{\xi})\mathbf{g}^*(\mathbf{x})}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}) da(\mathbf{x}) \\ &= \frac{1}{4\pi} \int_{S_U} \int_{S_U} \frac{\mathbf{g}^*(\boldsymbol{\xi})\mathbf{f}(\mathbf{x})}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}) da(\mathbf{x}) = (\mathbf{f}, K\mathbf{g}). \end{aligned} \quad (2.200)$$

The spectrum of K is real, while $i\kappa$ is purely imaginary. Therefore, the homogeneous equation $(I - i\kappa K)\mathbf{j} = \mathbf{0}$ has the trivial solution $\mathbf{j} = \mathbf{0}$. The operator $(I - i\kappa K)$ is invertible and has a bounded inverse $(I - i\kappa K)^{-1}$, for all possible values of κ . Thus, $(I - i\kappa K)\mathbf{j} = \mathbf{j}^s$ has a unique solution.

2.4 Summary

We have modeled a gradient coil by a set of infinitely thin copper strips on a cylindrical surface. We want to calculate the current density in the strips, due to a prescribed current through (groups of) strips. From the Maxwell equations, the boundary conditions, the constitutive equations and the prescription of the total current through the strips, an integral equation for the surface current is derived, given by (2.195). This integral equation is valid for an arbitrary configuration of strips on a cylindrical surface. This is the basis for the following chapters.

Mathematical Analysis

In this chapter, we describe the core of the mathematical analysis of our model. The problem defined in the previous chapter has led to an integral equation of the second kind; see (2.195). We analyze the properties of the integral equation, which form the basis of the numerical implementation.

In Section 3.1, we explain how the current can be expanded by Fourier modes, such that the integral equation is decomposed into a sequence of integral equations for each mode separately. For each integral equation the kernel is proved to be dominated by a logarithmic function of the coordinate perpendicular to the direction of the driving source.

To solve the linear integral equation, we propose the Galerkin method; see Section 3.2. The choice for the basis functions of the current distribution needed in the Galerkin method is based on the logarithmic behavior of the kernel. With this, we arrive at the central theme of this thesis. Namely, this logarithmic behavior reveals that the Legendre polynomials of the first kind are the most appropriate choice for the basis functions to describe one variable of the current distribution in the strips. The other variable is described by the Fourier modes. With respect to the local coordinates, this variable describes the preferable direction of the eddy currents induced by a source current.

In Section 3.3, we apply the Galerkin method to an integral equation of the second kind with a purely logarithmic kernel function. Legendre polynomials are used as basis functions and a matrix equation is derived to determine the coefficients of the Legendre polynomials.

In previous research [76], [77], a geometry of plane rectangular strips of infinite length has been investigated. This special case leads to an integral equation for which the kernel function is purely logarithmic. In Section 3.4, we describe the solution method used in this special case and show useful reference results for the chapters to come.

In Section 3.5, the summary of the present chapter is presented.

3.1 Leading integral equation, with logarithmic kernel

Numerous numerical methods exist to solve integral equations of the second kind. One of the problems that we encounter in the numerical approaches for the integral equations in (2.177) and (2.178) is that the kernel functions are singular in the point $(\varphi, z) = (0, 0)$, causing slow convergence and long computation times. Figures 3.1 (a) and (b) show the graphs of $\mathcal{K}_\varphi(\varphi, z)$ and $\mathcal{K}_z(\varphi, z)$, respectively, for $\varphi \in [-\pi, \pi]$ and $z \in [-5, 5]$. We search for a numerical method to solve the integral equation that is based on a full analysis of the integral equation.

Figure 3.1: The kernel functions with singularities in the point $(\varphi, z) = (0, 0)$. (a) $\mathcal{K}_\varphi(\varphi, z)$; (b) $\mathcal{K}_z(\varphi, z)$.

We consider the integral equation for the component of the current in the preferable direction. This integral equation is the leading integral equation and is discussed for each of the three types of strips, mentioned in Section 2.2: the circular loop around the cylinder, the circular strip on the surface of the cylinder and the patch on the cylinder. Theoretically, the three types of strips can be transformed into the following three standard shapes: a ring, a plane circular strip and a plane rectangular patch. The characteristic behavior of the currents is preserved after the transformation. For each of the three standard shapes, we prove that the leading integral equation has a kernel function with a dominant logarithmic part. This logarithmic part acts on the component, which is perpendicular to the preferable direction of the induction current.

3.1.1 Type one: The ring

Let S be the ring on the cylindrical surface defined in cylindrical coordinates φ and z by $[-\pi, \pi] \times [z_0, z_1]$; see Figure 3.2 (a). A current $\mathbf{j} = j_\varphi(\varphi, z)\mathbf{e}_\varphi + j_z(\varphi, z)\mathbf{e}_z$ in the ring satisfies $\nabla \cdot \mathbf{j} = 0$, is 2π -periodic in φ , and its z -component is zero at the edges.

Figure 3.2: The three standard shapes of strips. (a) The ring; (b) The plane circular strip; (c) The plane rectangular patch.

The source current can be applied to the ring itself or to a neighboring ring. For a ring, the main component of the source current is in the φ -direction, and then

$$\mathbf{j}^s(\varphi, z) = j_\varphi^s(\varphi, z)\mathbf{e}_\varphi + j_z^s(\varphi, z)\mathbf{e}_z \approx j_{\varphi,0}^s(z)\mathbf{e}_\varphi, \quad (3.1)$$

where

$$j_{\varphi,0}^s(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} j_\varphi^s(\varphi, z) d\varphi, \quad (3.2)$$

the φ -independent part of $\mathbf{j}^s$ (or, see (3.5), the zeroth Fourier component of $\mathbf{j}^s$). Note that $j_{\varphi,0}^s(z)\mathbf{e}_\varphi$ satisfies all three conditions mentioned above for $\mathbf{j}$.

The induced current on the ring then mainly flows in the direction opposite to the source current. The physical explanation for this is that the magnetic field caused by the source current induces an eddy current, which tries to oppose the field. This phenomenon can be detected in all examples in this thesis; see the results in the following chapters. The leading integral equation is now given by (see (2.177) and (2.179))

$$\frac{i\kappa}{4\pi} \int_{-\pi}^{\pi} \int_{z_0}^{z_1} \frac{\cos(\varphi - \theta) j_\varphi(\theta, \zeta)}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\theta d\zeta = j_\varphi(\varphi, z) - j_\varphi^s(\varphi, z). \quad (3.3)$$

The induced eddy current is related to the vector potential, according to $j_\varphi^e = i\kappa A_\varphi$, and is expressed by the left-hand side of (3.3).

The (source) current is 2π -periodic in the φ -direction and can therefore be expanded in a

Fourier series:

$$j_\varphi(\varphi, z) = j_{\varphi,0}(z) + \sum_{m=1}^{\infty} j_{\varphi,m}^{(c)}(z) \cos m\varphi + \sum_{m=1}^{\infty} j_{\varphi,m}^{(s)}(z) \sin m\varphi, \quad (3.4)$$

$$j_\varphi^s(\varphi, z) = j_{\varphi,0}^s(z) + \sum_{m=1}^{\infty} j_{\varphi,m}^{(c)}(z) \cos m\varphi + \sum_{m=1}^{\infty} j_{\varphi,m}^{(s)}(z) \sin m\varphi. \quad (3.5)$$

Each term of these series is called a *mode* from now on. We continue with a description for a sine mode. The results for the cosine modes are derived in the same way. Taking in (3.3) the inner product with $\sin n\varphi$, we obtain

$$\frac{i\kappa}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{z_0}^{z_1} \frac{\cos(\varphi - \theta) j_{\varphi,n}^{(s)}(\zeta) \sin n\theta \sin n\varphi}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\theta d\zeta d\varphi = \pi j_{\varphi,n}^{(s)}(z) - \pi j_{\varphi,n}^{(s)}(z). \quad (3.6)$$

The integrand on the left-hand side of (3.6) is singular in the point $(\theta, \zeta) = (\varphi, z)$. To be able to perform an analytical integration of at least the singular part of the integral in (3.6), we split $A_{\varphi,m}^{(s)}(\varphi, z)$ as follows (here, $A_{\varphi,m}^{(s)}(\varphi, z)$ is the m -th sine mode of the φ -component of the vector potential)

$$\begin{aligned} A_{\varphi,m}^{(s)}(\varphi, z) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{z_0}^{z_1} \frac{[\cos(\varphi - \theta) - \cos(\frac{\varphi - \theta}{2})] j_{\varphi,m}^{(s)}(\zeta) \sin m\theta}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\theta d\zeta \\ &\quad + \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{z_0}^{z_1} \frac{\cos(\frac{\varphi - \theta}{2}) j_{\varphi,m}^{(s)}(\zeta) \sin m\theta}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\theta d\zeta, \\ &= A_{\varphi,m}^{(s,1)}(\varphi, z) + A_{\varphi,m}^{(s,2)}(\varphi, z). \end{aligned} \quad (3.7)$$

Here, the first integral, represented by $A_{\varphi,m}^{(s,1)}(\varphi, z)$, is regular. Next, we split the remaining part, $A_{\varphi,m}^{(s,2)}(\varphi, z)$, according to

$$\begin{aligned} A_{\varphi,m}^{(s,2)}(\varphi, z) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{z_0}^{z_1} \frac{\cos(\frac{\varphi - \theta}{2}) j_{\varphi,m}^{(s)}(\zeta) [\sin m\theta - \sin m\varphi]}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\theta d\zeta \\ &\quad + \frac{\sin m\varphi}{4\pi} \int_{-\pi}^{\pi} \int_{z_0}^{z_1} \frac{\cos(\frac{\varphi - \theta}{2}) j_{\varphi,m}^{(s)}(\zeta)}{\sqrt{(z - \zeta)^2 + 4 \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\theta d\zeta, \end{aligned} \quad (3.8)$$

such that again the first integral is regular. In the second integral we can perform an analytical integration over θ , for which we make use of the formula

$$\frac{\partial}{\partial \varphi} \left[\operatorname{arcsinh} \left(\frac{2 \sin(\varphi/2)}{|z|} \right) \right] = \frac{\cos(\varphi/2)}{\sqrt{z^2 + 4 \sin^2(\varphi/2)}}. \quad (3.9)$$

Introducing

$$\mathcal{L}_1(b) = -\log |b|, \quad (3.10)$$

$$\mathcal{L}_2(a, b) = \log \left[|a| + \sqrt{a^2 + b^2} \right], \quad (3.11)$$

we can write

$$\operatorname{arcsinh}\left(\frac{|a|}{|b|}\right) = \mathcal{L}_1(b) + \mathcal{L}_2(a, b). \quad (3.12)$$

The function $\mathcal{L}_1(b)$ is singular in $b = 0$, whereas $\mathcal{L}_2(a, b)$ is singular in the point $(a, b) = (0, 0)$. Substituting $a = 2 \sin((\varphi - \theta)/2)$ and $b = z - \zeta$ into $\mathcal{L}_1(b)$ and $\mathcal{L}_2(a, b)$, and using (3.12) in (3.9), we obtain

$$\begin{aligned} \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{z_0}^{z_1} \frac{\cos\left(\frac{\varphi-\theta}{2}\right) j_{\varphi,m}^{(s)}(\zeta)}{\sqrt{(z-\zeta)^2 + 4 \sin^2\left(\frac{\varphi-\theta}{2}\right)}} d\theta d\zeta &= -\frac{1}{2\pi} \int_{z_0}^{z_1} \log |z - \zeta| j_{\varphi,m}^{(s)}(\zeta) d\zeta \\ &+ \frac{1}{2\pi} \int_{z_0}^{z_1} \mathcal{L}_2\left(2 \cos\left(\frac{\varphi}{2}\right), z - \zeta\right) j_{\varphi,m}^{(s)}(\zeta) d\zeta. \end{aligned} \quad (3.13)$$

The last integral in (3.13) is regular for all values of φ . Hence, the integral on the left-hand side of (3.13), and with that also $A_{\varphi,m}^{(s)}(\varphi, z)$, is dominated by an integral with a kernel that has logarithmic singularity at $\zeta = z$.

We can now rewrite (3.6) as

$$-\frac{i\kappa}{2\pi} \int_{z_0}^{z_1} \log |z - \zeta| j_{\varphi,m}^{(s)}(\zeta) d\zeta = j_{\varphi,m}^{(s)}(z) - j_{\varphi,m}^{s(s)}(z) + i\kappa(K_m^{reg} j_{\varphi,m}^{(s)})(z), \quad (3.14)$$

where K_m^{reg} is the operator that contains all the regular integrals that have been split off. These regular integrals can be integrated numerically. The effect of $K_m^{reg} j_{\varphi,m}^{(s)}$ to the solution for the current is small in comparison with the logarithmic integral in (3.14). This is verified later on in the numerical results.

Summing over all modes yields

$$-\frac{i\kappa}{2\pi} \int_{z_0}^{z_1} \log |z - \zeta| j_{\varphi}(\varphi, \zeta) d\zeta = j_{\varphi}(\varphi, z) - j_{\varphi}^s(\varphi, z) + i\kappa(K^{reg} j_{\varphi})(\varphi, z), \quad (3.15)$$

in which K^{reg} represents the total regular operator. So, the singular behavior is only in the z -direction and the singularity is logarithmic.

3.1.2 Type two: The plane circular strip

In case of a plane circular strip the approach is similar to the one of the ring. Let S be the surface of the circular strip in the plane defined in the polar coordinates r and φ by $[r_0, r_1] \times [-\pi, \pi]$; see Figure 3.2 (b). A divergence-free current $\mathbf{j} = j_r(r, \varphi)\mathbf{e}_r + j_{\varphi}(r, \varphi)\mathbf{e}_{\varphi}$ is flowing on the strip, which is 2π -periodic in φ and its r -component is zero at the edges.

The formula for the vector potential follows from (2.146), or more directly from (2.188) and

(2.193), as

$$A_r(r, \varphi) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{r_0}^{r_1} \frac{\sin(\varphi - \theta) j_{\varphi}(\rho, \theta) + \cos(\varphi - \theta) j_r(\rho, \theta)}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2(\frac{\varphi - \theta}{2})}} \rho \, d\rho \, d\theta, \quad (3.16)$$

$$A_{\varphi}(r, \varphi) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{r_0}^{r_1} \frac{\cos(\varphi - \theta) j_{\varphi}(\rho, \theta) - \sin(\varphi - \theta) j_r(\rho, \theta)}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2(\frac{\varphi - \theta}{2})}} \rho \, d\rho \, d\theta. \quad (3.17)$$

The main component of the source current is in the φ -direction. The field created by the source current induces an eddy current mainly in the opposite direction. The leading integral equation is given by

$$\frac{i\kappa}{4\pi} \int_{-\pi}^{\pi} \int_{r_0}^{r_1} \frac{\cos(\varphi - \theta) j_{\varphi}(\rho, \theta)}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2(\frac{\varphi - \theta}{2})}} \rho \, d\rho \, d\theta = j_{\varphi}(r, \varphi) - j_{\varphi}^s(r, \varphi). \quad (3.18)$$

The (source) current is 2π -periodic in the φ -direction and can therefore be expanded in a Fourier series:

$$j_{\varphi}(r, \varphi) = j_{\varphi,0}(r) + \sum_{m=1}^{\infty} j_{\varphi,m}^{(c)}(r) \cos m\varphi + \sum_{m=1}^{\infty} j_{\varphi,m}^{(s)}(r) \sin m\varphi, \quad (3.19)$$

$$j_{\varphi}^s(r, \varphi) = j_{\varphi,0}^s(r) + \sum_{m=1}^{\infty} j_{\varphi,m}^{(c)s}(r) \cos m\varphi + \sum_{m=1}^{\infty} j_{\varphi,m}^{(s)s}(r) \sin m\varphi. \quad (3.20)$$

Taking in (3.18) the inner product with $\sin n\varphi$, we obtain

$$\frac{i\kappa}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{r_0}^{r_1} \frac{\cos(\varphi - \theta) j_{\varphi,n}^{(s)}(\rho) \sin n\theta \sin n\varphi}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2(\frac{\varphi - \theta}{2})}} \, d\theta \, d\rho \, d\varphi = \pi j_{\varphi,n}^{(s)}(r) - \pi j_{\varphi,n}^{(s)s}(r). \quad (3.21)$$

We investigate the behavior of $A_{\varphi}(r, \varphi)$, given by (3.17), and in particular the part that corresponds to the m -th sine mode of the current, referred to as $A_{\varphi,m}^{(s)}(\varphi, z)$. The integrand is singular in the point $(\rho, \theta) = (r, \varphi)$. As in the example of a ring, we first split off a part containing $\cos((\varphi - \theta)/2)$, followed by a part containing $\sin m\varphi$. Moreover, we use

$$\frac{\partial}{\partial \varphi} \left[\frac{1}{\sqrt{r\rho}} \operatorname{arcsinh} \left(\frac{2\sqrt{r\rho} \sin(\varphi/2)}{|r - \rho|} \right) \right] = \frac{\cos(\varphi/2)}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2(\varphi/2)}}. \quad (3.22)$$

In this case, we substitute $a = 2\sqrt{r\rho} \sin((\varphi - \theta)/2)$ and $b = r - \rho$ into $\mathcal{L}_1(b)$ and $\mathcal{L}_2(a, b)$, as defined in (3.10) and (3.11), respectively. We obtain

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{r_0}^{r_1} \frac{\cos(\frac{\varphi - \theta}{2}) j_{\varphi,m}^{(s)}(\rho) \sin m\varphi}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2(\frac{\varphi - \theta}{2})}} \rho \, d\rho \, d\theta = -\frac{1}{2\pi} \int_{r_0}^{r_1} \log |r - \rho| \frac{j_{\varphi,m}^{(s)}(\rho)}{\sqrt{r\rho}} \rho \, d\rho$$

$$+ \frac{1}{2\pi} \int_{r_0}^{r_1} \mathcal{L}_2 \left(2\sqrt{r\rho} \cos\left(\frac{\varphi}{2}\right), r - \rho \right) \frac{j_{\varphi,m}^{(s)}(\rho)}{\sqrt{r\rho}} \rho \, d\rho. \quad (3.23)$$

The last integral in (3.23) is regular for all values of φ . The first integral contains a logarithmic singularity along the line $\rho = r$.

From (3.21) we derive

$$-\frac{i\kappa}{2\pi} \int_{r_0}^{r_1} \log |r - \rho| j_{\varphi,m}^{(s)}(\rho) \rho \, d\rho = j_{\varphi,m}^{(s)}(r) - j_{\varphi,m}^{s(s)}(r) + i\kappa(\mathbf{K}_m^{reg} j_{\varphi,m}^{(s)})(r), \quad (3.24)$$

with $\mathbf{K}_m^{reg}$ the operator containing all the regular integrals that have been split off. The effect of $\mathbf{K}_m^{reg} j_{\varphi,m}^{(s)}$ on the solution for the current is small in comparison with the logarithmic integral in (3.24).

Summing over all modes yields

$$-\frac{i\kappa}{2\pi} \int_{r_0}^{r_1} \log |r - \rho| j_{\varphi}(\rho, \varphi) \rho \, d\rho = j_{\varphi}(r, \varphi) - j_{\varphi}^s(r, \varphi) + i\kappa(\mathbf{K}^{reg} j_{\varphi})(r, \varphi), \quad (3.25)$$

in which $\mathbf{K}^{reg}$ represents the total regular operator. So, the singular behavior is only in the r -direction and the singularity is logarithmic.

3.1.3 Type three: The plane rectangular patch

Let S be the rectangular patch in the plane defined in Cartesian coordinates x and y by $[x_0, x_1] \times [y_0, y_1]$; see Figure 3.2 (c). A divergence free current $\mathbf{j} = j_x(x, y)\mathbf{e}_x + j_y(x, y)\mathbf{e}_y$ is flowing on the patch, having a zero normal component at the edges. In this case, the source current is not present in the patch itself. Eddy currents are induced by a source current through another strip near the patch. We assume that the source current flows in the x -direction. The preferable direction of the eddy current in the patch is then also the x -direction.

The leading integral equation is given by

$$\frac{i\kappa}{4\pi} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \frac{j_x(\xi, \eta)}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} \, d\xi \, d\eta = j_x(x, y) - j_x^s(x, y), \quad (3.26)$$

where $j_x^s(x, y)$ is the inductive source current defined on the patch. The induced current is related to the vector potential, according to $j_x = i\kappa A_x$, and is expressed by the left-hand side of (3.26). The integrand of this expression is singular in the point $(\xi, \eta) = (x, y)$. Next, we expand $j_x(x, y)$ and $j_x^s(x, y)$ into a Fourier sine series

$$j_x(x, y) = \sum_{m=1}^{\infty} j_{x,m}(y) \sin\left(\frac{m\pi(x - x_0)}{x_1 - x_0}\right), \quad (3.27)$$

$$j_x^s(x, y) = \sum_{m=1}^{\infty} j_{x,m}^s(y) \sin\left(\frac{m\pi(x - x_0)}{x_1 - x_0}\right). \quad (3.28)$$

In these series, every mode is equal to zero at the edges $x = x_0$ and $x = x_1$. Taking in (3.26) the inner product with $\sin(n\pi(x - x_0)/(x_1 - x_0))$, we obtain

$$\begin{aligned} \frac{i\kappa}{4\pi} \int_{x_0}^{x_1} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \sum_{m=1}^{\infty} \frac{j_{x,m}(\eta) \sin\left(\frac{m\pi(\xi-x_0)}{x_1-x_0}\right) \sin\left(\frac{n\pi(x-x_0)}{x_1-x_0}\right)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} d\xi d\eta dx \\ = \frac{1}{2}(x_1 - x_0)j_{x,n}(y) - \frac{1}{2}(x_1 - x_0)j_{x,n}^s(y). \end{aligned} \quad (3.29)$$

We investigate the vector potential A_x . Similar to the previous two examples, we split A_x according to

$$\begin{aligned} A_x(x, y) = \frac{1}{4\pi} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \sum_{m=1}^{\infty} \frac{j_{x,m}(\eta) \left(\sin\left(\frac{m\pi(\xi-x_0)}{x_1-x_0}\right) - \sin\left(\frac{m\pi(x-x_0)}{x_1-x_0}\right) \right)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} d\xi d\eta \\ + \frac{1}{4\pi} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi(x-x_0)}{x_1-x_0}\right) \int_{x_0}^{x_1} \int_{y_0}^{y_1} \frac{j_{x,m}(\eta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} d\xi d\eta. \end{aligned} \quad (3.30)$$

The first integral has a regular integrand, such that its numerical computation is straightforward. The characteristic behavior that is important for further investigations is hidden in the second integral. The integration over ξ can be performed analytically. We use

$$\frac{\partial}{\partial x} [\operatorname{arcsinh}\left(\frac{x}{|y|}\right)] = \frac{1}{\sqrt{x^2 + y^2}}, \quad (3.31)$$

and with $\mathcal{L}_1(b)$ and $\mathcal{L}_2(a, b)$, as defined in (3.10) and (3.11), where $a = x - \xi$ and $b = y - \eta$, we can write for each mode

$$\begin{aligned} \frac{1}{4\pi} \int_{x_0}^{x_1} \int_{y_0}^{y_1} \frac{j_{x,m}(\eta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} d\xi d\eta = -\frac{1}{2\pi} \int_{y_0}^{y_1} \log |y - \eta| j_{x,m}(\eta) d\eta \\ + \frac{1}{4\pi} \int_{y_0}^{y_1} [\mathcal{L}_2(x - x_1, y - \eta) + \mathcal{L}_2(x - x_0, y - \eta)] j_{x,m}(\eta) d\eta. \end{aligned} \quad (3.32)$$

The integral with $\mathcal{L}_2$ is regular because the only two possible singular points $(x, y) = (x_0, \eta)$ and $(x, y) = (x_1, \eta)$ are canceled by the current due to the boundary conditions $j_x(x_0, \eta) = j_x(x_1, \eta) = 0$. Hence, only the first integral contains a logarithmic singularity along the line $\eta = y$.

From (3.29) we derive

$$-\frac{i\kappa}{2\pi} \int_{y_0}^{y_1} \log |y - \eta| j_{x,m}(\eta) d\eta = j_{x,m}(y) - j_{x,m}^s(y) + i\kappa \sum_{n=1}^{\infty} (K_{mn}^{reg} j_{x,n})(y), \quad (3.33)$$

with K_{mn}^{reg} the operators containing all the regular integrals that have been split off. The effect of $K_{mn}^{reg} j_{x,m}$ to the solution for the current is small in comparison with the logarithmic integral in (3.33).

Summing over all modes yields

$$-\frac{i\kappa}{2\pi} \int_{y_0}^{y_1} \log |y - \eta| j_x(x, \eta) d\eta = j_x(x, y) - j_x^s(x, y) + i\kappa(K^{reg} j_x)(x, y), \quad (3.34)$$

in which K^{reg} represents the total regular operator. So, the singular behavior is only in the y -direction and the singularity is logarithmic.

3.2 Solution procedure

In the previous section, we showed that the general form of the integral equation for currents on a cylindrical surface is given by

$$-\frac{i\kappa}{2\pi} \int_{z_0}^{z_1} \log |z - \zeta| j_\varphi(\varphi, \zeta) d\zeta = j_\varphi(\varphi, z) - j_\varphi^s(\varphi, z) + i\kappa(K^{reg} j_\varphi)(\varphi, z). \quad (3.35)$$

In terms of the modes $j_{\varphi,m}(z)$, (3.35) is decomposed for closed loops of strips into

$$-\frac{i\kappa}{2\pi} \int_{z_0}^{z_1} \log |z - \zeta| j_{\varphi,m}(\zeta) d\zeta = j_{\varphi,m}(z) - j_{\varphi,m}^s(z) + i\kappa(K_m^{reg} j_{\varphi,m})(z), \quad (3.36)$$

for all m , and in case of a patch on a cylindrical surface into

$$-\frac{i\kappa}{2\pi} \int_{z_0}^{z_1} \log |z - \zeta| j_{\varphi,m}(\zeta) d\zeta = j_{\varphi,m}(z) - j_{\varphi,m}^s(z) + i\kappa \sum_{n=1}^{\infty} (K_{mn}^{reg} j_{\varphi,n})(z), \quad (3.37)$$

for all m . Note that in (3.37) all modes have a contribution to $j_{\varphi,m}(z)$, through the regular integral operators K_{mn}^{reg} . The number of modes is determined by the source current. Say that the source current is described by M modes. Then, the infinite series in (3.37) can be replaced by a finite series with M terms.

In operator form, (3.36) can be written as

$$(I - i\kappa K_m) j_{\varphi,m} = j_{\varphi,m}^s, \quad (3.38)$$

with operator K_m defined as

$$(K_m j_{\varphi,m})(z) = -\frac{1}{2\pi} \int_{z_0}^{z_1} \log |z - \zeta| j_{\varphi,m}(\zeta) d\zeta - (K_m^{reg} j_{\varphi,m})(z). \quad (3.39)$$

The operator form of (3.37) contains a series of all modes. For the explanation of the Galerkin method, we restrict ourselves to equation (3.38). The treatment of the patch is discussed in Chapter 5. For an operator form of (3.37), we refer to (5.43).

To solve (3.38), we choose a finite family of functions $j_{\varphi,m}(z)$, $m = 0, \dots, M$, with corresponding modes in the z -direction, such that the total current is divergence free on $S_\cup$ and has normal components equal to zero at the edges. The finite family leads to a projection method. The projection method we use to get the approximation of the current distribution is the (Petrov-) Galerkin method. In this section, we explain briefly how this method works and how we decide on the basis functions.

3.2.1 The Galerkin method

Let W be an inner product space with subspaces $W_{\parallel}$ and $W_{\perp}$, such that W is the orthogonal direct sum of $W_{\parallel}$ and $W_{\perp}$, i.e. $W = W_{\parallel} \oplus W_{\perp}$. This means that each element $j_{\varphi,m} \in W$ can be uniquely decomposed as

$$j_{\varphi,m} = j_{\parallel} + j_{\perp}, \quad j_{\parallel} \in W_{\parallel}, \quad j_{\perp} \in W_{\perp}. \quad (3.40)$$

We know then that a projection operator $\Pi : W \rightarrow W_{\parallel}$ exists, such that $\Pi j_{\varphi,m} = j_{\parallel} \in W_{\parallel}$ and $(I - \Pi)j_{\varphi,m} = j_{\perp} \in W_{\perp}$. The projection Π is self-adjoint and $\Pi^2 = \Pi$. In the projection method, Π is of finite rank and $\Pi j_{\varphi,m}$ is assumed to be a good approximation of $j_{\varphi,m}$. Let us first give an example of an approximation for a one-dimensional function.

A very important application is the least-squares approximation of a square integrable function by polynomials. Let $W = L^2(-1, 1)$, and W_K the space of polynomials of degree less than or equal to K . A known orthogonal polynomial basis for W is $\{P_k\}_{k \geq 0}$, consisting of the Legendre polynomials

$$P_k(z) = \frac{1}{2^k k!} \frac{d^k}{dz^k} [(z^2 - 1)^k], \quad k \geq 0. \quad (3.41)$$

For any $f \in W$, its least-squares approximation from W_K is given by

$$\Pi_K f = \sum_{k=0}^K \frac{(f, P_k)}{(P_k, P_k)} P_k = \sum_{k=0}^K \frac{2k+1}{2} (f, P_k) P_k. \quad (3.42)$$

The convergence criterion reads

$$\lim_{K \rightarrow \infty} \|f - \Pi_K f\|_2 = 0. \quad (3.43)$$

Given the source $j_{\varphi,m}^s$, we assume that there is a subspace W^0 of W , such that $j_{\varphi,m}^s \in W^0$. In W^0 we choose a finite-dimensional subspace W_K spanned by the basis $\{\phi_1, \dots, \phi_K\}$, with associated projection operator Π_K . Then we can write

$$j_{\varphi,m}^{(K)}(z) = \Pi_K j_{\varphi,m}(z) = \sum_{k=1}^K \alpha_k \phi_k(z). \quad (3.44)$$

This is substituted into (3.38). The coefficients $\{\alpha_1, \dots, \alpha_K\}$ are determined by forcing the equation to be almost exact in some sense. We now introduce the residual $r^{(K)}$ in the approximation when using $j_{\varphi,m} \approx j_{\varphi,m}^{(K)}$ in (3.38), as

$$r^{(K)} = (I - i\kappa K_m) j_{\varphi,m}^{(K)} - j_{\varphi,m}^s. \quad (3.45)$$

The coefficients $\{\alpha_1, \dots, \alpha_K\}$ are determined by using the requirement that the inner products of the residual $r^{(K)}$ with a set of test functions must equal zero. In the Galerkin method

the test functions are the same as the basis functions. This means that the α_k , $k = 1, \dots, K$, must be chosen such that

$$(r^{(K)}, \phi_l) = 0, \quad l = 1, \dots, K. \quad (3.46)$$

Thus, we obtain the linear system

$$\sum_{k=1}^K \alpha_k [(\phi_k, \phi_l) - i\kappa(\mathbf{K}_m \phi_k, \phi_l)] = (j_{\varphi, m}^s, \phi_l), \quad l = 1, \dots, K. \quad (3.47)$$

Note that, with use of Π_K , we can rewrite (3.46) as

$$\Pi_K r^{(K)} = 0, \quad (3.48)$$

or equivalently

$$\Pi_K (\mathbf{I} - i\kappa \mathbf{K}_m) j_{\varphi, m}^{(K)} = \Pi_K j_{\varphi, m}^s. \quad (3.49)$$

In case we would have used test functions different from the basis functions, but also defining a basis of a subspace of W , then the method is called the Petrov-Galerkin method. We apply the Petrov-Galerkin method in Chapter 6. For a more thorough description of the (Petrov-) Galerkin method we refer to [21], [23].

In our model of strips on a cylinder, we first require that every two basis functions on different strips are orthogonal. This is easy to achieve, because the strips are distinct. Namely, if we consider the basis functions to be defined only on the surface of the corresponding strip and to be equal to zero everywhere outside the strip, then every inner product of two basis functions of different strips is equal to zero. Second, every two basis functions on the same strip are also required to be orthogonal. Satisfying these requirements enables us to construct an orthogonal basis. The inner products (ϕ_k, ϕ_l) result in the Gram matrix. For an orthonormal basis, the Gram matrix is equal to the identity matrix. Equation (3.47) can be written as

$$(\mathbf{G} - i\kappa \mathbf{A}) \mathbf{a} = \mathbf{c}, \quad (3.50)$$

where $(k, l = 1, \dots, K)$

$$A_{lk} = (\phi_l, \mathbf{K}_m \phi_k), \quad (3.51)$$

$$G_{lk} = (\phi_l, \phi_k), \quad (3.52)$$

$$c_l = (\phi_l, j_{\varphi, m}^s), \quad (3.53)$$

$$a_k = \alpha_k. \quad (3.54)$$

Note that the vector $\mathbf{c}$ consists of the coefficients of the source current.

3.2.2 Choice of basis functions

To apply the Galerkin method to our model of thin strips on a cylindrical surface, we need to investigate what kind of basis functions are most suitable. An option could be to use local

basis functions, with a bounded support. The matrices will be sparse, but also of large size. Examples of such methods are e.g. the Finite Difference Method (FDM), the Finite Integration Technique (FIT), the Finite Element Method (FEM), the Boundary Element Method (BEM) and the Finite Volume Method (FVM).

For the geometries we use in this thesis, we can also construct global basis functions without any trouble. Global basis functions are defined on the whole domain of a strip, without having limited support. Matrices will be dense, but of small size. An appropriate choice of global basis functions is such that the resulting matrix is diagonally dominant.

For strips on a cylindrical surface, the currents are 2π -periodic in the φ -direction. In correspondence to the examples in Section 3.1, we expand j_φ and j_z into Fourier series, according to

$$j_\varphi(\varphi, z) = j_{\varphi,0}(z) + \sum_{m=1}^{\infty} j_{\varphi,m}^{(c)}(z) \cos m\varphi + \sum_{m=1}^{\infty} j_{\varphi,m}^{(s)}(z) \sin m\varphi, \quad (3.55)$$

$$j_z(\varphi, z) = j_{z,0}(z) + \sum_{m=1}^{\infty} j_{z,m}^{(c)}(z) \cos m\varphi + \sum_{m=1}^{\infty} j_{z,m}^{(s)}(z) \sin m\varphi. \quad (3.56)$$

For a patch on a cylindrical surface, bounded at $\varphi = \varphi_0, \varphi_1$ and $z = z_0, z_1$, we use (cf. the plane rectangular patch, discussed in Subsection 3.1.3)

$$j_\varphi(\varphi, z) = \sum_{m=1}^{\infty} j_{\varphi,m}(z) \sin\left(\frac{m\pi(\varphi - \varphi_0)}{\varphi_1 - \varphi_0}\right), \quad (3.57)$$

$$j_z(\varphi, z) = \sum_{m=1}^{\infty} j_{z,m}(z) \cos\left(\frac{m\pi(\varphi - \varphi_0)}{\varphi_1 - \varphi_0}\right). \quad (3.58)$$

Note that the modes $j_{\varphi,m}(z)$ and $j_{z,m}(z)$ must be chosen such that $\nabla \cdot \mathbf{j} = 0$ is satisfied.

The basis functions in the z -direction are not so straightforward to choose. For that, the geometry of the strips is important. The strips have a finite width, so in the width direction polynomial expansions could be opportune. Both Legendre polynomials and Chebyshev polynomials are commonly used in this case; see [9, p.11]. Very relevant here is the logarithmically singular behavior of the kernel function in the integral equation, as we found in Section 3.1. This logarithmically singular behavior occurs in the coordinate perpendicular to the preferable direction of the current, which is in most cases the coordinate representing the width direction of the strip. We will argue that the Legendre polynomials are a very appropriate choice for the basis functions in this direction.

For the m -th mode of the φ -component of the current density, denoted by $j_{\varphi,m}(z)$, an integral equation corresponding to (3.38) is derived, in which a logarithmically singular kernel function $\log|z - \zeta|$ is present. Expanding $j_{\varphi,m}(z)$ in terms of scaled and shifted Legendre

polynomials in the z -direction, we obtain

$$j_{\varphi,m}(z) = \sum_{k=1}^{\infty} \alpha_{km} P_k\left(\frac{z - c_z}{d_z}\right), \quad (3.59)$$

where

$$c_z = \frac{z_1 + z_0}{2}, \quad d_z = \frac{z_1 - z_0}{2}, \quad (3.60)$$

such that the interval $[z_0, z_1]$ is scaled to $[-1, 1]$. This brings us to an important result: We use Legendre polynomials because of the analytical relation

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 P_k(z) P_{k'}(\zeta) \log |z - \zeta| d\zeta dz \\ &= \begin{cases} \frac{8}{(k+k')(k+k'+2)[(k-k')^2 - 1]}, & \text{if } k+k' > 0 \text{ even,} \\ 0, & \text{if } k+k' \text{ odd,} \\ 4 \log 2 - 6, & \text{if } k=k'=0. \end{cases} \end{aligned} \quad (3.61)$$

For the derivation of this relation, we refer to Appendix A. The φ -component of the current in the patch is the sum of all modes in the φ -direction, so

$$j_{\varphi}(\varphi, z) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{km} P_k\left(\frac{z - c_z}{d_z}\right) \sin\left(\frac{m\pi(\varphi - \varphi_0)}{\varphi_1 - \varphi_0}\right). \quad (3.62)$$

The total z -component of the current, $j_z(\varphi, z)$, must satisfy the boundary conditions at the edges, $j_z(\varphi, z_0) = j_z(\varphi, z_1) = 0$. If we introduce, for $z \in [-1, 1]$,

$$Z_k(z) = \int_{-1}^z P_k(\zeta) d\zeta = \frac{1}{2k+1} [P_{k+1}(z) - P_{k-1}(z)], \quad k \geq 1, \quad (3.63)$$

then $Z'_k = P_k$ and $Z_k(\pm 1) = 0$. Moreover, the expansion for the current is term by term free of divergence, if we write $j_z(\varphi, z)$ as

$$j_z(\varphi, z) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{km} \frac{m\pi d_z}{\varphi_1 - \varphi_0} Z_k\left(\frac{z - c_z}{d_z}\right) \cos\left(\frac{m\pi(\varphi - \varphi_0)}{\varphi_1 - \varphi_0}\right). \quad (3.64)$$

The constants α_{km} are determined from the integral equation for $j_{\varphi}(\varphi, z)$ by applying the adapted Galerkin method.

3.3 A purely logarithmic kernel function

In Section 3.1, we have shown how for each of the three standard types of strips the excited current can be written as an expansion of modes. For each of the modes an integral equation

is derived, containing a logarithmic kernel. In this section, we investigate an integral equation for one variable in which the kernel function is purely logarithmic. We consider an integral equation of the second kind, as in (3.35), for the function $j(z)$ defined on the normalized interval $[-1, 1]$. In accordance with (3.35), the right-hand side is given by a function $j^s(z)$, representing a source. We write

$$j(z) + \frac{i\kappa}{2\pi} \int_{-1}^1 j(\zeta) \log |z - \zeta| d\zeta = j^s(z), \quad \text{for } z \in [-1, 1]. \quad (3.65)$$

Here, $\kappa > 0$ is a system parameter, as in (3.35).

In operator form, (3.65) can be written as

$$(I - i\kappa K)j = j^s, \quad (3.66)$$

where K denotes the integral operator on $L_2[-1, 1]$ defined by

$$(Kj)(z) = -\frac{1}{2\pi} \int_{-1}^1 j(\zeta) \log |z - \zeta| d\zeta, \quad z \in [-1, 1]. \quad (3.67)$$

The weakly singular kernel $\log |z - \zeta|$, with $-1 < z, \zeta < 1$, is square integrable and symmetric. Hence, K is a symmetric Hilbert-Schmidt operator on $L_2[-1, 1]$ and (3.65) has a unique solution for each right-hand side j^s . The equation (3.65) is a Fredholm integral equation of the second kind (see [3], [23]).

Since K is a Hilbert-Schmidt operator on $L_2[-1, 1]$ there exist countably many eigenvalues λ_n with corresponding normalized eigenfunctions u_n . Since K is symmetric the eigenfunctions u_n are orthogonal. There is a total orthonormal set of eigenfunctions, which means that for each f ,

$$f = \sum_{n=0}^{\infty} (f, u_n) u_n, \quad (3.68)$$

and

$$Kf = \sum_{n=0}^{\infty} \lambda_n (f, u_n) u_n. \quad (3.69)$$

Therefore, instead of integral equation (3.66), we can write

$$\sum_{n=0}^{\infty} (j, u_n) u_n - i\kappa \sum_{n=0}^{\infty} \lambda_n (j, u_n) u_n = \sum_{n=0}^{\infty} (j^s, u_n) u_n. \quad (3.70)$$

Thus, we find the explicit solution

$$j = \sum_{n=0}^{\infty} \frac{1}{1 - i\kappa \lambda_n} (j^s, u_n) u_n. \quad (3.71)$$

The problem is that the eigenfunctions u_n cannot be calculated analytically. We need a numerical approximation. Since K is a Hilbert-Schmidt operator, we may replace K by a finite rank operator $K^{(N)}$, such that (see [22])

$$\|K - K^{(N)}\| < \epsilon, \quad (3.72)$$

for $\epsilon \ll 1$. Operator theory tells us that the eigenvalues $\lambda_1^{(N)}, \dots, \lambda_N^{(N)}$ of $K^{(N)}$ satisfy

$$|\lambda_n - \lambda_n^{(N)}| = \mathcal{O}(\epsilon), \quad (3.73)$$

for $n = 1, \dots, N$. The eigenvalues $\lambda_n^{(N)}$ as well as the corresponding eigenfunctions $u_n^{(N)}$ can be calculated numerically. Thus, we obtain a numerical approximation $j^{(N)}$ of the true j , which reads

$$j^{(N)} = \sum_{n=0}^N \frac{1}{1 - i\kappa\lambda_n^{(N)}} (j^s, u_n^{(N)}) u_n^{(N)}. \quad (3.74)$$

The eigenvalues of K satisfy $\sum_{n=0}^{\infty} |\lambda_n|^2 < \infty$. In [50], Reade has proved that the operator with kernel function $\log |z - \zeta|$ defined on $[-1, 1]$ has negative eigenvalues μ_n , which are bounded by

$$-\pi \log 2 \leq \mu_0 \leq -\frac{\pi}{4}, \quad -\frac{\pi}{n} \leq \mu_n \leq -\frac{\pi}{4(n+1)}, \quad n \geq 1. \quad (3.75)$$

Therefore, the operator K in (3.67) is positive definite and its eigenvalues λ_n are bounded by

$$\frac{1}{8} \leq \lambda_0 \leq \frac{1}{2} \log 2, \quad \frac{1}{8(n+1)} \leq \lambda_n \leq \frac{1}{2n}, \quad n \geq 1. \quad (3.76)$$

This shows that the eigenvalues λ_n are of the order $1/n$.

In numerical practise, the method to solve the integral equation (3.65) by replacing the operator K by a finite rank operator $K^{(N)}$ is called the Galerkin method. Here, we take the source function uniform on the interval $[-1, 1]$, i.e. $j^s(z) = 1$. According to the analysis in [76] and [77], we obtain a solution $j(z)$ that is even when $j^s(z)$ is even. Moreover, after substituting $j^s(z) = 1$, we obtain

$$-\frac{i\kappa}{2\pi} PV \int_{-1}^1 \frac{j(\zeta)}{z - \zeta} d\zeta = j'(z), \quad z \in [-1, 1]. \quad (3.77)$$

In order to determine $j(z)$, we expand $j(z)$ in terms of Legendre polynomials,

$$j(z) = \sum_{k=0}^{\infty} \alpha_k P_{2k}(z), \quad (3.78)$$

where we used that $j(z)$ is even. Any solution of (3.77) can be multiplied by an arbitrary constant. Therefore, we fix $\alpha_0 = 1$ to obtain a solution $j_0(z)$. To match the condition $j^s(z) = 1$, we determine the constant C , such that $j(z) = Cj_0(z)$ satisfies the integral equation (3.65) with $j^s(z) = 1$.

Furthermore, using the Legendre functions of the second kind $Q_k(z)$, defined by (see e.g. [61, Eq.(8.11.1)])

$$Q_k(z) = \frac{1}{2} PV \int_{-1}^1 \frac{P_k(\zeta)}{z - \zeta} d\zeta, \quad z \in [-1, 1], \quad (3.79)$$

where $k \geq 0$ and substituting (3.78) and (3.79) into (3.77), we obtain the relations

$$-\frac{i\kappa}{\pi} \sum_{k=0}^{\infty} \alpha_k Q_{2k}(z) = \sum_{k=0}^{\infty} \alpha_k P'_{2k}(z), \quad (3.80)$$

from which the coefficients α_k , $k \geq 1$, have to be determined. Taking the inner products with the Legendre polynomials $P_{2l-1}(z)$ both on the left- and right-hand side of (3.80), we arrive at

$$\frac{2\pi i}{\kappa} \sum_{k=l}^{\infty} \alpha_k - \sum_{k=1}^{\infty} A_{lk} \alpha_k = \frac{1}{l(2l-1)}, \quad (3.81)$$

where

$$A_{lk} = \frac{1}{(2l-2k-1)(k+l)}, \quad k, l = 1, 2, \dots \quad (3.82)$$

In the derivation we used (see [2, Eq. (8.14.8)])

$$\int_{-1}^1 Q_{2k}(z) P_{2l-1}(z) dz = \frac{1}{(2l-2k-1)(k+l)}, \quad (3.83)$$

and

$$\int_{-1}^1 P'_{2k}(z) P_{2l-1}(z) dz = \begin{cases} 2, & k \geq l, \\ 0, & k < l, \end{cases} \quad (3.84)$$

We conclude that the solution for $j(z)$ on the interval $[-1, 1]$ has the form (3.78), where the coefficients α_k should be determined from the infinite system (3.81).

In the next section, we give an example in which the integral equation (3.65) appears and show how we deal numerically with the system (3.81). This example is concerned with the artificial case of infinitely long plane rectangular strips.

3.4 Special case: Plane rectangular strips

In this section, we discuss a special model that yields an integral operator with a purely logarithmic kernel. We consider a set of N ($N \geq 1$) parallel conducting strips of narrow rectangular cross-section and of infinite length. This is of course physically not feasible, but it forms the fundament of all mathematical modeling done in this thesis. We apply an affine transformation such that in width direction each strip is presented by the interval $[-1, 1]$. We show that the current in a strip is described by (3.65) and that the solution procedure presented in the previous section applies.

The current distribution in a set of parallel conducting strips has been investigated in [76] and [77]. We briefly recapitulate the methods from these two works, and present results that are of interest for the models specified in the following three chapters. One important aspect captured by the plane strip model is that it describes to a great extent the behavior of the currents in an arbitrary set of narrow conductors on a cylindrical surface. The local

interaction of two current carriers is well described by the interaction between two plane rectangular strips.

3.4.1 Model formulation

We consider a set of N parallel strips, which are flat, lying in one plane. The longitudinal direction of the strips is taken in the x -direction, while the z -axis is the width direction and the y -axis is the thickness direction; see Figure 3.3. All strips have thickness h , but their widths may differ: the n^{th} strip, $n \in \{1, \dots, N\}$, has width $z_1^{(n)} - z_0^{(n)}$. The thickness of the strips is always much smaller than their width. As before, the thickness of the strips is neglected, because the penetration depth is larger than half the thickness (see Section 2.3).

Figure 3.3: Configuration of a system of N infinitely long parallel sheets; cross-section in y - z -plane.

Through the strips an electric current is flowing in longitudinal direction. Strips are divided into groups and for each group the total current is prescribed. We search for a representation of the current distribution in the strips. Since we use a quasi-static approach, the current is uniform in the x -direction. The approximation to infinitely thin and infinitely long strips simplifies the boundary conditions considerably, in so far that we now have to consider conditions in the plane $y = 0$ only. At $y = 0$, jumps occur in $\mathbf{H}$ and $\mathbf{E}$ across the surface $S_\cup$, the joint surface of the strips, having its normal vector in the positive y -direction ($\mathbf{n} = \mathbf{e}_y$). The system of equations and boundary conditions for a set of infinitely long strips allows a solution that is independent of the longitudinal direction. As a consequence, the current $\mathbf{j} = h\mathbf{J}$ has a component in the x -direction depending on z only, i.e.

$$\mathbf{j} = j(z)\mathbf{e}_x, \quad z \in [z_0^{(n)}, z_1^{(n)}], \quad n = 1 \dots N. \quad (3.85)$$

Instead of $S_\cup$, which represents a surface, we introduce the one-dimensional set $S_\cup^z$ as the union of all intervals, denoting the positions of the strips,

$$S_\cup^z = \sum_{n=1}^N [z_0^{(n)}, z_1^{(n)}]. \quad (3.86)$$

As a direct consequence of (3.85), the only non-zero components of $\mathbf{E}$ and $\mathbf{H}$ are E_x , H_y and H_z . From now on, we restrict ourselves to the upper half plane $\{(y, z) \mid y > 0\}$, where the following equations hold for $E_x = E(y, z)$, $H_y = H_y(y, z)$, $H_z = H_z(y, z)$:

$$\frac{\partial E}{\partial z} = i\mu_0\omega H_y, \quad \frac{\partial E}{\partial y} = -i\mu_0\omega H_z, \quad (3.87)$$

$$\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} = 0, \quad (3.88)$$

together with the boundary conditions at $y = 0$,

for $z \in S_{\mathbb{U}}^z$

$$E(0, z) = \frac{1}{\sigma h} j(z) - \frac{1}{\sigma h} j^s(z), \quad H_z(0, z) = \frac{1}{2} j(z), \quad (3.89)$$

for $z \notin S_{\mathbb{U}}^z$

$$\frac{\partial E}{\partial y}(0, z) = 0, \quad H_z(0, z) = 0. \quad (3.90)$$

These conditions are complemented by the far field condition, stating that

$$H_y, H_z \rightarrow 0, \quad \text{for } \sqrt{y^2 + z^2} \rightarrow \infty. \quad (3.91)$$

From the relations (3.87) and (3.88), it follows that the field components H_y , H_z and E all satisfy the Laplace equation in the half-space $y > 0$. For H_z this leads to the following boundary value problem

$$\begin{aligned} \Delta H_z(y, z) &= 0, \quad \text{on } \{(y, z) \mid y > 0\}, \\ H_z(0, z) &= \frac{1}{2} j(z), \quad z \in S_{\mathbb{U}}^z, \\ H_z(0, z) &= 0, \quad z \notin S_{\mathbb{U}}^z, \\ H_z(x, z) &\rightarrow 0, \quad \sqrt{y^2 + z^2} \rightarrow \infty. \end{aligned} \quad (3.92)$$

The solution of this problem has the integral representation

$$H_z = \frac{1}{2\pi} \int_{S_{\mathbb{U}}^z} j(\zeta) \frac{y}{y^2 + (z - \zeta)^2} d\zeta. \quad (3.93)$$

This solution was found using Fourier transformation with respect to the variable z , see [76]. From this result, we obtain the following expression for the y -component of the magnetic field:

$$H_y = -\frac{1}{2\pi} \int_{S_{\mathbb{U}}^z} j(\zeta) \frac{z - \zeta}{y^2 + (z - \zeta)^2} d\zeta. \quad (3.94)$$

We introduce the vector potential $\mathbf{A}$ as defined by (2.30). By (3.93) and (3.94), $\mathbf{A} = A_x \mathbf{e}_x$, where $A_x = A(y, z)$ is given by

$$A(y, z) = -\frac{1}{4\pi} \int_{S_{\mathbb{U}}^z} j(\zeta) \log((z - \zeta)^2 + y^2) d\zeta. \quad (3.95)$$

The source current is represented by a function $j^s(z)$ that is uniform on each strip and zero outside the strips ($z \notin S_\cup^z$). In Section 2.3, the detailed description of the source current is given. Moreover, we assume that the strips are coupled in groups of one or more strips, where each group is connected to one source. This means that for each group, $j^s(z)$ has the same value on each strip of this group. For the notation, we assume that we have L different groups consisting of one or more strips, with total partial surface S_l^z , $l = 1, \dots, L$, and such that $S_\cup^z = \sum_{l=1}^L S_l^z$. On each group, we express $j^s(z)$ by the characteristic function $\psi_l(z) = \mathbf{1}_{[S_l^z]}$, being zero everywhere except on the strips of group S_l^z , where it has the value one. Denoting the uniform value of $j^s(z)$ on S_l^z by C_l , we write

$$j^s(z) = \sum_{l=1}^L C_l \psi_l(z). \quad (3.96)$$

The inner product of two characteristic functions ψ_l and $\psi_{l'}$, where $l, l' \in \{1, \dots, L\}$, satisfies

$$(\psi_l, \psi_{l'}) = \int_{-\infty}^{\infty} \psi_l(z) \psi_{l'}(z) dz = D_l \delta_{ll'}, \quad (3.97)$$

where $\delta_{ll'}$ is the Kronecker delta function, which is equal to one if $l = l'$ and zero if $l \neq l'$, and D_l is the sum of the widths of all strips of group l .

We still need L extra relations to determine the unknowns $C_1, \dots, C_L$. We demand that the total current in each group of strips is prescribed by

$$\int_{S_l^z} j(z) dz = \int_{S_l^z} j(z) \psi_l(z) dz = I_l, \quad \text{for } l = 1, \dots, L. \quad (3.98)$$

Introducing a typical length scale D^c and a typical current density j^c , we obtain (3.98) in a dimensionless form

$$\int_{S_l^z} j(z) \psi_l(z) dz = \frac{I_l}{D^c j^c}. \quad (3.99)$$

The dimensionless form of (3.97) is $(\psi_l, \psi_{l'}) = (D_l/D^c) \delta_{ll'}$.

Substituting (3.95) into Ohm's law (2.97), we arrive at the integral equation for $j(z)$,

$$j(z) + \frac{i\kappa}{2\pi} \int_{S_\cup^z} j(\zeta) \log|z - \zeta| d\zeta = \sum_{l=1}^L c_l \psi_l(z), \quad \text{for } z \in S_\cup^z. \quad (3.100)$$

where

$$\kappa = h\sigma\mu_0\omega D^c, \quad c_l = \frac{C_l}{j^c}, \quad j^c = \frac{\sum_{l=1}^L I_l}{\sum_{l=1}^L D_l}, \quad (3.101)$$

such that j^c is the average current over all strips. This integral equation is a Fredholm equation of the second kind with the logarithmic function $\log|z - \zeta|$ as kernel.

The procedure to solve (3.100) for N strips is similar to the procedure to solve the corresponding equation for a geometry of N rings; cf. Chapter 4. Here, we present the analysis

to determine the magnetic field and the resistance for one strip. In Section 3.3, the way to approximate the current in one strip by a series of Legendre polynomials has been presented. Choosing length scale D^c equal to half the width of the strip yields automatically the domain definition $z \in [-1, 1]$ and the uniqueness condition from (3.99)

$$\int_{-1}^1 j(z) dz = 2. \quad (3.102)$$

Using the result of (3.78), with known coefficients α_k , we can derive expressions for the magnetic field and the resistance. In (3.93) and (3.94), the external magnetic field components H_z and H_y are represented as integrals on $j(z)$. To evaluate these integral representations, we write

$$\frac{1}{\pi} \frac{y}{y^2 + (z - \zeta)^2} = \frac{1}{2\pi i} \left(\frac{1}{\zeta - \psi} - \frac{1}{\zeta - \psi^*} \right), \quad (3.103)$$

and

$$-\frac{1}{\pi} \frac{z - \zeta}{y^2 + (z - \zeta)^2} = \frac{1}{2\pi} \left(\frac{1}{\zeta - \psi} + \frac{1}{\zeta - \psi^*} \right), \quad (3.104)$$

where $\psi = z + iy$. Next, we define the linear mapping (Hilbert transformation; see e.g. [44, Sect.4.1], [70, Ch.5])

$$\mathbb{H}j(\psi) = \frac{1}{\pi} PV \int_{-1}^1 \frac{j(\zeta)}{\zeta - \psi} d\zeta, \quad \psi \in \mathbb{C}. \quad (3.105)$$

Then, the components of the magnetic field are presented in terms of $\mathbb{H}j$ by

$$H_z = \frac{1}{4i} ((\mathbb{H}j)(\psi) - (\mathbb{H}j)(\psi^*)) \quad (3.106)$$

and

$$H_y = \frac{1}{4} ((\mathbb{H}j)(\psi) + (\mathbb{H}j)(\psi^*)). \quad (3.107)$$

From the expression (3.78) for the current, we obtain

$$\mathbb{H}j(\psi) = \sum_{k=0}^{\infty} \frac{\alpha_k}{\pi} PV \int_{-1}^1 \frac{P_{2k}(\zeta)}{\zeta - \psi} d\zeta = -\frac{2}{\pi} \sum_{k=0}^{\infty} \alpha_k Q_{2k}(\psi), \quad (3.108)$$

where we used (3.79). Note that the functions $Q_{2k}(\psi)$ are analytic on $\mathbb{C} \setminus [-1, 1]$; so the mappings

$$(y, z) \mapsto Q_{2k}(z + iy) \quad \text{and} \quad (y, z) \mapsto Q_{2k}(z - iy) \quad (3.109)$$

are harmonic. Moreover, across the interval $[-1, 1]$, there is the jump relation (see [81, Sect.5.19])

$$\frac{1}{\pi i} [Q_k(z - i0) - Q_k(z + i0)] = P_k(z), \quad z \in [-1, 1], \quad (3.110)$$

and finally

$$\lim_{|\psi| \rightarrow \infty} Q_k(\psi) = 0. \quad (3.111)$$

Hence, the components of the magnetic field can be written as

$$H_z(y, z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \alpha_k [Q_{2k}(z - iy) - Q_{2k}(z + iy)], \quad (3.112)$$

and

$$H_y(y, z) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \alpha_k [Q_{2k}(z - iy) + Q_{2k}(z + iy)]. \quad (3.113)$$

An important quantity of practical interest is the energy loss due to the currents. Therefore, also the dissipated power will be expressed in terms of the coefficients α_k . In accordance with (2.60), the time-averaged dissipated power is given by

$$\bar{P}_{diss} = \frac{1}{2\sigma} \int_V \mathbf{J} \cdot \mathbf{J}^* dv. \quad (3.114)$$

For the infinitely long and thin strip we consider here, the dissipated power per unit of length $\bar{P}_{diss}^{(L)}$ (in Watt/m) is found by replacing the volume integral by one over the cross-sectional surface and, moreover, by using

$$\mathbf{J}(z) = \frac{j^c}{h} j(z) \mathbf{e}_x, \quad (3.115)$$

where $j(z)$ is the dimensionless current density. This yields

$$\bar{P}_{diss}^{(L)} = \frac{|j^c|^2 D^c}{2h\sigma} \int_{-1}^1 |j(z)|^2 dz. \quad (3.116)$$

Moreover, the power dissipated per unit length of the conductor can be expressed in terms of the total effective current I_e , see (2.65), and the resistance $\bar{R}^{(L)}$ per unit of length ($\bar{R}^{(L)}$ in Ohm/m), as

$$\bar{P}_{diss}^{(L)} = \bar{R}^{(L)} I_e^2 = \bar{R}^{(L)} \frac{(|j^c| D^c)^2}{2} \left| \int_{-1}^1 j(z) dz \right|^2 = 2\bar{R}^{(L)} (|j^c| D^c)^2, \quad (3.117)$$

where (3.102) is used. Combining (3.116) and (3.117), we get the following expression for the resistance

$$\bar{R}^{(L)} = \frac{1}{4D^c h\sigma} \int_{-1}^1 |j(z)|^2 dz. \quad (3.118)$$

Using the expansion (3.78) and orthogonality of Legendre polynomials, we arrive at

$$\bar{R}^{(L)} = \frac{1}{2D^c h\sigma} \sum_{k=0}^{\infty} \frac{|\alpha_k|^2}{4k+1}. \quad (3.119)$$

3.4.2 Display of results

In this section, we explain how our results are displayed in this thesis. The components of the surface current density $\mathbf{j}(\mathbf{x})$ are complex-valued functions and a complex-valued function can be presented in different ways. We here present results for the model of plane rectangular strips. Although this model is physically not feasible, it provides insight in the behavior of the currents due to induction effects. Moreover, it gives first approximations for other models, such as the ring model, as will be discussed in the next chapter.

We consider one strip of 4 cm width, positioned with its center at $z = 0$. Its thickness is 2.5 mm and the total amplitude of the current flowing through the cross-section of the strip is 600 A. The total current is expanded in Legendre polynomials, as in (3.78), up to an accuracy of 99 %, and the coefficients are calculated from (3.81). The length scale D^c is half the width of the strip, i.e. $D^c = 2$ cm.

Figure 3.4: One plane strip of 4 cm width with a source current of 600 A at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz (Δ). (a) Amplitude of the current distribution; (b) Phase-lag with respect to the source current.

Considering the current distribution, we believe that a presentation of the real and imaginary part is not very informative. On the other hand, the amplitude of the current density, $|j(z)|$, and the phase-lag with respect to the source, $\Delta\phi(z)$, contain all the information about how the current is distributed in the strips. Assuming that the harmonic source current has its maximum at $t = 0$, we write the time dependent surface current density as $j(z, t) = \Re\{j(z)e^{-i\omega t}\} = |j(z)| \cos(\omega t - \Delta\phi(z))$.

Figure 3.4 (a) shows the amplitude of the current distribution for four frequencies, $f = 100, 400, 700, 1000$ Hz. The current at the edges is higher than in the center. This *edge-effect*, which is due to the self-inductance of the strip, becomes stronger as the frequency

increases. For low frequencies, the current is distributed almost uniformly, whereas for high frequencies, the current is concentrated at the edges. The average current is equal to $1.5 \cdot 10^4$ A/m, which is the value for $j^c = I/(2D^c)$.

Besides the edge-effect, we observe a phase difference in the system. In Figure 3.4 (b), the phase-lag with respect to the source current is shown as function of the position, for the four frequencies $f = 100, 400, 700, 1000$ Hz. For each frequency only two points are in phase with the source. The currents near the edges are ahead in phase with respect to the source (i.e. negative phase-lag), whereas currents at the center are behind in phase. The phase difference varies with frequency.

The edge-effect and phase-lag of the current cause a distortion in the magnetic field. The magnitude of the field changes and the time behavior does not correspond to the one of the desired field. The quality of the images in MRI is very much susceptible to these effects. So, it is important for the design of a gradient coil to investigate to what extent a change in the current distribution is acceptable.

We note that the solution is linear with the current, which means that if we increase the total current I , then the current densities are increased proportionally. The shapes of the distributions in Figures 3.4 (a) and (b) remain the same, but scales are changing.

Figure 3.5: One strip of 4 cm width with a direct total current of 600 A. (a) Contour lines of $B_z(y, z)$; (b) Contour lines of $B_y(y, z)$.

Both $\mathbf{A}(y, z, t)$ and $\mathbf{B}(y, z, t)$ are vector fields in free space. One way of presenting vector fields is by use of contour lines. Contour lines connect the points in which the field has the same magnitude. However, the fields are changing in time and thus contour plots can only be shown for one instant. In this thesis, we show contour plots for $t = 0$ only, unless explicitly

stated otherwise. Also, if we want to consider a field component on a straight line, we choose to present the real part of the component at $t = 0$. For an example we refer to the next chapter, where rings on a cylinder are considered. There, the gradient of the magnetic field is plotted on the axis of symmetry of the cylinder.

In this example of one plane strip, we use (3.112) for $H_z(y, z)$ and (3.113) for $H_y(y, z)$. Multiplying these expressions by $\mu_0 j^c$, we obtain the dimensional representations of $B_z(y, z)$ and $B_y(y, z)$. The contour lines of $B_z(y, z)$ and $B_y(y, z)$ are shown in the Figures 3.5 (a) and (b), respectively.

An important aspect of the present research is the frequency-dependence of the solution. In order to describe this dependence, we investigate the behavior of the characteristic quantities $\bar{R}$ and $\bar{L}$. These characteristic quantities are essentially related to the current distribution in the strips.

Figure 3.6: One plane strip of 4 cm width with a total current of 600 A. (a) Resistance per unit of length $\bar{R}^{(L)}$ at frequencies up to $f = 10000$ Hz; (b) Resistance per unit of length $\bar{R}^{(L)}$ at frequencies up to $f = 400$ Hz.

For the plane strip we present the resistance per unit of length, computed according to (3.119), for different frequencies. In Figure 3.6 (a), $\bar{R}^{(L)}$ is shown for the frequency domain $[0, 10000]$ Hz. We observe that $\bar{R}^{(L)}$ increases with the frequency. The physical interpretation is that due to the stronger edge-effects the current flows through a narrower part of the strip, leading to a higher resistance. The resistance per unit of length at $f = 0$ Hz is verified analytically by taking it equal to one over the conductivity times the cross-sectional surface, that is

$$\bar{R}_{DC}^{(L)} = \frac{1}{2hD^c\sigma} \approx 1.7 \cdot 10^{-4} \Omega/\text{m}. \quad (3.120)$$

This result corresponds to a DC current, which is distributed uniformly, such that all coefficients $\alpha_k, k \geq 1$ are equal to zero and $\alpha_0 = 1$ in (3.119). In Figure 3.6 (b), the resistance per unit of length is shown for the smaller range $[0, 400]$ Hz, with a higher resolution than

used in the range $[0, 10000]$ Hz. We observe different behaviors for small ($f < f_{char}$) and large ($f > f_{char}$) frequencies: for $f < f_{char}$, the slope of $\bar{R}^{(L)}$ is increasing, whereas for $f > f_{char}$, this slope is decreasing. Hence, f_{char} represents a point of inflection. The value of the characteristic frequency f_{char} is about 170 Hz. In Chapter 4, we discuss appearance of this characteristic frequency in more detail and show how it can be calculated.

An interesting aspect for our further analysis is the spectrum of eigenvalues of the operator K from (3.67). The eigenvalues are computed using the matrix $\mathbf{A}$ given in (3.81). Figure 3.7 (a) shows the eigenvalues of the operator, λ_n , together with its lower and upper bound according to (3.76). We observe that the eigenvalues behave as $1/n$ for greater n . In Figure 3.7 (b), all values are multiplied by $n + 1$, such that we see how accurate the lower and upper bounds are.

Figure 3.7: (a) The eigenvalues of the operator, λ_n (+), with lower bound $1/(8(n+1))$ (○), for $n \geq 0$, and upper bound $1/(2n)$ (*), for $n \geq 1$ and $\log(2)/2$ for $n = 0$; (b) All values multiplied by $n + 1$.

In Figure 3.4 (a), we have seen how edge-effects appear in a time-varying system. To avoid misunderstandings: this is not a matter of repulsion of currents. To explain this, we consider the analogue of two wires carrying currents in the same direction. These currents generate an attracting Lorentz force. On the other hand, opposite currents in the two wires generate a repulsing force. In contrast, the edge-effects within a strip are due to the changing magnetic flux. The changing magnetic flux causes an electromotive force that generates an eddy current whose direction is such as to set up a magnetic flux opposing the change. At the center of the strip the effect of induction is stronger than near the edges of the strip. Therefore, at the center, changing magnetic flux is higher, eddy current density is higher and total current density is lower.

As a second example, we consider a system of two parallel plane strips of infinite length carrying a total current of 600 A. The currents in the two strips are in phase. The widths

Figure 3.8: Two plane strips of 4 cm width with total current 600 A, at frequencies $f = 100$ Hz (*), $f = 400$ Hz (○), $f = 700$ Hz (+), $f = 1000$ Hz (Δ). (a) Amplitude of the current distribution for sources in phase; (b) Phase-lag for sources in phase; (c) Amplitude of the current distribution for sources in anti-phase; (d) phase-lag for sources in anti-phase.

of the strips are 4 cm and the distance between the central lines of the strips is 8 cm. The results for the amplitude of the current density and the phase-lag are presented in Figures 3.8 (a) and (b), respectively. We observe edge-effects within the strips, as well as a global edge-effect over the system of strips. This global edge-effect is explained in the same way as the edge-effect within the strip; points at the edges experience a lower changing magnetic flux.

In case the currents in the two parallel strips are in anti-phase, the results are different; see Figures 3.8 (c) and (d). Here, the currents are more concentrated at the center of the system. This is explained as follows: the changing magnetic flux created by the current in the first strip generates eddy currents in the second strip that oppose this flux. The direction of the induced eddy currents is the same as the direction of the current applied to the second strip. Therefore, the induced eddy currents amplify the total current density. Currents closer to the first strip are amplified more than currents further away.

We conclude that Legendre polynomials are especially suited for use in the Galerkin method, when applied to the Fredholm integral equation of the second kind with logarithmic singular kernel (3.100). With the choice of Legendre polynomials we found basis functions with the following properties:

- Complete
- Rapid convergence
- Easy to compute

Complete refers to an accurate expression in the form of a series with respect to the basis functions. *Rapid convergence* refers to the number of basis functions needed to approximate the current distribution to within a certain tolerance. In the examples of this section, Legendre polynomials up to eighth order have been used to obtain an accuracy of one percent. *Easy to compute* refers to avoiding tedious numerical integration where possible and the use of the analytical formula (3.61).

3.5 Summary

In this chapter, we presented the mathematical analysis for the integral equation of the second kind obtained in Chapter 2. The integral equation results from the model for a current distribution on a cylindrical surface. The kernel function of the integral is singular in one point. This singular behaviour inspired us to develop a strategy, partly based on applied analysis and partly on numerical analysis.

The model analysis predicts that the induced eddy currents in the strips prefer to flow in the direction opposite to the applied source current, because the eddy currents tend to oppose the magnetic field caused by the source current. We consider the leading integral equation in terms of the component of the current in this preferable direction. The essential behaviour of

the kernel in this leading integral equation is logarithmic in the coordinate perpendicular to the preferable direction.

To solve the integral equation, the Galerkin method with global basis functions is applied to approximate the current distribution. In the angular direction of the cylinder we choose trigonometric functions to express periodicity. In width direction of the strips, we choose Legendre polynomials. With this choice we found basis functions that are complete, converge rapidly and are easy to compute.

From the special case of plane rectangular strips as described in Section 3.4, we conclude that Legendre polynomials are specially suited for use in the Galerkin method, when applied to the Fredholm integral equation of the second kind with logarithmic singular kernel.

Circular loops of strips

The z-coil consists of a long strip of copper wound around a cylinder. The width of the strip is a few centimeters and its thickness a few millimeters. The distance between two successive loops varies from a few millimeters to several centimeters. To model the z-coil, we consider a finite number, N , of circular loops of strips, which are parallel and have the same central axis. Hence, the ring-model is a specific example of the general model described in Chapter 2. In this chapter, we show how the current distribution is affected by parameters such as frequency of the excitation source, width of the rings, distances between the rings, and radius of the rings. We investigate the magnetic field, its gradient, the resistance and self-inductance of the rings, all depending on the frequency. We show the resemblance with the special case of plane rectangular strips, as discussed in Section 3.4.

In Section 4.1, we formulate the problem for the configuration of a set of rings, adjust the integral equations from the general model in Section 2.3, using the properties of the ring model, and investigate the behavior of the kernel function in the obtained integral equation for this specific case. The Galerkin method, as described in Section 3.2, is applied to approximate the current distribution. Section 4.2 contains the explanation of the implementation of this method. This method results in a set of linear equations. Numerical results are shown in Section 4.3. In Section 4.4, we present the summary of the present chapter.

4.1 Model formulation

4.1.1 Configuration

We consider a set of N ($N \geq 1$) coaxial circular loops of strips, or rings, as depicted in Figure 4.1. The geometry is described by cylindrical coordinates (r, φ, z) , where the z -axis coincides with the central axis of the rings. The radius of each ring is R , and each ring has its own uniform width. To keep the geometry as general as possible, we assume that not all the rings have the same width, and that the mutual distances are different. The thickness is

Figure 4.1: *The configuration of a set of parallel rings.*

assumed the same for all rings (i.e. h).

In Section 2.2, we have explained how strips on a cylinder can be categorized in three types. The configuration of the case under consideration is a specific example of the second type, i.e. the strip that forms a closed loop around the cylinder. The rings are mutually parallel and perpendicular to the central axis. Therefore, the configuration is axi-symmetric. For closed loops of strips we cannot apply a source that supplies a voltage difference. Instead, we model a primary current, which is injected in one point, as described in the general model in Section 2.3. The conductivity of the copper rings enables the current to flow through the rings. The current distribution that arises as such, i.e. without any inductive effects, is called the source current $\mathbf{J}^s$. Due to the axi-symmetric configuration the source current is distributed uniformly over the width of the rings, and has only a tangential component.

Through the rings, a time-harmonic current at angular frequency ω is flowing, with current distribution $\mathbf{J}(r, \varphi, z, t) = \Re\{\mathbf{J}(r, \varphi, z)e^{-i\omega t}\}$. The frequency is assumed to be low, i.e. $f = \omega/2\pi < 10^4$ Hz. The cross-section of a ring is a narrow rectangle, meaning that the thickness is small compared to the width. But more relevant for us is that the ratio of half the thickness of the ring, $h/2$, with respect to the penetration depth, δ , is smaller than one. Therefore, for the low frequencies we consider here, the current density is approximately uniform in the thickness direction. Consequently, we can assume that the rings are infinitely thin, if we replace the current density $\mathbf{J}$ (in A/m²) by the current per unit of length $\mathbf{j}$ (in A/m), defined as $\mathbf{j} = h\mathbf{J}$. From now on, the current distribution in the rings is independent of r and the component of the current in the r -direction is neglected. Moreover, for an axi-symmetric geometry, the current does not depend on φ , and flows only in the φ -direction, such that $\mathbf{j} = j_\varphi(z)\mathbf{e}_\varphi$.

In accordance with the general model described in Section 2.2, the rings occupy a surface $S_\cup$ on the cylinder S_c . The set of rings is subdivided into L groups, each of which having a prescribed total current $I_l(t) = \Re\{I_l e^{-i\omega t}\}$, $I_l \in \mathbb{C}$, $l = 1, \dots, L$ driven by a separate source current. The sum of the widths of all rings in group l is denoted by D_l . Let each group

l consist of N_l rings, then $S_n^{(l)}$ can be defined as the surface of the n -th ring within group l , with $n \in \{1, \dots, N_l\}$ and $l \in \{1, \dots, L\}$, such that

$$S_{\cup}^{(l)} = \sum_{n=1}^{N_l} S_n^{(l)}, \quad \text{and} \quad S_{\cup} = \sum_{l=1}^L S_{\cup}^{(l)} = \sum_{l=1}^L \sum_{n=1}^{N_l} S_n^{(l)}, \quad (4.1)$$

and

$$S_n^{(l)} = \{(r, \varphi, z) \mid r = 1, -\pi \leq \varphi \leq \pi, z_0^{(n)} \leq z \leq z_1^{(n)}\}. \quad (4.2)$$

Here, $\sum$ denotes the disjoint union of sets. The index n represents the number of the ring within group l . The dimensional quantities are incorporated directly, meaning that we have used a characteristic current j^c and a characteristic distance D^c , defined by

$$D^c = R, \quad j^c = \frac{\sum_{l=1}^L I_l}{\sum_{l=1}^L D_l}, \quad (4.3)$$

such that the distances are scaled by the radius of the cylinder (yielding $r = 1$ on the cylinder), and the current is scaled by the average amplitude of the current through all rings. Note that $I_l \in \mathbb{C}$, so also $j^c \in \mathbb{C}$ is a complex characteristic current.

For currents that do not depend on φ , it is more convenient to introduce a notation for the z -intervals in which the rings are positioned. Instead of $S_{\cup}$, which represents a surface, we introduce the one-dimensional set $S_{\cup}^z$ as the collection of all intervals, denoting the axial positions of the rings:

$$S_{\cup}^z = \sum_{n=1}^N [z_0^{(n)}, z_1^{(n)}]. \quad (4.4)$$

The dimensional total width D_{tot} of all rings together is defined as

$$D_{tot} = R \sum_{n=1}^N (z_1^{(n)} - z_0^{(n)}), \quad (4.5)$$

which is, of course, equal to the sum of the widths D_l of all groups $l = 1, \dots, L$.

In terms of the coordinate system $\{\mathbf{e}_s, \mathbf{e}_t, \mathbf{e}_n\}$, as introduced in Section 2.2, $\mathbf{e}_s$ represents the direction of the central line of the rings, $\mathbf{e}_t$ denotes the width direction of the rings, and $\mathbf{e}_n$ is the outward directed normal on the rings. It is evident that for a set of rings $\mathbf{e}_s$, $\mathbf{e}_t$ and $\mathbf{e}_n$ correspond with $\mathbf{e}_\varphi$, $\mathbf{e}_z$ and $\mathbf{e}_r$, respectively.

4.1.2 Adjustment of the integral equation

In Chapter 2, an integral equation of the second kind is derived for a general distribution of thin conductors on a cylinder. The resulting equation is (2.195). For the specific case of rings only, we use the integral equation for the φ -component of the current (2.177), with kernel given by (2.175). We substitute a current that only depends on the z -coordinate, $\mathbf{j} = j_\varphi(z)\mathbf{e}_\varphi$,

into (2.177). After integration over θ from $-\pi$ to π , only the term $k = 0$ in the infinite series (2.175) yields a contribution, so that we obtain

$$\begin{aligned}\mathcal{K}_\varphi(\varphi, z) &= \mathcal{K}(z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} [I_{-1}(|p|)K_{-1}(|p|) + I_1(|p|)K_1(|p|)] e^{ipz} dp \\ &= \frac{1}{\pi} \int_0^{\infty} I_1(p)K_1(p) \cos pz \, dp.\end{aligned}\quad (4.6)$$

With this kernel function $\mathcal{K}(z)$, we obtain the integral equation of the second kind

$$i\kappa \int_{S_\cup^z} \mathcal{K}(z - \zeta) j_\varphi(\zeta) \, d\zeta = j_\varphi(z) - j_\varphi^s(z), \quad z \in S_\cup^z, \quad (4.7)$$

where $\kappa = h\sigma\mu_0\omega R$. For further computations we need to investigate the behavior of the kernel function. For this, we can use either of the following expressions:

$$\begin{aligned}\mathcal{K}(z) &= \frac{1}{\pi} \int_0^{\infty} I_1(p)K_1(p) \cos pz \, dp = \frac{1}{2} \int_0^{\infty} J_1^2(p) e^{-pz} \, dp \\ &= \frac{1}{4|z|^3} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 3; -\frac{4}{z^2}\right) = \frac{1}{2\pi k} \left[(2 - k^2)K(k) - 2E(k) \right].\end{aligned}\quad (4.8)$$

Here, J_1 is the Bessel function of order one of the first kind (see [82]), ${}_2F_1$ is the hypergeometric function (see [49]), K and E are the complete elliptic integrals of the first and second kind, respectively (see [10]), and

$$k^2 = \frac{4}{4 + z^2}. \quad (4.9)$$

The reason for presenting these four expressions is that all of them are used in literature and that different authors have different preferences. Apart from the ones in (4.8), other expressions are used as well. However, the expressions in (4.8) are the more common in the symbolic toolboxes of programs, such as Mathematica and Maple.

We note that the expression with the elliptic functions is similar to the formula for the mutual inductance between two circular loops of wires; see e.g. [31], [48]. In [31], the author shows that the model of a set of circular loops of wires provides results for discrete currents, which describe the qualitative behavior of the continuous current distribution in rings with a similar geometry. The wire model yields some insight in the problem, but cannot be used as a proper quantitative validation, because the thickness and the number of wires are arbitrary.

Investigation of the behavior of the kernel function $\mathcal{K}(z)$ reveals that it has a singularity at $z = 0$. From a numerical point of view, it is important to find an approximate formula that is valid for small z . The asymptotic expansion of $\mathcal{K}(z)$ for $z \rightarrow 0$ up to terms of $\mathcal{O}(z^2)$ is of the form

$$\mathcal{K}(z) = -\frac{1}{2\pi} \left(\log|z| - \log 2 + \Psi^{(0)}\left(\frac{3}{2}\right) + \gamma \right) + \mathcal{O}(z^2), \quad (4.10)$$

where $\Psi^{(0)}(3/2) \approx 0.03649$ ($\Psi^{(0)}$ is the polygamma function; see [2, Eq.(6.4.4)]), and $\gamma \approx 0.57722$ (Euler's constant; see [2, Eq.(6.1.3)]). In Figure 4.2(a), both the kernel function

(solid line) and the asymptotic expansion (dashed line) are drawn for $z \in [-4, 4]$. The difference between $\mathcal{K}(z)$ and the asymptotic expansion is a regular function; see Figure 4.2(b). In the numerical computations, we make use of the approximation for $\mathcal{K}(z)$ according to (4.10).

Figure 4.2: (a) The kernel $\mathcal{K}(z)$ (solid line) and the approximate function (dashed line); (b) The difference between $\mathcal{K}(z)$ and its approximation.

The possibility to decompose the kernel into a logarithmic and a regular part enables us to investigate the equivalence between a set of parallel rings and a set of infinitely long parallel strips in one plane. For the strips in one plane, the integral formulation is shown in Section 3.4 and it has a kernel that is purely logarithmic. Therefore, according to (4.10), the solution of the ring problem is partly accounted for by the solution of the plane-strip problem. The remaining part of the solution is determined from the regular difference function, i.e. the difference between $\mathcal{K}(z)$ and the $\log |z|$ term.

We note that the solutions of the integral equations for the two problems of rings and plane strips become equal if the radii of the rings tend to infinity, i.e. $R/D_{tot} \gg 1$. As the influence of the currents in two different points on one ring is only local, we see that for $R/D_{tot} \gg 1$, the current distribution in one point on a ring becomes equal to the current distribution in an infinitely long plane strip tangent to the ring in that point; see Figure 4.3. This is corroborated by the behavior of $\mathcal{K}(z)$ for $z \ll 1$ (note that $z \equiv z/R < D_{tot}/R \ll 1$), as given by (4.10). For $D_{tot}/R \ll 1$, only the logarithmic kernel remains, which is equivalent to the kernel in the plane-strip case. We come back to this in the discussion of the numerical results; see Section 4.3.

Another way of interpreting the behavior of the kernel function $\mathcal{K}(z)$ is that, locally, the curvature of the strips can be neglected, and that, globally, the current in a point is not affected by currents on a large distance. In Section 4.3, we check whether these interpretations are correct.

Figure 4.3: For a large radius of the ring (2), the current distribution in point P becomes equal to the current distribution in an infinitely long plane strip (1) tangent to the ring in point P .

4.2 Composition of the matrices

In this section, we explain how we apply the Galerkin method, as described in Section 3.2, to the problem of rings. The current distribution is approximated by a series of global basis functions. Since the currents are independent of the tangential direction φ , it suffices to use z -dependent basis functions only. We have seen that the current density function has to satisfy (4.7). Both the functions $j_\varphi(z)$ and $j_\varphi^s(z)$ must be expanded in a series of global basis functions. For the latter, we know the analytical solution. Namely, the static current distribution in a set of rings is uniform in the z -direction. Thus, in the dynamic case, an appropriate description for the source current is a set of basis functions that are uniform in z on each ring.

In the Galerkin method we search for a family of basis functions, such that a linear combination of these basis functions gives a good approximation of the current distribution. The best way to deal with the source current $j_\varphi^s(z)$ is to assimilate it in the basis. Then $j_\varphi^s(z)$ can be described by one basis function, or by a linear combination of merely a few basis functions. Due to the projection method, the source current disappears after one or merely a few projections. In the case we consider, $j_\varphi^s(z)$ is a stepwise uniform function in z and can therefore easily be included in the family of basis functions. We show how the projection on basis functions introduced in (4.11) makes the source current disappear from the formulation, which eventually results in equation (4.22).

Let $\psi_l(z)$ be the characteristic function of group l , defined by

$$\psi_l(z) = \sum_{n=1}^{N_l} \mathbf{1}_{[z_0^{(n)}, z_1^{(n)}]}, \quad (4.11)$$

where N_l is the number of rings in group l . By $\mathbf{1}_{[a,b]}$ we mean the characteristic function of the interval $[a, b]$. Note that $[z_0^{(n)}, z_1^{(n)}]$ is a dimensionless interval, scaled on R . Due to disjointness of the intervals, the characteristic functions $\psi_l(z)$ satisfy

$$(\psi_l, \psi_{l'}) = \frac{D_l}{R} \delta_{ll'}, \quad (4.12)$$

where D_l is width of group l and $\delta_{ll'}$ is the Kronecker delta. The source current $j_\varphi^s(z)$ can be expressed as a linear combination of the characteristic functions $\psi_l(z)$

$$j_\varphi^s(z) = \sum_{l=1}^L C_l \psi_l(z), \quad (4.13)$$

and the integral equation (4.7) becomes

$$j_\varphi(z) - i\kappa \int_{S_\cup^z} \mathcal{K}(z - \zeta) j_\varphi(\zeta) d\zeta = \sum_{l=1}^L C_l \psi_l(z), \quad z \in S_\cup^z. \quad (4.14)$$

Moreover, the total current per group l is prescribed, so $j_\varphi(z)$ must satisfy

$$(j_\varphi, \psi_l) = \int_{S_\cup^z} j_\varphi(z) \psi_l(z) dz = \hat{I}_l, \quad l \in \{1, \dots, L\}, \quad (4.15)$$

where $\hat{I}_l = I_l / (j^c R)$, with j^c according to (4.3).

We introduce the operator $\mathbf{K}$ by

$$(\mathbf{K}f)(z) = \int_{S_\cup^z} \mathcal{K}(z - \zeta) f(\zeta) d\zeta, \quad (4.16)$$

such that (4.14) can be written in operator form as

$$(\mathbf{I} - i\kappa \mathbf{K}) j_\varphi = \sum_{l=1}^L C_l \psi_l. \quad (4.17)$$

We define the projection Π on the linear span of the characteristic functions ψ_l , $l \in \{1, \dots, L\}$ by

$$(\Pi f)(z) = \sum_{l=1}^L \frac{(f, \psi_l)}{(\psi_l, \psi_l)} \psi_l(z). \quad (4.18)$$

The projection Π applied to j_φ yields, with use of (4.12) and (4.15),

$$\Pi j_\varphi = \sum_{l=1}^L \frac{R}{D_l} (j_\varphi, \psi_l) \psi_l = \sum_{l=1}^L \frac{R \hat{I}_l}{D_l} \psi_l. \quad (4.19)$$

Next, j_φ is split into two parts according to

$$j_\varphi = \Pi j_\varphi + (\mathbf{I} - \Pi) j_\varphi = \sum_{l=1}^L \frac{R \hat{I}_l}{D_l} \psi_l + j_\perp. \quad (4.20)$$

Here, $j_\perp = (I - \Pi)j_\varphi$ lies in the orthoplement of the range of Π , i.e. $(j_\perp, \psi_l) = 0$, for $l = 1, \dots, L$. Multiplying (4.17) by $(I - \Pi)$, we obtain

$$j_\perp - i\kappa(I - \Pi)\mathbf{K}j_\varphi = 0, \quad (4.21)$$

or equivalently,

$$j_\perp - i\kappa(I - \Pi)\mathbf{K}j_\perp = i\kappa \sum_{l=1}^L \frac{R\hat{I}_l}{D_l} (I - \Pi)\mathbf{K}\psi_l. \quad (4.22)$$

We have now eliminated j_φ in favor of $j_\perp$ and the constants C_l have disappeared in favor of the prescribed currents $\hat{I}_l$. So, the source current $j_\varphi^s(z)$ is completely covered by the family of basis functions $\psi_l(z)$.

We approximate $j_\perp$ by a finite series of M mutually orthogonal basis functions $\phi_m(z)$, according to

$$j_\perp(z) \doteq \sum_{m=1}^M \alpha_m \phi_m, \quad \phi_m \in \text{ran}(I - \Pi). \quad (4.23)$$

Substituting (4.23) into (4.22), taking the inner product of this relation with ϕ_n , and using the self-adjointness of the operator $(I - \Pi)$, we obtain

$$\sum_{m=1}^M \alpha_m (\phi_m, \phi_n) - i\kappa \sum_{m=1}^M \alpha_m (\mathbf{K}\phi_m, \phi_n) = i\kappa \sum_{l=1}^L \frac{R\hat{I}_l}{D_l} (\mathbf{K}\psi_l, \phi_n), \quad (4.24)$$

for $n = 1, 2, \dots, M$. From this system of M linear equations, the M unknown coefficients α_m can be determined. For convenience of notation, we write (4.24) in a matrix formulation as

$$(\mathbf{G} - i\kappa\mathbf{A})\mathbf{a} = i\kappa\mathbf{B}\mathbf{f}, \quad (4.25)$$

where $(m, n = 1, \dots, M; l = 1, \dots, L)$

$$\begin{aligned} A(n, m) &= (\mathbf{K}\phi_m, \phi_n), \\ G(n, m) &= (\phi_m, \phi_n), \\ a(m) &= \alpha_m, \\ B(n, l) &= (\mathbf{K}\psi_l, \phi_n), \\ f(l) &= \frac{R\hat{I}_l}{D_l}. \end{aligned} \quad (4.26)$$

In Subsection 4.1.2, we have shown that the kernel $\mathcal{K}(z)$ has a logarithmic singularity in $z = 0$; see (4.10). This corresponds with the explanation of the Fourier modes in the general configuration of Section 3.1, where we have proved that the essential behavior of the kernels is logarithmic. Moreover, it turned out in Section 3.2 that a very appropriate choice for the basis functions is to use Legendre polynomials, $P_s(z)$. Then, for the inner products in (4.26) the analytic expression (3.61) can be used. Due to the simple analytical computations, the

simulations are fast, and moreover we shall see that only a few basis functions are needed to provide a good approximation. Other relevant properties of the Legendre polynomials $P_s(z)$, $s \in \mathbb{N}$, are

$$\int_{-1}^1 P_0(z) dz = 2, \quad \int_{-1}^1 P_s(z) dz = 0, \quad (4.27)$$

for all $s \geq 1$, and the orthogonality property

$$\int_{-1}^1 P_s(z) P_{s'}(z) dz = \frac{2}{2s+1} \delta_{ss'}. \quad (4.28)$$

For the current distribution $j_\perp(z)$, the pertinent intervals are $[z_0^{(q)}, z_1^{(q)}]$, $q = 1, \dots, N$, instead of $[-1, 1]$. Therefore, we introduce shifted and scaled Legendre polynomials, denoted by $P_{s,q}(z; c_q; d_q)$, of order s , defined on ring q as

$$P_{s,q}(z; c_q; d_q) = P_s\left(\frac{z - c_q}{d_q}\right), \quad z \in [z_0^{(q)}, z_1^{(q)}], \quad (4.29)$$

where

$$c_q = \frac{z_1^{(q)} + z_0^{(q)}}{2}, \quad d_q = \frac{z_1^{(q)} - z_0^{(q)}}{2}, \quad q = 1, \dots, N. \quad (4.30)$$

For $z \notin [z_0^{(q)}, z_1^{(q)}]$, $P_{s,q}(z; c_q; d_q) = 0$. Equation (4.28) then transforms into

$$\int_{S_\cup^z} P_{s,q}(z; c_q; d_q) P_{s',q'}(z; c_{q'}; d_{q'}) dz = \frac{2d_q}{2s+1} \delta_{ss'} \delta_{qq'}. \quad (4.31)$$

The shifted and scaled Legendre polynomials are used to construct the basis functions. We create a series expansion in S different basis functions for each ring, separately. Hence,

$$j_\perp = \sum_{s=1}^S \sum_{q=1}^N \alpha_{s,q} \phi_{s,q} + \sum_{q=1}^{N-L} \alpha_{0,q} \phi_{0,q}, \quad (4.32)$$

where S denotes the number of degrees of basis functions that is included, and N is the number of rings. The reason for splitting the series into two sums – one with $\phi_{s,q}$ and one with $\phi_{0,q}$ – follows below. We note that the total number of basis functions is $(S+1)N - L$, which will be set equal to the number M used in (4.23). A requirement on the basis functions is that they are mutually orthogonal as well as orthogonal to the functions $\psi_l(z)$.

According to (4.27), Legendre polynomials satisfy the first requirement. Moreover, for Legendre polynomials of degree one and higher, we have

$$\int_{S_\cup^z} P_{s,q}(z; c_q; d_q) \psi_l(z) dz = 0, \quad (4.33)$$

for $q = 1, \dots, N$, $l = 1, \dots, L$, and $s \geq 1$. Therefore, we choose the basis functions of degree one and higher equal to the shifted and scaled Legendre polynomials

$$\phi_{s,q} = P_{s,q}(z; c_q; d_q), \quad q = 1, \dots, N, \quad s = 1, \dots, S. \quad (4.34)$$

The Legendre polynomial of degree zero is constant. The total current $j_\varphi(z)$ is approximated for a part by L piecewise constant functions $\psi_l(z)$. So, for $j_\perp(z)$, $N - L$ piecewise constant functions are left, which can differentiate the current distributions in the rings within a group. Here, $\phi_{0,q}$ denotes the constant basis function on ring q , $q = 1, \dots, N$. This explains why we split off the sum with the $N - L$ terms in (4.32).

For the description of the basis functions of degree zero, we first need to define a basis of characteristic functions $\{\mathbf{1}_{[z_0^{(1)}, z_1^{(1)}]}, \dots, \mathbf{1}_{[z_0^{(N)}, z_1^{(N)}]}\}$, where the order is such that the characteristic functions of each group follow successively. The functions $\psi_l(z)$ can be constructed by N_l elements of this basis, where N_l is the number of rings of group l (clearly, $\sum_{l=1}^L N_l = N$). For example, the function $\psi_1(z)$ can be written as a vector that starts with N_1 ones, followed by $N - N_1$ zeros.

For the basis functions $\phi_{0,q}(z)$, we need $N_l - 1$ linear combinations of the basis elements for each group l , chosen in such a way that each $\phi_{0,q}(z)$ is orthogonal to all $\psi_l(z)$ and $\phi_{0,q'}(z)$, for $q \neq q'$. Each function $\phi_{0,q}$, for $q \in \{1, \dots, N_l - 1\}$, can be expressed as a vector that contains the coefficients of the corresponding linear combination. Note that for the L groups together, we thus obtain $\sum_{l=1}^L (N_l - 1) = N - L$ basis functions.

To visualize the $N_l - 1$ basis functions for each group, we introduce the $N \times N_l$ matrices Γ_l , $l = 1, \dots, L$, the columns of which correspond to the vector representations of ψ_l and $\phi_{0,q}$, where q is the number of the ring in group l , $q \in \{1, \dots, N_l - 1\}$. Furthermore, we define the $N \times N$ matrix Γ as a row of sub-matrices Γ_l , according to $\Gamma = [\Gamma_1, \dots, \Gamma_L]$. Note that each sub-matrix Γ_l contains a block of $N - N_l$ rows with only zeros. So, two columns from different sub-matrices always have a zero inner product.

It suffices to explain each sub-matrix Γ_l separately. The construction of these sub-matrices can be achieved in several ways. For our numerical simulations we choose here a simple algorithm. In (4.35), we show the non-zero part of Γ_l , a sub-matrix of size $N_l \times N_l$, which clarifies the algorithm:

$$\begin{pmatrix} \gamma_1 & -1 & -\frac{1}{2} & -\frac{1}{3} & \cdots & -\frac{1}{N_l-1} \\ 1 & \gamma_2 & -\frac{1}{2} & -\frac{1}{3} & \cdots & -\frac{1}{N_l-1} \\ 1 & 0 & \gamma_3 & -\frac{1}{3} & \cdots & -\frac{1}{N_l-1} \\ 1 & 0 & 0 & \gamma_4 & \cdots & -\frac{1}{N_l-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & \gamma_{N_l} \end{pmatrix} \quad (4.35)$$

The first column of the matrix in (4.35) represents the function ψ_l and the other columns are the vector representations of $\phi_{0,q}$. All columns are orthogonal. On the diagonal the values are equal to (for d_i , see (4.30))

$$\gamma_1 = 1, \quad \gamma_j = \frac{1}{(j-1)d_j} \sum_{i=1}^{j-1} d_i, \quad j = 2, \dots, N_l. \quad (4.36)$$

Because the Legendre polynomials are shifted and scaled in a different way on each ring, it is easier to transform all intervals to $[-1, 1]$. We express the entries of the matrix equation (4.25) for Legendre polynomials of order larger than zero. The zeroth-order Legendre polynomials are not included, but they yield a combination of the solutions for $B(n, l)$ in (4.37), using the coefficients of (4.35). For $s, s' = 1, \dots, S$ and $q, q' = 1, \dots, N$ and $l = 1, \dots, L$, we obtain

$$\begin{aligned}
 A(n, m) &= d_q d_{q'} \int_{-1}^1 \int_{-1}^1 P_{s'}(z) P_s(\zeta) \mathcal{K}(d_{q'} z - d_q \zeta + c_{q'} - c_q) d\zeta dz, \\
 G(n, m) &= \frac{2d_q}{2s+1} \delta_{mn}, \\
 a(m) &= \alpha_m, \\
 B(n, l) &= d_q d_{q'} \int_{-1}^1 \int_{-1}^1 P_{s'}(z) \mathcal{K}(d_{q'} z - d_q \zeta + c_{q'} - c_q) d\zeta dz, \\
 f(l) &= \frac{R \hat{I}_l}{D_l},
 \end{aligned} \tag{4.37}$$

with $m = N - L + (s - 1)N + q$ and $n = N - L + (s' - 1)N + q'$, such that $m, n \in \{N - L + 1, \dots, (S + 1)N - L\}$.

The elements of the matrices **A** and **B** can be determined from the integrals in (4.37), whose integrands are logarithmically singular if $q = q'$. We split off the logarithmic part, given by (4.10), and are left with a regular difference function. This difference function in combination with the Legendre polynomials is integrated numerically. The Gauss-Legendre quadrature rule is used as integration method. For the logarithmic part together with the Legendre polynomials, we use (3.61).

4.3 Numerical results

In this section, we present the results of the numerical simulations of three different configurations: one ring, the Maxwell pair, consisting of two rings, and a realistic geometry of a z-coil, consisting of 24 rings. All simulations make use of the method described in this chapter; see also [33]. A special software tool has been developed, called **Eddy**, which can compute the current distributions and the corresponding electromagnetic fields for any arbitrary set of parallel rings (both on a cylinder and in a plane). In **Eddy**, the positions of the centers of the rings have to be defined, as well as the widths and the thicknesses of the rings. We can assign which rings are connected in a group, and what the total current through each group is, the frequency of the source, the radius of the cylinder, and the number of basis functions. Furthermore, extra options can be chosen for postprocessing, such as plotting the amplitude of the current density, the phase lag of the current density with respect to the source, and contour lines, flux lines or values along a line of components of vector fields, such as the magnetic field or the vector potential.

In **Eddy**, first the coefficients $\alpha_{s,q}$ and $\alpha_{0,q}$ from (4.32) are determined, which results in $j_\perp$.

Subsequently, the total current j_φ is determined from (4.20).

4.3.1 One ring

We first consider one ring of width $D = 4$ cm, positioned with its center at $z = 0$. The thickness is $h = 2.5$ mm, the radius is $R = 35$ cm and the amplitude of the total current through the ring is $I = 600$ A. All values are typical for a ring in a z-coil of an MRI-scanner. The current distribution is expanded in a series of Legendre polynomials, with coefficients calculated from the linear set of equations (4.25).

Figure 4.4: One ring of 4 cm width with a total current of 600 A at frequencies $f = 100$ Hz (*), $f = 400$ Hz ($\circ$), $f = 700$ Hz (+), $f = 1000$ Hz (Δ). (a) Amplitude of the current distribution; (b) Phase-lag with respect to the source current.

Figure 4.4 (a) shows the amplitude of the current distribution in the ring for the four frequencies $f = 100, 400, 700, 1000$ Hz. We observe that the current at the edges is higher than in the center. This *edge-effect*, which is due to the self-inductance of the ring, becomes stronger as the frequency increases. For low frequencies, the current is distributed almost uniformly, whereas for high frequencies the current is more concentrated at the edges. The average current is always equal to $1.5 \cdot 10^4$ A/m, which is the value for $j^c = I/D$.

To obtain the time-dependent result for the currents, we multiply the current density by $e^{-i\omega t}$ and take the real part. Simulating this case, we observe a phase difference in the system. In Figure 4.4 (b), the phase-lag $\Delta\phi$ with respect to the source current is shown as function of the position for the four frequencies $f = 100, 400, 700, 1000$ Hz. For each frequency only two points are in phase with the source (i.e. $\Delta\phi = 0$). The currents near the edges are ahead in phase with respect to the source (i.e. negative phase-lag), whereas currents at the center are behind in phase. The phase difference also varies with the frequency.

We note that the system is linear with respect to the magnitude of the current, which means

that if we increase the total current I , then the current densities are increased proportionally. The shapes of the distributions in Figures 4.4 (a) and (b) remain the same, but scales are changing.

Figure 4.5: One ring of 4 cm width with a total current of 600 A at $f = 1000$ Hz. (a) Flux lines of the magnetic field; (b) Contour lines of B_z ; (c) B_z on z -axis; (d) Difference between B_z for $f = 1000$ Hz and B_z for $f = 0$ Hz on z -axis.

For the determination of the magnetic field, we use the law of Biot-Savart as formulated in (2.46). In Figure 4.5 (a), the flux lines of the vector potential are shown in the (y, z) -plane. These flux lines represent the direction of the magnetic field and encircle the ring that carries the current. In the context of gradient coils, we are mainly interested in the axial component of the magnetic field, B_z . Figure 4.5 (b) shows the contour lines of the B_z -field in the (y, z) -plane. The contour lines connect points of the same B_z -value. Here, the differences between the values of two successive contour lines are always the same. Therefore, from a figure like

this, the steepness of the gradient of the B_z -field can be determined from the distance between the contour lines. Hence, equally spaced contour lines represent a linear gradient. We observe that in the case of one ring, the gradient is far from linear. A frequency of $f = 1000$ Hz is used for the situations in Figures 4.5 (a) and (b).

In Figure 4.5 (c), the axial component of the magnetic field, B_z , along the central axis of the ring is shown. The frequency of the excitation source is $f = 1000$ Hz. For lower frequencies, the differences are too small to be observed in Figure 4.5 (c). Therefore, we consider the difference between the magnetic field caused by an alternating current and the magnetic field caused by a direct current. In Figure 4.5 (d), the difference is shown for the alternating current having a frequency $f = 1000$ Hz. Note that the difference is less than one micro Tesla, or $\Delta B_z/B_z < 10^{-3}$.

(a)

(b)

(c)

(d)

Figure 4.6: One ring of 4 cm width with a total current of 600 A. (a) Resistance $\bar{R}$ at frequencies up to $f = 10000$ Hz; (b) Resistance $\bar{R}$ at frequencies up to $f = 400$ Hz; (c) Self-inductance $\bar{L}$ at frequencies up to $f = 10000$ Hz; (d) Self-inductance $\bar{L}$ at frequencies up to $f = 400$ Hz.

The first characteristic quantity we present, is the resistance $\bar{R}$ as a function of the frequency f ; see (2.66). In Figure 4.6 (a), $\bar{R}$ is shown for the frequency range $[0, 10000]$ Hz. We observe that $\bar{R}$ increases with the frequency. The physical interpretation is that due to the stronger edge-effects the current flows through a narrower part of the ring, which corresponds to a higher resistance. The value of the resistance $\bar{R}_{DC}$ at $f = 0$ Hz is verified analytically by taking the circumferential length divided by the conductivity times the cross-sectional surface, i.e.

$$\bar{R}_{DC} = \frac{2\pi R}{hD\sigma} \approx 3.74 \cdot 10^{-4} \Omega. \quad (4.38)$$

We observe a decay in the slope of the resistance. In Smythe [59, pp.368–371], it is shown how a similar behavior occurs in the situation of a conducting medium filling a half-space, with a harmonic varying source current flowing along the surface. The induced current causes a resistance that increases with the square root of the frequency. This can be explained by the decrease of the order of the edge effect for increasing frequency. In the case of a thin ring, the decay of the slope is stronger. The graph in Figure 4.6 (a) can be fitted by a function that is proportional to the frequency to the power one eighth.

In Figure 4.6 (b), the resistance is shown for the smaller range of frequencies up to $f = 400$ Hz, with a higher resolution than used in the range from 0 to 10000 Hz. We observe that the behavior of $\bar{R}$ changes from low to high frequency range. We observe a point of inflection f_{char} close to 170 Hz. Such a point of inflection is also observed in the next example, a ring of 30 cm width. We discuss this phenomenon at the end of this subsection.

In Figure 4.6 (c), the self-inductance $\bar{L}$ is shown for frequencies up to $f = 10000$ Hz. We see that $\bar{L}$ decreases with the frequency. Thus, if the current is jostled near the edges of the ring, as happens in the case of higher frequencies, then the inductive effects decrease. Or, conversely, if the current changes more rapidly in time, then the self-inductance has to be lower. For that, the amplitude of the current near the edges of the ring is higher than at the center.

For low frequency range, from 0 to 400 Hz, we observe that the self-inductance $\bar{L}$ changes behavior as frequencies increase; see Figure 4.6 (d). The inflection is at the same characteristic frequency f_{char} as found for the resistance.

Apart from a dependence on the frequency, the results of our simulations also depend on the width D of the ring. To illustrate this, we repeat the computations for a ring of 30 cm width. Figures 4.7 (a) and (b) show the amplitude of the current density and the phase-lag in this ring for the four frequencies $f = 100, 400, 700, 1000$ Hz. Here, the average current is equal to $2.0 \cdot 10^3$ A/m, i.e. the value for $j^c = I/D$. We observe that the current is more concentrated near the edges than in the 4 cm ring.

Let us take a closer look at the integral equation

$$(\mathbf{I} - i\kappa\mathbf{K})j_\varphi = j_\varphi^s. \quad (4.39)$$

Figure 4.7: One ring of 30 cm width with a total current of 600 A. (a) Amplitude of the current distribution at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz (Δ); (b) Phase-lag with respect to the source current at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz (Δ); (c) Amplitude of the current distribution at frequencies $f = 13.3$ Hz (*), $f = 53.3$ Hz (o), $f = 93.3$ Hz (+), $f = 133.3$ Hz (Δ); (d) Phase-lag with respect to the source current at frequencies $f = 13.3$ Hz (*), $f = 53.3$ Hz (o), $f = 93.3$ Hz (+), $f = 133.3$ Hz (Δ).

Here, $\kappa = h\sigma\mu_0\omega R$ is a system parameter, which represents the contribution of the inductive effects. A numerical approximation $j_\varphi^{(N)}$ of the true j_φ is given by

$$j_\varphi^{(N)} = \sum_{n=0}^N \frac{1}{1 - i\kappa\lambda_n^{(N)}} (j_\varphi^s, u_n^{(N)}) u_n^{(N)}, \quad (4.40)$$

according to (3.74). Here, $\lambda_n^{(N)}$ and $u_n^{(N)}$ are the eigenvalues and eigenfunctions, respectively, of the finite rank operator $K^{(N)}$ that is used to replace K . The series in (4.40) can also be written as

$$j_\varphi^{(N)} = \sum_{n=0}^N \left[\frac{1}{1 + (\kappa\lambda_n^{(N)})^2} + i \frac{\kappa\lambda_n^{(N)}}{1 + (\kappa\lambda_n^{(N)})^2} \right] (j_\varphi^s, u_n^{(N)}) u_n^{(N)}. \quad (4.41)$$

In order for the inductive effects (represented by the imaginary part) to dominate the resistive effects (represented by the real part), $\kappa\lambda_n^{(N)}$ must be greater than one.

Figure 4.8: (a) The first 40 scaled eigenvalues of the operator K , $2R\lambda_n/D$ (+), compared with a lower bound $1/(8(n+1))$ (o), for $n \geq 0$, and an upper bound $1/(2n)$ (*), for $n \geq 1$ and $\log(2)/2$ for $n = 0$; (b) All values multiplied by $n+1$.

We can approximate the eigenvalues of the operator K , by using the matrix $\mathbf{A}$ from (4.25), which represents the finite rank operator $K^{(N)}$. This means that we have to compute the eigenvalues of $\mathbf{G}^{-1}\mathbf{A}$, where the Gram matrix $\mathbf{G}$ is used to orthonormalize the basis functions. Furthermore, due to scaling of the distances on radius R , the integral operator K is related to the integration interval $[-D/(2R), D/(2R)]$. In order to compare the results with the plane strip case, we scale the integration interval further, such that we integrate from -1 to 1. Consequently, we obtain an additional factor $D/(2R)$. If we scale the eigenvalues λ_n , $n = 0, 1, 2, \dots$, of K with the same factor, then they can be compared to the eigenvalues of

the plane strip. Figures 4.8 (a) and (b) show that the eigenvalues are constrained by

$$\frac{1}{8} \leq \frac{2R}{D} \lambda_0 \leq \frac{1}{2} \log 2, \quad \frac{1}{8(n+1)} \leq \frac{2R}{D} \lambda_n \leq \frac{1}{2n}, \quad n \geq 1, \quad (4.42)$$

where the upper and lower bounds are obtained from the plane strip case in (3.76) and hold here as well. The eigenvalues here are very much in conformity with the ones from the plane strip case.

The dominant eigenvalue $\lambda_0^{(N)}$ is equal to

$$\lambda_0^{(N)} = 0.257 \frac{D}{2R}. \quad (4.43)$$

The dependence of κ on R is not a characteristic feature of the system. But $\hat{\kappa}$, defined by

$$\hat{\kappa} = \lambda_0^{(N)} \kappa = 0.257 \frac{h\sigma\mu_0\omega D}{2} = 0.807 h\sigma\mu_0 f D, \quad (4.44)$$

is a characteristic parameter of the system.

From (4.44), we see now that multiplying the width of the ring by a factor yields the same $\hat{\kappa}$ as multiplying the frequency with the same factor. This means that for a ring of 30 cm width instead of one that is 7.5 times narrower (4 cm), the frequency should be 7.5 times less than in the 4 cm case in order to find the same current distribution. In Figures 4.7 (c) and (d), the amplitude of the current and the phase-lag are shown for a ring of 30 cm width, with frequencies $f = 13.3, 53.3, 93.3, 113.3$ Hz. Comparing the results with the 4 cm case at the frequencies $f = 100, 400, 700, 1000$ Hz – see Figures 4.4 (a) and (b) – we see that the distributions are indeed the same.

Next, we define the characteristic frequency

$$f_{char} = \frac{1}{0.807 h\sigma\mu_0 D} = \frac{1.234}{h\sigma\mu_0 D}, \quad (4.45)$$

such that

$$\hat{\kappa} = \frac{f}{f_{char}}. \quad (4.46)$$

Then, $f = f_{char}$ represents the frequency at which prevailing resistive effects with respect to inductive effects change into prevailing inductive effects. This characteristic frequency can be observed clearly in the graphs of the resistance and the self-inductance against the frequency.

In Figure 4.9 (a), the resistance $\bar{R}$ of a 30 cm ring is shown for the frequency domain $f \in [0, 10000]$ Hz. The resistance of the ring at $f = 0$ Hz is equal to

$$\bar{R}_{DC} = \frac{2\pi R}{hD\sigma} \approx 4.99 \cdot 10^{-5} \Omega. \quad (4.47)$$

The resistance increases with the frequency. In Figure 4.9 (b), the resistance is shown for frequencies up to $f = 50$ Hz. We observe that at low frequencies the behavior of $\bar{R}$ is

different from the one at high frequencies. The point of inflection in Figure 4.9 (b) is about 22 Hz. This value corresponds very well to the theoretical value $f_{char} = 22.3$ Hz, following from (4.45). Note that for a 4 cm ring the characteristic frequency is $f_{char} = 167.0$ Hz, according to (4.45). Again, this value corresponds well to the point of inflection that can be observed in Figure 4.6.

For the self-inductance we find the same result: at the characteristic frequency $f_{char} = 22.3$ Hz, the behavior of $\bar{L}$ changes; see Figures 4.9 (c) and (d). Comparing this with the 4 cm case, we see again that the value is 7.5 times smaller.

Figure 4.9: One ring of 30 cm width with a total current of 600 A. (a) Resistance $\bar{R}$ at frequencies up to $f = 10000$ Hz; (b) $\bar{R}$ at frequencies up to $f = 50$ Hz. (c) Self-inductance $\bar{L}$ at frequencies up to $f = 10000$ Hz; (d) $\bar{L}$ at frequencies up to $f = 50$ Hz.

4.3.2 The Maxwell pair

The second example we consider, is a special configuration of two rings. With a pair of rings we are able to create a gradient in the axial component of the magnetic field. In principle, a gradient coil is based on the properties of a so-called Maxwell pair. A Maxwell pair consists of two circular loops of wires, both of radius R , placed coaxially and parallel, and with the z -axis as the central axis. If both wires are at equal distance from the origin and carry a current in anti-phase, then the B_z -field has a gradient in the origin. It has been proved that for a mutual distance between the wires equal to $R\sqrt{3}$, the field has no second, third and fourth order z -dependence; see Jin [28, Sect.2.4]. In other words, at this distance the optimal linear field is achieved.

In the design of gradient coils, one uses a stream function approach, which results in a pattern of streamlines; see e.g. Peeren [46], [47], Tomasi [71]. For the manufacture of the coils, copper strips are placed on this pattern. For low frequencies and narrow strips, the currents cause a magnetic field that approximates the design field. However, increasing the frequencies or increasing the widths of the strips causes errors, which lead to unacceptable deviations in the gradient of the magnetic field.

Figure 4.10: Two rings of 4 cm width (○) carrying 1526.0 A in anti-phase and two rings of 30 cm width (*) carrying 1841.6 A in anti-phase, both at frequency $f = 1000$ Hz. (a) Amplitude of the current distribution; (b) Magnetic field B_z along the z -axis.

In this subsection we consider a pair of rings of 4 cm width and a pair of rings of 30 cm width. For a uniform distribution of the current in the rings (comparable with the DC situation) the distance between the rings for an optimal linearity of the B_z -field can be determined. We find an optimal distance equal to 60.8 cm for the 4 cm case and 71.4 cm for the 30 cm case. In order to obtain a desired gradient of $1.0 \cdot 10^4 \mu\text{T/m}$, the current in the 4 cm rings must be 1526.0 A and in the 30 cm rings 1841.6 A, and the currents in both rings are in anti-phase.

In Figures 4.10 (a) and (b), the amplitudes of the current distributions and the z -component of the magnetic field at the z -axis are shown for $f = 1000$ Hz, for both cases. The gradient of the field is approximately $1.0 \cdot 10^4 \mu\text{T/m}$ here, but this value is not exact. The difference, however, is not visible on the scale of this figure. A closer look at the difference is needed and is presented in Figure 4.11.

In Figure 4.11 (a), the difference between the B_z -field at the z -axis at $f = 0$ Hz and at $f = 100, 400, 700, 1000$ Hz is shown for the 4 cm case. We are especially interested in the deviations in the gradient at the origin. We observe that the error, i.e. the deviation from the ideal value of $1.0 \cdot 10^4 \mu\text{T/m}$, increases with the frequency. For a frequency of 1000 Hz we observe an error in the gradient of approximately $3 \mu\text{T/m}$, meaning that the relative error is of the order 10^{-4} . In Figure 4.11 (b), the difference between the B_z -field at the z -axis at $f = 0$ Hz and at $f = 100, 400, 700, 1000$ Hz is shown for the 30 cm case. The relative error is in this case almost 10 times larger, i.e. of the order 10^{-3} . Thus, the use of wider rings entails larger errors in the gradient of the field.

Figure 4.11: Difference between the B_z -field at $f = 0$ Hz and $f = 100$ Hz (*), $f = 400$ Hz (◦), $f = 700$ Hz (+), $f = 1000$ Hz (Δ). (a) Two rings of 4 cm width carrying 1526.0 A in anti-phase; (b) Two rings of 30 cm width carrying 1841.6 A in anti-phase.

We see in Figure 4.11 how the eddy currents affect the magnetic field. Due to the edge-effects, the gradient of B_z at the origin has increased. In MRI, this deviation causes errors in the images. So, a time varying current in the rings gives not the same field as a DC current, for which the pair has been designed. This error can be compensated by the current supply. If we know the error of the field as a percentage of the desired field, the applied current should be decreased with the same frequency. Due to linearity of the system, the error in the gradient is then compensated. Note that this does not mean that higher-order effects are not amplified.

In the error analysis, we only consider a small area around $z = 0$. In a larger area, errors increase. We see that we cannot realize a linear magnetic field within a large region of interest with a Maxwell pair; more turns are needed. Another reason to use more turns is that the total dissipated power can be decreased and that a lower current needs to be supplied. In the next subsection we consider a realistic z-coil, modeled by 24 rings on the cylinder.

Figure 4.12: Contour lines of the B_z -field in the (y, z) -plane. (a) Two rings of 4 cm width carrying 1526.0 A in anti-phase at $f = 1000$ Hz; (b) Two rings of 30 cm width carrying 1841.6 A in anti-phase at $f = 1000$ Hz; (c) Difference between the B_z -field at $f = 0$ Hz and $f = 1000$ Hz for two rings of 4 cm width carrying 1526.0 A in anti-phase; (d) Difference between the B_z -field at $f = 0$ Hz and $f = 1000$ Hz for two rings of 30 cm width carrying 1526.0 A in anti-phase.

Apart from the z -axis, we can present the magnetic field in the whole (y, z) -plane by contour lines. In Figure 4.12 (a), the contour lines of the B_z -field are shown for two rings of 4 cm width carrying 1526.0 A, while in Figure 4.12 (b) the contour lines of the B_z -field are shown

for two rings of 30 cm width carrying 1841.6 A. The currents are in anti-phase at frequency $f = 1000$ Hz. The linearity of the field is characterized by parallel contour lines. In both figures we observe a region around the origin in which the field is fairly linear. The errors of the fields with respect to the DC-fields are presented in Figures 4.12 (c) and (d), for the 4 and 30 cm case, respectively. In these figures, the third order effects are visible and regions with a dense distribution of contour lines correspond to a rapidly increasing error. The deviations in the gradient field are problematic, because they cannot be compensated for by additional currents.

Figure 4.13: Two rings of 4 cm width carrying 1526.0 A in anti-phase. (a) Resistance $\bar{R}$ at frequencies up to $f = 10000$ Hz; (b) $\bar{R}$ at frequencies up to $f = 50$ Hz. (c) Self-inductance $\bar{L}$ at frequencies up to $f = 10000$ Hz; (d) $\bar{L}$ at frequencies up to $f = 50$ Hz.

The resistance of a system of two rings is determined from (2.66), i.e. the total dissipated power divided by the squared effective current. In Figure 4.13 (a), the resistance $\bar{R}$ is shown for the 4 cm case for the frequency range $[0, 10000]$ Hz. We observe that $\bar{R}$ increases with

the frequency. The resistance $\bar{R}_{DC}$ at $f = 0$ Hz is verified analytically by taking double the circumferential length divided by the conductivity times the cross-sectional surface, i.e.

$$\bar{R}_{DC} = \frac{4\pi R}{hD\sigma} \approx 7.48 \cdot 10^{-4} \Omega. \quad (4.48)$$

In Figure 4.13 (b), the resistance is shown for the smaller range of frequencies up to $f = 400$ Hz, with a higher resolution than used in the range from 0 to 10000 Hz. We observe a point of inflection at the same characteristic frequency as in the example of one ring, i.e. $f_{char} = 167.0$ Hz. This corresponds to the formula for f_{char} in (4.45), in which the width of one ring is used.

The self-inductance of a system of two rings is determined from (2.67), i.e. the total magnetic energy divided by the squared effective current. In Figure 4.13 (c), the self-inductance $\bar{L}$ is shown for the 4 cm case for frequencies up to 10000 Hz. We see that $\bar{L}$ decreases with the frequency. For low frequency range, from 0 to 400 Hz, the self-inductance shows a point of inflection at $f_{char} = 167.0$ Hz; see Figure 4.13 (d). The inflection is at the same characteristic frequency as found for the resistance and the example of one ring.

In Figure 4.14, the resistance and the self-inductance are shown for two rings of 30 cm width. We observe in both $\bar{R}$ and $\bar{L}$ a characteristic frequency $f_{char} = 22.3$ Hz. This value corresponds to f_{char} of one ring of 30 cm width and is verified analytically by (4.45).

Figure 4.14: Two rings of 30 cm width carrying 1841.6 A in anti-phase. (a) Resistance $\bar{R}$ at frequencies up to $f = 10000$ Hz; (b) $\bar{R}$ at frequencies up to $f = 50$ Hz. (c) Self-inductance $\bar{L}$ at frequencies up to $f = 10000$ Hz; (d) $\bar{L}$ at frequencies up to $f = 50$ Hz.

4.3.3 A realistic z-coil

The third example we consider, is a set of 24 rings. This set describes a realistic z-coil. All 24 rings have a width of 2 cm and a radius of 35 cm. The positions are determined by a gradient coil design program of Philips Medical Systems. In this program, the magnetic energy is minimized, subject to the requirements that the gradient of the magnetic field at the origin is $1.0 \cdot 10^4 \mu\text{T/m}$, the second, third, and fourth order z -dependence are zero, and all currents are on the radius $R = 0.35$ m. The design program determines a pattern of streamlines of the current. From this pattern a set of 24 rings is selected. We assign to each ring a width of 2 cm and compute again the field caused by the discretized currents. The error of the discretized current, in terms of the first- and third-order z -dependence, is compensated by scaling of the geometry. The resulting begin and end positions are listed in Table 4.1.

Table 4.1: *Positions of rings and directions of currents in realistic z-coil.*

Ring	Left	Right	Dir.	Ring	Left	Right	Dir.
1	-0.5277	-0.5077	-1	13	0.0306	0.0506	-1
2	-0.4267	-0.4067	-1	14	0.0738	0.0938	-1
3	-0.3753	-0.3553	-1	15	0.1572	0.1772	1
4	-0.3386	-0.3186	-1	16	0.1896	0.2096	1
5	-0.3090	-0.2890	-1	17	0.2151	0.2351	1
6	-0.2830	-0.2630	-1	18	0.2389	0.2589	1
7	-0.2589	-0.2389	-1	19	0.2630	0.2830	1
8	-0.2351	-0.2151	-1	20	0.2890	0.3090	1
9	-0.2096	-0.1896	-1	21	0.3186	0.3386	1
10	-0.1772	-0.1572	-1	22	0.3553	0.3753	1
11	-0.0938	-0.0738	1	23	0.4067	0.4267	1
12	-0.0506	-0.0306	1	24	0.5077	0.5277	1

The total current in each ring is 180.52 A, but not all currents are in phase. In Table 4.1, the phases are represented by the column called direction. A one stands for a current in phase and a minus one stands for a current in anti-phase. We see that the rings are not equally distributed and that the two halves do not exactly have opposite phase. This all has to do with the magnetic energy that is minimized.

In Figure 4.15, the amplitude of the current density is shown for an applied frequency of 1000 Hz. The average current density in each ring is equal to

$$j^c = \frac{\sum_{l=1}^L I_l}{\sum_{l=1}^L D_l} = 9.03 \cdot 10^3 \text{ A/m.} \quad (4.49)$$

We observe local edge-effects in the rings as well as a global edge-effect in the whole system. We see how opposite currents amplify each other (e.g. in the two halves of the coil), whereas currents in the same direction damp each other (e.g. within one half of the coil).

Figure 4.15: Amplitude of the current distribution in a set of 24 rings representing a z -coil. Each ring carries a current of 180.52 A at frequency $f = 1000$ Hz.

In Figure 4.16, the magnetic field of the set of 24 rings is presented. The contour lines of the B_z field in the (y, z) -plane are shown in Figure 4.16 (a). The frequency used is $f = 1000$ Hz. The linearity of the field is characterized by parallel contour lines. We observe a region around the origin in which the field is linear. In Figure 4.16 (b), the difference of the field with the DC field is presented by contour lines. This figure gives an indication of the third-order effects.

In Figure 4.16 (c), the magnetic field on the z -axis is shown, for the frequencies $f = 100, 400, 700, 1000$ Hz. The gradient of the field is approximately $1.0 \cdot 10^4 \mu\text{T/m}$, but this value is not exact. The difference is not visible on the scale of the figure. For that, we consider the difference between the field caused by alternating currents and a direct current. In Figure 4.16 (d), the results are shown for the frequencies $f = 100, 400, 700, 1000$ Hz. The error of the gradient at $f = 1000$ Hz is approximately 0.5% of the desired gradient. Compensation of the gradient is possible when the applied current in the rings is increased by 0.5%. Higher-order effects are still present.

The resistance of a system of 24 rings is determined from (2.66). In Figure 4.17 (a), the resistance $\bar{R}$ is shown for the frequency range $[0, 10000]$ Hz. We observe that $\bar{R}$ increases with the frequency. The resistance $\bar{R}_{DC}$ at $f = 0$ Hz is verified analytically according to

$$\bar{R}_{DC} = \frac{48\pi R}{hD\sigma} \approx 1.79 \cdot 10^{-2} \Omega. \quad (4.50)$$

Figure 4.16: Magnetic field of a set of 24 rings representing a z -coil. (a) Contour lines of B_z at $f = 1000$ Hz; (b) Contour lines of the difference between B_z at $f = 1000$ Hz and B_z at $f = 0$ Hz; (c) B_z on z -axis at $f = 1000$ Hz; (d) Difference between B_z at $f = 100$ Hz (*), $f = 400$ Hz ($\circ$), $f = 700$ Hz (+), $f = 1000$ Hz (Δ) and B_z at $f = 0$ Hz on z -axis.

In Figure 4.17 (b), $\bar{R}$ is shown for the smaller range of frequencies up to $f = 800$ Hz, with a higher resolution than used in the range from 0 to 10000 Hz. We observe a point of inflection at the characteristic frequency $f_{char} = 334.0$ Hz. This corresponds to the formula for f_{char} in (4.45), in which a width of 2 cm has to be substituted.

Figure 4.17: Resistance and self-inductance of a set of 24 rings representing a z-coil. (a) $\bar{R}$ at frequencies up to $f = 10000$ Hz; (b) $\bar{R}$ at frequencies up to $f = 400$ Hz; (c) $\bar{L}$ at frequencies up to $f = 10000$ Hz; (d) $\bar{L}$ at frequencies up to $f = 400$ Hz.

The self-inductance of a system of 24 rings is determined from (2.67). In Figure 4.17 (c), $\bar{L}$ is shown for frequencies up to 10000 Hz. The self-inductance decreases with the frequency. For low frequency range, from 0 to 800 Hz, the self-inductance shows a point of inflection at $f_{char} = 334.0$ Hz; see Figure 4.17 (d). The inflection is at the same characteristic frequency as found for the resistance.

The total dissipated power is less in a set of 24 rings than in a set of 2 rings. Namely, if we consider the Maxwell pair consisting of two rings of 4 cm width, then $\bar{R}$ and $\bar{L}$ can be

obtained from Figure 4.13. At a frequency of 1000 Hz (for example), we find

$$\bar{R} = 9.6 \cdot 10^{-4} \Omega, \quad \bar{L} = 3.09 \cdot 10^{-6} H. \quad (4.51)$$

With a current $I = 1526.0$ A, the time-averaged magnetic energy is determined from (2.67):

$$\bar{U}_m = \frac{1}{2} \bar{L} I^2 = 3.60 \text{ J}. \quad (4.52)$$

The dissipated power follows from (2.66):

$$\bar{P}_{diss} = I^2 \bar{R} = 2.24 \cdot 10^3 \text{ W}. \quad (4.53)$$

For the set of 24 rings, we determine from Figure 4.17 that at $f = 1000$ Hz

$$\bar{R} = 2.6 \cdot 10^{-2} \Omega, \quad \bar{L} = 1.32 \cdot 10^{-4} H. \quad (4.54)$$

From this, we obtain for currents equal to $I = 180.52$ A,

$$\bar{U}_m = 2.15 \text{ J}, \quad \bar{P}_{diss} = 8.47 \cdot 10^2 \text{ W}. \quad (4.55)$$

Hence, the total dissipated power in a set of 24 rings is less than in 2 rings, while in both situations the same gradient in the magnetic field is created. The physical explanation for this is that the field is more confined within a dense coil. The overall advantage is that the current that has to be supplied to the 24 rings is almost ten times less than the one needed in the 2 rings to create the same gradient.

4.3.4 Validation

The configurations used in this chapter consist of thin copper rings placed on a cylinder. Due to the penetration depth, which is greater than half the thickness of the rings, the modeling step to an infinitely thin ring could be made. The current is represented by a surface current for which the integral equation (4.7) holds. The Galerkin method is applied to solve this integral equation. The basis functions are defined on the whole domain of a ring, without having limited support. The resulting matrices are of small size, but dense.

Another option to solve the integral equation for the current distribution is by using local basis functions in the Galerkin method, with local support. The matrices are then sparse, but of large size. We have performed such computations with the finite element package SYSIPHOS. The field-circuit coupling that is implemented, is described in [20], the solver is described in [19], and the necessary adaptive mesh refinement is described in [43]. The advantage of this 2D solver is that only a grid of elements needs to be defined on one symmetry plane of the system. Axi-symmetry such as in our system of rings can be indicated. Nevertheless, in order to obtain a good approximation of the current, a very fine mesh needs to be generated. First of all, the thicknesses of the rings are very small. For an accurate solution of the current distribution, the elements have a size in the order of tenths of a millimeter. Next,

the air inside the rings must be meshed, because the vector potential is valid everywhere. Also the air around the system has to be meshed up to a distance where the vector potential is sufficiently small. For rings of 35 cm radius and a mesh that is nicely connected to the tiny mesh in the rings, a lot of elements are defined. Finally, a discretization of the symmetry plane in the axial direction must be defined. We are interested in the current distribution in that direction, so we want a fine mesh in that direction as well. Thus, a matrix system with millions of elements is easily obtained.

However, for a validation of the results of our approach of global basis functions, it has been worthwhile to do the finite element computations. We obtained exactly the same current distributions with the two methods. Also the resistances and self-inductances were computed with the FEM method, yielding the same results. Additional data obtained from the FEM method is the distribution of the current over the thickness of the rings. This distribution is not uniform, but the variation of the current is very small. From the discrete distribution of currents we determine the coefficient of variance. In Figure 4.18, the coefficient of variance of the current in both r - and z -direction is shown for a ring of 4 cm width, 2.5 mm thickness, and 50 cm radius. We observe the small variation in the r -direction. Hence, the assumption that the currents is uniformly distributed in that direction is permitted. Note that the variation in the z -direction changes in the same way as the resistance. The correspondence follows directly from the formulas.

Figure 4.18: Coefficient of variance (c.v.) of the current distribution in axial direction (full line) and radial direction (dash-dotted line), plotted against the frequency.

The methods using Finite Difference Time Domain (FDTD), see e.g. Taflov and Hagness [66], or the Finite Integration Technique (FIT), see e.g. Clemens et al. [13] and Van Rienen [52], also have a drawback here. For the thin conductors, which have curved shapes, local mesh refinements are necessary. In these methods local refinements are not standard, so

a small mesh in the conductor means also a redundant small mesh in the air.

With global basis function, we are able to determine the characteristic behavior of the system. With a few coefficients (not more than ten in the examples of the previous sections), we computed parameters, such as resistance, self-inductance and characteristic frequency. But also the displacement of the current is determined accurately, whereas with local basis functions very small elements are necessary. We remark that for complicated geometries, numerical methods with local basis functions have to be used. In the following chapters we describe other geometries for which global basis functions are possible. So, the remarks about the numerical validation are the same there, and will therefore not be repeated.

The forces on the rings are of the order of 1 N, which is negligibly small compared to the magnetic force caused by the main magnet and the gravitational force. Moreover, the gradient coil is fixed very firmly in the scanner, such that it cannot be displaced by the strong magnetic forces of the main field. Thus, the assumption that there are no magneto-mechanical is valid.

The results of the magnetic field are obtained by the law of Biot-Savart (2.46). The current distribution is known and the magnetic field follows after numerical integration. In [67], Tas has developed an analytical method to determine the magnetic field of a system of coaxial rings. In this work, the magnetic field is approximated by spherical harmonics and by a Taylor series expansion. For this, the coefficients of the currents are used that are computed by Eddy. Besides the same results as presented in this chapter, Tas has also investigated the higher order field distortions. For higher frequencies (above 1000 Hz), they start to play a role, i.e. they are not negligibly small compared to the first order coefficient of the field.

Furthermore, from [67] it follows that a superposition of results in case of multiple frequencies is allowed. The dynamic effects of a trapezoidal acquisition sequence are obtained from the approximation of this sequence of pulses by a series of harmonics. According to our assumption, only a few frequencies are needed and the results show the decay of the eddy currents.

4.4 Summary

In a set of circular loops of strips, the current flows in the tangential direction and depends only on the axial coordinate. The two-dimensional integral equation derived in Chapter 2, for arbitrary geometries, is reduced to a one-dimensional integral equation. The kernel can be expressed in terms of elliptic integrals. From the asymptotic expansion of these elliptic integrals near the singularity point, the logarithmic function appears, as predicted by the mathematical analysis presented in Chapter 3.

With the logarithmic part of the kernel, we are able to use the analytical formula derived in Section 3.2. Legendre polynomials are used as basis functions. The remaining part of the kernel is regular and is computed numerically.

The coefficients of the Legendre polynomials are determined by our software tool Eddy. The resulting current distribution in the rings shows edge-effects, and a phase-lag with respect to the source. This causes distortions in the magnetic field. A change in the gradient of the field can be compensated by additional source currents. However, the higher order effects do not disappear.

Rings and islands

In the previous chapter, the z-coil has been modeled as a set of parallel coaxial rings. In that case, the current distribution is independent of the tangential coordinate, i.e. the solution is axi-symmetric. However, in a z-coil embedded in a system of other coils and magnets, possibly in different directions, eddy currents are induced, so that the current distribution in the z-coil is no longer axi-symmetric. Even without the other coils, a realistic model of the z-coil should contain the tangential coordinate dependence. To make the model of the z-coil more realistic we include *islands*, which present slits in the copper. Islands are thin pieces of (copper) strips positioned between coaxial parallel rings; they are on the same cylindrical surface as these rings. The current through the rings induces eddy currents in the islands and vice versa.

Islands are introduced to come to a more realistic model of the z-coil. The reason why we include them in the model is the following. In the design of a gradient coil, a pattern of streamlines of the current is computed in such a way that a desired magnetic field is created. On this pattern, copper strips are placed, such that the currents are forced to flow in the direction of the streamlines. However, at places where the streamlines diverge, unfilled spaces appear. These spaces are then filled by pieces of copper for a better heat conduction and a better stability of the coil.

These additional pieces of copper are called islands. In the islands, eddy currents are induced. They distort the magnetic field and give the coil a higher resistance. By introducing islands in our model, we want to investigate the effect of the currents in the islands on the magnetic field and on the resistance of the coil.

Eddy currents are present in all parts of the coil and have a negative effect on the performance as a whole. To reduce this effect, one cuts slits in the copper; see Figure 5.1. These slits can be regarded as loose pieces of copper, i.e. the islands. On the islands, the current is in the form of eddies. If the islands are small, only at high frequencies eddy currents are induced,

because the resistance in a small island is large. Moreover, for large resistances, the decay of the eddy currents is fast.

In Section 5.1, we describe the geometry of the model and the derivation of the integral equation. The solution procedure is presented in Section 5.2 and numerical results are shown in Section 5.3. Finally, in Section 5.4, the summary of this chapter is presented.

Figure 5.1: A gradient coil in which slits are cut to reduce the eddy currents.

5.1 Model formulation

In this chapter, we consider a set of N_r parallel coaxial circular strips, or rings, and N_i islands. The model is the extension of the one presented in the previous chapter, where we considered a system of rings only. Hence, the conductors lie on the same virtual cylinder S_c , defined in terms of cylindrical coordinates as

$$S_c = \{(r, \varphi, z) \mid r = R, -\pi \leq \varphi \leq \pi, -\infty < z < \infty\}; \quad (5.1)$$

one of the central lines of each island is parallel to the rings. The z -axis coincides with the central axis of the cylinder. Each separate ring or island is of uniform width, and the thickness h is the same for every conductor. The strips occupy the surface $S_\cup = S_r + S_i$, where S_r is the total surface of the rings and S_i is that of the islands. We indicate the surface of one ring or one island by S_q , where we number the rings and the islands successively by $q = 1, 2, \dots, N_r$, for the rings, and $q = N_r + 1, N_r + 2, \dots, N_r + N_i$, for the islands. Hence,

$$S_r = \sum_{q=1}^{N_r} S_q, \quad S_i = \sum_{q=1}^{N_i} S_{N_r+q}, \quad (5.2)$$

where

$$S_q = \{(r, \varphi, z) \mid r = R, -\pi \leq \varphi \leq \pi, z_0^{(q)} \leq z \leq z_1^{(q)}\}, \quad (5.3)$$

for $q = 1, 2, \dots, N_r$, and

$$S_q = \{(r, \varphi, z) \mid r = R, \varphi_0^{(q)} \leq \varphi \leq \varphi_1^{(q)}, z_0^{(q)} \leq z \leq z_1^{(q)}\}, \quad (5.4)$$

for $q = N_r + 1, N_r + 2, \dots, N_r + N_i$. More islands can be present between two rings. A configuration of three rings and four islands is depicted in Figure 5.2. In this example, the islands have the same axial begin and end positions $z_0^{(q)}$ and $z_1^{(q)}$. This model mimics the situation of a ring with four slits in the width direction. With respect to the three possible types of strips that are discussed in Section 2.2, we are dealing here with the patch (island) and the closed loop around the cylinder (ring).

Figure 5.2: The geometry of a set of three rings and four islands; here, the islands have the same axial begin and end positions.

The source currents are applied to the rings. They change harmonically in time with an angular frequency ω . Though not applied to the islands, these source currents will excite the whole system.

The difference between a ring and an island is that the current on a ring will flow completely around the cylinder, whereas on an island it cannot. A ring forms a closed loop, so the current density must be periodic. On the other hand, in the z -direction the rings have a finite width and the normal component of the current density on the edges $z = z_0^{(q)}$ and $z = z_1^{(q)}$ must be equal to zero. An island has a begin and an end in both φ - and z -direction, which requires that the normal components of the current density are equal to zero at the edges $z = z_0^{(q)}, z_1^{(q)}$ and $\varphi = \varphi_0^{(q)}, \varphi_1^{(q)}$. Together with the requirement that the current distribution is divergence-free, the only possible way for a current to flow on an island is to form closed cycles within the island. In the results of Section 5.3, we show the occurrence of such eddies.

In Chapter 2, an integral equation has been derived for a general set of thin conductors on a cylinder. The geometry of rings and islands is a specific example of this general model. For the model under consideration, the integral equation comes down to (see (2.177))

$$i\kappa \int_{S_\cup} \mathcal{K}_\varphi(\varphi - \theta, z - \zeta) j_\varphi(\theta, \zeta) da(\theta, \zeta) = j_\varphi(\varphi, z) - j_\varphi^s(\varphi, z), \quad (5.5)$$

where $\kappa = h\sigma\mu_0\omega R$, and $j_\varphi(\varphi, z)$, $j_\varphi^s(\varphi, z)$ denote the φ -components of the source and total current density, respectively. The kernel function is given by (2.175), or in reduced form by (2.185),

$$\mathcal{K}_\varphi(\varphi, z) = \frac{\cos(\varphi)}{4\pi\sqrt{z^2 + 4\sin^2(\frac{\varphi}{2})}}. \quad (5.6)$$

The source current has only a φ -component, so that we can restrict to solving the integral equation for j_φ and thereafter calculate j_z from the divergence-free relation. For the rings, apart from the periodicity condition, this implies for $q \in \{1, \dots, N_r\}$,

$$j_z(\varphi, z_0^{(q)}) = j_z(\varphi, z_1^{(q)}) = 0, \quad \frac{\partial j_\varphi}{\partial \varphi} = -\frac{\partial j_z}{\partial z}, \quad (5.7)$$

whereas for the islands, $q \in \{N_r + 1, \dots, N_r + N_i\}$, we have besides (5.7) the additional condition

$$j_\varphi(\varphi_0^{(q)}, z) = j_\varphi(\varphi_1^{(q)}, z) = 0. \quad (5.8)$$

As in the model presented in Chapter 4, the set of rings is subdivided into L groups, each of which has a prescribed total current I_l , $l = 1, \dots, L$ and is driven by a separate source current. The sum of the widths of all rings in group l is denoted by D_l . The currents and distances are scaled by the factors j^c and D^c , respectively. These scale factors are defined by

$$j^c = \frac{\sum_{l=1}^L I_l}{\sum_{l=1}^L D_l}, \quad D^c = R. \quad (5.9)$$

The integral equation for the current distribution is solved using the Galerkin method, as explained in the next section.

5.2 Solution procedure

In this section, we explain how we solve $j_\varphi(\varphi, z)$ from (5.5), taking care of (5.7) and (5.8). We apply the Galerkin method. So, we have to choose appropriate basis functions. Therefore, we first investigate the behavior of the kernel function (5.6), which has a singularity in the point $(\varphi, z) = (0, 0)$. We have shown in Section 3.1 that, given a preferred direction of the current, the singularity is logarithmic in the coordinate perpendicular to this preferred direction.

5.2.1 Rewriting the kernel function

In the present example of rings and islands, the preferred direction is the φ -direction. According to Section 3.1, the logarithmic behavior then occurs in the width direction, i.e. the z -direction. For further computations, we want to split off the logarithmic part. Below, we derive the Fourier cosine series representation of $\mathcal{K}_\varphi(\varphi, z)$:

$$\mathcal{K}_\varphi(\varphi, z) = \frac{1}{4\pi^2} \left(Q_{\frac{1}{2}}(\chi) + \sum_{p=1}^{\infty} \cos(p\varphi) (Q_{p-\frac{3}{2}}(\chi) + Q_{p+\frac{1}{2}}(\chi)) \right), \quad (5.10)$$

where

$$\chi = \frac{2 + z^2}{2}. \quad (5.11)$$

The functions $Q_{p-\frac{1}{2}}(\chi)$, $p \geq 0$, are the Legendre functions of the second kind of odd-half-integer order, which have a logarithmic singularity in $\chi = 1$, i.e. $z = 0$, as we will show at the end of this subsection. By splitting off this logarithmic part, we are able to solve the dominant contributions of the inner products in our problem analytically. The remaining part for each cosine mode is regular and computed numerically.

We now present the steps of the derivation of expression (5.10) and show that the functions $Q_{p-\frac{1}{2}}(\chi)$ have indeed a logarithmic singularity in $z = 0$. We go back to the first expression in (2.185), i.e.

$$\mathcal{K}_\varphi(\varphi, z) = \frac{\cos(\varphi)}{4\pi} \sum_{p=-\infty}^{\infty} e^{ip\varphi} \int_0^\infty J_p^2(s) e^{-sz} ds. \quad (5.12)$$

Formula (13.22.2) from Watson [82] tells us that

$$\int_0^\infty e^{-sz} J_p^2(s) ds = \frac{1}{\pi} Q_{p-\frac{1}{2}}\left(\frac{2+z^2}{2}\right). \quad (5.13)$$

Substitution of (5.13) into (5.12) yields

$$\mathcal{K}_\varphi(\varphi, z) = \frac{\cos(\varphi)}{4\pi^2} \left(Q_{-\frac{1}{2}}(\chi) + 2 \sum_{p=1}^{\infty} \cos(p\varphi) Q_{p-\frac{1}{2}}(\chi) \right), \quad (5.14)$$

where we used that $Q_{-p-\frac{1}{2}}(\chi) = Q_{p-\frac{1}{2}}(\chi)$ and $\chi = (2 + z^2)/2$. From (5.14), we arrive at the Fourier cosine series of $\mathcal{K}_\varphi(\varphi, z)$, as given in (5.10).

Expressions for $Q_{-\frac{1}{2}}(\chi)$ and $Q_{\frac{1}{2}}(\chi)$ can be obtained from [2], namely

$$Q_{-\frac{1}{2}}(\chi) = kK(k), \quad (5.15)$$

$$Q_{\frac{1}{2}}(\chi) = \frac{1}{k}[(2 - k^2)K(k) - 2E(k)], \quad (5.16)$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, and

$$k^2 = \frac{4}{4 + z^2}. \quad (5.17)$$

The function $Q_{\frac{1}{2}}(\chi)$ in (5.16) represents the kernel in the integral equation for rings in Chapter 4; see (4.8). In the computation of higher-order Legendre functions of the second kind of odd-half-integer order, we use the recurrence relation

$$Q_{p-\frac{1}{2}}(\chi) = 4\chi \frac{p-1}{2p-1} Q_{p-\frac{3}{2}}(\chi) - \frac{2p-3}{2p-1} Q_{p-\frac{5}{2}}(\chi), \quad p \geq 2. \quad (5.18)$$

For all $p \geq 0$, the functions $Q_{p-\frac{1}{2}}(\chi)$ are singular in $\chi = 1$ (i.e. $z = 0$), as can be seen from the asymptotic expansion of $Q_{p-\frac{1}{2}}(\chi)$ for $z \rightarrow 0$; see e.g. [39, Eq.(4.8.2)], [62, Eq.(59.9.2)],

$$Q_{p-\frac{1}{2}}(\chi) = -\ln|z| - \gamma + \ln 2 - \Psi^{(0)}\left(\frac{2p+1}{2}\right) + \mathcal{O}(z^2), \quad (5.19)$$

for all $p \geq 0$. Here, γ is Euler's constant and $\Psi^{(0)}$ is the polygamma function; see [2]. Note that for $p = 1$ the expansion is the same as in (4.10). We conclude that each cosine mode in (5.10) has a logarithmic singularity at $z = 0$, which is of the same magnitude for all $p \geq 0$.

5.2.2 Basis functions

We want to use global basis functions, i.e. defined globally on the rings and islands. Due to the preferred direction of the current, the logarithmic singularity of the kernel function is in the z -coordinate and we choose Legendre polynomials of the first kind for this direction. We can then use the analytical solutions for the singular integrals as demonstrated in (3.61). In φ -direction, we must use 2π -periodic functions.

The current distribution $\mathbf{j}$ is decomposed into $\mathbf{j}_r$ for the rings and $\mathbf{j}_i$ for the islands, such that in vector form

$$\mathbf{j} = \begin{pmatrix} \mathbf{j}_r \\ \mathbf{j}_i \end{pmatrix}. \quad (5.20)$$

Then, we can present integral equation (5.5) as an operator matrix equation, according to

$$\begin{pmatrix} \mathbf{j}_r \\ \mathbf{j}_i \end{pmatrix} - i\kappa \begin{pmatrix} K_{rr} & K_{ri} \\ K_{ir} & K_{ii} \end{pmatrix} \begin{pmatrix} \mathbf{j}_r \\ \mathbf{j}_i \end{pmatrix} = \begin{pmatrix} \mathbf{j}_r^s \\ \mathbf{0} \end{pmatrix}. \quad (5.21)$$

The operator K_{rr} represents the inductive effects between the rings mutually. In the same way, K_{ii} represents the inductive effects between the islands mutually, whereas K_{ri} and K_{ir} represent the inductive effects between rings and islands. In case of rings only, the matrix equation reduces to

$$(\mathbf{I} - i\kappa K_{rr})\mathbf{j}_r = \mathbf{j}_r^s, \quad (5.22)$$

cf. equation (4.7). The current in the rings, computed from (5.21), and the current in the rings, computed from (4.7), will differ a few promille (observed from the results; see Section 5.3).

Taking into account the boundary conditions $(\mathbf{j} \cdot \mathbf{n}) = 0$ and the divergence-free current relation $\nabla \cdot \mathbf{j} = 0$, we come to the following expansion for the (dimensionless) current density:

On the rings, numbered q , with $q \in \{1, \dots, N_r\}$, for $z \in [z_0^{(q)}, z_1^{(q)}]$:

$$\begin{aligned} j_\varphi^{(q)}(\varphi, z) &= \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\alpha_{mn}^{(q)} \cos m\varphi + \beta_{mn}^{(q)} \sin m\varphi) P_n \left(\frac{z - c_z^{(q)}}{d_z^{(q)}} \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \alpha_{0n}^{(q)} P_n \left(\frac{z - c_z^{(q)}}{d_z^{(q)}} \right) + \alpha_{00}^{(q)} \right\} \mathbf{1}_{[z_0^{(q)}, z_1^{(q)}]}, \end{aligned} \quad (5.23)$$

$$j_z^{(q)}(\varphi, z) = \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m d_z^{(q)} (-\beta_{mn}^{(q)} \cos m\varphi + \alpha_{mn}^{(q)} \sin m\varphi) Z_n \left(\frac{z - c_z^{(q)}}{d_z^{(q)}} \right) \right\} \mathbf{1}_{[z_0^{(q)}, z_1^{(q)}]}, \quad (5.24)$$

where

$$c_z^{(q)} = \frac{z_1^{(q)} + z_0^{(q)}}{2}, \quad d_z^{(q)} = \frac{z_1^{(q)} - z_0^{(q)}}{2}, \quad (5.25)$$

$$Z_n(z) = \int_{-1}^z P_n(\zeta) d\zeta = \frac{1}{2n+1} [P_{n+1}(z) - P_{n-1}(z)]. \quad (5.26)$$

Here, $\mathbf{1}_{[z_0^{(q)}, z_1^{(q)}]}$ is the characteristic function, i.e. the function equal to one on the interval $[z_0^{(q)}, z_1^{(q)}]$ and zero otherwise. The terms in (5.23) with $m = 0$ correspond to the basis functions used for the rings only case in the previous chapter and correspond to divergence-free currents without a j_z -component. The terms with $m > 0$ account for currents with a j_z -component. They occur due to interaction between rings and islands. Note that the coefficients are ring dependent. Moreover, $Z_n(\pm 1) = 0$ for $n \geq 1$, such that the representation (5.24) for the z -component of the current automatically satisfies the boundary conditions at the z -edges at the edges (5.7)₁.

On the islands, numbered q , $q \in \{N_r + 1, \dots, N_r + N_i\}$, for $\varphi \in [\varphi_0^{(q)}, \varphi_1^{(q)}]$ and $z \in [z_0^{(q)}, z_1^{(q)}]$:

$$j_\varphi^{(q)}(\varphi, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn}^{(q)} \sin\left(\frac{m\pi(\varphi - \varphi_0^{(q)})}{2d_\varphi^{(q)}}\right) P_n\left(\frac{z - c_z^{(q)}}{d_z^{(q)}}\right) \mathbf{1}_{[z_0^{(q)}, z_1^{(q)}]} \mathbf{1}_{[\varphi_0^{(q)}, \varphi_1^{(q)}]}, \quad (5.27)$$

$$j_z^{(q)}(\varphi, z) = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m\pi d_z^{(q)}}{2d_\varphi^{(q)}} \alpha_{mn}^{(q)} \cos\left(\frac{m\pi(\varphi - \varphi_0^{(q)})}{2d_\varphi^{(q)}}\right) Z_n\left(\frac{z - c_z^{(q)}}{d_z^{(q)}}\right) \mathbf{1}_{[z_0^{(q)}, z_1^{(q)}]} \mathbf{1}_{[\varphi_0^{(q)}, \varphi_1^{(q)}]}, \quad (5.28)$$

where

$$d_\varphi^{(q)} = \frac{\varphi_1^{(q)} - \varphi_0^{(q)}}{2}. \quad (5.29)$$

Here, $\mathbf{1}_{[\varphi_0^{(q)}, \varphi_1^{(q)}]}$ is the characteristic function of the interval $[\varphi_0^{(q)}, \varphi_1^{(q)}]$. The absence of the terms with $n = 0$ is due to the fact that only induced currents and no source currents are assumed on the islands.

Again, we note that the representations (5.27) and (5.28) for j_φ and j_z automatically satisfy the boundary conditions for zero normal current at the edges of the islands (5.7)₁ and (5.8). Moreover, because $Z'_n = P_n$, the divergence-free restriction (5.7)₂ is also automatically satisfied by these representations.

We have determined the current $\mathbf{j}$ once the coefficients $\alpha_{mn}^{(q)}$ and $\beta_{mn}^{(q)}$ are known. Because the source current acts on the rings only, we apply the same technique with projections as applied in Chapter 4. Thus, as in (4.11), we introduce basis functions $\psi_l(z)$ that are equal to one on the rings of group l , and zero otherwise. Analogous to (4.19), projection of the current $j_\varphi(\varphi, z)$ on these basis functions applies only to $j_{r,\varphi}$, the φ -component of the current in the

rings, and yields

$$\Pi j_{r,\varphi} = \sum_{l=1}^L \frac{(j_{r,\varphi}, \psi_l)}{(\psi_l, \psi_l)} \psi_l = \sum_{l=1}^L \frac{R\hat{I}_l}{D_l} \psi_l, \quad (5.30)$$

where Π is the projection operator and $\hat{I}_l = I_l / (j^c R)$. Following the procedure starting at (4.20), $j_{r,\varphi}$ is decomposed into two parts according to

$$j_{r,\varphi} = \Pi j_{r,\varphi} + (I - \Pi) j_{r,\varphi} = \sum_{l=1}^L \frac{R\hat{I}_l}{D_l} \psi_l + j_{\perp}. \quad (5.31)$$

Here, $j_{\perp}$ is in the orthoplement of the range of Π , i.e. $(j_{\perp}, \psi_l) = 0$, for $l = 1, \dots, L$. From the operator matrix equation (5.21), we deduce

$$\begin{pmatrix} j_{r,\varphi} \\ j_{i,\varphi} \end{pmatrix} - i\kappa \begin{pmatrix} K_{rr} & K_{ri} \\ K_{ir} & K_{ii} \end{pmatrix} \begin{pmatrix} j_{r,\varphi} \\ j_{i,\varphi} \end{pmatrix} = \begin{pmatrix} j_{r,\varphi}^s \\ 0 \end{pmatrix}. \quad (5.32)$$

Multiplying the first row in (5.32) by $(I - \Pi)$, we obtain

$$\begin{pmatrix} j_{\perp} \\ j_{i,\varphi} \end{pmatrix} - i\kappa \begin{pmatrix} (I - \Pi)K_{rr} & (I - \Pi)K_{ri} \\ K_{ir} & K_{ii} \end{pmatrix} \begin{pmatrix} j_{r,\varphi} \\ j_{i,\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (5.33)$$

or equivalently,

$$\begin{pmatrix} j_{\perp} \\ j_{i,\varphi} \end{pmatrix} - i\kappa \begin{pmatrix} (I - \Pi)K_{rr} & (I - \Pi)K_{ri} \\ K_{ir} & K_{ii} \end{pmatrix} \begin{pmatrix} j_{\perp} \\ j_{i,\varphi} \end{pmatrix} = i\kappa \begin{pmatrix} \sum_{l=1}^L \frac{R\hat{I}_l}{D_l} (I - \Pi)K\psi_l \\ 0 \end{pmatrix}. \quad (5.34)$$

We have eliminated $j_{r,\varphi}$ in favor of $j_{\perp}$ and the constants C_l have disappeared in favor of the prescribed currents I_l . The source current $j_{\varphi}^s(z)$ is completely covered by the family of basis functions $\psi_l(z)$.

The coefficient $\alpha_{00}^{(q)}$ in (5.23) represents the total current through a ring, because all other basis functions integrate to zero. Within one group of rings the coefficients $\alpha_{00}^{(q)}$ can still be different for each ring. We denote the highest degree of the basis functions on the q^{th} ring or island in z -direction and in φ -direction by $M_z^{(q)}$ and $M_{\varphi}^{(q)}$, respectively. To distinguish between rings in a group, we need an additional number of $N_r - L$ basis functions. The construction of these basis functions has been presented in Chapter 4, at the end of Section 4.2. The number of basis functions on the rings, denoted by M_r , is thus

$$M_r = \sum_{q=1}^{N_r} M_z^{(q)} (2M_{\varphi}^{(q)} + 1) + N_r - L, \quad (5.35)$$

whereas the number of basis functions on the islands, denoted by M_i , is

$$M_i = \sum_{q=1}^{N_i} M_z^{(N_r+q)} M_{\varphi}^{(N_r+q)}. \quad (5.36)$$

The total number of basis functions is equal to $M = M_r + M_i$.

The current $j_\perp(\varphi, z)$ is approximated by a finite series of basis functions as in Chapter 4. For the current $j_{i,\varphi}$ we act accordingly. We denote each basis function by $\phi_\mu(\varphi, z)$, $\mu = 1, \dots, M$, and the coefficients by $\alpha_{r,\mu}$ for the rings and $\alpha_{i,\mu}$ for the islands, such that

$$j_\perp(\varphi, z) \doteq \sum_{\mu=1}^{M_r} \alpha_{r,\mu} \phi_\mu(\varphi, z), \quad j_{i,\varphi}(\varphi, z) \doteq \sum_{\mu=M_r+1}^M \alpha_{i,\mu} \phi_\mu(\varphi, z). \quad (5.37)$$

For ϕ_μ the basis functions of (5.23) and (5.27) are used with an arrangement that is convenient for numerical implementation. The uniform basis functions on the rings are represented by $\phi_1, \dots, \phi_{N_r-L}$. Within the set of non-uniform basis functions on the rings, first the basis functions with the same z -order are clustered, and then the basis functions with the same φ -order. The same holds for the islands.

Substituting (5.37) into (5.34), computing the inner products with all separate basis functions and using the self-adjointness of the operator $(I - \Pi)$, we obtain

$$\begin{aligned} & \sum_{\mu=1}^{M_r} \alpha_{r,\mu} (\phi_\mu, \phi_\nu) - i\kappa \sum_{\mu=1}^{M_r} \alpha_{r,\mu} (\mathbf{K}_{rr} \phi_\mu, \phi_\nu) - i\kappa \sum_{\mu=M_r+1}^M \alpha_{i,\mu} (\mathbf{K}_{ri} \phi_\mu, \phi_\nu) \\ &= i\kappa \sum_{l=1}^L \frac{R \hat{I}_l}{D_l} (\mathbf{K}_{rr} \psi_l, \phi_\nu), \end{aligned} \quad (5.38)$$

$$\begin{aligned} & \sum_{\mu=M_r+1}^M \alpha_{i,\mu} (\phi_\mu, \phi_\nu) - i\kappa \sum_{\mu=M_r+1}^M \alpha_{i,\mu} (\mathbf{K}_{ii} \phi_\mu, \phi_\nu) - i\kappa \sum_{\mu=1}^{M_r} \alpha_{r,\mu} (\mathbf{K}_{ir} \phi_\mu, \phi_\nu) \\ &= 0. \end{aligned} \quad (5.39)$$

Two basis functions on different rings (islands) are always orthogonal, because they are defined for distinct z -values. Moreover, two different basis functions on the same ring (island) are orthogonal, because of the orthogonality of Legendre polynomials and goniometric functions.

From (5.38) - (5.39), we obtain a linear set of equations from which the coefficients $\alpha_{r,\mu}$, $\alpha_{i,\mu}$ can be determined. This will be elaborated in the following subsection. The solution procedure consists of the following steps:

1. Calculate $\alpha_{r,\mu}$, $\alpha_{i,\mu}$ from (5.38) - (5.39); for this, the results of Subsection 5.2.3 (where it is indicated how the inner products in (5.38) - (5.39) can be calculated) are needed.
2. Calculate $j_\perp(\varphi, z)$, $j_{i,\varphi}(\varphi, z)$ from (5.37).
3. Calculate $j_\varphi(\varphi, z)$ from (5.31).

This solution procedure is implemented in Eddy, the software tool developed for this research.

5.2.3 Calculation of the matrix elements

For the determination of the unknown coefficients $\alpha_{r,\mu}$, $\alpha_{i,\mu}$ from (5.38) - (5.39), we need to calculate the matrix elements occurring in this equation. For these matrix elements we distinguish the following three possible cases:

1. Matrix elements from basis functions of rings mutually, $(K_{rr}\phi_\mu, \phi_\nu)$.
2. Matrix elements from basis functions of a ring and an island, $(K_{ri}\phi_\mu, \phi_\nu) = (K_{ir}\phi_\mu, \phi_\nu)$.
3. Matrix elements from basis functions of islands mutually, $(K_{ii}\phi_\mu, \phi_\nu)$.

For each case, we follow the same procedure: we expand the current distribution on each ring and each island in a Fourier series in the φ -direction, after which we use the cosine-series for K_φ according to (5.10), to evaluate $K_{rr}j_\perp$ etc., using the orthogonality of the $\cos(n\varphi)$ and $\sin(n\varphi)$ functions, and that the operators are bounded. Thus, we start with the Fourier series for the current distribution on an arbitrary ring or island,

$$j_\varphi(\varphi, z) = j_0(z) + \sum_{m=1}^{\infty} j_m^{(c)}(z) \cos(m\varphi) + \sum_{m=1}^{\infty} j_m^{(s)}(z) \sin(m\varphi), \quad (5.40)$$

where the coefficients $j_m^{(c)}(z)$ and $j_m^{(s)}(z)$ are still unknown and will be written in terms of Legendre polynomials further on.

In (5.10), we expressed the kernel K_φ as a Fourier cosine series. Accordingly, we define ($m \geq 1$)

$$k_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_\varphi(\varphi, z) d\varphi = \frac{1}{4\pi^2} Q_{\frac{1}{2}}(\chi), \quad (5.41)$$

$$k_m(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_\varphi(\varphi, z) \cos(m\varphi) d\varphi = \frac{1}{4\pi^2} (Q_{m-\frac{3}{2}}(\chi) + Q_{m+\frac{1}{2}}(\chi)), \quad (5.42)$$

where we used (5.14). Thus, we obtain

$$Kj_\varphi = (k_0 * j_0) + \sum_{m=1}^{\infty} (k_m * j_m^{(c)}) \cos(m\varphi) + \sum_{m=1}^{\infty} (k_m * j_m^{(s)}) \sin(m\varphi), \quad (5.43)$$

with $*$ denoting the convolution with respect to the z -coordinate.

We proceed with the computation of $(K_{rr}\phi_\mu, \phi_\nu)$. We consider two rings, q_1 and q_2 , situated at $[z_0^{(q_1)}, z_1^{(q_1)}]$ and $[z_0^{(q_2)}, z_1^{(q_2)}]$, respectively. For this case, the basis functions are defined in (5.23). Define $B_{m_1, n_1}^{(q_1)}$ as a basis function on q_1 and $B_{m_2, n_2}^{(q_2)}$ as a basis function on q_2 , where $m_1, m_2 \geq 0$ correspond to the order of the trigonometric function in the φ -direction and $n_1, n_2 \geq 0$ correspond to the order of the Legendre polynomial in the z -direction. The inner products needed in (5.38) then become

$$(KB_{m_1, n_1}^{(q_1)}, B_{m_2, n_2}^{(q_2)}) = (1 + \delta_{m_1 0}) \delta_{m_1 m_2} \pi \hat{A}_{m_1, n_1, n_2}, \quad (5.44)$$

with $\hat{A}_{m_1, n_1, n_2}$ defined by

$$\hat{A}_{m_1, n_1, n_2} = \int_{z_0^{(q_2)}}^{z_1^{(q_2)}} (k_{m_1} * P_{n_1}) P_{n_2} dz. \quad (5.45)$$

For $m_1 = m_2 \geq 1$ the inner product $(KB_{m_1, n_1}^{(q_1)}, B_{m_2, n_2}^{(q_2)})$ only gives a nonzero solution if m_1 and m_2 refer either both to a cosine function, or both to a sine function. We recall that the Legendre polynomials in (5.45) are the scaled and shifted versions as introduced in (5.23). The computation of $\hat{A}_{m_1, n_1, n_2}$ is discussed in (5.48).

We continue with the second case of a ring q_1 and an island q_2 , for the determination of $(K_{ri}\phi_\mu, \phi_\nu)$ and $(K_{ir}\phi_\mu, \phi_\nu)$. On the island q_2 , positioned at $[z_0^{(q_2)}, z_1^{(q_2)}]$, we denote a basis function as $B_{m_2, n_2}^{(q_2)}$, where m_2 is the order in the φ -direction and n_2 is the order in the z -direction. In order to get 2π periodic functions, we expand $B_{m_2, n_2}^{(q_2)}$ as follows:

$$B_{m_2, n_2}^{(q_2)}(\varphi, z) = \left(a_{m_2, 0}^{(q_2)} + \sum_{p=1}^{\infty} a_{m_2, p}^{(q_2)} \cos(p\varphi) + \sum_{p=1}^{\infty} b_{m_2, p}^{(q_2)} \sin(p\varphi) \right) P_{n_2} \left(\frac{z - c_z^{(q_2)}}{d_z^{(q_2)}} \right), \quad (5.46)$$

in which $a_{m_2, p}$, $b_{m_2, p}$ are the Fourier coefficients referring to the φ -dependent basis functions $\sin[m_2\pi(\varphi - \varphi_0^{(q_2)})/2d_\varphi^{(q_2)}] \mathbf{1}_{[\varphi_0^{(q_2)}, \varphi_1^{(q_2)}]}$.

The inner products $(KB_{m_1, n_1}^{(q_1)}, B_{m_2, n_2}^{(q_2)})$ become

$$(KB_{m_1, n_1}^{(q_1)}, B_{m_2, n_2}^{(q_2)}) = \begin{cases} (1 + \delta_{m_1 0})\pi a_{m_2, m_1}^{(q_2)} \hat{A}_{m_1, n_1, n_2}, & \text{if } m_1 \geq 0 \text{ and 'cos'}, \\ \pi b_{m_2, m_1}^{(q_2)} \hat{A}_{m_1, n_1, n_2}, & \text{if } m_1 \geq 1 \text{ and 'sin'}. \end{cases}$$

Finally, for the determination of $(K_{ii}\phi_\mu, \phi_\nu)$, we consider two islands q_1 and q_2 to obtain

$$\begin{aligned} (KB_{m_1, n_1}^{(q_1)}, B_{m_2, n_2}^{(q_2)}) &= 2\pi a_{m_1, 0}^{(q_1)} a_{m_2, 0}^{(q_2)} \hat{A}_{0, n_1, n_2} \\ &+ \pi \sum_{p=1}^{\infty} (a_{m_1, p}^{(q_1)} a_{m_2, p}^{(q_2)} + b_{m_1, p}^{(q_1)} b_{m_2, p}^{(q_2)}) \hat{A}_{p, n_1, n_2}. \end{aligned} \quad (5.47)$$

For all inner products derived above, integrals have to be computed in which only Legendre polynomials appear. Let q_1 and q_2 be two strips, which can be either a ring or an island. In each of the three cases, we obtain integrals of the following general form

$$\hat{A}_{p, n_1, n_2} = \int_{z_0^{(q_2)}}^{z_1^{(q_2)}} (k_p * P_{n_1}) P_{n_2} dz = d_z^{(q_1)} d_z^{(q_2)} \int_{-1}^1 \int_{-1}^1 f(\chi, p) P_{n_1}(\zeta) P_{n_2}(z) d\zeta dz, \quad (5.48)$$

where, from (5.41) - (5.42),

$$f(\chi, p) = \begin{cases} \frac{1}{4\pi^2} Q_{\frac{1}{2}}(\chi), & \text{if } p = 0, \\ \frac{1}{4\pi^2} (Q_{p-\frac{3}{2}}(\chi) + Q_{p+\frac{1}{2}}(\chi)), & \text{if } p \geq 1, \end{cases}$$

and, from (5.11) after transformation to the scaled and shifted variables,

$$\chi = \frac{2 + (d_z^{(q_1)} z - d_z^{(q_2)} \zeta + c_z^{(q_1)} - c_z^{(q_2)})^2}{2}. \quad (5.49)$$

For simplicity of notation, we have used here χ , where a fully consistent notation would have been $\chi = \chi^{(q_1, q_2)}(z, \zeta)$.

In Subsection 5.2.1, we have derived the asymptotic expansion of $Q_{p-\frac{1}{2}}(\chi)$, $p \geq 0$, for $\chi \rightarrow 1$. Accordingly, $f(\chi, p)$ has a logarithmic singularity at $\chi = 1$, for all $p \geq 0$, given by (5.19). For the computation of the double integral in (5.48) we apply the same procedure as in the previous chapter. First, we extract the logarithmic part. The remaining part is regular and can be computed numerically in a straightforward way. For the logarithmic part we use formula (3.61). This analytical formula enables us to perform fast computations and to obtain accurate results.

The basis functions have not been normalized, so the Gram matrix is not equal to the identity matrix. However, due to the orthogonality of the basis functions ϕ_μ , the Gram-matrix $\mathbf{G}$ is a diagonal matrix. For basis functions on rings, see (5.23), we obtain for $\mu, \nu \in \{N_r - L + 1, \dots, M_r\}$ and corresponding m, n

$$\begin{aligned} G_{\mu\nu} &= (\phi_\mu, \phi_\nu) = \delta_{\mu\nu} \int_{z_0^{(q)}}^{z_1^{(q)}} \int_{-\pi}^{\pi} \cos^2(m\varphi) P_n^2\left(\frac{z - c_z^{(q)}}{d_z^{(q)}}\right) d\varphi dz \\ &= \frac{2\pi d_z^{(q)}}{2n+1} (\delta_{m0} + 1) \delta_{\mu\nu}. \end{aligned} \quad (5.50)$$

Here, cosine functions are used, with $m \geq 0$, $n \geq 1$. In case of sine functions with $m \geq 1$, $n \geq 1$, we obtain the same result. The first $N_r - L$ elements on the diagonal of $\mathbf{G}$ are obtained from the procedure described in Section 4.2, making use of the matrix in (4.35)

As rings and islands are disjoint, their basis functions yield zeros in the Gram matrix. The elements of the Gram-matrix corresponding to basis functions of islands, see (5.27), are for $\mu, \nu \in \{M_r + 1, \dots, M\}$

$$\begin{aligned} G_{\mu\nu} &= (\phi_\mu, \phi_\nu) = \delta_{\mu\nu} \int_{z_0^{(q)}}^{z_1^{(q)}} \int_{\varphi_0^{(q)}}^{\varphi_1^{(q)}} \sin^2\left(\frac{m\pi(\varphi - \varphi_0^{(q)})}{2d_\varphi^{(q)}}\right) P_n^2\left(\frac{z - c_z^{(q)}}{d_z^{(q)}}\right) d\varphi dz \\ &= \frac{2d_z^{(q)} d_\varphi^{(q)}}{2n+1} \delta_{\mu\nu}. \end{aligned} \quad (5.51)$$

5.3 Numerical results

In this section, we present the results of the numerical simulations of three different configurations: one ring and one island, two rings and one island, and two rings and four islands between the two rings. All simulations make use of the method described in this chapter; see

also [32]. We use our software tool Eddy to compute the current distributions. In Eddy, the central z -positions of the rings and islands have to be defined as well as the central φ -positions for the islands. Other geometrical inputs are the widths of the rings and islands, the lengths of the islands, the thickness h and the radius R of the cylinder. Source currents are only applied to the rings. We can assign which rings are connected in a group, what the total current through each group is, the frequency of the source, and the number of basis functions. Furthermore, extra options can be chosen for postprocessing, such as plotting the amplitude of the current density and the streamlines of the currents in the rings and islands.

In Eddy, first the coefficients $\alpha_{r,\mu}$, $\alpha_{i,\mu}$ are determined from (5.38) - (5.39), then $j_\perp$, $j_{i,\varphi}$ from (5.37) and finally, the total current j_φ from (5.31).

5.3.1 One ring and one island

The first configuration we consider, consists of one ring together with one island, both on a cylinder of radius $R = 0.35$ m. The width of the ring is 0.04 m and its center is positioned at $z = -0.04$ m. It carries a total current of 600 A at frequency f . The island has a width of 2 cm and its length $l := (\varphi_1^{(1)} - \varphi_0^{(1)})R$ is a quarter of the circumference of the cylinder, i.e. $l = 2\pi R/4 \approx 0.55$ m. The center of the island is positioned at $(\varphi, z) = (0, 0)$. Due to the time-dependent source current in the ring, eddy currents are induced in both ring and island. The coefficients of the current distributions are computed by Eddy.

Figure 5.3: One ring of 4 cm width with a total current of 600 A and one island of 2 cm width and 55 cm length. (a) Amplitude of the current density in φ -direction on the line $\varphi = 0$, at frequencies $f = 100$ Hz (*), $f = 400$ Hz ($\circ$), $f = 700$ Hz (+), $f = 1000$ Hz ($\triangle$); (b) Streamlines of the current in the island at frequency $f = 1000$ Hz, with starting points at $\varphi = 0$ and z such that $|j_\varphi(0, z)| = 1000, 2000, \dots, 10000$ A/m.

In Figure 5.3 (a), the amplitude of the current density in φ -direction, $|j_\varphi|$, on the line $\varphi = 0$ is plotted as a function of z . We observe a distribution over the width of the ring that is slightly different from the one in the configuration of only one ring; see Figure 4.4 (a). The reason for this is that an eddy current has been induced in the island, which at his turn affects the distribution in the ring. The current in the island, on the side of the ring, is in such a direction that it opposes the magnetic field caused by the ring. Thus, this direction is opposite to the one in the ring. We have seen in Section 3.4 and Section 4.3 that opposite currents amplify each other. Therefore, the amplitude of the current at $z = -0.02$ is higher than at $z = -0.06$.

In the island, the current has to be free of divergence. This implies for instance that the total normal current across the line $\varphi = 0$ is zero. Thus, $j_\varphi(0, z)$ integrated from $z = -0.01$ to $z = 0.01$ must be equal to zero. In Figure 5.3 (a), the amplitude $|j_\varphi(0, z)|$ is shown, which is greater than zero everywhere, but the sign of $j_\varphi(0, z)$ changes at the point where $|j_\varphi(0, z)| = 0$. Integration over z indeed yields a zero total normal current. We observe an edge-effect in the patch and a stronger amplitude on the side of the ring, amplified by the opposite current in the ring.

In Figure 5.3 (b), the streamlines of the current in the island are shown. We observe the streamlines forming closed cycles, revealing that the currents are divergence-free. Hence, the induced eddy currents are visualized as eddies in this figure. Note that the eye (i.e. the center) of the eddy is not at the center of the island, but closer to the ring. This is because on that side the magnetic field and, thus, the induced current are stronger.

To each streamline, a start position has to be assigned. Here, we have chosen the start positions from Figure 5.3 (a): along the line $\varphi = 0$ the amplitude of the current, $|j_\varphi(0, z)|$, has been divided in equidistant steps of 1000 A/m; the corresponding z -coordinates yield the start positions of the streamlines. In this way, the edge-effect is represented in Figure 5.3 (b) by the density of the streamlines.

The current in the ring is mainly in the φ -direction. In contrast to the configuration of one ring only, the current here also has a z -component. This z -component is induced by the island. However, compared to j_φ , j_z is very small. Therefore, the streamlines in the ring would be straight lines, on the scale of this figure, encircling the ring (cf. Figure 5.6 (b), further on).

5.3.2 Two rings and one island

In the second configuration we consider, we put an additional ring to the right of the island. This ring also has a width of 4 cm, is positioned with its center at $z = 0.04$ m, and carries a current of 600 A. The coefficients of the current distributions are computed from Eddy.

First, we take the currents in the rings in anti-phase. In Figure 5.4 (a), the amplitude of the current in the φ -direction on the line $\varphi = 0$ is shown. The current in the rings is quite similar to the current in two plane rectangular strips of infinite length; see Figure 3.8 (a). This shows once more that the current in plane rectangular strips provides a good approximation for the

Figure 5.4: Two rings of 4 cm width with a total current of 600 A in anti-phase and one island of 2 cm width and 55 cm length between them. (a) Amplitude of the current density in φ -direction on the line $\varphi = 0$, at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz (Δ); (b) Streamlines of the current in the island at frequency $f = 1000$ Hz, with starting points at $\varphi = 0$ and z such that $|j_\varphi(0, z)| = 2000, 4000, \dots, 20000$ A/m.

current in rings. The small difference between Figure 5.4 (a) and Figure 3.8 (a) (only to observe at the edges nearest to the island and only for high frequencies) is caused by the island. An eddy current is induced in the island, which at its turn induces eddy currents in the rings. Therefore, the currents $|j_\varphi(0, z)|$ at $z = \pm 0.02$ in Figure 5.4 (a) are a bit larger than those in Figure 3.8 (a). Comparing Figure 5.4 (a) with Figure 5.3 (a), we see that the induced current in the island is now symmetric and twice as strong, in conformity with our expectations.

In Figure 5.4 (b), streamlines of the current in the island are shown. The start positions of the streamlines are on the line $\varphi = 0$, with z -coordinates such that $|j_\varphi(0, z)|$ makes steps of 2000 A/m. We observe one eddy with its eye at the center of the island; the geometry and, thus, the current distribution are anti-symmetric in both φ and z .

In Figure 5.5, the results are shown for the situation in which the currents in the rings are in phase. The amplitude of the current is shown in Figure 5.5 (a). We observe local edge-effects in the rings and in the island, and a global edge-effect in the whole system. The current in the rings is again comparable to the current in two plane rectangular strips of infinite length; see Figure 3.8 (c). No difference can be observed on the scale of these figures; this is clear because the amplitude of the eddy current in the island is much smaller than in the anti-phase case.

The reason that the current in the island is in this case smaller than in the previous example

Figure 5.5: Two rings of 4 cm width with a total current of 600 A in phase and one island of 2 cm width and 55 cm length between them. (a) Amplitude of the current density in φ -direction on the line $\varphi = 0$, at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz (Δ); (b) Streamlines of the current in the island at frequency $f = 1000$ Hz, with starting points at $\varphi = 0$ rad and z such that $|j_\varphi(0, z)| = 300, 600, \dots, 3000$ A/m.

is that now two eddies are induced. The two eddies are visualized by the streamlines in Figure 5.5 (b). The start positions are on the line $\varphi = 0$ with z -coordinates such that $|j_\varphi(0, z)|$ makes steps of 300 A/m. The edge-effects are also visible in this figure: we observe the eddies with eyes close to the edge of the island.

5.3.3 Two rings and four islands

In the third configuration, we consider two rings at the same positions as in the previous example, but now with four islands in between. The four islands are with their centers at the line $z = 0$ and each island has a length of 50 cm. This means that all together they almost form a complete ring. So, we consider a geometry here that models a ring, which is cut in width-direction in four pieces. This mimics a slitted ring, as can occur in real coils. The source currents are applied to the rings, are in anti-phase, and have an intensity of 600 A.

In Figure 5.6 (a), the amplitude of the current in the φ -direction on the line $\varphi = \pi/4$ is shown. This line crosses the center of one of the islands. Due to symmetry the current distributions are the same in all four islands. The current in the rings is similar to the current in the previous example with one island; see Figure 5.4 (a). Compared with the distribution in one island, the currents in the four islands have a higher amplitude. The reason for this is that the currents in the islands also amplify each other.

In Figure 5.6 (b), streamlines of the currents in the rings and in the islands are shown. The

Figure 5.6: Two rings of 4 cm width with a total current of 600 A in anti-phase and four islands of 2 cm width and 50 cm length between them. (a) Amplitude of the current density in φ -direction on the line $\varphi = \pi/4$, at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz (Δ); (b) Streamlines of the currents at frequency $f = 1000$ Hz, with starting points at central φ positions and z such that $|j_\varphi(0, z)|$ makes steps of $6.0 \cdot 10^3$ A/m.

streamlines in the rings are straight, indicating that the induced z -component of the current is much smaller than the φ -component. The same behavior can be observed for the preceding two examples. We observe that in each island one eddy appears. For the islands, the start positions of the streamlines are chosen on the central lines $\varphi = -3\pi/4, -\pi/4, \pi/4, 3\pi/4$, with z -coordinates such that $|j_\varphi(\varphi, z)|$ makes steps of $6.0 \cdot 10^3$ A/m. The eyes of the eddies are at the centers of the islands due to symmetry. For the rings, the start positions are chosen on the line $\varphi = -\pi$, with z -coordinates such that $|j_\varphi(-\pi, z)|$ makes steps of $6.0 \cdot 10^3$ A/m.

An interesting aspect in this example is that the four islands behave qualitatively as one ring. Namely, on the edges between two neighboring islands, the currents are in opposite direction, such that the resulting magnetic fields from these currents are canceled. Moreover, on the lines $z = -0.01$ and $z = 0.01$, the currents are in the same direction. So, from a certain distance, the currents are observed as distributed in a ring. Consequently, the magnetic field at some distance from the surface of the cylinder (e.g. near the central axis) is affected as if a ring, instead of four islands, is present on the cylinder.

Apart from the magnetic field, the islands also affect the resistance and the self-inductance of the system. For the computation of the resistance and the self-inductance, we introduce the matrices $\mathbf{G}$ and $\mathbf{A}$ and the column vector $\mathbf{a}$. Here, $\mathbf{G}$ denotes the diagonal matrix with the inner products of the separate basis functions (ϕ_μ, ϕ_ν) as diagonal elements. The elements of matrix $\mathbf{A}$ are represented by the inner products $(K_{rr}\phi_\mu, \phi_\nu)$, $(K_{ri}\phi_\mu, \phi_\nu)$, $(K_{ir}\phi_\mu, \phi_\nu)$,

(a)

(b)

(c)

(d)

Figure 5.7: Resistance and self-inductance of a set of two rings (*), a set of two rings with one island in between ($\circ$), a set of two rings with four islands in between (+), and a set of three rings ($\triangle$). (a) $\bar{R}$ at frequencies up to $f = 10000$ Hz; (b) $\bar{R}$ at frequencies up to $f = 400$ Hz; (c) $\bar{L}$ at frequencies up to $f = 10000$ Hz; (d) $\bar{L}$ at frequencies up to $f = 400$ Hz.

$(\mathbf{K}_{ii}\phi_\mu, \phi_\nu)$. The vector $\mathbf{a}$ consists of the coefficients belonging to the basis functions. The time-averaged dissipated power is defined by (2.151), and can be expressed in terms of $\mathbf{G}$ and $\mathbf{a}$ as

$$\bar{P}_{diss} = \frac{R^2 |j^c|^2}{2h\sigma} \int_{S_U} \mathbf{j}(\mathbf{x}) \cdot \mathbf{j}^*(\mathbf{x}) da(\mathbf{x}) = \frac{R^2 |j^c|^2}{2h\sigma} \mathbf{a}^* \mathbf{G} \mathbf{a}. \quad (5.52)$$

Moreover, $I_e^2 = II^*/2 = 2|j^c|^2 R^2 d_z^2$, with d_z here denoting half the (dimensionless) width of a ring, such that we obtain from (2.66):

$$\bar{R} = \frac{1}{4h\sigma d_z^2} \mathbf{a}^* \mathbf{G} \mathbf{a}. \quad (5.53)$$

The time-averaged magnetic energy is defined by (2.152), and can be expressed in terms of $\mathbf{A}$ and $\mathbf{a}$ as

$$\bar{U}_m = \frac{\mu_0 R^3 |j^c|^2}{16\pi} \int_{S_U} \int_{S_U} \frac{\mathbf{j}(\boldsymbol{\xi}) \cdot \mathbf{j}^*(\mathbf{x})}{|\mathbf{x} - \boldsymbol{\xi}|} da(\boldsymbol{\xi}) da(\mathbf{x}) = \frac{\mu_0 R^3 |j^c|^2}{4} \mathbf{a}^* \mathbf{A} \mathbf{a}. \quad (5.54)$$

Using (2.67), we obtain

$$\bar{L} = \frac{\mu_0 R}{4d_z^2} \mathbf{a}^* \mathbf{A} \mathbf{a}. \quad (5.55)$$

In Figure 5.7, we show the resistance and the self-inductance as a function of the frequency, for four different configurations.

First, two rings are considered with a width of 4 cm and a distance between the centers of 8 cm. The currents in the rings are in anti-phase. The resistance $\bar{R}_{DC}$ at $f = 0$ Hz can be computed analytically,

$$\bar{R}_{DC} = \frac{4\pi R}{hD\sigma} \approx 7.48 \cdot 10^{-4} \Omega. \quad (5.56)$$

The resistance increases with the frequency; see the lowest line in Figure 5.7 (a), where the frequency range is up to $f = 10000$ Hz.

Second, an island is placed between the two rings. This configuration is the same as in the example of Subsection 5.3.2. The currents in the rings are still in anti-phase. In case of a stationary current, no eddy currents are induced. In other words, the system does not notice the presence of the island. Therefore, the resistance $\bar{R}_{DC}$ at $f = 0$ Hz is the same as for two rings only. The resistance increases a bit more with the frequency than in the two rings case; see second lowest line in Figure 5.7 (a).

Third, the configuration of this subsection is considered, consisting of two rings and four islands. The resistance $\bar{R}_{DC}$ at $f = 0$ Hz is the same as in the previous two configurations. For higher frequencies, eddy currents are induced in all four islands. Together, they oppose the magnetic field more than the eddies in one island, such that the total resistance of the system is higher. This is shown by the second highest line in Figure 5.7 (a).

The fourth configuration consists of the two rings from the first case, with an additional ring between them, not connected to any source. Again, for a stationary current in the two rings

that are connected to the source, no eddy currents are induced. Thus, $\bar{R}_{DC} = 7.48 \cdot 10^{-4} \Omega$. For higher frequencies, a current is induced in the additional ring that forms a closed cycle around the cylinder. Due to the anti-symmetry in the system (applied currents are in anti-phase), the total current in the additional ring is equal to zero. However, due to the closed cycles of the current, the magnetic field is opposed more than in the previous configurations. Therefore, the total resistance of the system is higher, as can be observed from the highest line in Figure 5.7 (a).

In Figure 5.7 (b), the resistance is shown for the smaller range of frequencies up to $f = 400$ Hz, with a higher resolution than used in the range from 0 to 10000 Hz. All points of inflection in the resistances of the four configurations lie in this range. In Figure 5.7 (c), the self-inductances of the four configurations are shown for the frequency range up to $f = 10000$ Hz. We observe that the self-inductances decrease with the frequency and that the order of the lines is opposite to the order in the figure of the resistances, i.e. the two ring case is represented by the highest line and the three ring case is represented by the lowest line. The self-inductances in the smaller range from 0 to 400 Hz are shown in Figure 5.7 (d). All points of inflection in the self-inductance of the four configurations lie in this range.

5.4 Summary

In this chapter, we have modeled a set of rings and islands on a cylindrical surface. The islands represent the pieces of copper that are present in a gradient coil to fill up empty spaces for better heat transfer in the coil and higher stability of the coil. Another application of this island model is when slits have been cut in the copper strips for the reduction of the eddy currents. Due to the islands the current distribution is not rotational symmetric, in contrast to the current in a set of rings, as presented in the previous chapter.

In the set of rings and islands, the source current is applied only to the rings. Therefore, the preferable direction of the eddy currents induced in the islands is the tangential direction. As described in Chapter 3, the kernel then has a logarithmic singularity in the axial coordinate. In Section 5.2, the kernel is expressed in terms of Legendre functions of the second kind of odd-half-integer order. The asymptotic expansion of these functions near the singularity point contains the logarithmic function.

We split off the logarithmic part of the kernel and the resulting regular part is solved by numerical integration. For the logarithmic part, we can use the analytical formula derived in Section 3.2 and the Legendre polynomials are used as basis functions. In the tangential coordinate, Fourier modes are used, because of the 2π -periodicity and the orthogonality with the trigonometric functions in the expansion of the kernel function. In the resulting set of equations, the mutual relation between the rings and the islands is explicitly expressed by the coupling between the modes.

The coefficients of the basis functions are determined by our software tool `Eddy`. The result-

ing current distribution shows edge-effects in both rings and islands. We have shown that the total resistance increases when an island is added to the system. The opposite holds for the self-inductance of the system; it decreases when an island is added. In case a ring is added to the system, the resistance increases even more, whereas the self-inductance decreases even more. This additional ring is well approximated by a set of islands, as such describing a ring that is cut in pieces.

Plane circular strips

From the three possible types of strips on a cylinder, as we have defined in Section 2.2, we have discussed examples of the closed loops around the cylinder (rings) and the patches on the cylinder (islands) in the previous two chapters. In this chapter, we consider strips of the third type; the closed loops on top of the cylinder. With this geometry of strips, we present a model for the x-coil and the y-coil (see e.g. Figure 2.1 (a)).

According to Section 3.1, an integral equation of the second kind can be derived for such a configuration, with a kernel showing a logarithmic behavior in the width direction of the strip. Exploiting this logarithmic behavior, we construct a series expansion for the current distribution in terms of Legendre polynomials.

In Section 6.1, we describe the geometry of the model and the derivation of the integral equation. The solution method is presented in Section 6.2 and numerical results are shown in Section 6.3. Finally, in Section 6.4, the summary of this chapter is presented.

6.1 Problem formulation

In this chapter, we consider the case of one or more closed circular loops of strips on top of a cylinder of radius R . The geometrical description of such a loop is given at the end of Section 2.2; we will recapitulate the main results here. As always, the loops are assumed to be infinitely thin strips of uniform width D . The central line of the loop is described by the position vector $\mathbf{c}(u)$, where u is the tangential coordinate along the central line. Mathematically, we can imagine this central line as the intersection of the basic cylinder of radius R and another virtual cylinder of radius R_s . The axis of the latter cylinder is normal to that of the first cylinder, which is the z -axis. To make things more concrete, we assume that the axis of the second cylinder coincides with the y -axis of our Cartesian system of coordinates. In order to obtain a closed intersection curve, we must require $R_s < R$. In that case, $\mathbf{c}(u)$ is as given in Section 2.2, equation (2.84). Here, we restrict ourselves to $R_s < R$, but loops with

$R_s > R$ can be dealt with analogously. As in the preceding chapters, besides the case of one circular loop, we consider the case of a set of concentric loops of different radii.

To the central line of a loop, a set of local unit coordinate vectors $\{\mathbf{e}_r(u), \mathbf{e}_u(u), \mathbf{e}_v(u)\}$ is connected, as given by (2.85) - (2.87). The vector $\mathbf{e}_u$ is tangent to the central line, $\mathbf{e}_r$ normal to the basic cylinder (thus coinciding with the r -direction of a cylindrical coordinate system), and $\mathbf{e}_v$ points towards the center of the circular strip. Let us denote, for the moment, the coordinate in the width direction of the strip by v , so $v \in [-D/2, D/2]$.

Figure 6.1: *The configuration of a set of plane circular strips.*

For the system we consider, we generally have $D \ll R$. This makes the following approximation possible. As we have already noticed at the end of Section 4.1, for $D \ll R$, we can locally neglect the curvature of the strips. Globally, the current in a point is not affected by currents on a large distance. This means that, in a point of the loop, thus locally, we may neglect the curvature $1/R$ of the basic cylinder. We can locally approximate the curved loop by a plane loop; see also Figure 4.3. We should realize that this absolutely does not mean that we approximate the whole curved circular loop by a plane loop. As the current in a point is only affected by the currents in its vicinity, as far as the current distribution in this point concerns, it does not matter how the rest of the loops is shaped. Thus, as we are in first place interested in this current distribution, we may just as well replace the current circular loop (on the cylinder) by a plane one. However, it must be emphasized that this does not hold when we want to calculate the magnetic field induced by the currents in the curved circular loops on top of the cylinder.

All this holds irrespectively of the value of R_s , as long as $R_s < R$. Even if R_s becomes small with respect to R , say in the order of D , this approximation may still be applied.

Hence, since in this chapter we will only calculate the current distributions in the loops, we can approximate them by plane circular loops; the configuration of a set of plane circular loops is depicted in Figure 6.1. In this plane configuration, we replace the width coordinate v (measured as the distance from the central line) by a radial coordinate r , measured from the center of the ring (the notation r should not be confused with the r -coordinate of the cylindrical coordinates). Moreover, we replace the tangential coordinate u by the angle φ , as introduced in Figure 6.1.

We consider a set of N ($N \geq 1$) circular strips, which are lying concentrically in a plane; see Figure 6.1. The plane is described in terms of the polar coordinates r and φ . Each separate strip is of uniform width. Due to the symmetry in the geometry, the current does not depend on φ and flows only in the φ -direction, so $\mathbf{j} = j_\varphi(r)\mathbf{e}_\varphi$. The current distribution $j_\varphi(r)$ is the unknown we want to calculate here.

The loops occupy a surface $S_\cup$ on the plane. The set of loops is subdivided into L groups, each of which has a prescribed total current $I_l(t) = \Re\{I_l e^{-i\omega t}\}$, $l = 1, \dots, L$, with $I_l \in \mathbb{C}$ a complex constant, and is driven by a separate source current. The sum of the widths of all loops in group l is denoted by D_l . Let each group l consist of N_l loops, then $S_n^{(l)}$ can be defined as the surface of the n -th loop within group l , with $n \in \{1, \dots, N_l\}$ and $l \in \{1, \dots, L\}$, such that

$$S_\cup^{(l)} = \sum_{n=1}^{N_l} S_n^{(l)}, \quad \text{and} \quad S_\cup = \sum_{l=1}^L S_\cup^{(l)} = \sum_{l=1}^L \sum_{n=1}^{N_l} S_n^{(l)}, \quad (6.1)$$

and

$$S_n^{(l)} = \{(r, \varphi) \mid r_0^{(n)} \leq r \leq r_1^{(n)}, -\pi \leq \varphi \leq \pi\}. \quad (6.2)$$

Since the currents do not depend on the tangential direction φ , it is more convenient to introduce a notation for the r -intervals in which the loops are positioned. Instead of $S_\cup$, which represents a surface, we introduce the one-dimensional set $S_\cup^r$ as the collection of all radial intervals of the loops:

$$S_\cup^r = \sum_{n=1}^N [r_0^{(n)}, r_1^{(n)}]. \quad (6.3)$$

We can derive the set of equations in the same way as explained in Chapter 2, using Maxwell's equations, the boundary conditions, and the assumptions valid in our model. However, instead of showing all the steps that lead to the resulting integral equation, we present a shorter derivation, which makes use of the general formula for the vector potential, see (2.146).

Let j^c be the characteristic value for the surface current through the strips, and D^c a charac-

teristic distance, defined by

$$D^c = r_0^{(1)}, \quad j^c = \frac{\sum_{l=1}^L I_l}{\sum_{l=1}^L D_l}. \quad (6.4)$$

Thus, the distances are scaled by the inner radius of the first loop, i.e. the most inner one, and the current is scaled by the average current through all strips. We can write the vector potential $\mathbf{A}$ at position $\mathbf{x}$ in a dimensionless form $\tilde{\mathbf{A}}$, according to

$$\mathbf{A}(\mathbf{x}) = \mu_0 D^c j^c \tilde{\mathbf{A}}(\tilde{\mathbf{x}}), \quad (6.5)$$

as

$$\tilde{\mathbf{A}}(\tilde{\mathbf{x}}) = \frac{1}{4\pi} \int_{S_U} \frac{\tilde{\mathbf{j}}(\tilde{\boldsymbol{\xi}})}{|\tilde{\mathbf{x}} - \tilde{\boldsymbol{\xi}}|} d\mathbf{a}(\tilde{\boldsymbol{\xi}}), \quad (6.6)$$

where $\tilde{\mathbf{x}}$ is the dimensionless position, scaled by D^c , and $\tilde{\mathbf{j}}$ is the dimensionless current, scaled by j^c . The tildes are omitted from now on. We use the general setting as introduced in (2.188). On the plane circular strips we denote the observation point $\mathcal{P}$ and the source point $\mathcal{Q}$ by vectors $\mathbf{p}$ and $\mathbf{q}$, in polar coordinates given by

$$\mathbf{p} = (r, \varphi), \quad \mathbf{q} = (\rho, \theta), \quad (6.7)$$

respectively. The distance between the points is $\mathcal{R}(\mathcal{P}, \mathcal{Q})$. The vector potential can be computed from

$$\mathbf{A}|_{\mathcal{P}} = T_p^{-1} \frac{1}{4\pi} \int_{S_U} \frac{1}{\mathcal{R}(\mathcal{P}, \mathcal{Q})} T_q \mathbf{j}|_{\mathcal{Q}} d\mathbf{a}(\mathcal{Q}), \quad (6.8)$$

where T_p^{-1} is given by (2.191) and T_q by (2.192). The transformations T_p^{-1} and T_q can be applied successively, as in (2.193). The distance between the two points is

$$\mathcal{R}(\mathcal{P}, \mathcal{Q}) = \sqrt{(r - \rho)^2 + 4r\rho \sin^2\left(\frac{\varphi - \theta}{2}\right)}. \quad (6.9)$$

For $\mathbf{j} = j_\varphi(r) \mathbf{e}_\varphi$, we then obtain

$$A_r(r, \varphi) = \frac{1}{4\pi} \int_{S_U} \frac{\sin(\varphi - \theta) j_\varphi(\rho)}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\mathbf{a}(\rho, \theta), \quad (6.10)$$

$$A_\varphi(r, \varphi) = \frac{1}{4\pi} \int_{S_U} \frac{\cos(\varphi - \theta) j_\varphi(\rho)}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\mathbf{a}(\rho, \theta). \quad (6.11)$$

Using Ohm's law, we obtain the integral equation of the second kind

$$j_\varphi(r) - \frac{i\kappa}{4\pi} \int_{S_U} \frac{\cos(\varphi - \theta) j_\varphi(\rho)}{\sqrt{(r - \rho)^2 + 4r\rho \sin^2\left(\frac{\varphi - \theta}{2}\right)}} d\mathbf{a}(\rho, \theta) = j_\varphi^s(r), \quad (6.12)$$

where $j_\varphi^s(r)$ is the source current and

$$\kappa = h\sigma\mu_0\omega D^c. \quad (6.13)$$

The current is independent of φ , so integrating (6.12) over φ from $-\pi$ to π , makes the φ -dependence in the kernel disappear, resulting in

$$\begin{aligned} j_\varphi(r) - j_\varphi^s(r) &= \frac{i\kappa}{8\pi^2} \int_{S_U^r} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\cos(\varphi - \theta) j_\varphi(\rho)}{\sqrt{r\rho} \sqrt{\frac{(r-\rho)^2}{r\rho} + 4 \sin^2(\frac{\varphi-\theta}{2})}} \rho \, d\rho \, d\theta \, d\varphi \\ &= \frac{i\kappa}{2\pi} \int_{S_U^r} \frac{j_\varphi(\rho)}{k\sqrt{r\rho}} [(2 - k^2)K(k) - 2E(k)] \rho \, d\rho. \end{aligned} \quad (6.14)$$

Here, $K(k)$ and $E(k)$ are the complete elliptic integral of the first and second kind, respectively, and

$$k^2 = \frac{4}{4 + (\frac{r-\rho}{\sqrt{r\rho}})^2} = \frac{4r\rho}{(r + \rho)^2}. \quad (6.15)$$

6.2 Solution method

6.2.1 Construction of a linear set of equations

According to Section 3.2, the Galerkin method could be applied to solve the integral equation (6.14). The kernel function in the integral equation is dominated by a logarithmic singularity in $(r - \rho)$, multiplied by r . This would lead to the choice of basis functions to be Legendre polynomials of the first kind, divided by an extra term $\sqrt{r}$. This extra term $\sqrt{r}$ is a consequence of the definition of the inner product of r -dependent functions, given by $L_2(S_U^r, r \, dr)$. However, since we have more information about the behavior of the currents, we adjust the solution method and use the Petrov-Galerkin method. For the plane circular loop, the extra information follows from the known solution of the direct current distribution. We explain briefly how the adjustment is carried out for one strip. The case of more strips has the same treatment and is explained further on.

Let the current distribution $j_\varphi(r)$ be expanded in a series of basis functions $\phi_m(r)$, with corresponding coefficients α_m , according to

$$j_\varphi(r) = \sum_{m=0}^{\infty} \alpha_m \phi_m(r). \quad (6.16)$$

The source current $j_\varphi^s(r)$ is expanded in the same basis functions $\phi_m(r)$, with coefficient β_m ,

$$j_\varphi^s(r) = \sum_{m=0}^{\infty} \beta_m \phi_m(r). \quad (6.17)$$

For an appropriate choice of the basis functions, we are led by the solution for the DC-situation. For $\omega \rightarrow 0$ (equivalently, $\kappa \rightarrow 0$), the inductive effects disappear, and $j_\varphi(r) \rightarrow j_\varphi^s(r)$. The analytical solution for the current distribution in one plane circular loop, carrying a total current I , is given by

$$j_\varphi(r) = \frac{I}{r D^c j^c \log(\frac{r_1}{r_0})}. \quad (6.18)$$

Consequently, the current is inversely proportional to r . Therefore, we introduce new functions $\hat{j}_\varphi$ and $\hat{\phi}_m$, such that the current and the basis functions become

$$j_\varphi(r) = \frac{\hat{j}_\varphi(r)}{r}, \quad \phi_m(r) = \frac{\hat{\phi}_m(r)}{r}. \quad (6.19)$$

Substituting $\hat{j}_\varphi(r)$ into (6.14) and multiplying by r , we obtain

$$\hat{j}_\varphi(r) - \hat{j}_\varphi^s(r) = \frac{i\kappa}{2\pi} \int_{r_0}^{r_1} \sqrt{\frac{r}{\rho}} \frac{\hat{j}_\varphi(\rho)}{k} [(2 - k^2)K(k) - 2E(k)] d\rho. \quad (6.20)$$

The new kernel has a logarithmic singularity in $r = \rho$ (i.e. $k = 1$). So, we decide to approximate $\hat{j}_\varphi(r)$ by a series of scaled and shifted Legendre polynomials:

$$\hat{j}_\varphi(r) \doteq \sum_{m=0}^M \alpha_m P_m\left(\frac{r - c_r}{d_r}\right), \quad (6.21)$$

$$\hat{j}_\varphi^s(r) \doteq \sum_{m=0}^M \beta_m P_m\left(\frac{r - c_r}{d_r}\right), \quad (6.22)$$

where

$$c_r = \frac{r_1 + r_0}{2}, \quad d_r = \frac{r_1 - r_0}{2}. \quad (6.23)$$

As in the examples of the previous chapters, the spatial behavior of the source current is the same for all frequencies ω , including $\omega = 0$. This yields

$$\beta_0 = \frac{I}{D^c j^c \log\left(\frac{r_1}{r_0}\right)}, \quad \beta_m = 0, \quad m \geq 1. \quad (6.24)$$

The coefficients α_m , $m \geq 0$, depend on the frequency ω . Only for $\omega = 0$, we know them a priori. Namely, $\alpha_m = 0$, $m \geq 1$, $\alpha_0 = \beta_0$, because for $\omega = 0$ there are no inductive effects. Moreover, in the previous chapters, the total current through a strip has always been accounted for by α_0 , because the higher-order polynomials P_m , $m \geq 1$, do not contribute to the total current. However, in the case of plane circular strips, the higher-order polynomials do contribute. The total current condition for one strip says

$$\int_{r_0}^{r_1} \frac{\hat{j}_\varphi(\rho)}{\rho} d\rho = \alpha_0 \ln\left(\frac{r_1}{r_0}\right) + d_r \sum_{m=1}^M \alpha_m \int_{-1}^1 \frac{P_m(r)}{d_r r + c_r} dr = \frac{I}{D^c j^c}. \quad (6.25)$$

In contrast to the Galerkin method, in which the test functions are the same as the basis functions, we apply a projection method with slightly different test functions. This brings us to choose the following definition for the inner product in this chapter:

$$(f, g) = \int_{S_\cap^r} f(r)g(r) r dr. \quad (6.26)$$

Whereas the basis functions are Legendre polynomials of the first kind divided by r , for the test functions we choose the Legendre polynomials themselves. This choice is based on the

relation for the inner products

$$\int_{r_0}^{r_1} \frac{P_m(\frac{\rho-c_r}{d_r})}{\rho} P_n(\frac{\rho-c_r}{d_r}) \rho \, d\rho = \frac{2d_r}{2m+1} \delta_{mn}. \quad (6.27)$$

Furthermore, the source current only yields a non-zero inner product for the zeroth-order test function P_0 . This is the main reason for the adjustment of the method, because it enables us to embed the source current in the family of basis functions, such that we can decompose the total current density into components that comprise the source current density.

We can now generalize the method to the situation of two or more plane circular loops. Let $\psi_l(r)$ be the characteristic function of group l , defined by

$$\psi_l(r) = \sum_{n=1}^{N_l} \mathbf{1}_{[r_0^{(n)}, r_1^{(n)}]}, \quad (6.28)$$

where N_l is the number of rings in group l and $\mathbf{1}_{[r_0^{(n)}, r_1^{(n)}]}$ represents the characteristic function, which is one on each interval $[r_0^{(n)}, r_1^{(n)}]$, and zero otherwise. Due to disjointness of the intervals, the characteristic functions $\psi_l(r)$ satisfy

$$(\frac{\psi_{l_1}}{r}, \psi_{l_2}) = \frac{D_{l_1}}{D^c} \delta_{l_1 l_2}, \quad (6.29)$$

where D_{l_1} is the sum of the widths of all rings in group l_1 . The source current $j_\varphi^s(r)$ is in every loop inversely proportional to r and can therefore be written as

$$j_\varphi^s(r) = \sum_{l=1}^L C_l \frac{\psi_l(r)}{r}, \quad (6.30)$$

so that the integral equation (6.14) becomes

$$j_\varphi(r) - i\kappa \int_{S_\cup^r} \mathcal{K}(r, \rho) j_\varphi(\rho) \rho \, d\rho = \sum_{l=1}^L C_l \frac{\psi_l(r)}{r}, \quad r \in S_\cup^r, \quad (6.31)$$

where

$$\mathcal{K}(r, \rho) = \frac{1}{2\pi k \sqrt{r\rho}} [(2 - k^2)K(k) - 2E(k)], \quad k^2 = \frac{4r\rho}{(r + \rho)^2}. \quad (6.32)$$

Moreover, the total current per group is prescribed, so $j_\varphi(r)$ must satisfy

$$\int_{S_\cup} j_\varphi(\rho) \psi_l(\rho) \, d\rho = (\frac{j_\varphi}{r}, \psi_l) = \hat{I}_l, \quad l \in \{1, \dots, L\}, \quad (6.33)$$

where $\hat{I}_l = I_l / (j^c D^c)$, with j^c according to (4.3).

We define the operator $\mathbf{K}$ by

$$(\mathbf{K}f)(r) = \int_{S_\cup} f(\rho) \mathcal{K}(r - \rho) \rho \, d\rho, \quad (6.34)$$

such that (6.31) can be written in operator form as

$$(\mathbf{I} - i\kappa\mathbf{K})j_\varphi = \sum_{l=1}^L C_l \frac{\psi_l}{r}. \quad (6.35)$$

We define the projection Π on the linear span of the characteristic functions ψ_l/r , $l \in \{1, \dots, L\}$, by

$$(\Pi f)(r) = \sum_{l=1}^L \frac{(f, \psi_l)}{(\frac{\psi_l}{r}, \psi_l)} \frac{\psi_l(r)}{r}. \quad (6.36)$$

The projection Π applied to j_φ yields, with use of (6.29),

$$\Pi j_\varphi = \sum_{l=1}^L D^c \frac{(j_\varphi, \psi_l)}{D_l} \frac{\psi_l}{r} = \sum_{l=1}^L \alpha_0^{(l)} \frac{\psi_l}{r}, \quad (6.37)$$

where the coefficients $\alpha_0^{(l)}$ are defined by $\alpha_0^{(l)} = D^c(j_\varphi, \psi_l)/D_l$, $l = 1, \dots, L$. Note that (6.33) cannot be used here in the same way as in Chapter 4, where the coefficients of the basis functions ψ_l disappeared in favor of the total currents $\hat{I}_l$. Hence, we cannot eliminate the unknown coefficients $\alpha_0^{(l)}$ here.

Next, j_φ is split into two parts according to

$$j_\varphi = \Pi j_\varphi + (\mathbf{I} - \Pi)j_\varphi = \sum_{l=1}^L \alpha_0^{(l)} \frac{\psi_l}{r} + j_\perp. \quad (6.38)$$

Here, $j_\perp = (\mathbf{I} - \Pi)j_\varphi$ is in the orthoplement of the range of Π , i.e. $(j_\perp, \psi_l) = 0$, for $l = 1, \dots, L$. Applying the operator $(\mathbf{I} - \Pi)$ to (6.35), we obtain

$$j_\perp - i\kappa(\mathbf{I} - \Pi)\mathbf{K}j_\varphi = 0, \quad (6.39)$$

or equivalently,

$$j_\perp - i\kappa(\mathbf{I} - \Pi)\mathbf{K}j_\perp = i\kappa \sum_{l=1}^L \alpha_0^{(l)} (\mathbf{I} - \Pi)\mathbf{K} \frac{\psi_l}{r}. \quad (6.40)$$

So, the constants C_l have disappeared, but they are replaced by the coefficients $\alpha_0^{(l)}$, which are also not known a priori. Up to now, we have used the same method as for the set of coaxial rings; see Chapter 4. In the example of coaxial rings the coefficients $\alpha_0^{(l)}$ were completely accounted for by the prescribed currents $\hat{I}_l$ and were therefore known a priori. So, from this point on, the method for the present example of plane circular strips needs an extension. Apart from the L unknowns C_l , $l = 1, \dots, L$, we have an additional number of L unknowns $\alpha_0^{(l)}$, $l = 1, \dots, L$. Moreover, we have not used the L known values $\hat{I}_l$, $l = 1, \dots, L$ yet. Therefore, we need an additional set of L equations. These equations can be obtained from the inner products of relation (6.35) with ψ_l , $l = 1, \dots, L$.

We approximate $j_\perp$ by a finite series of basis functions $\phi_m(r)$. As explained before, the basis functions are inversely proportional to r . Therefore, we introduce the functions $\hat{\phi}_m(r) = r\phi_m(r)$, such that

$$j_\perp(z) \doteq \sum_{m=1}^M \alpha_m \frac{\hat{\phi}_m}{r}, \quad \frac{\hat{\phi}_m}{r} = \phi_m \in \text{ran}(I - \Pi). \quad (6.41)$$

Taking successively on both sides of (6.35) the inner product with ψ_l and on both sides of equation (6.40) with $\hat{\phi}_n$, we arrive at

$$\alpha_0^{(l)} \frac{D_l}{D^c} - i\kappa \sum_{k=1}^L \alpha_0^{(k)} (\mathbf{K} \frac{\psi_k}{r}, \psi_l) - i\kappa \sum_{m=1}^M \alpha_m (\mathbf{K} \frac{\hat{\phi}_m}{r}, \psi_l) = C_l \frac{D_l}{D^c}, \quad (6.42)$$

$$\sum_{m=1}^M \alpha_m (\frac{\hat{\phi}_m}{r}, \hat{\phi}_n) - i\kappa \sum_{l=1}^L \alpha_0^{(l)} (\mathbf{K} \frac{\psi_l}{r}, \hat{\phi}_n) - i\kappa \sum_{m=1}^M \alpha_m (\mathbf{K} \frac{\hat{\phi}_m}{r}, \hat{\phi}_n) = 0, \quad (6.43)$$

for $l = 1, \dots, L$ and $n = 1, 2, \dots, M$. For convenience of notation, we write the equations (6.42) and (6.43) in matrix form as

$$(\mathbf{G} - i\kappa \mathbf{A}) \mathbf{a} = \mathbf{c}, \quad (6.44)$$

where the block matrices $\mathbf{G}$, $\mathbf{A}$ and the column vectors $\mathbf{a}$, $\mathbf{c}$ are defined by

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{22} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix}, \quad (6.45)$$

with elements ($m, n = 1, \dots, M; k, l = 1, \dots, L$)

$$A_{11}(l, k) = (\mathbf{K} \frac{\psi_k}{r}, \psi_l), \quad A_{12}(l, m) = (\mathbf{K} \frac{\hat{\phi}_m}{r}, \psi_l),$$

$$A_{21}(n, l) = (\mathbf{K} \frac{\psi_l}{r}, \hat{\phi}_n), \quad A_{22}(n, m) = (\mathbf{K} \frac{\hat{\phi}_m}{r}, \hat{\phi}_n),$$

$$G_{11}(l, k) = \frac{D_l}{D^c} \delta_{kl}, \quad G_{22}(n, m) = (\frac{\hat{\phi}_m}{r}, \hat{\phi}_n),$$

$$a_1(l) = \alpha_0^{(l)}, \quad a_2(m) = \alpha_m,$$

$$c_1(l) = \frac{D_l}{D^c} C_l, \quad c_2(m) = 0. \quad (6.46)$$

The way to solve this matrix equation will be explained in Section 6.2.2. We first show how the entries of the matrices are computed.

The functions $\hat{\phi}_m$ are expressed in Legendre polynomials of the first kind. For the current distribution $j_\perp(r)$, the pertinent combination of intervals is $S_\cup^r$ instead of $[-1, 1]$. Therefore,

we introduce shifted and scaled Legendre polynomials, $P_{s,q}(r; c_q; d_q)$, of order s , defined on strip q ($q = 1, \dots, N$) as

$$P_{s,q}(r; c_q; d_q) = P_s\left(\frac{r - c_q}{d_q}\right), \quad r \in [r_0^{(q)}, r_1^{(q)}], \quad (6.47)$$

where

$$c_q = \frac{r_1^{(q)} + r_0^{(q)}}{2}, \quad d_q = \frac{r_1^{(q)} - r_0^{(q)}}{2}, \quad q = 1, \dots, N. \quad (6.48)$$

For $r \notin [r_0^{(q)}, r_1^{(q)}]$, $P_{s,q}(r; c_q; d_q) = 0$. Orthogonality of Legendre polynomials yields

$$\int_{S_q^+} \frac{P_{s,q}(\rho; c_q; d_q)}{\rho} P_{s',q'}(\rho; c_{q'}; d_{q'}) \rho \, d\rho = \frac{2d_q}{2s+1} \delta_{ss'} \delta_{qq'}. \quad (6.49)$$

The shifted and scaled Legendre polynomials are used to construct the basis functions. In analogy with (4.32), we create a series expansion in different basis functions for each strip, separately. Hence,

$$j_\perp = \sum_{s=1}^S \sum_{q=1}^N \alpha_{s,q} \frac{\hat{\phi}_{s,q}}{r} + \sum_{q=1}^{N-L} \alpha_{0,q} \frac{\hat{\phi}_{0,q}}{r}, \quad (6.50)$$

where S denotes the number of degrees of basis functions that is included, and N is the number of strips. The $N - L$ constant functions $\hat{\phi}_{0,q}$ differentiate the current distributions in the strips within each group.

A requirement for the basis functions is that they must be orthogonal with respect to each other as well as to the functions $\psi_l(r)$. For Legendre polynomials of degree one and higher, we have

$$\int_{S_q^+} \frac{P_{s,q}(\rho; c_q; d_q)}{\rho} \psi_l(\rho) \rho \, d\rho = 0, \quad (6.51)$$

for $q = 1, \dots, N$, $l = 1, \dots, L$, and $s \geq 1$. Therefore, we can choose the basis functions of degree one and higher equal to the shifted and scaled Legendre polynomials:

$$\phi_{s,q} = P_{s,q}(r; c_q; d_q), \quad q = 1, \dots, N, \quad s = 1, \dots, S. \quad (6.52)$$

The functions $\hat{\phi}_{0,q}$ are constructed in the same way as in the example of rings, explained in Chapter 4.

Because the Legendre polynomials are shifted and scaled in a different way on each strip, it is convenient to transform all intervals to $[-1, 1]$. For the entries of $\mathbf{A}$, we have to compute integrals of the form

$$d_q d_{q'} \int_{-1}^1 \int_{-1}^1 P_{s'}(r) P_s(\rho) \mathcal{K}(d_{q'} r + c_{q'}, d_q \rho + c_q) (d_{q'} r + c_{q'}) \, d\rho \, dr, \quad (6.53)$$

where $s, s' = 0$ are substituted for the treatment of ψ_l . The entries of $\mathbf{G}$ follow directly from (6.49). We remark that the integrands are logarithmically singular if $q = q'$. In that case, we

extract the logarithmic part, which is obtained explicitly from the asymptotic expansion of $\mathcal{K}(r, \rho)$ for r close to ρ :

$$r\mathcal{K}(r, \rho) \approx -\frac{1}{2\pi}(\log|r - \rho| - \log\sqrt{r\rho} - \log 2 + \Psi^{(0)}(\frac{3}{2}) + \gamma), \quad (6.54)$$

where $\Psi^{(0)}(3/2) \approx 0.03649$ ($\Gamma^{(0)}$ is the polygamma function), and $\gamma \approx 0.57722$ (Euler's constant); see [2]. The difference between $r\mathcal{K}(r, \rho)$ and the series expansion is a regular function. For the numerical computations, we use (3.61) for the logarithmic part of (6.54). The remaining difference function is regular and can therefore be integrated numerically. We will use the Gauss-Legendre quadrature rule as integration method.

6.2.2 Solving the linear set of equations

In this section, we discuss the solution procedure for the matrix equation (6.44). In this matrix equation, the unknowns are α_m , $m = 1, \dots, M$, and $\alpha_0^{(l)}$, $l = 1, \dots, L$. The coefficients C_l , introduced in (6.30), are also not known yet. They should be related to the L prescribed values of the total currents $\hat{I}_l$ through groups $l = 1, \dots, L$. The parameter κ has a known value and the entries of $\mathbf{G}$ and $\mathbf{A}$ can be determined from (6.46). Additionally, we know the total currents $\hat{I}_1, \dots, \hat{I}_L$ in the L groups, which are given by

$$\hat{I}_l = \int_{S_l^+} j_\varphi(\rho) \psi_l(\rho) d\rho = \alpha_0^{(l)} \left(\frac{\psi_l}{r}, \frac{\psi_l}{r} \right) + \sum_{m=1}^M \alpha_m \left(\frac{\hat{\phi}_m}{r}, \frac{\psi_l}{r} \right). \quad (6.55)$$

We use this relation to eliminate the L unknown coefficients C_l . To this end, we define the $L \times (L + M)$ matrix $\mathbf{S}$ by

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \end{pmatrix}, \quad (6.56)$$

with elements ($m = 1, \dots, M; k, l = 1, \dots, L$)

$$\begin{aligned} S_{11}(l, k) &= \left(\frac{\psi_l}{r}, \frac{\psi_l}{r} \right) \delta_{kl}, \\ S_{12}(l, m) &= \left(\frac{\psi_l}{r}, \frac{\hat{\phi}_m}{r} \right), \end{aligned} \quad (6.57)$$

where the latter elements are only non-zero if $\hat{\phi}_m$ acts on a ring in group l . Moreover, let $\hat{\mathbf{I}}$ be the L -vector consisting of all $\hat{I}_l$, $l = 1, \dots, L$. Then, (6.55) can be written in matrix form

$$\hat{\mathbf{I}} = \mathbf{S}\mathbf{a}. \quad (6.58)$$

Thus, the vector $\mathbf{a}$ of length $L + M$ is transformed into vector $\hat{\mathbf{I}}$ of length L . The vector $\mathbf{c}$ in the right-hand side of (6.44) also has length $L + M$. However, only the first L values are unknown (representing C_l); the other entries are equal to zero. To reduce also the vector $\mathbf{c}$ to a vector of length L , we introduce the $(L + M) \times L$ matrix $\mathbf{B}$ and the L -vector $\hat{\mathbf{c}}$, by

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{pmatrix}, \quad \mathbf{c} = \mathbf{B}\hat{\mathbf{c}}, \quad (6.59)$$

where $(m = 1, \dots, M; k, l = 1, \dots, L)$

$$B_{11}(l, k) = \frac{D_l}{D^c} \delta_{kl}, \quad B_{21}(m, k) = 0, \quad (6.60)$$

and

$$\hat{c}(l) = C_l. \quad (6.61)$$

We know that $\mathbf{G} - i\kappa\mathbf{A}$ is invertible, and so the matrix equation (6.44) reveals that $\mathbf{a} = (\mathbf{G} - i\kappa\mathbf{A})^{-1}\mathbf{c}$. According to (6.58) and (6.59)₂, the two L -vectors $\hat{\mathbf{I}}$ and $\hat{\mathbf{c}}$ are related to each other by

$$\hat{\mathbf{I}} = \mathbf{S}(\mathbf{G} - i\kappa\mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{c}}, \quad (6.62)$$

or

$$\hat{\mathbf{c}} = [\mathbf{S}(\mathbf{G} - i\kappa\mathbf{A})^{-1}\mathbf{B}]^{-1}\hat{\mathbf{I}}, \quad (6.63)$$

since also $\mathbf{S}(\mathbf{G} - i\kappa\mathbf{A})^{-1}\mathbf{B}$ is invertible. Here, we are mainly interested in the current distribution, for which we need $\mathbf{a}$. This vector now follows from

$$\mathbf{a} = (\mathbf{G} - i\kappa\mathbf{A})^{-1}\mathbf{B}[\mathbf{S}(\mathbf{G} - i\kappa\mathbf{A})^{-1}\mathbf{B}]^{-1}\hat{\mathbf{I}}. \quad (6.64)$$

We have obtained a linear set of equations that can be solved numerically.

6.3 Numerical results

We consider configurations such as they follow from an approach that is used in the design of x-coils and y-coils. For the design, a stream function is computed with specific software. The stream functions has a shape that is well approximated by a cosine function. Here, we define the stream function $s(r)$ as

$$s(r) = \begin{cases} I_{max} \cos\left(\frac{\pi r}{2R_{ref}}\right), & \text{if } |r| < R_{ref}, \\ 0, & \text{if } |r| > R_{ref}, \end{cases} \quad (6.65)$$

where R_{ref} is the radius that defines the support domain of $s(r)$, and I_{max} represents the maximum value of $s(r)$. The stream function is discretized by use of streamlines, which form circles at fixed values of r . Each streamline forms the central line of a plane circular loop. The number of streamlines and the widths of the loops are such that the strips do not overlap.

In case we want to specify N plane circular loops with central lines at positions $r = r^{(n)}$, $n = 1, \dots, N$, and with each loop carrying the same total current, the discretization can be made according to equidistant steps in the stream function. Then, each loop has a total current of $I^{(n)} = I_{max}/N$, and the positions $r^{(n)}$ are determined from

$$\cos\left(\frac{\pi r^{(n)}}{2R_{ref}}\right) = \frac{n - \frac{1}{2}}{N}, \quad n = 1, \dots, N, \quad (6.66)$$

such that

$$r^{(n)} = \frac{2R_{ref}}{\pi} \arccos\left(\frac{2n-1}{2N}\right). \quad (6.67)$$

The two configurations we consider here, consist successively of one and ten plane circular loops. They all have a width of 2 cm, so that they do not overlap. The maximum value of the stream function is chosen as $I_{max} = 600$ A, and its domain is limited by $R_{ref} = 0.5$ m.

6.3.1 One plane circular strip

We first consider one plane circular loop of width $D = 0.02$ m. The thickness is $h = 2.5$ mm and the amplitude of the total current flowing through the loop is $I = 600$ A. According to (6.67), we find that the central line forms a circle at $r = 2R_{ref}/3 = 33.33$ cm. The current is expanded in a series of Legendre polynomials divided by r , with coefficients calculated from the linear set of equations (6.64). These calculations are implemented in our software tool Eddy.

Figure 6.2: Amplitude of the current distribution in one plane circular loop of 2 cm width with a total current of 600 A at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz ($\triangle$).

In Figure 6.2, the amplitude of the current distribution in the strip is shown for the four frequencies $f = 100, 400, 700, 1000$ Hz. For low frequencies, the current tends to the DC solution, which can be determined analytically according to (6.18). Note that in (6.18) the variables j_φ and r are dimensionless. In dimensional form we obtain

$$j_\varphi(r) = \frac{I}{r \log\left(\frac{r_1}{r_0}\right)} = \frac{9.996 \cdot 10^3}{r} \text{ A/m}, \quad (6.68)$$

where $r_0 = 32.33$ cm and $r_1 = 34.33$ cm are the inner and outer radius of the loop,

respectively. The value of j^c in (6.18) represents the average current, which means that $j^c = I/D = 3.0 \cdot 10^4$ A/m.

On the scale used in Figure 6.2, the current distribution at $f = 100$ Hz is very close to the DC solution. We observe that for higher frequencies edge-effects occur. In contrast to the situation of one ring, of which the current distribution is shown in Figure 4.4 (a), the edge-effects are not symmetric in the plane circular loop. The current near the inner edge of the loop is higher than near the outer edge. This is an important observation. Namely, it means that the 'center of gravity' of the current is shifted towards the inner edge of the loop. Consequently, the magnetic field along the axis of symmetry increases and deviates from the desired field.

Thus, in comparison with the uniform current distribution that is assumed in the design approach using streamlines, two effects occur that play a role in the distortion of the desired field. First, the current flows more near the inner edge of the loop than near the outer edge, because the circumferential length is smaller there. We call this effect the $1/r$ -effect, and it also occurs in the DC situation. The second effect is the edge-effect, which occurs in a time-varying situation and is caused by induction.

6.3.2 Ten plane circular strips

In the second example, we consider a configuration of $N = 10$ plane, concentric, circular loops. They all have a width of 2 cm and a thickness of 2.5 mm. We determine the radii of the central lines according to (6.67). The inner radii $r_0^{(n)}$ and the outer radii $r_1^{(n)}$ are listed in Table 6.1, for $n = 1, \dots, 10$. The total current in each ring is $I^{(n)} = I_{max}/N = 60$ A. The applied currents have a frequency f and are in phase. The resulting current distributions within the loops are calculated by Eddy, in which the linear set of equations (6.64) is implemented.

Table 6.1: Inner and outer radii of the set of 10 plane circular strips.

Ring	Inner radius	Outer radius
1	9.11 cm	11.11 cm
2	16.66 cm	18.66 cm
3	22.01 cm	24.01 cm
4	26.48 cm	28.48 cm
5	30.46 cm	32.46 cm
6	34.14 cm	36.14 cm
7	37.62 cm	39.62 cm
8	40.96 cm	42.96 cm
9	44.21 cm	46.21 cm
10	47.41 cm	49.41 cm

In Figure 6.3, the amplitude of the current distribution in the ten loops is shown for the four frequencies $f = 100, 400, 700, 1000$ Hz. The average current in each ring is $j^c = 3.0 \cdot 10^3$ A/m. In the DC situation, the current is distributed according to the analytical formula

$$j_\varphi(r) = \sum_{n=1}^N \frac{I^{(n)}}{\log\left(\frac{r_1^{(n)}}{r_0^{(n)}}\right)} \frac{\psi_n(r)}{r}, \quad (6.69)$$

where $\psi_n(r)$ is the characteristic function of loop n .

On the scale used in Figure 6.3, the current distribution at $f = 100$ Hz is very close to the DC solution. We observe that for higher frequencies once again edge-effects occur. Moreover, considering the envelope of the graph, a global edge-effect is observed, as is the case in a set of concentric rings. However, the currents are in almost all plane loops more pronounced near the inner edges. Therefore, the 'centers of gravity' of the currents are shifted towards the inner edges, resulting in an amplified magnetic field along the axis of symmetry.

Figure 6.3: Amplitude of the current distribution in a set of ten plane circular loops of 2 cm width, each with a total current of 60 A at frequencies $f = 100$ Hz (*), $f = 400$ Hz (o), $f = 700$ Hz (+), $f = 1000$ Hz (Δ).

6.4 Summary

In this chapter, we have modeled the x-coil and y-coil by a set of plane circular loops. Being in one plane instead of on a cylindrical surface has a negligibly small effect on the current distribution in the loops. The reason is that the currents are only affected locally. For the magnetic field this model does not hold. However, for the current distribution in the strips on the cylindrical surface, we use the distribution calculated from the plane strip model. Subsequently, Biot-Savart's law can be used to compute the magnetic field.

In a set of plane circular loops, the current flows in the tangential direction and depends only on the radial coordinate. The two-dimensional integral equation derived in Chapter 2, for arbitrary geometries, is reduced to a one-dimensional integral equation. The kernel can be expressed in terms of elliptic integrals. From the asymptotic expansion of this kernel near the singularity point, the logarithmic function appears, as predicted by the mathematical analysis presented in Chapter 3.

The distribution of the source current is of great importance for the total current distribution. In plane circular loops, the source current features a decay in the radial component. For this reason, not the Galerkin method is applied, but the Petrov-Galerkin method, in which the basis functions are Legendre polynomials divided by the radial coordinate, and the test functions are the Legendre polynomials themselves. For the logarithmic part of the kernel, we use the analytical formula derived in Section 3.2. The remaining part of the kernel is regular and is computed numerically.

The coefficients of the basis functions are determined by our software tool Eddy. The resulting current distribution in the loops shows two effects that influence the magnetic field. First, a decay in the current density in the radial direction occurs, called the $1/r$ -effect. Second, edge-effects occur due to time-varying fields. The eddy currents that forms these edge-effects are more pronounced near the inner edge of the circular strip. Thus, in total, the currents are shifted towards the center of the system, such that their centers of gravity do not coincide with the streamlines anymore.

Aspects and conclusions of the design

This thesis is concerned with the development of a software tool that assists in the design of gradient coils in an MRI-scanner. In this software tool, which we called Eddy, we have implemented a model that describes the current distribution in the copper strips of a gradient coil in an MRI-scanner together with the resulting magnetic field. This model has been analyzed mathematically and simulations can be executed for four specific configurations: a parallel set of plane rectangular strips, a parallel set of ring-shaped strips, a set of rings and islands, and a set of circular loops of strips on a cylindrical surface.

With Eddy, inductive effects in the coils are simulated and distortions in the magnetic field are predicted. The inductive effects are characterized by induced eddy currents, which appearance depends on the frequency of the applied source, the geometry of the conductors, the distance between the conductors, and the conductivity. From the analysis and the simulations, the influence of the eddy currents on the gradient of the magnetic field, dissipated energy, stored energy, resistance and self-inductance of a coil has been determined.

7.1 Aspects of the design

Modeling

The gradient coils in an MRI-scanner are designed to create a uniform gradient in the magnetic field in the direction of the central axis of the coil. Coils consist of copper strips, wound around a cylinder. The main goal of this thesis was a detailed analysis and simulation of the eddy currents that are present in the strips of a gradient coil. We reached our objective by developing a software tool that supports the overall design of gradient coils at Philips Medical Systems (PMS). For the software tool development, we have constructed a mathematical model describing the current distribution in the strips of a gradient coil. The model

incorporates time-dependent fields and mutual electromagnetic field coupling.

In its most general form, the geometric model consists of an arbitrary set of conducting strips on a cylindrical surface. The strips can be combined in separate groups. Each group is connected to its own source, such that the total currents through the groups are not necessarily the same. For the mathematical model, we use the Maxwell equations, together with the associated boundary conditions and constitutive equations. To reduce the set of equations, we assume that the electromagnetic fields are time-harmonic, the only driving source is a current source, the media are copper and air, a quasi-static approach can be used, the thicknesses of the strips can be neglected, and the conductors are rigid. The current is transformed into a surface current and the model leads to an integral equation of the second kind for the current density. This integral equation forms the basis for our further mathematical analysis and our numerical simulations for the four specific configurations that are used in this thesis. We demonstrate our method by means of explicit numerical results.

The first specific configuration consists of a parallel set of infinitely long plane rectangular strips. Although this geometry is physically not feasible, it forms the basis of all mathematical modeling done in this thesis. Second, we consider a parallel set of rings. The rings form the geometric model for the z-coil. Third, the geometry of rings is extended with islands, all placed on the same cylindrical surface and in between the rings. The islands represent the pieces of copper that are present in a gradient coil to fill up empty spaces for better heat transfer in the coil and higher stability of the coil. Another application of this model comes into sight when slits have been cut in the copper strips for the reduction of the eddy currents. The fourth specific configuration models the x-coil and y-coil by a set of plane circular strips. Being in one plane instead of on a cylindrical surface has a negligibly small effect on the current distribution in the strips. The reason is that the currents affect each other only locally.

Analysis

The analysis in this thesis provides insight in the behavior of eddy currents in a gradient coil. Characteristic quantities of a system are derived, in particular self-inductance, resistance, characteristic frequency, phase-lag with respect to the source, dissipated and stored energy, variation of the magnetic field, the linearity of the gradient field, and the error of the achieved field in comparison with the desired field. For all these characteristic quantities, we have investigated their dependence on the frequency of the applied source, the shape of the conductors, the distance between the conductors, and the conductivity.

From our mathematical analysis, we have inferred that the kernel function of the integral is singular in one point. This singular behavior inspired us to develop a strategy, partly based on applied analysis, partly based on numerical analysis.

The model analysis predicts that the induced eddy currents in the strips prefer to flow in the direction opposite to the applied source current. The physical explanation is that eddy cur-

rents tend to oppose the magnetic field caused by the source current. We have considered the leading integral equation in terms of the component of the current in this preferred direction, imposed by the source current. The essential behavior of the kernel in this leading integral equation is logarithmic in the coordinate perpendicular to the preferred direction.

To solve the integral equation, the Galerkin method with global basis functions has been applied to approximate the current distribution. In the preferred direction we have chosen trigonometric functions to express periodicity. Thus, the current is expanded in Fourier modes and via the inner products we have derived a direct coupling between the modes. In the width direction of the strips, we have chosen Legendre polynomials. With this choice we found basis functions that are complete, converge rapidly and are easy to compute. Hence, this method is especially dedicated to the models considered in this thesis.

Results

The main result of the project is the software tool *Eddy*. With *Eddy*, we are able to investigate the characteristics of the current distributions in the strips and the corresponding magnetic field. All these characteristics are related to the coefficients with respect to the Legendre polynomials. Thus, the core part of *Eddy* is the calculation of the Legendre coefficients.

The global distribution of the current is accounted for by the source current, which represents the static distribution. In the examples of plane rectangular strips, rings, and rings with islands, the source current has a uniform distribution. In plane circular strips, the source current shows a decay in the radial coordinate. This decay is called the $1/r$ -effect.

In a time-varying system, eddy currents are induced. The resulting current densities show edge-effects and a phase-lag with respect to the source. The term edge-effect indicates that the current at the edges is higher than in the center. In our results, we have observed local edge-effects, which are caused by the self-inductance of a strip, and global edge-effects, which are caused by the mutual inductances among the strips. Both edge-effects and phase-lags result in distorted magnetic fields, such that the desired field is not accomplished.

The current distributions depend strongly on the frequency; edge-effects become stronger when the frequency is increased, and the errors in the magnetic field increase accordingly. At a distance from the strips, the magnetic field seems to experience a displacement of the center of gravity of the current. Especially in plane circular loops of strips, the center of gravity is shifted, because the currents are more pronounced near the inner edges of the loops.

The eddy currents also affect the resistance and self-inductance of a coil. We have shown that for every configuration the resistance increases with the frequency, whereas the self-inductance decreases with the frequency. Furthermore, both quantities show a point of inflection at a characteristic frequency. We showed that this characteristic frequency is fixed by an analytical formula.

In case of one strip, or only a few strips, distortions in the magnetic field are small, i.e. for MRI purposes not significant. However, for many strips, the global edge-effect becomes stronger, leading to noticeable errors. We have seen that in a set of 24 rings the error is of the order of one percent of the desired field. A way to reduce the effects of eddy currents is by use of pre-emphasis currents, shielding, or cutting slits in the conductors.

To mimic slits in the strips, we have developed the model with rings and islands. We have shown that the total resistance increases when an island is added to the system. The opposite holds for the self-inductance of the system; it decreases when an island is added. In case a loose ring is added to the system, the resistance increases even more, whereas the self-inductance decreases. Comparable results are obtained from a set of islands that approximate the loose ring, as such describing a ring that is cut in pieces.

Recommendations for model reduction

The numerical simulations for the four specific configurations have been programmed initially in `Matlab`. For compatibility reasons at PMS, the programs are translated to a C++ code. In this language, `Eddy` has been written. For all simulations performed for the used examples, the computation times were in the order of one second (on a PC with an Intel Pentium 3 processor at 797 MHz, and 256 MB of RAM). However, if one wants to use `Eddy` for a large set of strips, we recommend the following modifications:

- The Galerkin matrix $\mathbf{A}$ is dense, but diagonally dominant. We may replace $\mathbf{A}$ by $\mathbf{A}_\tau$, where $\mathbf{A}_\tau$ is obtained by dropping all elements smaller than τ , i.e. $|A_{ij}| < \tau$ (τ is often called a drop tolerance), thus eliminating much of the set-up time. The reason that the off-diagonal elements become smaller when they are further removed from the diagonal is that the distance between the two respective strips increases, resulting in smaller mutual inductions.
- For all configurations, the kernel function in the integral equation is logarithmically singular. Using only the logarithmic part, i.e. with the logarithmic function as kernel, we obtain the principal part of the solution. The resulting integration can be done fully analytically, leading to a simple formula. The remaining parts are regular and have been computed numerically. These computations regard numerical integrations, which become time consuming if the order of the basis functions is high. The regular parts have been computed for the completeness of the solution. However, if we neglect them, then the errors will be within one percent of the total solution. Moreover, the computation time will be reduced by more than ninety percent.
- The current distribution is constructed by Fourier modes in the angular direction of the cylinder. Modes can also be formed in the width direction of the strips, as a combination of Legendre polynomials. This combination can be obtained from the eigenvalues and eigenfunctions of a system. The advantage of the calculation of the eigenvalues is

that they only have to be computed once for a system, and can afterwards be used for computations for different frequencies.

- Based on the fact that the currents are, in magnitude, mainly determined by the source current, one could develop an iterative method for the mode-coupling among the strips. A certain mode of the current in one strip induces an eddy current in another strip of the same mode. The intensity of the induced current is small compared to the source current, because of the mutual distance. Additionally, the induced current causes on its turn an eddy current in the first strip, still having the same mode. Due to the distance, the intensity is decreased again, indicating that the iterative procedure will converge.

7.2 Achievements

In this thesis, we have considered four examples of strips in which a current is flowing: a parallel set of plane rectangular strips, a parallel set of ring-shaped strips, a set of islands, and a set of circular loops of strips on a cylindrical surface. For each example specifically, we have reached to the following achievements:

- From the special case of plane rectangular strips as described in Section 3.4, we have obtained a Fredholm integral equation of the second kind with logarithmic singular kernel for the current distribution. The Legendre polynomials are specially suited as basis functions in the Galerkin method, when applied to this integral equation. Moreover, an analytical formula for the inner products in the Galerkin method has been derived.
- In a set of rings, the current flows in the tangential direction and depends only on the axial coordinate. The two-dimensional integral equation derived for arbitrary geometries, is reduced to a one-dimensional integral equation. The kernel can be expressed in terms of elliptic integrals. The asymptotic expansion of these elliptic integrals near the singularity point shows the logarithmic behaviour, as predicted by the mathematical analysis for general geometries.

For the logarithmic part of the kernel, we are able to use the analytical formula for the inner products with Legendre polynomials as basis functions. The remaining part of the kernel is regular and matrix entries are computed numerically.

- In a set of rings and islands, the source current is applied only to the rings. Therefore, for the eddy currents induced in the islands the tangential direction is preferable. According to our mathematical analysis, the kernel has a logarithmic singularity in the axial coordinate and can be expressed in terms of Legendre functions of the second kind of odd-half-integer order. The asymptotic expansion of all these functions near the singularity point contains the logarithmic function. For the logarithmic part, we have used the analytical formula derived if Legendre polynomials are chosen as basis

functions. In the tangential coordinate, Fourier modes are chosen, because of the 2π -periodicity and the orthogonality with the trigonometric functions in the expansion of the kernel function. In the resulting set of equations, the mutual relation between the rings and the islands is explicitly expressed by the coupling between the modes.

- In a set of circular loops of strips on a cylindrical surface, we have first of all explained how and why we may approximate them by a set of plane circular strips. Here, the current flows in the tangential direction and depends only on the radial coordinate. The current satisfies an integral equation with a kernel that is expressed in elliptic integrals. Also, for this geometrical setting, the kernel has a logarithmic singularity which follows from the asymptotics of the elliptic integrals, and was predicted by our general mathematical analysis in Chapter 3.

The distribution of the source current is of great importance for the total current distribution. In plane circular strips, the source current features a decay in the radial component. For this reason, not the Galerkin method is applied, but the Petrov-Galerkin method, in which the basis functions are Legendre polynomials divided by the radial coordinate, and the test functions are the Legendre polynomials themselves.

7.3 Conclusions

In this section, we recapitulate the main conclusions of this thesis:

1. The current in the geometry of thin strips, which appear in the gradient coils designed at PMS, can be modeled as surface currents by averaging with respect to the thickness. The mathematical model is an integral equation of the second kind for the current density.
2. The kernel of the integral equation is logarithmically singular in the coordinate perpendicular to the preferred direction of the current.
3. The Galerkin method with global basis functions is the dedicated method for the considered configurations. Legendre polynomials are the most appropriate choice to describe the dependence of the current with respect to the transverse direction.
4. The logarithmic part of the kernel describes the principal behavior of the currents. The contribution of the regular part of the kernel is small, in the order of one percent of the total. For designers, this is an acceptable degree of accuracy.
5. The integral equation for a set of plane rectangular strips represents a model that describes qualitatively the current distribution and characteristic effects for all configurations.
6. The use of applied analysis, in particular explicit formulae for the special functions, makes our implementation very efficient from a computational point of view.

7. The method proposed in this thesis provides a fast and accurate approximation of the current distribution. The implementation makes use of little memory, this in contrast to numerical packages in which a lot of elements (memory) is needed to come to a high accuracy.
8. Edge-effects become stronger when the frequency increases. Together with the phase-lag, distortions in the magnetic field are explained.
9. Use of Legendre polynomials yields rapid convergence. For most of the computations, only eight Legendre polynomials are needed to obtain an accuracy of 99 %. For high frequencies (close to 10^4 Hz), we propose to use up to twelve polynomials.
10. The simulations confirm that the resistance of a system increases with the frequency, whereas the self-inductance decreases with the frequency.

Summarizing, we have developed a software tool that simulates the current distribution and the magnetic field in a gradient coil. This software tool is based on a mathematical analysis of the governing equations. It provides accurate results, with short computation times and limited memory utilization. The software tool can be used for every design concept to investigate the qualitative effects of model parameters on the magnetic field. Due to the local induction effects, the standard shapes as presented in this thesis can be used for any arbitrary shape of conductors on a cylindrical surface.

Derivation of (3.61)

For currents on a cylindrical surface, an integral equation is derived in Section 2.3. The kernel in this equation is singular. In Section 3.1, we show that, given a preferred direction of the current, the singularity is logarithmic in the coordinate perpendicular to this preferred direction. Using Legendre polynomials as basis functions in the Galerkin method, we obtain inner products dominated by the logarithmic part. Each dominant part is given by the following double integral:

$$\int_{-1}^1 \int_{-1}^1 \log |z - \zeta| P_m(z) P_n(\zeta) d\zeta dz. \quad (\text{A.1})$$

To solve this integral, we first introduce

$$Z_m(z) = \int_{-1}^z P_m(\zeta) d\zeta = \frac{1}{2m+1} [P_{m+1}(z) - P_{m-1}(z)], \quad m \geq 1. \quad (\text{A.2})$$

Then, $Z'_m(z) = P_m(z)$ and $Z_m(\pm 1) = 0$. Multiplication by the Legendre function of the second kind $Q_n(z)$ and integration over z yields

$$\begin{aligned} \int_{-1}^1 Q_n(z) Z_m(z) dz &= \int_{-1}^1 \left[\frac{1}{2} PV \int_{-1}^1 \frac{P_n(\zeta) Z_m(z)}{z - \zeta} d\zeta \right] dz \\ &= \frac{1}{2} \int_{-1}^1 [\log |z - \zeta| P_n(\zeta) Z_m(z)]_{-1}^1 d\zeta \\ &\quad - \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \log |z - \zeta| P_n(\zeta) P_m(z) d\zeta dz, \end{aligned} \quad (\text{A.3})$$

in which interchange of the integrals is allowed. Furthermore, we know (see [16, Sect.3.12])

$$\int_{-1}^1 Q_n(z) P_m(z) dz = \frac{1 - (-1)^{m+n}}{(m-n)(m+n+1)}, \quad (\text{A.4})$$

for all $m, n \in \mathbb{N}$. In case $m = n$, the result is equal to zero. The integral in (A.3) is then also calculated by

$$\begin{aligned}
 & \int_{-1}^1 Q_n(z) Z_m(z) \, dz \\
 &= \frac{1}{2m+1} \int_{-1}^1 Q_n(z) [P_{m+1}(z) - P_{m-1}(z)] \, dz \\
 &= \frac{1}{2m+1} \left[\frac{1 - (-1)^{m+n+1}}{(m-n+1)(m+n+2)} - \frac{1 - (-1)^{m+n-1}}{(m-n-1)(m+n)} \right] \\
 &= \begin{cases} \frac{-4}{(m-n+1)(m+n+2)(m-n-1)(m+n)}, & \text{if } m+n > 0 \text{ even,} \\ 0, & \text{if } m+n \text{ odd.} \end{cases} \quad (\text{A.5})
 \end{aligned}$$

Thus, from (A.3) and (A.5), it follows that for $m \geq 1, n \geq 0$

$$\begin{aligned}
 & \int_{-1}^1 \int_{-1}^1 \log |z - \zeta| P_n(\zeta) P_m(z) \, d\zeta \, dz \\
 &= \begin{cases} \frac{8}{(m+n)(m+n+2)[(m-n)^2 - 1]}, & \text{if } m+n > 0 \text{ even,} \\ 0, & \text{if } m+n \text{ odd.} \end{cases} \quad (\text{A.6})
 \end{aligned}$$

Moreover, n and m are interchangeable in (A.1), such that (A.6) also holds for $n \geq 1, m \geq 0$. Only to the case $m = n = 0$ we need to pay more attention. We use a similar approach, but now with $Z_m(z)$ replaced by $P_1(z)$ and $n = 0$. We obtain

$$\begin{aligned}
 \int_{-1}^1 Q_0(z) P_1(z) \, dz &= \int_{-1}^1 \left[\frac{1}{2} P V \int_{-1}^1 \frac{P_0(\zeta) P_1(z)}{z - \zeta} \, d\zeta \right] \, dz \\
 &= \frac{1}{2} \int_{-1}^1 [\log |z - \zeta| P_0(\zeta) P_1(z)]_{-1}^1 \, d\zeta \\
 &\quad - \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \log |z - \zeta| P_0(\zeta) P_0(z) \, d\zeta \, dz \\
 &= 2 \log 2 - 2 - \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \log |z - \zeta| \, d\zeta \, dz, \quad (\text{A.7})
 \end{aligned}$$

and from (A.4)

$$\int_{-1}^1 Q_0(z) P_1(z) \, dz = 1, \quad (\text{A.8})$$

such that

$$\int_{-1}^1 \int_{-1}^1 \log |z - \zeta| \, d\zeta \, dz = 4 \log 2 - 6. \quad (\text{A.9})$$

Equations (A.6) and (A.9) together form the solution of (3.61).

Bibliography

- [1] ABDOL, M.A. Fredholm-Volterra integral equation with singular kernel. *Applied Mathematics and Computation* **137** (2003), 231–243.
- [2] ABRAMOWITZ, M., AND STEGUN, I.A. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Dover Publications, New York, 1968.
- [3] ATKINSON, K.E. *A survey of numerical methods for the solution of Fredholm integral equations of the second kind*. Society for Industrial and Applied Mathematics, Philadelphia, 1976.
- [4] ATKINSON, K.E. *The numerical solution of integral equations of the second type*. Cambridge University Press, Cambridge, 1997.
- [5] BEKERS, D.J. *Finite Antenna Arrays: An Eigencurrent Approach*. PhD thesis, Eindhoven University of Technology, 2004.
- [6] BEKERS, D.J., EIJNDHOVEN, S.J.L. VAN, AND VEN, A.A.F. VAN DE. Modelling and analysis of a long thin conducting stripline. *Journal of Engineering Mathematics* **49** (2004), 373–390.
- [7] BEURDEN, M.C. VAN. *Integro-differential equations for electromagnetic scattering*. PhD thesis, Eindhoven University of Technology, 2003.
- [8] BOWTELL, R., AND MANSFIELD, P. Gradient coil design using active magnetic screening. *Magnetic Resonance in Medicine* **17** (1998), 15–21.
- [9] BOYD, J.P. *Chebyshev and Fourier spectral methods*. Dover Publications, New York, 2000.
- [10] BYRD, P.F., AND FRIEDMAN, M.D. *Handbook of Elliptic Integrals for Engineers and Scientists*. Springer-Verlag, Berlin, 1971.
- [11] CHEN, C.N., AND HOULT, D.I. *Biomedical magnetic resonance technology*. Hilger, Bristol, 1989.

- [12] CHRONIK, B., AND RUTH, B.K. Constrained length minimum inductance gradient coil design. *Magnetic Resonance in Medicine* **39** (1998), 270–278.
- [13] CLEMENS, M., THOMA, P., WEILAND, T., AND RIENEN, U. VAN. Computational electromagnetic-field calculation with the Finite Integration Method. *Surveys on Mathematics in Industry* **8** (1999), 213–232.
- [14] COLLIN, R.E. *Foundations for microwave engineering*. McGraw-Hill, New York, 1992.
- [15] CROZIER, S., AND DODDRELL, D.M. Gradient coil design by simulated annealing. *Journal of Magnetic Resonance* **103** (1993), 354–357.
- [16] ERDÉLYI, A., AND BATEMAN, H. *Higher transcendental functions*. Mc. Graw-Hill, New York, 1953.
- [17] FRENKIEL, T.A., JASINSKI, A., AND MORRIS, P.G. Apparatus for generation of magnetic field gradient wave forms. *Journal of Physics E: Scientific Instrumentation* **21** (1988), 374–377.
- [18] GENENKO, Y.A., SNEZHKO, A.V., AND USOSKIN, A. Finite magnetic shields effect on the transport current in superconducting strips. *Physica C* **401** (2004), 236–240.
- [19] GERSEM, H. DE, LAHAYE, D., VANDEWALLE, S., AND HAMEYER, K. Comparison of quasi minimal residual and bi-conjugate gradient iterative methods to solve complex symmetric systems arising from time-harmonic magnetic simulations. *COMPEL* **18** (1999), 298–310.
- [20] GERSEM, H. DE, MERTENS, R., PAHNER, U., BELMANS, R., AND HAMEYER, K. A topological method used for field-circuit coupling. *IEEE Transactions on Magnetics* **34** (1998), 3190–3193.
- [21] GOLBERG, M.A., AND CHEN, C.S. *Discrete projection methods for integral equations*. Computational Mechanics Publications, Southampton, 1997.
- [22] GROETSCH, C.W. *Elements of applicable functional analysis*. Computational Mechanics Publications, Southampton, 1980.
- [23] HACKBUSCH, W. *Integral equations: theory and numerical treatment*. Birkhauser, Basel, 1995.
- [24] HARBERTS, D.W. *Dissipation and ringing of CRT deflection coils*. PhD thesis Eindhoven University of Technology, Eindhoven, 2001.
- [25] HARGREAVES, B.A., CUNNINGHAM, C.H., NISHIMURA, D.G., AND CONOLLY, S.M. Variable-rate selective excitation for rapid MRI sequences. *Magnetic Resonance in Medicine* **52** (2004), 590–597.

- [26] HERES, P.J., NIEHOF, J., AND SCHILDERS, W.H.A. Implementation of MOR for time domain simulation on real-life interconnect structures. In *Proceedings of the 8th IEEE Workshop on Signal Propagation on Interconnect, Heidelberg, Germany, May 9-12 (2004)*.
- [27] HUTTER, K., AND VEN, A.A.F. VAN DE. *Field matter interactions in thermoelastic solids*. Springer-Verlag, Berlin, 1978.
- [28] JIN, J.M. *Electromagnetic analysis and design in magnetic resonance imaging*. CRC Press, London, 1999.
- [29] KLINKENBUSCH, L. *Theorie der sphärischen Absorberkammer und des mehrschaligen Kugelschirmes*. Habilitationsschrift Ruhr-Universität Bochum, Bochum, 1996.
- [30] KOVETZ, A. *The principles of electromagnetic theory*. Cambridge University Press, Cambridge, 1990.
- [31] KROOT, J.M.B. *Current distribution in a gradient coil, modeled as circular loops of strips*. Master's thesis for the postgraduate program Mathematics for Industry, Eindhoven University of Technology, 2002.
- [32] KROOT, J.M.B. Analysis of eddy currents in a gradient coil. In *Proceedings of the 5th international workshop on scientific computing in electrical engineering, Capo d'Orlando, Sicily, Italy, September 5-9 (2004)*.
- [33] KROOT, J.M.B., EIJNDHOVEN, S.J.L. VAN, AND VEN, A.A.F. VAN DE. Eddy currents in a gradient coil, modeled as circular loops of strips. Submitted to Journal of Engineering Mathematics.
- [34] KROOT, J.M.B., EIJNDHOVEN, S.J.L. VAN, AND VEN, A.A.F. VAN DE. Eddy currents in a gradient coil, modeled as circular loops of strips. In *Proceedings of the 3rd international conference on computational electromagnetics and its applications, Beijing, China, November 2-5 (2004)*.
- [35] LAMMERANER, J., AND STAFL, M. *Eddy currents*. Hiffe books, London, 1966.
- [36] LANDAU, L.D., AND LIFSHITZ, E.M. *Electrodynamics of continuous media*. Pergamon Press, Oxford, 1960.
- [37] LI, W., MECHEFSKE, C.K., GAZDZINSKI, C., AND RUTT, B.K. Acoustic noise and prediction in a 4-T MRI scanner. *Concepts in Magnetic Resonance Part B - Magnetic Resonance Engineering* **21B** (2004), 19–25.
- [38] LUKE, Y.L. *Integrals of Bessel functions*. Mc. Graw-Hill, New York, 1962.
- [39] MAGNUS, W., OBERHETTINGER, F., AND SONI, R.P. *Formulas and Theorems for the Special Functions of Mathematical Physics*. Sproinger-Verlag, Berlin, 1966.

- [40] MANSFIELD, P., BOWLEY, R.M., AND HAYWOOD, B. Controlled E-field gradient coils. *Magnetic Resonance Materials in Physics Biology and Medicine* **16** (2003), 113–120.
- [41] MANSFIELD, P., AND CHAPMAN, B. Multishield active magnetic screening of coil structures in NMR. *Journal of magnetic resonance* **72** (1987), 211–223.
- [42] MANSFIELD, P., HAYWOOD, B., AND COXON, R. Active acoustic control in gradient coils for MRI. *Magnetic Resonance in Medicine* **46** (2001), 807–818.
- [43] MERTENS, R., PAHNER, U., GERSEM, H.D., BELMANS, R., AND HAMEYER, K. Improving the overall solver speed: A fast, reliable and simple adaptive mesh refinement scheme. In *The 4th International workshop on Electric and Magnetic Fields, Marseille, May 12-15*.
- [44] NEDELEC, J. *Acoustic and electromagnetic equations: integral representation for harmonic problems*. Springer-Verlag, Berlin, 2001.
- [45] NGUYEN, U.D., BROWN, J.S., CHANG, I.A., KRYCIA, J., AND MIROZNIK, M.S. Numerical evaluation of heating of the human head due to magnetic resonance imaging. *IEEE Transactions on Biomedical Engineering* **51** (2004), 1301–1309.
- [46] PEEREN, G.N. *Stream function approach for determining optimal surface currents*. PhD thesis, Eindhoven University of Technology, 2003.
- [47] PEEREN, G.N. Stream function approach for determining optimal surface currents. *Journal of Computational Physics* **191** (2003), 305–321.
- [48] PLONUS, M.A. *Applied electromagnetics*. Mc. Graw-Hill, London, 1978.
- [49] PRUDNIKOV, A.P. *Integrals and series*. USSR Academy of Sciences, Moscow, 1986.
- [50] READE, J.B. Asymptotic behaviour of eigen-values of certain integral equations. *Proceedings of the Edinburgh Mathematical Society* **22** (1979), 137–144.
- [51] REM, P.C. *Eddy current separation*. Eburon, Delft, 1999.
- [52] RIENEN, U. VAN. *Numerical methods in computational electrodynamics: linear systems in practical applications*. Springer Verlag, Berlin, 2001.
- [53] ROMEO, F., AND HOULT, D.I. Magnetic field profiling: Analysis and correcting coil design. *Magnetic Resonance in Medicine* **1** (1984), 44–65.
- [54] SCHILDERS, W.H.A., AND MEURIS, P. Coupled device, circuit and interconnect simulation. In *Proceedings of the ESSDERC/ESSCIRC 2003 conference, Estoril, Portugal, September 16-18*.

- [55] SCHOENMAKER, W., AND MEURIS, P. Electromagnetic Interconnects and Passives Modeling: Software Implementation Issues. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* **21** (2002), 534–543.
- [56] SERGEANT, P., DUPRÉ, L., MELKEBEEK, J., AND VANDENBOSSCHE, L. Magnetic field computation for optimized shielding of induction heaters. *Journal of Computational and Applied Mathematics* **168** (2004), 437–446.
- [57] SIEBOLD, H. Gradient field coils for MR imaging with high spectral purity. *IEEE Transaction on Magnetics* **26** (1990), 897–900.
- [58] SIMEU, E., AND GEORGES, D. Modeling and control of an eddy current brake. *Control Engineering Practice* **4** (1996), 19–26.
- [59] SMYTHE, W.R. *Static and dynamic electricity*. Mc. Graw-Hill, New York, 1968.
- [60] SNEDDON, I.N. *Fourier transforms*. Mc. Graw-Hill, New York, 1951.
- [61] SPAIN, B., AND SMITH, M.G. *Functions of mathematical physics*. Van Nostrand Reinhold, London, 1970.
- [62] SPANIER, J., AND OLDHAM, K.B. *An Atlas of Functions*. Sproinger-Verlag, Berlin, 1987.
- [63] STOLL, R.L. *The analysis of eddy currents*. Clarendon Press, Oxford, 1974.
- [64] STRATTON, J.A. *Electromagnetic theory*. Mc. Graw-Hill, London, 1941.
- [65] SUITS, H., AND WILKEN, D.E. Improving magnetic field gradient coils for NMR imaging. *Journal of Physics E: Instrumentation* **22** (1989), 565–573.
- [66] TAFLOVE, A., AND HAGNESS, S.C. *Computational electrodynamics: the finite-difference time-domain method*. Artech House, London, 2000.
- [67] TAS, J.M.C. Modelling and analyzing the z-gradient coil in an MRI scanner. Master's thesis, Eindhoven University of Technology, 2005.
- [68] TEGOPOULOS, J.A., AND KRIEZIS, M. *Eddy currents in linear conducting media*. Elsevier, Amsterdam, 1985.
- [69] TERAMOTO, T., NAGAI, T., UTSUGI, Y., AND SAITO, M. Numerical analysis on fracture behavior of a single-edge-cracked plate subjected to electromagnetic force. *Fusion Engineering and Design* **42** (1998), 389–394.
- [70] TITSCHMARSCH, E.C. *Introduction to the theory of Fourier integrals*. Chelsea, New York, 1986.

- [71] TOMASI, D. Stream function optimization for gradient coil design. *Magnetic Resonance in Medicine* **45** (2001), 505–512.
- [72] TURNER, R. Passive screening of switched magnetic field gradients. *Journal of physics E: Scientific instrumentation* **19** (1986), 876–879.
- [73] TURNER, R. A target field approach for optimal coil design. *Journal of physics D: Applied physics* **19** (1986), 147–151.
- [74] TURNER, R. Minimum inductance coils. *Journal of physics E: Scientific instrumentation* **21** (1988), 948–952.
- [75] TURNER, R. Gradient coil design: A review of methods. *Magnetic Resonance Imaging* **11** (1993), 903–920.
- [76] ULICEVIC, T. *Skin effect in a gradient coil*. Master’s thesis for the postgraduate program Mathematics for Industry, Eindhoven University of Technology, 2001.
- [77] ULICEVIC, T., KROOT, J.M.B., EIJNDHOVEN, S.J.L. VAN, AND VEN, A.A.F. VAN DE. Current distribution in a parallel set of conducting strips. Accepted for publication in *Journal of Engineering Mathematics*.
- [78] VERBEEK, M.E. *Iterative solvers and preconditioning for electromagnetic boundary integral equations*. PhD thesis, Universiteit Utrecht, 2001.
- [79] VLAARDINGERBROEK, M.T., AND BOER, J.A. DEN. *Magnetic Resonance Imaging*. Springer-Verlag, Berlin, 1999.
- [80] VOSSEN, S.H.J.A. *A two-wire antenna system for detecting objects in a homogeneous dielectric half space*. PhD thesis, Eindhoven University of Technology, 2003.
- [81] WANG, Z.X., AND GUO, D. *Special functions*. World Scientific, London, 1989.
- [82] WATSON, G.N. *A treatise on the theory of Bessel functions: tables of Bessel functions*. Cambridge University Press, Cambridge, 1922.
- [83] WHILE, P.T., AND FORBES, L.K. Electromagnetic fields in the human body due to switched transverse gradient coils in MRI. *Physics in Medicine and Biology* **49** (2004), 2779–2798.
- [84] WONG, E., AND JESMANOWICZ, A. Coil optimization for MRI by conjugate gradient descent. *Magnetic Resonance in Medicine* **21** (1991), 39–48.

Index

Ampère's law, 17

Bessel function, 41

Biot-Savart's law, 23

blurring, 9

Chebyshev polynomial, 56

coil constant, 7

continuity of charge, 18

convolution principle, 40

Coulomb force, 26

Coulomb gauge, 22

dielectric displacement, 17

dissipated power, 24

eddy current, 8, 31

edge-effect, 66

effective current, 25

electric charge, 17

electric conductivity, 20

electric current, 18

electric field, 18

electric permittivity, 20

electric scalar potential, 21

electric susceptibility, 20

elliptic integral, 76

energy balance, 23

Euler's constant, 76

Faraday's law, 18

Fourier integral transform, 39

Fourier modes, 48

Fourier series expansion, 38

Fredholm integral equation, 58

Galerkin method, 54

Gauss' law, 17

ghosting, 9

global basis function, 56

Golay coil, 8

gradient coil, 6

gradient homogeneity, 7

Gram matrix, 55

gyromagnetic ratio, 3

Hilbert transformation, 64

Hilbert-Schmidt operator, 58

hypergeometric function, 76

impedance, 26

inductive reactance, 26

islands, 107

jump condition, 19

Larmor frequency, 3

Larmor precession, 2

Legendre function of the second kind, 59

Legendre polynomial, 54

longitudinal relaxation, 4

Lorentz force, 26

Lorentz gauge, 22

magnetic energy, 25

magnetic induction, 18

magnetic permeability, 20
Magnetic Resonance Imaging, 1
magnetic susceptibility, 20
magnetic vector potential, 21
magnetization field, 20
main magnet, 4
Maxwell pair, 8, 92
Maxwell's equations, 17
mode, 48
modified Bessel function, 39

nucleus, 2

Ohm's law, 21

penetration depth, 32
Petrov-Galerkin method, 55
polarization field, 20
polygamma function, 76
Poynting vector, 24
primary field, 20

quasi-static approach, 23, 32

region of interest, 7
resistance, 25
RF-coils, 4
RF-pulse, 3

secondary field, 20
self-eddies, 32
self-eddy current, 32
self-inductance, 25
slits, 107
source current, 31
spherical harmonics, 104
stream function, 8
streamline, 8
surface charge density, 19
surface current density, 19

transverse relaxation, 4

voltage source, 31

Summary

Magnetic Resonance Imaging (MRI) is an imaging technique that plays an important role in the medical community. It provides images of cross-sections of a body, taken from any angle. The principle of MRI is based on the reaction of the body on a magnetic field. Hydrogen protons are stimulated by a strong external magnetic field and additional radio pulses, resulting in small electromagnetic signals emitted by the protons. The emitted signals are received by an acquisition system and processed to become an image with contrast differences.

The selection of a slice is realized by the so-called gradient coils. A gradient coil consists of copper strips wrapped around a cylinder. Due to mutual magnetic coupling, eddy currents are induced, resulting in a non-uniform distribution of the current. The eddy currents cause a distortion in the desired magnetic field. Spatial non-linearity of the gradient results in blurred images. Moreover, the presence of eddy currents increases the resistance of the coil and, consequently, the power dissipation. In order to reduce the energy costs, the dissipated power has to be minimized. Meanwhile, the self-inductance of the coil decreases due to the eddy currents, resulting in the need for a smaller voltage supply.

For analysis and design of gradient coils, finite element packages are used. However, these packages cannot always provide sufficient insight in the characteristics describing the qualitative behavior of the distribution of the currents, relating the geometry to typical parameters such as edge-effects, mutual coupling and heat dissipation. In this thesis, a detailed analysis of the eddy currents in gradient coils is presented. In particular, the analysis has led to the design of a software tool that simulates the current distribution and the electromagnetic fields inside the scanner quantitatively. Both the analysis and the software tool support the overall design of gradient coils. In the simulation, special attention is devoted to time effects (different frequencies) and spatial effects (space-dependent magnetic fields). Moreover, characteristic quantities of a system are derived, in particular resistance, self-inductance, characteristic frequencies, and linearity of the gradient field. For all these characteristic quantities, their dependence on the frequency of the applied source, the shape of the conductors, the distance between the conductors, and the conductivity is investigated.

For the mathematical model, Maxwell's equations are used, together with the associated boundary conditions and constitutive equations. Assumptions to reduce the set of equations

are that the electromagnetic fields are time-harmonic, the only driving source is a current source, the media are copper and air, a quasi-static approach can be used, the thicknesses of the strips can be neglected, and the conductors are rigid. The current is transformed into a surface current and the model leads to an integral equation of the second kind for the current density.

The model analysis predicts that the induced eddy currents in the strips prefer to flow in the direction opposite to the applied source current. The physical explanation is that eddy currents tend to oppose the magnetic field caused by the source current. The leading integral equation is formulated in terms of the component of the current in this preferred direction, imposed by the source current. The essential behavior of the kernel in this leading integral equation is logarithmic in the coordinate perpendicular to the preferred direction.

To solve the integral equation, the Galerkin method with global basis functions is applied to approximate the current distribution. In the preferred direction, trigonometric functions are chosen to express periodicity. Thus, the current is expanded in Fourier modes and via the inner products a direct coupling between the modes is achieved. In the width direction of the strips, Legendre polynomials are chosen. With this choice, basis functions are found that are complete, converge rapidly and the resulting inner products are easy to compute. Hence, this method is especially dedicated to the problems considered in this thesis.

The simulations show how the eddy currents are characterized by edge-effects. Edge-effects become stronger when the frequency is increased, and the errors in the magnetic field increase accordingly. The eddy currents also affect the resistance and self-inductance of a coil. For every configuration, the resistance increases with the frequency, whereas the self-inductance decreases with the frequency. Furthermore, both quantities show a point of inflection at a characteristic frequency. This characteristic frequency is expressed by an analytical formula.

The software tool has been designed for strips of different types of gradient coils and to model slits in the strips. The implementation makes use of limited memory, this in contrast to numerical packages, in which a lot of elements (memory) is needed to come to a high accuracy. Moreover, a fast approximation of the current distribution is achieved, because of appropriate basis functions and the use of explicit analytical results.

Samenvatting

Magnetic Resonance Imaging (MRI) speelt tegenwoordig een belangrijke rol in de medische wereld. Met deze techniek kunnen afbeeldingen gemaakt worden van dwarsdoorsneden van een lichaam, vanuit alle hoeken. Het principe van MRI is gebaseerd op de reactie van het lichaam op een magnetisch veld. Waterstof protonen worden gestimuleerd door een sterk extern magnetisch veld en hoogfrequente pulsen. Er ontstaan kleine electro-magnetische signalen, die door de protonen worden uitgestraald. Deze signalen worden opgevangen en verwerkt tot een afbeelding met contrastverschillen.

Voor het selecteren van een doorsnede worden de zogeheten gradiëntspoelen gebruikt. Een gradiëntspoel is gemaakt van koperen strips die om een cylinder zijn gewikkeld. Door onderlinge magnetische koppeling worden wervelstromen geïnduceerd, die leiden tot een niet-uniforme verdeling van de stroom. Deze wervelstromen veroorzaken een verstoring in het gewenste magnetisch veld. Ruimtelijke niet-lineariteit van het gradiëntveld leidt tot wazige afbeeldingen. Tevens neemt de weerstand toe als er wervelstromen zijn, en daarmee ook het gedissipeerd vermogen. Om de energiekosten te verlagen, moet het gedissipeerd vermogen worden geminimaliseerd. Tegelijkertijd neemt de zelfinductie van de spoel af, waardoor met een lager voltage kan worden volstaan.

Voor de analyse en het ontwerp van gradiëntspoelen worden eindige elementen pakketten gebruikt. Deze pakketten verschaffen echter niet altijd voldoende inzicht in de karakteristieken die het kwalitatieve gedrag van de stroomverdeling beschrijven, zoals het verband tussen de geometrie van de spoel en typische parameters als randeffecten, onderlinge koppeling en warmtedissipatie. In dit proefontwerp wordt een gedetailleerde analyse gegeven van de wervelstromen in de gradiëntspoelen. Deze analyse heeft geleid tot een software-ontwerp dat de stroomverdeling en de electro-magnetische velden in de scanner simuleert. De analyse en het software-ontwerp dragen bij aan het totale ontwerp van gradiëntspoelen. In het ontwerp is speciale aandacht besteed aan tijdseffecten (verschillende frequenties) en ruimtelijke effecten (plaatsafhankelijke magnetische velden). Verder zijn er karakteristieke grootheden van het systeem afgeleid, zoals in het bijzonder weerstand, zelfinductie, karakteristieke frequenties en lineariteit van het gradiëntveld. Voor al deze karakteristieke grootheden is onderzocht hoe ze afhangen van de frequentie van de bron, de vorm van de geleiders, de

afstand tussen de geleiders en de geleidingscoëfficiënt.

Voor het wiskundige model zijn de vergelijkingen van Maxwell gebruikt, samen met de bijbehorende randvoorwaarden en constitutieve vergelijkingen. De aannames die gemaakt zijn om het stelsel vergelijkingen te reduceren, zijn dat de electro-magnetische velden tijds-harmonisch zijn, de enige aandrijvende kracht een stroombron is, de enige media koper en lucht zijn, een quasi-statische benadering toegepast kan worden, de dikte van een strip verwaarloosbaar dun is en dat de geleiders niet kunnen bewegen. De stroom is gemodelleerd als een oppervlaktestroom en het model leidt tot een integraalvergelijking van de tweede soort voor deze stroomdichtheid.

De analyse van het model voorspelt dat de geïnduceerde wervelstromen de neiging hebben om in de richting tegengesteld aan de aangebrachte bron te stromen. De fysische verklaring hiervoor is dat de wervelstromen het magnetisch veld dat door de stroombron veroorzaakt wordt, tegenwerken. De leidende integraalvergelijking is uitgedrukt in de component van de stroom in deze voorkeursrichting. Het essentiële gedrag van de kernfunctie in deze leidende integraalvergelijking is logaritmisch in de coördinaat die loodrecht op de voorkeursrichting staat.

Om de integraalvergelijking op te lossen, wordt de Galerkin methode toegepast met globale basisfuncties. In de voorkeursrichting zijn goniometrische functies gekozen vanwege de periodiciteit. De stroom is dus ontwikkeld in Fourier modes en via de inwendige produkten wordt een directe koppeling tussen de modes aangetoond. In de breedte-richting van de strips zijn Legendre-polynomen gekozen. Met deze keuze zijn basisfuncties gevonden die volledig zijn, snel convergeren, en inwendige produkten opleveren die eenvoudig te berekenen zijn. Dus deze methode is uitermate geschikt voor de problemen in dit proefontwerp.

De simulaties tonen aan hoe de wervelstromen worden gekenmerkt door randeffecten. De randeffecten nemen toe als de frequentie wordt verhoogd, net als de fouten in het magnetisch veld. De wervelstromen beïnvloeden ook de weerstand en de zelfinductie van een spoel. Voor elke geometrie neemt de weerstand toe met de frequentie, terwijl de zelfinductie afneemt met de frequentie. Verder vertonen beide grootheden een buigpunt bij een karakteristieke frequentie. Voor deze karakteristieke frequentie is een analytische uitdrukking afgeleid.

Het software-ontwerp is geschikt voor verschillende soorten gradiëntspoelen en voor het modelleren van snedes in de strips. De simulaties hebben weinig computergeheugen nodig, in tegenstelling tot numerieke pakketten, en leiden tot een hoge nauwkeurigheid. Verder wordt een snelle convergentie naar de juiste stroomverdeling bereikt, vanwege de geschikte basisfuncties en het gebruik van expliciete analytische resultaten.

Curriculum vitae

Jan Kroot was born in Loon op Zand, The Netherlands, on October 28th 1975. He completed his pre-university education (gymnasium) at the Dr. Moller College in Waalwijk in 1994 and started his Applied Mathematics studies at Eindhoven University of Technology in the same year. During his studies, he was a trainee at Océ Technologies in Venlo, The Netherlands. His master's project was carried out at Axxicon Moulds Eindhoven and resulted in the thesis with the title "Shrinkage at Microscopic Level in the Injection Moulding of Compact Discs". In 1999 he graduated with distinction. From March 2000 until March 2002, he followed the postgraduate program Mathematics for Industry at the Stan Ackermans Institute of the Eindhoven University of Technology. Within the framework of this program, he carried out several projects for companies, such as Océ Technologies, The Netherlands, and Rsscan International, Belgium. In the months April - June 2001, he joined the Department of Engineering Mathematics at the University of Bristol, United Kingdom, where he developed a car-following model of highway traffic. The final project was carried out for Philips Medical Systems in Best, The Netherlands, after which he received the PDEng (Professional Doctorate in Engineering) degree. This final project was continued as a PhD on design. From March 2002 until June 2005, he was a PhD student in the CASA group of the Mathematics Department at the Eindhoven University of Technology and carried out the research at the company Philips Medical Systems in Best, The Netherlands.