A note on polychomoty

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Published: 01/01/1989

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Download date: 02. Jan. 2019
RANA 89-27
December 1989
A NOTE
ON POLYCHOTOMY
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A note on polychotomy

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Supported in Part by a grant from the Netherlands Organisation for Scientific Research (NWO).
A note on polychotomy

Dedicated to professor H.J.Stetter on the occasion of his sixtieth birthday

ABSTRACT

Previous analyses on bounds for the growth of solutions of BVP were dealing either with two point or multipoint conditions. For BVP where some of the two point boundary conditions are separated from the rest one could only give a cruder polychotomy result for a multipoint case as such. In this note we show that such (decoupled) two point condition actually induces a dichotomic subspace. This is done by reconsidering the notions of dichotomy and polychotomy and deriving appropriate projection mappings which discribe the solution space structure.
1 Introduction

For a solution $y$ of the ODE

$$\mathcal{L}y = \frac{dy}{dt} - Ay = f,$$  \hspace{1cm} (1.1)

one can specify a variety of conditions to make the solution unique, so the problem well-posed. Of practical (in particular numerical) interest is whether such a solution can be bounded in terms of the source function $f$, (cf.[1]). For an initial condition, i.e. $y$ specified at some time $t_0$, this gives rise to (Lyapunov) stability notions. In the latter case one actually would like the homogeneous problem

$$\mathcal{L}y = 0,$$

to be such that the solution space contains nonincreasing modes only.

For boundary value problems, defined on a finite interval $[a, b]$ one can show a more or less corresponding result. If one has a two point boundary condition (specified in terms of values at $t = a$ and $t = b$) the solution space consists of modes that are either nonincreasing ("controlled" from $t = a$) or nondecreasing ("controlled" from $t = b$), called dichotomy (see [4]). If there are more such points were conditions are being prescribed, including the limit case of an integral condition, then it has been shown in [5] that an even more general structure is possible; modes are "controlled" from one of these condition points, i.e. they do not increase away from such a point; this is called polychotomy.

In cases where one has a two point condition separated from the remaining (multipoint) condition, the above theoretical result does not say anything in particular for the subspace induced by the former conditions. In this note we like to clear up this situation, showing that one actually has a dichotomic solution subspace of the same dimension as the order of the decoupled two point condition. Much to our surprise the proof of this intuitively straightforward result turned out to be far from trivial. Yet the result itself is important for understanding even more complex situations (like parameter or eigenvalue problems).

In the next section we first give some preliminary results in order not to burden the actual proofs of the theorems that follow. In section three we first reconsider the notions of dichotomy and polychotomy (for two point and multipoint situations respectively). Because of this we give a different (and short!) proof of well conditioning implying these properties, as compared to the ones given in [4,5]. The main result is a theorem where a conditioning bound is related to a so called partial dichotomy bound; the latter notion precisely describes the solution space as a direct sum of a dichotomic and polychotomic subspaces.
2 Preliminaries

2.1 Notation

In the following we consider $L^p_{\infty}$, $L^1_p$ and $C^p$, the spaces of matrix valued functions (of dimension $p \times q$) whose components are in $L_\infty(0,1)$, $L_1(0,1)$ and $C[0,1]$ respectively. We use the usual Euclidean norm

$$|a| = \sqrt{a^T a}, \quad a \in \mathbb{R}^q$$

and its induced norm

$$|a| = \|a\|_2, \quad a \in \mathbb{R}^q$$

and its induced norm

$$|A| = \sup_{0 \neq a \in \mathbb{R}^q} \frac{|Aa|}{|a|}, \quad A \in \mathbb{R}^{p \times q}$$

Furthermore, we define

$$\|Z\|_\infty = \begin{cases} \sup \{ |Z(t)|, \quad Z \in L^p_{\infty} \} \\ \max \{ |Z(t)|, \quad Z \in C^p \} \end{cases}$$

and

$$\|Z\|_1 = \int_0^1 |Z(t)| dt, \quad Z \in L^1_p.$$ 

From the singular value decomposition of matrices cf.[3], a matrix $A \in \mathbb{R}^{p \times q}$ of rank $r$ has a decomposition

$$A = UDV^T$$

where

$$UU^T = I, \quad U \in \mathbb{R}^{p \times r}$$

$$D = \text{diag}(\sigma_1(A), \ldots, \sigma_r(A))$$

$$VV^T = I, \quad V \in \mathbb{R}^{q \times r}$$

and $\sigma_i(A) > 0$, $i = 1, \ldots, r$ are the singular values of $A$. We denote by $A^+$ the generalized inverse of $A$ which is defined by

$$A^+ = VD^{-1}U^T.$$
2.2 Auerbach's Lemma

Let \( \mathcal{V} \) be a normed linear space of dimension \( r \) with norm denoted by \( \| \cdot \| \) and let \( \mathcal{V}^* \) be the space of all linear functionals from \( \mathcal{V} \to \mathbb{R} \).

Define a norm on \( \mathcal{V}^* \) by
\[
\| y^* \| = \sup_{x \in \mathcal{V}} \frac{y^*(x)}{\| x \|}, \quad y^* \in \mathcal{V}^*.
\]

**Definition 2.1** A boundary of \( \mathcal{V} \) is any set
\[
\mathcal{D} \subseteq \{ y^* \in \mathcal{V}^* | \| y^* \| \leq 1 \}
\]
such that
\[
\| x \| = \sup_{y^* \in \mathcal{D}} y^*(x) \quad \forall x \in \mathcal{V}.
\]

As in [5] we shall use Auerbach's Lemma which is the following

**Lemma 2.1** (Auerbach's lemma see [2, Lemma 4]). If \( \mathcal{D} \) is a closed boundary of \( \mathcal{V} \) then there exist \( y_i^* \in \mathcal{D}, y_j \in \mathcal{V} \); \( i, j = 1, \ldots, r \) such that
\[
y_i^*(y_j) = \delta_{ij}, \quad \| y_i^* \| = 1, \quad \| y_j \| = 1, \quad i, j = 1, \ldots, r.
\]
where
\[
\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\]
Since \( \{ y^* \in \mathcal{V}^* | \| y^* \| \leq 1 \} \) is a closed boundary, the Corollary below follows immediately.

**Corollary 2.1** There exist \( y_i^* \in \mathcal{V}^*, y_j \in \mathcal{V}; i, j = 1, \ldots, r \) such that
\[
y_i^*(y_j) = \delta_{ij}, \quad \| y_i^* \| = 1, \quad \| y_j \| = 1, \quad i, j = 1, \ldots, r.
\]

From this we derive two important basic results.

**Lemma 2.2** Let \( \mathcal{S} \) be an \( r \)-dimensional subspace of \( C^{n \times 1} \). Then there is a basis \( z_i \in \mathcal{S}, i = 1, \ldots, r \) and points \( 0 \leq t_1 \leq \ldots \leq t_r \leq 1 \) such that

(i) \( z_j^T(t_j)z_i(t_j) = \delta_{ij} \)

(ii) \( \| z_i \|_{\infty} = 1 \)

**Proof** Let \( \| y \| = \| y \|_{\infty} \) for \( y \in \mathcal{S} \). Clearly, \( \mathcal{S} \) equipped with the norm \( \| \cdot \| \) is a normed linear space of dimension \( r \). Furthermore,
\[
\mathcal{D} = \{ y^* \in \mathcal{S}^* | y^*(y) = c^T y(t), c \in \mathbb{R}^n, 0 \leq t \leq 1 \}
\]
is a closed boundary for $S^*$. The result now follows from Corollary 2.1.

Lemma 2.3 Let $Z \in C^{n \times q}$,

$$S = \{Zc \mid c \in \mathbb{R}^q\}.$$ 

and

$$\dim(S) = r.$$ 

Then, there are points $0 \leq t_1 \leq \ldots \leq t_r \leq 1$ and rank-1 matrices $E_j \in \mathbb{R}^{q \times n}, j = 1, \ldots , r$ such that

(i) $\sum_{j=1}^r E_j Z(t_j) = P$ where $P$ is an orthogonal projection

(ii) $P E_j = E_j, j = 1, \ldots , r$

(iii) $ZP = Z$

(iv) $\sum_{j=1}^r \|ZE_j\|_\infty \leq r.$

Proof $S$ is an $r$ dimensional subspace of $C^{n \times 1}$ and from lemma 2.2 there is a basis $z_i \in S, i = 1, \ldots , r$ and points $0 \leq t_1 \leq \ldots \leq t_r \leq 1$ such that

$$z_j^T(t_j)z_i(t_j) = \delta_{ij}, \quad i, j = 1, \ldots , r.$$ 

$$\|z_i\|_\infty = 1, \quad i = 1, \ldots , r$$

Let

$$Z_1 = [z_1, \ldots , z_r] \in C^{n \times r}.$$ 

Then

$$Z = Z_1 B, \quad B \in \mathbb{R}^{r \times q}$$

for some matrix $B$ with rank $(B) = r$.

Let

$$B^+ = [\tilde{b}_1 \ldots \tilde{b}_r],$$

where

$$\tilde{b}_j \in \mathbb{R}^q, \quad j = 1, \ldots , r.$$ 

It is now a simple matter to verify that (i) - (iii) hold with

$$E_i = \tilde{b}_i z_i^T(t_i), \quad i = 1, \ldots , r$$

and

$$P = B^+ B.$$

\qed
2.3 Linear Algebra

As further tools in proving our main result we need some linear algebraic properties.

Lemma 2.4 Let

\[ Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad Q_1, Q_2 \in \mathbb{R}^{n \times p} \]

where \( p \leq n \) and

\[ Q_1^T Q_1 + Q_2^T Q_2 = I. \]

Then

\[ Q_1 = U_1 D_1 V^T \]
\[ Q_2 = U_2 D_2 V^T \]

where

\[
U_1 U_1^T = U_2 U_2^T = I \quad U_1, U_2 \in \mathbb{R}^{n \times p} \\
D_1 = \text{diag} (\cos \theta_1, \ldots, \cos \theta_p) \\
D_2 = \text{diag} (\sin \theta_1, \ldots, \sin \theta_p) \\
0 \leq \theta_j \leq \pi/2, \quad j = 1, \ldots, p
\]

and

\[ V V^T = I, \quad V \in \mathbb{R}^{p \times p} \]

Proof. From the singular value decomposition of matrices and the fact that the singular values of \( Q_1 \) are less than 1,

\[ Q_1 = U_1 D_1 V^T \]

where \( U_1 \in \mathbb{R}^{n \times p}, D_1 \in \mathbb{R}^{p \times p} \) and \( V \in \mathbb{R}^{p \times p} \) are of the form described in the statement of the lemma. Furthermore,

\[ (Q_2 V)^T (Q_2 V) = I - D_2^2 \]

and from the singular value decomposition of matrices we find that

\[ Q_2 V = U_2 D_2 \]

where

\[ D_2 = (I - D_2^2)^{1/2} \]

and

\[ U_2^T U_2 = I, \quad U_2 \in \mathbb{R}^{n \times p}. \]

Thus,

\[ Q_2 = U_2 D_2 V^T. \]
Lemma 2.5 Let $A \in \mathbb{R}^{p \times q}$, $B \in \mathbb{R}^{q \times m}$. Then $\sigma_i(B) \geq \sigma_i(AB)/\sigma_1(A), i = 1, \ldots, \text{rank}(B)$.

Proof The result follows immediately from the mini-max characterization of singular values. \hfill \Box

Lemma 2.6 Let $A \in \mathbb{R}^{p \times q}$ and $P \in \mathbb{R}^{q \times q}$ such that

(i) $A = BP$, for some $B \in \mathbb{R}^{p \times q}$

(ii) $\text{rank}(A) = \text{rank}(P)$ and $P = P^T = P^2$.

Then

$A^+A = P$

Proof Clearly, the nullspace of $A$ and $P$ are the same. The result therefore follows from the uniqueness of orthogonal projections. \hfill \Box

Lemma 2.7 Let

\[ X = \begin{bmatrix} A \\ B \end{bmatrix} \quad A, B \in \mathbb{R}^{n \times p}, \]

$\text{rank}(X) = q \leq n$ and $\sigma_i(X) \geq 1, i = 1, \ldots, q$. Then, there exist matrices $C, E \in \mathbb{R}^{p \times n}$ such that

(i) $CA + EB = X^+X$ (a projection)

(ii) $\|C\|, \|E\| \leq \sqrt{2}$

(iii) $\text{rank}(C) + \text{rank}(E) = q$.

Proof. Let

\[ X = UDV^T \]

where

\[ D = \text{diag}(\sigma_1(X), \ldots, \sigma_q(X)) \]
\[ U^TU = V^TV = I \quad U \in \mathbb{R}^{2n \times q}, V \in \mathbb{R}^{p \times q} \]

From the singular value decomposition of matrices with orthonormal columns (Lemma 2.4), it follows that

\[ U = \begin{bmatrix} U_1D_1 \\ U_2D_2 \end{bmatrix} V_1^T \]
where
\[ U_1^T U_1 = U_2^T U_2 = I; \quad U_1, U_2 \in \mathbb{R}^{n \times q} \]
\[ V_1^T V_1 = V_1 V_1^T = I; \quad V_1 \in \mathbb{R}^{q \times q} \]
and
\[ D_1 = \text{diag} (\cos \theta_1, \ldots, \cos \theta_q) \]
\[ D_2 = \text{diag} (\sin \theta_1, \ldots, \sin \theta_q) \]
with
\[ 0 \leq \theta_1 \leq \ldots \leq \theta_q \leq \pi/4. \]

Now let \( r \) be the integer such that
\[ \theta_r \leq \pi/4, \quad \theta_{r+1} > \pi/4 \]
and define
\[ \tilde{D}_1 = \text{diag} (\cos \theta_1, \ldots, \cos \theta_r, 0, \ldots, 0) \]
\[ \tilde{D}_2 = \text{diag} (0, \ldots, 0, \sin \theta_{r+1}, \ldots, \sin \theta_q) \]
The result clearly follows with
\[ C = V D^{-1} V_1 \tilde{D}_1^+ U_1^T \]
\[ E = V D^{-1} V_1 \tilde{D}_2^+ U_2^T \]
3 Dichotomy and (Partial) Polychotomy

Consider the differential equation
\[ \mathcal{L}y = y' - Ay = f, \]  
(3.1)
where \( f \in L^1_{n \times 1} \) and \( A \in L^1_{n \times n} \). Clearly, \( y \in C_{n \times 1} \) and \( y' \in L^1_{n \times 1} \). We associate with the equation a fundamental matrix \( Y \) which satisfies
\[ \mathcal{L}Y = 0, \quad Y(0) = I \]
and a solution space
\[ S = \{ Yc | c \in \mathbb{R}^n \}. \]

As we showed in [5] the stability constant related to the BC can be related to the bound on the Green’s function for well-conditioned multipoint problems (with a finite number of such internal or boundary points). This makes that it is sufficient for such problems to consider homogeneous BC only. Unfortunately this is not necessarily sufficient for e.g. integral BC. For assessing properties of the fundamental solution, however, we only need to consider the bounds on the Green’s functions. Therefor we conclude that we may take the BC homogeneous and need to require boundedness of solutions in terms of source terms only.

Our interest is in dichotomy and polychotomy of the space \( S \). We define

**Definition 3.1** The solution space \( S \) has a dichotomic constant \( \kappa \) if there exist matrices \( B_0, B_1 \in \mathbb{R}^{n \times n} \) such that

(i) \( \text{rank}(B_0) + \text{rank}(B_1) = n \)

(ii) there is a unique solution of (3.1) that also satisfies
\[ B_0y(0) + B_1y(1) = 0 \]
and this solution satisfies
\[ \|y\|_{\infty} \leq \kappa \|f\|_1 \]
for every \( f \in L^1_{n \times 1} \).

**Definition 3.2** The solution space has a polychotomic constant \( \kappa \) if there are points \( 0 < t_1 \leq \ldots \leq t_n \leq 1 \) and matrices \( E_j \in \mathbb{R}^{n \times n}, j = 1, \ldots, n \) such that

(i) \( \sum_{j=1}^n \text{rank}(E_j) = n \)
(ii) there is a unique solution of (3.1) that also satisfies

\[ \sum_{j=1}^{n} E_{j}y(t_{j}) = 0 \]

and this solution satisfies

\[ \|y\|_{\infty} \leq \kappa \|f\|_{1} \]

for every \( f \in L_{T}^{n \times 1} \).

It should be noted that the above definitions are somewhat different, though equivalent to those in [4], [5].

This follows e.g. for the dichotomy case by taking for \( f \) \( \delta \)-functions centered at 0 or at 1, and writing

\[ y(t) = \int_{0}^{1} G(t, s) f(s) \, ds, \]

where \( G(t, s) \) is the Green's function.

With this different dichotomy definitions 3.1 we can prove the following theorem which is equivalent to tho 3.16 in [4]. The construction here is somewhat different and the proof is included to show the more general approach here.

**Theorem 3.1** Suppose that there are matrices \( \tilde{B}_{0}, \tilde{B}_{1} \in \mathbb{R}^{n \times n} \) such that there is a unique solution of (3.1) that also satisfies

\[ \tilde{B}_{0}Y(0) + \tilde{B}_{1}Y(1) = 0 \]

(3.2)

If the solution of (3.1) and (3.2) satisfies

\[ \|y\|_{\infty} \leq \lambda \|f\|_{1} \]

for every \( f \in L_{T}^{n \times 1} \), then \( S \) has a dichotomic constant

\[ \kappa \leq \lambda(1 + 4\lambda) \]

**Proof** Without loss of generality we assume that

\[ \tilde{B}_{0}\tilde{B}_{0}^{T} + \tilde{B}_{1}\tilde{B}_{1}^{T} = I \]

(3.3)

and define

\[ \Phi(t) = Y(t) \left[ \tilde{B}_{0}Y(0) + \tilde{B}_{1}Y(1) \right]^{-1}. \]

Clearly

\[ \tilde{B}_{0}\Phi(0) + \tilde{B}_{1}\Phi(1) = I. \]

and it follows from lemmas 2.5 and 2.7 that there exist matrices \( B_{0}, B_{1} \) such that
(i) $B_0\Phi(0) + B_1\Phi(1) = I$

(ii) $\text{rank}(B_0) + \text{rank}(B_1) = n$

(iii) $|B_0|, |B_1| \leq \sqrt{2}$.

Thus, the mapping $\mathcal{P} : C^{n\times 1} \rightarrow S$ defined by

$$(\mathcal{P} u)(t) = \Phi(t)(B_0 u(0) + B_1 u(1))$$

is a projection and

$$\|\mathcal{P}u\|_{\infty} \leq \|\Phi\|_{\infty} 2\sqrt{2}\|u\|_{\infty} \quad (3.4)$$

Since the solution of (3.1), (3.2) has the form

$$\tilde{y}(t) = \int_0^1 \tilde{G}(t,s) f(s) ds \quad (3.5)$$

where

$$\tilde{G}(t,s) = \begin{cases} \Phi(t)\tilde{B}_0\Phi(0)\Phi^{-1}(s) & t > s \\ -\Phi(t)\tilde{B}_1\Phi(1)\Phi^{-1}(s) & t < s \end{cases}$$

it follows from the hypothesis that

$$|\tilde{G}(t,0)c_0 + \tilde{G}(t,1)c_1| \leq \lambda \{ |c_0| + |c_1| \}$$

$$\leq \sqrt{2}\lambda(\|c_0\|^2 + |c_1|^2)^{1/2}$$

for every $c_0, c_1, \in \mathbb{R}^n$. In particular, if

$$c_0 = B_0^T c, \quad c_1 = B_1^T c$$

we obtain, on using (3.3) and the form of $\tilde{G}(t,s)$ above,

$$|\Phi(t)c| \leq \sqrt{2}\lambda|c| \quad (3.6)$$

On combining (4) and (6) we obtain

$$\|\mathcal{P}u\|_{\infty} \leq 4\lambda\|u\|_{\infty}.$$

Now let

$$y = \tilde{y} - \mathcal{P}\tilde{y}$$

where $\tilde{y}$ is defined by (3.5). Then, it is easy to verify that $y$ is the unique solution to (3.1) and

$$B_0 y(0) + B_1 y(1) = 0$$

Furthermore,

$$|y| \leq |\tilde{y}| + |\mathcal{P}\tilde{y}|$$

$$\leq \lambda(1 + 4\lambda)\|f\|_1.$$
for every $f \in L_{1}^{n \times 1}$.

The next result on polychotomy is slightly stronger than corollary 5.1 in [5].

**Theorem 3.2** For every $f \in L_{1}^{n \times 1}$, let (3.1) have a solution $\tilde{y}$ satisfying

$$\|\tilde{y}\|_{\infty} \leq \lambda \|f\|_1.$$ 

Then, $S$ has a polychotomic constant satisfying

$$\kappa \leq (n + 1)\lambda.$$ 

**Proof** From lemma 2.2, there are points $0 \leq t_1 \leq \ldots \leq t_n \leq 1$ and rank 1 matrices $E_j \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, n$ such that

(i) $\sum_{j=1}^{n} E_j Y(t_j) = I$

(ii) $\sum_{j=1}^{n} |Y(t)E_j| \leq n.$

Clearly, the mapping $P : C^{n \times 1} \rightarrow S$ defined by

$$(Pu)(t) = Y(t) \sum_{j=1}^{n} E_j u(t_j)$$

is a projection and

$$\|Pu\|_{\infty} \leq n \|u\|_{\infty}.$$ 

If $\tilde{y}$ is a solution of (3.1) then

$$y = \tilde{y} - P\tilde{y}$$

is the unique solution of (3.1) and

$$\sum_{j=1}^{n} E_j \tilde{y}(t_j) = 0.$$ 

Furthermore,

$$\|y\|_{\infty} \leq (1 + n)\|\tilde{y}\|_{\infty}$$

for every solution $\tilde{y}$ of (3.1) and in particular for the $\tilde{y}$ that satisfies

$$\|\tilde{y}\|_{\infty} \leq \lambda \|f\|_1.$$ 

Thus,

$$\|y\|_{\infty} \leq (n + 1)\lambda \|f\|_1$$

and the result follows. \qed
We now come to our main result:
If precisely \( p \) of the BC constitute a twopoint condition (and the rest are of multipoint or of integral form) then we expect these to induce a dichotomic subspace of order \( p \). This gives rise to the following

**Definition 3.3** The solution space has a *partial dichotomic* constant \( \kappa \) of order \( p \) if there are matrices \( B_0, B_1 \in \mathbb{R}^{n \times n} \), points \( 0 \leq t_1 \leq \ldots \leq t_{n-p} \leq 1 \) and rank 1 matrices \( E_j, j = 1, \ldots, n-p \) such that

(i) \( \text{rank}(B_0) + \text{rank}(B_1) = p \)

(ii) there is a unique solution of (3.1) and

\[
B_0 y(0) + B_1 y(1) + \sum_{j=1}^{n-p} E_j y(t_j) = 0
\]

and it satisfies

\[
\|y\|_{\infty} \leq \kappa \|f\|_1
\]

for every \( f \in L_1^{n \times 1} \).

**Theorem 3.3** Suppose that for every \( f \in L_1^{n \times 1} \) there is a solution \( \tilde{y} \) of (3.1) that also satisfies

\[
\|\tilde{y}\|_{\infty} \leq \lambda \|f\|_1
\]

where

\[
\text{rank}(\tilde{B}_0 Y(0) + \tilde{B}_1 Y(1)) = p
\]

Then, \( S \) has a partial dichotomic constant \( \kappa \) of order \( p \) that satisfies

\[
\kappa \leq (1 + n - p)\lambda(1 + 4(1 + n - p)\lambda).
\]

**Proof** Assume without loss of generality that

\[
\tilde{B}_0, \tilde{B}_1 \in \mathbb{R}^{p \times n}
\]

and that

\[
\tilde{B}_0 \tilde{B}_0^T + \tilde{B}_1 \tilde{B}_1^T = I.
\]  \hspace{1cm} (3.7)

Now define

\[
\tilde{P} = [\tilde{B}_0 Y(0) + \tilde{B}_1 Y(1)]^* [\tilde{B}_0 Y(0) + \tilde{B}_1 Y(1)]
\]

and

\[
\tilde{S}_2 = \{ y(t)(I - \tilde{P}) c | c \in \mathbb{R}^n \}.
\]
Clearly, \( \dim(\hat{\mathcal{S}}_2) = n - p \) and it follows from lemmas 2.3 and 2.6 that there are points \( 0 < t_1 < \ldots < t_{n-p} < 1 \) and rank-1 matrices \( \hat{E}_j, j = 1, \ldots, n - p \) such that

\[
\sum_{j=1}^{n-p} \hat{E}_j Y(t_j)(I - \hat{P}) = I - \hat{P}
\]

\[
\hat{E}_j = (I - \hat{P})E_j, \quad j = 1, \ldots, n - p
\]

and

\[
\sum_{j=1}^{n-p} |Y(t)\hat{E}_j| \leq n - p.
\]

Thus, the mapping \( \hat{P}_2 : C^{n \times 1} \to \hat{\mathcal{S}}_2 \) defined by

\[
(\hat{P}_2 u)(t) = Y(t) \sum_{j=1}^{n-p} \hat{E}_j u(t_j)
\]

is a projection; we find

\[
\|\hat{P}_2 u\|_{\infty} \leq (n - p)\|u\|_{\infty}
\]

and

\[
(I - \hat{P}_2)Y(I - \hat{P}) = 0
\]

We now turn to solution of (3.1) that also satisfies

\[
\hat{B}_0 y(0) + \hat{B}_1 y(1) = 0.
\]

It is easy to verify that such solutions have the form

\[
y(t) = Y(t)(I - \hat{P})c + \int_0^t Y(t)(I - \hat{P})Y^{-1}(s)f(s)ds
\]

\[
+ \int_0^1 \hat{G}(t, s)f(s)ds
\]

where \( c \in \mathbb{R}^n \) is arbitrary and

\[
\hat{G}(t, s) = \begin{cases} \hat{\Phi}_1(t)\hat{B}_0 Y(0)Y^{-1}(s), & t > s \\ -\hat{\Phi}_1(t)\hat{B}_1 Y(1)Y^{-1}(s), & t < s \end{cases}
\]

and

\[
\hat{\Phi}_1(t) = Y(t) [\hat{B}_0 Y(0) + \hat{B}_1 Y(1)]^+.
\]

By a suitable choice of \( \delta \)-functions at 0 and at 1 it follows from the hypothesis that for every \( \eta, \beta \in \mathbb{R}^n \) there are \( c_0, c_1 \in \mathbb{R}^n \) such that

\[
\|\hat{G}(t, 0)\eta - Y(t)(I - \hat{P})c_0\|_{\infty} \leq \lambda|\eta|
\]

and

\[
\| - \hat{G}(t, 1)\beta - Y(t)(I - \hat{P})c_1\|_{\infty} \leq \lambda|\beta|.
\]
Thus, from (3.8), (3.9) and the definition of $\tilde{G}$,

$$
\|(I - \tilde{P}_2)\Phi_1(\tilde{B}_0 \eta + \tilde{B}_1 \beta)\|_\infty \\
\leq \|(I - \tilde{P}_2)[\Phi_1\tilde{B}_0 \eta - Y(t)(I - \tilde{P})c_0] \|_\infty \\
+ \|(I - \tilde{P}_2)[\Phi_1\tilde{B}_1 \beta - Y(t)(I - \tilde{P})c_1] \|_\infty \\
\leq (1 + n + p)\lambda (|\eta| + |\beta|) \\
\leq \sqrt{2}(1 + n + p)\lambda (|\eta|^2 + |\beta|^2)^{1/2}
$$

If we now take $\eta = \tilde{B}_0^T c$, $\beta = \tilde{B}_1^T c$ and note that (3.7) holds, we obtain

$$
\|\Phi_1 c\|_\infty \leq \sqrt{2}(1 + n + p)\lambda |c|
$$

where

$$
\Phi_1 = (I - \tilde{P}_2)\tilde{\Phi}_1.
$$

That is,

$$
\|\Phi_1\|_\infty \leq \sqrt{2}(1 + n + p)\lambda.
$$

(3.10)

In addition, using the relationship

$$
[\tilde{B}_0 Y(0) + \tilde{B}_1 Y(1)] [\tilde{B}_0 Y(0) + \tilde{B}_1 Y(1)]^+ = I,
$$

it is easy to verify that

$$
\tilde{B}_0 \Phi_1(0) + \tilde{B}_1 \Phi_1(1) = I.
$$

(3.11)

Now define

$$
X = \left[ \frac{\Phi_1(0)}{\Phi_1(1)} \right].
$$

From (3.7), (3.11) and lemma 2.5, $\sigma_i(X) \geq 1, i = 1, \ldots, p$ and it therefore follows from lemma 2.7 that there are matrices $\tilde{B}_0, \tilde{B}_1, \in \mathbb{R}^{p \times n}$ such that

$$
\tilde{B}_0 \Phi_1(0) + \tilde{B}_1 \Phi_1(1) = I
$$

and

$$
|\tilde{B}_0|, |\tilde{B}_1| \leq \sqrt{2}
$$

(3.12)

and

$$
\text{rank}(\tilde{B}_0) + \text{rank}(\tilde{B}_1) = p.
$$

Now define

$$
S_1 = \{\Phi_1(t)c|c \in \mathbb{R}^p\}
$$

and the mapping $P_1 : C^{n \times 1} \to S_1$ by

$$
(P_1 u)(t) = \Phi_1(t)(\tilde{B}_0 u(0) + \tilde{B}_1 u(1))
$$

Clearly, $P_1$ is a projection and from (3.10) and (3.12),

$$
\|P_1 u\|_\infty \leq 4(n + 1 - p)\lambda \|u\|_\infty.
$$
In addition, \[ \tilde{P}_2 P_1 = 0. \]

We also define
\[
\begin{align*}
\Phi_2 &= (I - P_1)Y(I - \tilde{P}) \\
S_2 &= \{\Phi_2(t)c|c \in \mathbb{R}^n\}
\end{align*}
\]

and
\[ P_2 = (I - P_1)\tilde{P}_2. \]

It is easy to show that \( P_2 : C^n \rightarrow S_2 \) is a projection, that
\[ P_1 P_2 = 0, \quad P_2 P_1 = 0 \]

and
\[ (P_1 + P_2)Y = Y. \]

Thus, on taking
\[ S = \{Yc|c \in \mathbb{R}^n\} \]

and
\[ P = P_1 + P_2, \]

we find that \( P : C^n \rightarrow S \) is a projection satisfying
\[
\begin{align*}
\|P u\|_\infty &\leq [4(n+1-p)\lambda + (1 + 4(n+1-p)\lambda)(n-p)]\|u\|_\infty \\
&= [(n-p) + 4(n+1-p)^2] \lambda\|u\|_\infty
\end{align*}
\]

Now if \( \tilde{y} \) is any solution of (3.1), then
\[ y = (I - P)\tilde{y} \]

is also a solution. For every \( f \in L^1^n \) there is a solution \( \tilde{y} \) that satisfies
\[ \|\tilde{y}\|_\infty \leq \lambda\|f\|_1 \]

and hence
\[
\begin{align*}
\|y\|_\infty &= \|(I - P)\tilde{y}\|_\infty \\
&\leq (n+1-p)(1 + 4(n+1-p)\lambda)\lambda\|f\|_1.
\end{align*}
\]

In addition, it is easy to verify that \( y \) is the unique solution to (3.1) and
\[ B_0 y(0) + B_1 y(1) + \sum_{j=1}^{n} E_j y(t_j) = 0 \]

where
\[ B_k = \left( I - \sum_{j=1}^{n-p} \tilde{E}_j Y(t_j) \right) \left[ \tilde{B}_0 Y(0) + \tilde{B}_1 Y(1) \right]^{+} \tilde{B}_k, \quad k = 0, 1 \]
and
\[ E_j = \left[ I - [\hat{B}_0 Y(0) + \hat{B}_1 Y(1)] \right] \tilde{E}_j, \quad j = 1, \ldots, n - p. \]

Clearly, \( E_j, \quad j = 1, \ldots, n - p \) are rank-1 matrices and
\[ \text{rank}(B_0) + \text{rank}(B_1) = p. \]

The general result of theorem 3.3 implies bounds for the limiting cases \( p = n \) ("dichotomy") and \( p = 0 \) ("polychotomy"). Apparently the bound for \( p = n \) is the same as obtained in theorem 3.1. For \( p = 0 \) we get a cruder bound than in theorem 3.2, i.e. an extra term \( 4(n + 1)\lambda^2 \|f\|_1 \). This difference is a mere consequence of the technique by which we proved the theorem. Actually, we conjecture that the \( \lambda^2 \) term might be omitted entirely, also in the dichotomy case (but we have been unable to prove this thus far, see also [4]).
References


