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The cooling of molten glass in a mould

by

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Abstract
The actual formation of glass jars consists of two stages, viz. pressing and blowing. In the first stage the hot (liquid) glass is pressed in a mould to form the 'parison'. The heat transfer in glass and mould can be solved numerically by using a combined BEM/FEM discretisation. In particular, we use a FEM approach for the discretisation of Stokes' equations and the stationary convection diffusion equation in the glass, and a Galerkin BEM approach for the diffusion equation in the mould. This way, one uses the optimal properties of both techniques. The eventual objective is to consider the full nonstationary problem.

1 Introduction
The coupling of different numerical techniques has become increasingly popular. When PDEs have to be solved on various subdomains, it can be advantageous to use an appropriate technique on each subdomain. So, e.g. the Finite Element Method (FEM) and the Boundary Element Method (BEM) may be combined as we do in this paper. In order to have a consistent formulation on common BEM/FEM boundaries, we use a Galerkin BEM formulation.

The problem at hand comes from Dutch glass industry which is interested in reducing energy usage. Their goal is to make thinner glass jars, which both satisfy certain strength criteria and give only limited loss in the production process. This production process consists of two stages. First, a glass 'gob' is pressed to become a 'parison'. Then this parison is blown to the final shape of the glass jar. We concentrate on the formation of the
parison. In this paper, a discretisation of the energy equation is developed. Since it is very difficult to measure temperatures inside the mould, appropriate boundary conditions for the energy equation on the glass-mould surfaces are hard to determine. One approach is described in [6], where an analytical solution of the 1-D situation is used to determine approximate boundary conditions for the 3-D computations. We will solve the problem with a pure numerical model, in which the mould is part of the computational domain. This enables us to prescribe realistic boundary (and initial) conditions, in particular at the outer walls of the mould.

During the pressing of the glass temperature differences of the order of $800^\circ C$ occur but the time scale of the process is very short [6]. A consequence of this is that the variation of the temperature in the mould will be limited to a thin boundary layer only. The temperature in the mould as such is not of interest; only its effect on the glass viscosity is important for the process. Also, the volume of the mould is much larger than the gob volume. By reducing the number of unknowns in a BEM formulation on the mould and the plunger, we hope to accelerate the computation, in which a large number of time steps will be needed.

The geometry and the material properties of the mould and plunger do not change in time. So the discretisation there has to be performed only once. The extra computational costs of determining the coefficients of the matrices, arising from the Galerkin BEM discretisation, has to be paid only once. For the convection diffusion equation on the glass domain, a FEM discretisation is more appropriate because the equation is the strongly convection dominated. Since remeshing on the glass domain will be necessary, and the viscosity changes in time and place, the FEM matrices need to be updated quite often. The velocity field, used in the convection diffusion equation comes from the solution of Stokes’ equations on the glass domain. These latter equations are solved with standard FEM techniques and this will be discussed only briefly.

In Section 2, the dimensionless equations are given and boundary conditions are chosen. Section 3 deals with the discretisation methods and the discrete coupling conditions. In Section 4 we present some numerical results for the stationary equations. Remarks and further outlooks are the subject of section 5.

2 Equations and boundary conditions

The formation of a parison starts with a piece of liquid viscous glass, the gob, being cut from a flow of glass produced in a glass oven. This gob has a temperature of about $1200^\circ C$ and is dropped in a mould where it is pressed to become a parison, see figure (1).
When the parison leaves the first mould to be blown to its final shape in the second mould, it has a mean temperature of approximately 600°C. The initial temperature of the mould is 400°C, so very large differences in temperature occur. Glass above the glass temperature $T_g$ behaves like a Newtonian fluid [8]. For temperatures above the glass temperature $T_g$, the viscosity of glass $\eta$ as a function of the temperature $T$ can be described by the so termed Vogel-Fulcher-Tamman relation [8]:

$$\eta = K \exp\left(\frac{E_0}{T - T_0}\right).$$  \hspace{1cm} (1)

Here, $K$ is a constant, $E_0$ is the viscosity activation energy and $T_0$ is a fixed temperature. These three parameters are fitted in a least squares sense to certain measurements.

The following abbreviations are introduced

$\Omega_1$ the mould
$\Omega_2$ the glass domain
$\Omega_3$ the plunger

$\Gamma_i$ boundary of $\Omega_i$
$\Gamma_{i,j}$ $\Gamma_i \cap \Gamma_j$
$\Gamma_{i,ext}$ $\Gamma_i \setminus \Gamma_j$, $j \neq i$

2.1 Equations

We will use an Eulerian formulation on the glass domain, and a Lagrangian formulation on the two mould domains. The coupling conditions (9) will then assure continuity of the global solution and of the heat flux. The advantage of this approach is that the movement of the plunger does not give rise to a convection term in the energy equation. Thus, for a material point, let us denote $X$ for its position on time $t = 0$. Assume there is a smooth invertible function $\phi$ such that

$$x = \phi(X, t), \ t \geq 0.$$  \hspace{1cm} (2)

Here, $x = x(t)$ is the position of the material point at time $t$. Now we relate the functions $T$ and $\dot{T}$ by

$$T(x, t) = T(\phi(X, t), t) = \dot{T}(X, t)$$  \hspace{1cm} (3)
On the glass domain \( \Omega_2 \), the process can be described by the following dimensionless equations

\[
-\nabla p + \nabla \cdot \eta \left( \text{grad} \, v + \left( \text{grad} \, v \right)^T \right) = 0 \\
\nabla \cdot v = 0
\]

where the dimensionless parameters are defined as

\[
p = p(x, t) \quad \text{the pressure} \\
v = v(x, t) \quad \text{the velocity} \\
x = (x_1, x_2)^T \quad \text{the position} \\
t \quad \text{time} \\
T = T(x, t) \quad \text{the temperature} \\
\eta = \eta(T) \quad \text{the dynamic viscosity} \\
\alpha = \frac{k_g}{\rho_g c_g} \quad \text{the glass diffusivity} \quad \text{which is assumed to be a constant material parameter. Here,} \quad k_g \quad \text{is the thermal conductivity of the glass,} \quad \rho_g \quad \text{is the mass density of the glass,} \quad c_g \quad \text{is the specific heat of the glass;} \quad v_k \quad \text{and} \quad L \quad \text{are a characteristic velocity and a characteristic length respectively.}
\]

On the plunger \( \Omega_3 \) and the fixed outer mould \( \Omega_1 \), the equation is given in Lagrangian form by

\[
P_{em} \frac{\partial \hat{T}}{\partial t} - \nabla \cdot \hat{T} = 0.
\]

\( P_{em} = \frac{v_L}{\alpha} \) is the Peclet number on the mould, which is defined like \( P_{eg} \), with the subscript \( m \) denoting mould.

\[
\frac{\partial \hat{T}}{\partial t} = \frac{DT}{Dt} = \frac{\partial T}{\partial t} + (v_m \cdot \nabla T)
\]

where \( v_m = 0 \) on the outer mould, and \( v_m = (0, v_m)^T \) is the velocity of the plunger.

### 2.2 The coupling conditions

In order to couple the BEM and FEM solutions, we require both continuity of the temperature and of the heat flux on common boundaries:

\[
\begin{align*}
\hat{T}_{bem}(\phi^{-1}(x)) &= T_{fem}(x) \\
k_m \frac{\partial \hat{T}_{bem}(\phi^{-1}(x))}{\partial n} &= -k_g \frac{\partial T_{fem}(x)}{\partial n}
\end{align*}
\]

\( x \in \Gamma_{1,2} \cup \Gamma_{2,3} \).
In order to model the heat transfer, due to radiation inside the glass, we use the so called Rossland approximation [8]. In this approximation, the thermal conductivity $k_g$ is enlarged to account for radiation heat transfer. Let us denote $k_d$ for the thermal conductivity due to heat conduction in the glass, and $k_r$ for the 'thermal conductivity' due to radiation heat transfer. Then in the Rossland approximation $k_g$ satisfies $k_g = k_d + k_r$, where

$$k_r = \frac{16 n^2 \sigma T_m^3}{3 \alpha},$$

(10)

with $T_m$ being the mean temperature, $n$ is the refraction index, $\sigma$ is the Stefan-Boltzmann radiation constant and $\alpha$ is the heat absorption coefficient of the glass. We must emphasize that the Rossland approximation is not fully justified here, as we intend to simulate the production of transparent glass. The condition $\alpha^{-1} \ll L$ for the validity of the Rossland approximation is violated in this case.

### 2.3 Boundary conditions

Stokes' equations (5) are solved only on $\Omega_2$. We assume a no-slip condition on the glass-mould contact lines. Thus we impose the following Dirichlet boundary conditions

$$v = 0 \text{ for } x \in \Gamma_{1,2} \text{ and } v = (0, v_m)^T \text{ for } x \in \Gamma_{2,3}.$$

(11)

On the free surface of the moving glass $\Gamma_{2,ext}$, the normal stress is equal to the air pressure which we set to zero, thus

$$T \cdot n = 0,$$

(12)

where the stress tensor $T$ is defined by $T := -p I + 2\eta \Sigma$, [10]. $I$ is the identity tensor and $\Sigma$ is the rate of deformation tensor:

$$\Sigma_{ij} := \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

(13)

Surface tension and gravity forces may be neglected.

For the energy equations (6) and (7), at this stage, Dirichlet conditions are chosen at $\Gamma_{1,ext}$ and $\Gamma_{3,ext}$. A homogeneous Neumann condition is imposed on $\Gamma_{2,ext}$.

### 3 Formulation of the discrete problem

First, we give the discrete mixed FEM formulation of the Stokes problem (5). Then, the FEM and BEM formulation for the stationary forms of equations (6) and (7) together with the discrete coupling conditions of both formulations are given.
3.1 The momentum and continuity equations

The glass domain is subdivided into triangular finite elements. We use modified $P_2^+ - P_1$ Crouzeix-Raviart elements, cf. [3]. In matrix form, the discrete form of equation (5) is then, cf. [4],

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = 0 ,$$

where the vectors $v$ and $p$ contain the values of the velocities and the pressure in the nodal points respectively. The velocity is thus approximated by piecewise quadratic shape functions; the pressure is approximated by piecewise linear shape functions.

3.2 The energy equation

The same mesh of triangular finite elements is used for the discretisation of the convection-diffusion equation (6). Thus, the temperature $T$ in the glass is approximated by

$$T = \sum_i \alpha_i \phi_i(x), \ x \in \Omega_2 ,$$

where the $\phi_i$ are the same piecewise quadratic 2-D shape functions as those used for the velocity.

On glass-mould boundaries $\Gamma_{1,2}$ and $\Gamma_{2,3}$, no boundary conditions are given. We approximate the normal derivative of the temperature by shape functions $\lambda_j$. These latter shape functions can for instance be the restriction of $\phi_i$ to $\Gamma_{1,2}$ and $\Gamma_{2,3}$, cf. [2],

$$q(x) = \frac{\partial T(x)}{\partial n} = \sum_i \beta_i \lambda_i(x), \ x \in \Gamma_{1,2} \cup \Gamma_{2,3} .$$

Application of the usual Galerkin weighting procedure leads to the following FEM system

$$\begin{pmatrix} K^1 & K^{1,2} & 0 \\ K^{2,1} & K^2 & K^{2,3} \\ 0 & K^{3,2} & K^3 \end{pmatrix} \begin{pmatrix} \alpha^{1,2} \\ \alpha^2 \\ \alpha^{2,3} \end{pmatrix} + \begin{pmatrix} Q^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q^3 \end{pmatrix} \begin{pmatrix} \beta^{1,2} \\ \beta^2 \beta^{2,3} \end{pmatrix} = 0 .$$

The FEM matrices satisfy

$$Pe_g K_{ij} = \int_{\Omega_2} [(v \cdot \nabla \phi_j) \phi_i + (\nabla \phi_j \cdot \nabla \phi_i)] d\Omega_2 - \int_{\Gamma_{2,ext}} \frac{\partial \phi_j}{\partial n} \phi_i d\Gamma_2 ,$$

$$Q^1_{ij} = \int_{\Gamma_{1,2}} \phi_i \lambda_j d\Gamma_2 ,$$

$$Q^3_{ij} = \int_{\Gamma_{2,3}} \phi_i \lambda_j d\Gamma_3 ,$$

(18) (19) (20)
where $\phi_i$ is the shape function belonging to node $x_i$. The vector $\alpha^2$ contains $T$ on $\Omega_2 \setminus (\Gamma_{1,2} \cup \Gamma_{2,3})$, whereas $\alpha^{1,2}$, $\beta^{1,2}$, $\alpha^{2,3}$ and $\beta^{2,3}$ contain $T$ and $q$ on $\Gamma_{1,2}$ and $\Gamma_{2,3}$.

Usage of BEM necessitates a discretisation of the boundaries only. The restriction of the piecewise quadratic shape functions in $\Omega_2$ are piecewise quadratic on $\Gamma_2$. Therefore, we use piecewise quadratic shape functions $\phi_i(x)$ as the basis for our BEM approximation. In addition to the function values, the normal derivatives of the function have to be approximated in BEM. Thus, we seek a solution $T(x) = \hat{T}(X)$ on the boundaries of the BEM domain in the following form

$$
T(x) = \sum_i \alpha_i \phi_i(x) \ , \ q(x) = \frac{\partial T(x)}{\partial n} = \sum_i \beta_i \lambda_i(x) \ , \ x \in \Gamma_1 \ , \ \Gamma_3 \ .
$$

Application of the Galerkin BEM to the stationary form of equation (7) on the outer mould $\Omega_1$ yields, c.f. [5],

$$
\left( \begin{array}{c}
H^1 \\
H^{1,2}
\end{array} \right) \left( \begin{array}{c}
\alpha^1 \\
\alpha^{1,2}
\end{array} \right) = \left( \begin{array}{c}
G^1 \\
G^{1,2}
\end{array} \right) \left( \begin{array}{c}
\beta^1 \\
\beta^{1,2}
\end{array} \right) \ .
\label{eq:22}
$$

The BEM matrices satisfy

$$
H_{ij} = \int_{\Gamma_x} \left( \frac{1}{2} \phi_i(x) + \int_{\Gamma_y} \phi_j(y) q^* d\Gamma_y \right) \phi_i(x) d\Gamma x 
$$

$$
G_{ij} = \int_{\Gamma_x} \int_{\Gamma_y} \lambda_j(y) u^* d\Gamma_y \phi_i(x) d\Gamma x \ ,
$$

where $\phi_i$ is the 1-D shape function belonging to node $x_i$. The vectors $\alpha^1$ and $\beta^1$ contain $T$ and $q$ on $\Gamma_1 \setminus \Gamma_{1,2}$ respectively, whereas $\alpha^{1,2}$ and $\beta^{1,2}$ contain those on $\Gamma_{1,2}$.

$u^* = u^*(x,y)$ is the fundamental solution in 2-D of the Laplace equation, and $q^*$ is the normal derivative of $u^*$ with respect to $y$:

$$
\begin{align*}
u^*(x,y) &= -\frac{1}{2\pi} \log || x - y || \\ q^*(x,y) &= \frac{\partial u^*}{\partial n_y} = n_y \cdot (\frac{\partial u^*}{\partial y_1}, \frac{\partial u^*}{\partial y_2})^T
\end{align*}
$$

The norm $|| \cdot ||$ denotes the Euclidian norm. $\Gamma = \Gamma_1$, and by the vector $n_y$ we mean the outward pointing normal at a boundary point $y$.

Similarly to equation (22), the discrete system on $\Omega_3$ is

$$
\left( \begin{array}{c}
H^3 \\
H^{2,3}
\end{array} \right) \left( \begin{array}{c}
\alpha^3 \\
\alpha^{2,3}
\end{array} \right) = \left( \begin{array}{c}
G^3 \\
G^{2,3}
\end{array} \right) \left( \begin{array}{c}
\beta^3 \\
\beta^{2,3}
\end{array} \right) \ .
\label{eq:26}
$$
The assembled system of the discrete stationary energy equation can
then be written as follows:

\[
\begin{pmatrix}
H^1 & H^{1,2} & -G^1 & -\kappa G^{1,2} & 0 & \cdots & 0 \\
0 & \cdots & 0 & H^3 & H^{2,3} & -G^3 & -\kappa G^{2,3} & 0 \\
0 & K^1 & 0 & Q^1 & 0 & \cdots & 0 & K^{1,2} \\
0 & \cdots & 0 & K^3 & 0 & Q^3 & K^{2,3} & 0 \\
0 & K^{2,1} & 0 & 0 & 0 & K^{2,3} & 0 & 0 & K^2
\end{pmatrix}
\begin{pmatrix}
\alpha^1 \\
\alpha^{1,2} \\
\beta^1 \\
\beta^{1,2} \\
\alpha^3 \\
\beta^{2,3} \\
\beta^3 \\
\alpha^2 \\
\gamma^2
\end{pmatrix} = 0
\]  

(27)

We have multiplied the BEM matrices \(G^{1,2}\) and \(G^{2,3}\) by the factor \(\kappa = \frac{k_s}{k_m}\) to ensure the continuity of the heat flux, see equation (9). In this way, the vectors \(\beta^{1,2}\) and \(\beta^{2,3}\) contain the values of the normal derivative of \(T\) with respect to \(\Omega_2\).

4 Some numerical results

For our computations, we use the Finite Element program \textit{SEPRAN}. The BEM subregions are considered as FEM superelements, so that the BEM matrices can be generated in our own element subroutine. The boundaries of the BEM domains are thus represented by line elements, whereas the FEM elements are 2-D triangular elements. For the generation of the \(Q\) matrices (see equation (17)), also line elements are used. The shape functions \(\lambda_i\) for \(q\) are chosen to be the same as the shape functions \(\phi_i\) for \(T\).

The weakly singular integrals \(G_{ii}\) in equation (24) are difficult to evaluate analytically because of the quadratic nature of the BEM elements (see also [5], pg. 333). Therefore, we have developed special cubature formulae of the kind

\[
\int_{-1}^{1} \int_{-1}^{1} w(\xi, \zeta) f(\xi, \zeta) d\xi d\zeta = \sum_{i=1}^{N^2} A_i f(\xi_i, \zeta_i) ,
\]

(28)

for the weight function \(w(\xi, \zeta) = -\log \| \xi - \zeta \|\) (see [9]). The function \(f\) depends on the Jacobian of the transformation to the standard element (including the curvature) and of the shape functions \(\phi(x(\xi))\) and \(\phi(y(\zeta))\).

The mesh we use (see figure (2)) consists of, in total, 503 nodes. The number of extra nodes we added for the additional discretisation of the mould and the plunger, is 113.
For our computations, we have chosen the dimensionless quantities as \( \eta = 0.02, Pe_g = 100 \) and \( Pe_m = 30 \). This is based on a glass temperature of \( T = 800^\circ C \), a plunger velocity \( (v_m)_1 = v_k = 0.05 \frac{m}{s} \) and a length \( L = 0.01 m \). The following boundary conditions were imposed: \( T = 800^\circ C \) at \( \Gamma_{3,ext} \), \( T = 960^\circ C \) at \( \Gamma_{1,ext} \) and \( q = 0 \) at \( \Gamma_{2,ext} \). Stokes’ equations are solved by the Penalty Method [3]. In these computations, we assumed that \( \frac{\partial T}{\partial t} = 0 \).

Figure (3) shows results for the velocity field and the stationary temperature distribution.
5 Remarks and further outlook

In our paper, we formulated a discretisation method for a domain, consisting of a subdomain where a convection diffusion equation has to be solved, and two other subdomains where a diffusion equation has to be solved. Galerkin BEM together with FEM were used in the combined problem.

We plan to extend our model to instationary problems. For the FEM discretisation, this is quite standard as is shown by the extensive use of the Method of Lines. BEM can be used for these time dependent problems, too, if a fundamental solution of the instationary equation is known, as is the case for the heat equation [1]. Yet, combination of the Method of Lines together with a spatial FEM discretisation on one hand, and time dependent BEM on the other, appears to be difficult. Therefore, we consider the use of the Dual Reciprocity BEM, where a similar matrix system is obtained as is in the Method of Lines [7]. This will enable us to combine BEM and FEM, also for the actual time dependent problem.

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