1. Introduction. Recently a team of five authors\(^1\) published a collection of over 400 geometric inequalities, most of them dealing with triangles. The majority of the latter can be rewritten in the form \( P(a, b, c) > 0 \) or \( P(a, b, c) \geq 0 \) where \( P(a, b, c) \) is a symmetric and homogeneous polynomial in the real variables \( a, b, c \), representing the sides of a triangle. In GI a great number of discrete polynomials \( P(a, b, c) \) is given. In this paper we determine the complete set of symmetric and homogeneous polynomials of order \( n \leq 3 \) that give rise to a correct geometric inequality and give some partial results for \( n = 4 \).

2. Preliminary remarks. If \( P(a, b, c) > 0 \) or \( P(a, b, c) \geq 0 \) is a geometric inequality and if \( P \) is symmetric and homogeneous we will call it an inequality polynomial or I.P. Many I.P.'s published in GI have the special property that they vanish identically for equilateral triangles. In such a case \( P \) will be called a special I.P. Now the symmetric and homogeneous polynomials of order \( n \) form a vector space \( V_n \) of finite dimension\(^2\). If \( P_1 \) and \( P_2 \) are I.P.'s then also \( \lambda_1 P_1 + \lambda_2 P_2 \) is one when \( \lambda_1 \) and \( \lambda_2 \) are non-negative, not both zero. So these polynomials form a convex subset of \( V_n \) which is the inner part \( C_n \) of a semicone.

The polynomial

\[
P_{pqr} = a^p b^q c^r + a^q b^r c^p + a^r b^p c^q + a^p b^r c^q + a^q b^p c^r + a^r b^q c^p
\]

where \( p \geq q \geq r \) is supposed, is a symmetric and homogeneous polynomial of order \( n = p + q + r \). Any symmetric and homogeneous polynomial of order \( n \) can be written as a linear combination \( \sum \lambda_{pqr} P_{pqr} \) of such polynomials. Each of these polynomials takes the value 6 in the point \((1, 1, 1)\). So the special I.P.'s all lie in a hyperplane \( H_n \) with equation \( \sum \lambda_{pqr} = 0 \). The set of special I.P.'s is a convex and semiconic subset \( C_n^* \) of this hyperplane; we have \( C_n^* = C_n \cap H_n \).

\* Presented October 1, 1970 by O. Bottema.
\(^1\) O. Bottema, R. Z. Dorevici, R. R. Janic, D. S. Mitrinovic, P. M. Vasic: Geometric Inequalities. Groningen 1969. It will be denoted GI in this paper.
\(^2\) For \( n = 6k \) this dimension is \( 3k^2 + 3k + 1 \), for \( n = 6k + i \) it is \( (k + 1)(3k + i) \); \( i = 1, 2, 3, 4, 5 \).
We order the polynomials $P_{pqr}$ by writing their leading terms in alphabetic order. Then the polynomials $P'_{pqr}$ obtained by subtracting its successor from each $P_{pqr}$ but the last one form a basis of $H_n$.

If $a > 0$ then $P(a, b, c)$ and $P(p a, q b, p c)$ have equal signs because $P$ is homogeneous. Therefore we need only to consider classes of similar triples. The classes of similar triples with positive elements form the inner part of the triangle $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ in the projective plane. The coordinates $a, b, c$ have to satisfy $a > b + c, b > c + a, c > a + b$. This reduces the part of the plane to be considered to the inner part of the triangle $(1, 1, 0), (0, 1, 1), (1, 0, 1)$.

Because $P$ is symmetric, a permutation of $a, b, c$ does not change its value. Hence without loss of generality we may assume $a \geq b \geq c$. This reduces the part of the plane to be considered to the inner part of the triangle $\Delta(1, 1, 1), (1, 1, 0), (2, 1, 1)$. Occasionally we will choose $b = 1, a = 1 + \gamma, c = 1 - \gamma$, and study the values of $P$ on the euclidean triangle $T: a, \gamma \geq 0, a + \gamma < 1$. This will not lead to confusion because points of $\Delta$ are denoted with three and points of $T$ with two coordinates.

3. I.P.'s of order 1. Here $X' = \frac{1}{2} P_{100} = a + b + c = 2s$ is a basis of $V_1$. The I.P. semicone is the set $x_1 X'$ with $x_1 > 0$. The set of special I.P.'s of order 1 is empty.

4. I.P.'s of order 2. $P_{200} = 2a^2 + 2b^2 + 2c^2$ and $P_{110} = 2ab + 2bc + 2ca$ form a basis of $V_2$, while $P_{200} = Q = (a - b)^2 + (b - c)^2 + (c - a)^2$ is a basis of $H_2$. We write $X'_2 = P_{200}$.

Then the semicone of special I.P.'s is the set $x_1 X'_2$ with $x_1 > 0$. Another basis of $V_2$ is given by $X_1$ and

$$X'_2 = \frac{1}{2} (P_{110} - P_{200}) = 2ab + 2bc + 2ca - a^2 - b^2 - c^2.$$ 

Here also $X'_2$ is an I.P. for $X'_2 = a^2 - (b - c)^2 + b^2 - (c - a)^2 + c^2 - (a - b)^2 > 0$. The I.P. semicone contains the set $S$: $x_1 X'_2 + x_2 X'_2; x_1 \geq 0, x_2 \geq 0$; not $x_1 = x_2 = 0$. On the other hand, if $P = x_1 X'_1 + x_2 X'_2$, then $P(1, 1, 0) = 2x_1$ and $P(1, 1, 1) = 3$ so if $P$ is an I.P. then both $x_1$ and $x_2$ are nonnegative.

So the I.P. semicone is exactly the set $S$.

5. I.P.'s of order 3. A basis for $H_3$ is given by

$$X'_2 = P'_{300} - P'_{210} = (a - b)^2 (a + b - c) + (b - c)^2 (b + c - a) + (c - a)^2 (c + a - l)$$

and

$$X'_3 = 3P'_{210} - P'_{300} = (a - b)^2 (3c - a - b) + (b - c)^2 (3a - b - c) + (c - a)^2 (3b - a)$$

Evidently, $X'_3$ is an I.P. Also $X'_2$ is one; we have

$$X'_2 = (a - b)^2 (3c - a - b) + (b - c)^2 (3a - b - c) + (a - b)^2 (3b - a - c) = 2(a - b)^2 (b + c - a) + 2(b - c)^2 (a + b - c) + 2(a - b) (b - c) (3b - a - c)$$

We write $X'_3 = P'_{300} - P'_{210} = (a - b)^2 (a + b - c) + (b - c)^2 (b + c - a) + (c - a)^2 (c + a - l)$.
because
\[ 3b-a-c > (3b-c)-(b+c) = 2(b-c). \]

Further, if \( P = x_1 X_1^3 + x_2 X_2^3 \) then \( P(2, 1, 1) = 4x_1 \) and \( P(1, 1, 0) = 4x_2 \). So in an I.P. neither \( x_1 \) nor \( x_2 \) can be negative.

The semicone of special I.P.'s therefore is the set \( x_1 X_1^3 + x_2 X_2^3, x_1 \geq 0, x_2 \geq 0 \), not \( x_1 = x_2 = 0 \).

As for the other I.P.'s certainly \( X_3^3 = (a+b-c)(b+c-a)(c+a-b) \) is one. Consider the set
\[ \{ P | P = x_1 X_1^3 + x_2 X_2^3 + x_3 X_3^3 \}. \]

We have \( P(2, 1, 1) = 4x_1; P(1, 1, 0) = 4x_2; P(1, 1, 1) = x_3 \). So the I.P. semicone is the above set under the condition \( x_1, x_2, x_3 \geq 0 \), not \( x_1 = x_2 = x_3 = 0 \).

6. I.P.'s of order 4. \( P_{400}, P_{310}, P_{220}, P_{211} \) form a basis of \( V_4 \); \( P'_{400}, P'_{310}, P'_{220} \) form a basis of \( H_4 \). Another basis of the latter space is given by

\[ X_1^4 = \frac{1}{2} (P_{400}' - P_{310}' - P_{220}') \]
\[ = \frac{1}{2} \{(a-b)^2 (a^2 + b^2 - c^2) + (b-c)^2 (b^2 + c^2 - a^2) + (c-a)^2 (c^2 + a^2 - b^2)\} \]
\[ = a^2 (a-b)^2 + c^2 (b-c)^2 + (a-c) (b-c) (c^2 + a^2 - b^2), \]

\[ X_2^4 = \frac{1}{2} (P_{400}' - 5P_{310}' + 3P_{220}') \]
\[ = \frac{1}{2} \{(a-b)^2 (a^2 + b^2 + 3c^2 - 4ab) + (b-c)^2 (b^2 + c^2 + 3a^2 - 4bc) \]
\[ + (c-a)^2 (c^2 + a^2 + 3b^2 - 4ac)\} \]
\[ = \frac{1}{2} (a-b)^2 \{(a-2b)^2 + (a-2c)^2\} + \frac{1}{2} (b-c)^2 \{(2a-c)^2 + (2b-c)^2\} \]
\[ + (a-b)(b-c)(c^2 + a^2 + 3b^2 - 4ac) \]

\[ X_3^4 = \frac{1}{2} (-P_{400}' + 3P_{410}' + P_{220}') \]
\[ = \frac{1}{2} \{(a-b)^2 (c^2 -(a-b)^2) + (b-c)^2 (a^2 -(b-c)^2) + (c-a)^2 (b^2 -(c-a)^2)\} \]

Evidently \( X_1^4 \) and \( X_2^4 \) are I.P.'s. \( X_3^4 \) is another one because
\[ c^2 + a^2 + 3b^2 - 4ac = (a-2c)^2 + 3 (b^2-c^2). \]

Now, let
\[ P = x_1 X_1^4 + x_2 X_2^4 + x_3 X_3^4. \]

We have \( P(2, 1, 1) = 4x_1; P(1, 1, 0) = 4x_2; \) so in an I.P. both \( x_1 \) and \( x_2 \) to be nonnegative.

For \( x_1, x_2 \) positive, \( x_3 \) negative, \( x_1 - x_2 \geq 0 \) we have
\[ (x_1 + x_2 - x_3) P(0, \gamma) = [\gamma^2 (x_1 + x_2 - x_3) - \gamma (x_1 - x_2)]^2 \gamma^2 (4x_1 x_2 - x_3^2). \]
Hence

\[ P \left( 0, \frac{x_1-x_2}{x_1+x_2-x_3} \right) = \frac{(4x_1x_2-x_3^2)(x_1-x_2)^2}{(x_1+x_2-x_3)^3}, \]

and, since \(0 \leq \frac{x_1-x_2}{x_1+x_2-x_3} < 1\), for an I.P. \( x_3^2 \leq 4x_1x_2 \) is required.

For \( x_1, x_2 \) positive, \( x_3 \) negative, \( x_2-x_1 \geq 0 \), we have

\[ (x_1+x_2-x_3)P(\alpha, 0) = [\alpha^2(x_1+x_2-x_3)-\alpha(x_2-x_3)]^2 + \alpha^2(4x_1x_2-x_3^2). \]

Apparently also here \( P \) can be an I.P. but if \( x_3^2 \leq 4x_1x_2 \),

Now consider the points in \( H_4 \) for which \( x_3^2 = 4x_1x_2, x_3 < 0 \); i.e., consider the polynomials \( P = t^2X_1 + X_2 - 2tX_3, \ t > 0 \). Then

\[
(a^2 + \alpha \gamma + \gamma^2)P = [(a^2 + \alpha \gamma + \gamma^2)(t-1) + (a^3 - \gamma^3)(t+1) + (a^2 \gamma - \alpha \gamma^2)(t+2)]^2 + 3a^2 \gamma^2 (a + \gamma)^2 (t+2)^2 \geq 0,
\]

with equality only for \( a = 0, \gamma = 0 \), for any \( t \); for \( a = 0, \gamma = \frac{t-1}{t+1} \) if \( t > 1 \); for \( \gamma = 0, a = \frac{t-1}{t+1} \) if \( t < 1 \). So in \( H_4 \) the semicone \( C_4^* \) of special I.P.'s is bounded by the cone \( x_3^2 = 4x_1x_2 \) and the tangent planes \( x_1 = 0, x_2 = 0 \).

In \( V_4 \) we can take as a basis the set \( \{X_1^4, X_2^2, X_3^4, X_4^1\} \) where

\[ X_4^4 = F^2 = s(s-a)(s-b)(s-c). \]

In an I.P. of the form \( P = x_1X_1^4 + x_2X_2^2 + x_3X_3^4 + x_4X_4^1 \) we must have \( x_1 \geq 0, \ x_2 \geq 0, \ x_4 \geq 0 \), since \( P(2, 1, 1) = 4x_1; \ P(1, 1, 0) = 4x_2; \ P(1, 1, 1) = 3x_4/16 \). For fixed \( x_1, x_2, x_3 > 0 \) we have to find the minimal value of \( x_3 \) for which \( P \) is still nonnegative definite on \( \Delta \) or \( T \). In that case there is a point \( Q \) of \( \Delta \) for which \( P \) vanishes. If this point is an inner point we must have

\[ \sum x_i \frac{\partial X_i}{\partial a} = \sum x_i \frac{\partial X_i}{\partial b} \sum x_i \frac{\partial X_i}{\partial c} = 0. \]

We obtain 3 homogeneous linear equations in \( x_1, x_2, x_3, x_4 \) which are independent because in an inner point of \( \Delta \) we have \( a \neq b \neq c \neq a \).

We will not carry out this computation here but we give the result in the form of the following

**Theorem.** If \( \Delta_0 = (a_0, b_0, c_0) \) is any triangle, then the polynomial

\[ \varphi(a, b, c) = 2(a_0^2 + b_0^2 + c_0^2)(a_0 + b_0, c_0)^2(ab + bc + ca)^2 \]

\[ + (a_0b_0 + b_0c_0 + c_0a_0)(a_0 + b_0 + c_0)^2(a^2 + b^2 + c^2)^2 \]

\[ - (a_0^2 + b_0^2 + c_0^2)(a_0b_0 + b_0c_0 + c_0a_0)(a + b + c)^4 \]

\[ \text{for all positive } t \text{ the semidefinite form vanishes for the class of equilateral triangles and in addition to that for exactly one class of similar isosceles triangles; conversely, for each class of similar isosceles triangles it is possible to construct a special I.P. that vanishes just for that class.} \]

\[ \text{So for all positive } t \]
is an I.P. vanishing for all points inside \( \Delta \) lying on the conic
\[
\left( a_o^2 + b_o^2 + c_o^2 \right) (ab + bc + ca) = (a_o b_o + b_o c_o + c_o a_o) \left( a^2 + b^2 + c^2 \right).
\]
It passes through \( \Delta_0 \).

**Proof.** Without loss of generality we may assume \( a + b + c = a_o + b_o + c_o \).
Let \( a_o^2 + b_o^2 + c_o^2 = u, \ a_o b_o + b_o c_o + c_o a_o = v, \ ab + bc + ca = v + w, \) then
\[
a^2 + b^2 + c^2 = u - 2w.
\]
We have
\[
\varphi = (a_o + b_o + c_o)^2 \left( 2u (v + w)^2 + v (u - 2w)^2 - uv (u + 2v) \right)
= 2 (a_o + b_o + c_o)^4 w^2 \geq 0.
\]
If in \( \varphi \) we determine the coefficients \( x_1, x_2, x_3, x_4 \) we obtain
\[
x_1 = (5u - 6v)^2, \ x_2 = (u - 2v)^2, \ x_3 = (u - 2v) (14u - 12v), \ x_4 = 48 (u - v)^2.
\]
Indeed \( x_1 \geq 0, \ x_2 \geq 0, \ x_4 \geq 0, \ x_3 \leq 0 \) since \( v \leq u < 2v \).

Since the two proportions \( x_1 : x_2 : x_4 \) depend on one parameter \( u/v \) only the I.P.'s of this type exist for special triples \( (x_1, x_2, x_4) \) only. Indeed we have
\[
48 uv = -18 x_1 + 66 x_2 + 8 x_4; \ 48 v^2 = -12 x_1 + 60 x_2 + 5 x_4;
\]
\[
48 u^2 = -24 x_1 + 72 x_2 + 12 x_4;
\]
so \( x_1, x_2, x_4 \) must satisfy the quadratic relation
\[
(-18 x_1 + 66 x_2 + 8 x_4)^2 = (-24 x_1 + 72 x_2 + 12 x_4) (-12 x_1 + 60 x_2 + 5 x_4),
\]
or
\[
9 x_1^2 - 18 x_1 x_2 + 9 x_2^2 - 6 x_1 x_4 - 6 x_2 x_4 + x_4^2 = 0.
\]
In the space spanned by \( X_1^4, X_2^4, X_4^4 \) this represents a cone inscribed in the trihedral angle bounded by \( x_1 = 0, \ x_2 = 0, \ x_4 = 0 \).

For all other vectors \( (x_1, x_2, x_4) \) the corresponding I.P. vanishes in a boundary point of \( \Delta \) which represents an isosceles triangle with vertical angle \( \frac{\pi}{3} \) (if on the segment between \( (1, 1, 0) \) and \( (1, 1, 1) \)); an isosceles triangle with vertical angle \( \frac{\pi}{3} \) (if on the segment between \( (1, 1, 1) \) and \( (2, 1, 1) \)); or a degenerate triangle (if on the segment between \( (2, 1, 1) \) and \( (1, 1, 0) \)).