Optimal segmentations

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Optimal segmentations

by

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OPTIMAL SEGMENTATIONS

Introduction

In programming methodology the attention gradually shifts from specific problems towards classes of problems, their characterization and theorems for their solutions. A classification of segment problems is in progress and several solution schemes may be viewed as theorems. A type of problems not too distant from the segment problems is that of partitionings. Given a sequence (or set) construct a partition, possibly an extremal partition, whose members all satisfy certain conditions. E.g. partition a list into segments that satisfy a certain “nice” predicate, give a construction of a partition with as few members as possible; such a partition may be called an optimal segmentation. I’ll derive conditions on the predicate involved that guarantee efficient algorithms modulo the predicate calculations (i.e. evaluation of predicates is assumed to take constant time). Moreover, it is shown that the proposed algorithms are greedy.

Notation and concepts

One of the alleged disadvantages of predicate calculus notation is indexitis. This is often circumvented by introduction of abbreviations and ad hoc notations. A more compact, sometimes even too compact, notation is the so-called Bird-Meertens formalism (with APL rudiments, see [BD]). Just as an experiment, I incorporate some of the BM features in predicate notation.

For a set (type) $\alpha$, the triple $(\alpha^*, \oplus, [])$ denotes the monoid of lists over $\alpha$. Lists are denoted as sequences between brackets. The catenation ($\oplus$)\(^1\) and the unit ([$$], the empty list) are polymorphic. So lists ($\alpha^*$) as well as lists of lists ($\alpha^{**}$) are both considered with the same symbols for catenation and unit, the distinction may be seen from the choice of identifiers:

$a \in \alpha$

$u, v, \ldots, z \in \alpha^*$

$us, vs, \ldots, zs \in \alpha^{**}$

I’ll use reduction (just $\oplus$ or flatten) and filter ($<$) as in BM. The functions inits, tails and segs are considered in the set-valued versions of those in BM, e.g.:

$tails.xs = \{vs \mid (Eus :: xs = us \oplus vs)\}$.

The segmentation concepts are formalized as follows:

\(^1\)As an experiment, $\oplus$ will be given the highest priority: $f.x \oplus y = f.(x \oplus y)$.
Let \( Q : \alpha^* \rightarrow \text{Bool} \) be a predicate on \( \alpha \)-lists.

Define the relations \( P, OP \subseteq \alpha^* \times \alpha^* \) and the function \( N : \alpha^* \rightarrow \mathbb{N} \) by

\[
\begin{align*}
xsP x & \equiv \not/xs = x \land Q \not xs = xs \\
N.x & = (\downarrow xs : xsP x : \#xs) \\
xsOP x & \equiv xsP x \land N.x = \#xs
\end{align*}
\]

Then \( xs(O)P x \) may be paraphrased as: \( xs \) is an (optimal) \( Q \)-segmentation for \( x \).

Note that optimal \( Q \)-segmentations need not be unique.

**Some properties**

It is good practice to collect, prior to the derivation, some properties of the concepts involved. The easy proofs are left as exercises:

1. \( []P[] \), hence \( N.[] = 0 \) and \( []OP[] \)
2. \( xsP x \land ysPy \Rightarrow xsP+y \)
3. \( xsOP x \land us \in \text{segs} xs \Rightarrow usP+u \)
4. \( xsP[[[]]]+ysP x \Rightarrow xsP+y \)
5. \( \) Note that by (4), empty segments may be discarded in considering optimal segmentations. If necessary one may consider \( Q' \) with \( Q'. x \equiv Q.x \land x \not \equiv [] \) instead of \( Q \).

Life would have been a lot easier (although very dull) if the \( OP \) version of (1) were true, quod non. Since the \( P \)-part of \( OP \) behaves nicely, an investigation of \( N \) is in order. It seems interesting to see whether some recurrence is lurking around. Indeed

\[
N.x+[a] = (\downarrow z, w : wP z = x \land Q.w+[a] : N.w+1)
\]

For: \( N.x+[a] \)

\[
= \{ \text{def } N \}
= (\downarrow ys : ysP x+[a] : \#ys)
= \{ \not/ys = x+[a] \Rightarrow ys \not \equiv [] \}
= (\downarrow zs, z : zsP x+[a] : \#zs + 1)
= \{ \text{def } P \}
= (\downarrow zs, z : (\not/zsP z = x+[a] \land Q \not zs = zs \land Q.z : \#zs + 1) \)
\[
= \{\text{one point rule}\}
\]
\[
(\downarrow z, z, w : w \curlyrightarrow z = x \curlyrightarrow [a] \land w = \curlyrightarrow / z s \land Q \angle z s = z s \land Q . z : \# z s + 1)
\]
\[
= \{\text{def } P\}
\]
\[
(\downarrow z, z, w : w \curlyrightarrow z = x \curlyrightarrow [a] \land z s P w \land Q . z : \# z s + 1)
\]
\[
= \{\text{promotion}\}
\]
\[
(\downarrow z, w : w \curlyrightarrow z = x \curlyrightarrow [a] \land Q . z : (\downarrow z s : z s P w : \# z s + 1))
\]
\[
= \{\text{def } N, \ p inf + 1 = p inf\}
\]
\[
(\downarrow z, w : w \curlyrightarrow z = x \curlyrightarrow [a] \land Q . z : N . w + 1)
\]
\[
= \{\text{split off } z = [\], \text{without loss of generality } \neg Q . [\] (5)\}
\]
\[
(\downarrow z, w : w \curlyrightarrow z = x \land Q . z \curlyrightarrow [a] : N . w + 1)
\]

Note that, thanks to the rule \(p inf + 1 = p inf\), the validity of the recurrence relation is independent of the existence of \(Q\)-segmentations. Nonexistence is rather unsatisfactory, so I propose an easy way out: assume

\[
(7) \quad Q . [a] \text{ for every } a \in \alpha
\]

Hence the exotic rule \(p inf + 1 = p inf\) is superfluous.

**Thinning out the quantification**

Since in the recurrence relation a quantification over all postfixes of \(x\) occurs, the resulting algorithm is quadratic modulo \(Q\)-calculations. Efficiency improvement is to be expected if only a small subset of the postfixes of \(x\) suffices. Given an optimal \(Q\)-segmentation \(x s\) for \(x\) an interesting subset of the postfixes of \(x\) is given by

\[
\{\# / v s \mid v s \in \text{tails} . x s\} \ (=: T)
\]

In order to restrict the quantification in the right-hand side of (6) to \(z \in T\), there should be reasons to discard \(z \not\in T\). Consider the following Setting (S)

\[
\{\text{Setting } (S)\}
\]

(i) \(z = \# / z s \land x = w \curlyrightarrow z \land z \not\in T\)

(ii) \(x s O P z \land Q . z \curlyrightarrow [a]\)

By (i), there are \(u s, v s, u, v\) such that

\[
x s = u s \curlyrightarrow [u \curlyrightarrow v] \curlyrightarrow v s \quad \text{and}\]
\[
w = (\# / u s) \curlyrightarrow u \land z = v \curlyrightarrow (\# / v s) \land u \not\in [\] \land v \not\in [\].
One may forget about this $z$ in the quantification of (6) if there is a $Q$-segmentation $zs$ of $x \vdash [a]$ such that
- $\text{last} \cdot zs = p \vdash [a]$ for some $p \in T$
- $\#zs \leq N \cdot w + 1$

Given setting $(S)$, two obvious candidates for $zs$ can be constructed from the $Q$-segmentation $xs$, such that $\text{last} \cdot zs = p \vdash [a]$ for some $p \in T$:

(c0) $zs = us \vdash [u \vdash v \vdash (\vdash / vs) \vdash [a]]$

(c1) $zs = us \vdash [u \vdash v] \vdash [(\vdash / vs) \vdash [a]]$

These candidates are $Q$-segmentations if:
- ad (c0): $Q \cdot u \vdash v \vdash (\vdash / vs) \vdash [a]$  
  Since $u \vdash v$ in $xs$ and $xsPx$, certainly $Q \cdot u \vdash v$.
  By (ii), $Q \cdot z \vdash [a]$, while $z = v \vdash (\vdash / vs)$ and $v \notin [] ((S))$.
  Hence overlap closedness of $Q$ is sufficient.  
  (I.e. $Q \cdot k \vdash l \land Q \cdot l \vdash m \land l \notin [] \Rightarrow Q \cdot k \vdash l \vdash m$.)

- ad (c1): $Q \cdot (\vdash / vs) \vdash [a]$  
  Since $Q \cdot z \vdash [a]$, while $z = v \vdash (\vdash / vs)$, it is sufficient to require $Q$ to be postfix closed.  
  (I.e. $Q \cdot k \vdash l \Rightarrow Q \cdot l$. Indeed a weaker requirement could be $Q \cdot k \vdash l \land Q \cdot l \vdash m \land l \notin [] \Rightarrow Q \cdot m$,  
  which seems a somewhat awkward property.)

With respect to the last requirement:

$\#zs \leq N \cdot w + 1$

$= \{ \#zs = \#us + 1 + j \text{ for candidate (cj)} \}$

$\#us \leq N \cdot w - j$

$\Leftarrow \{ \text{In setting (S): } us \sqsubseteq xs \land \vdash / us \sqsubseteq w \sqsubseteq \vdash / xs \}$

(OSj) $(A_{us'}, w' : us' \sqsubseteq xs \land \vdash / us' \sqsubseteq w' \sqsubseteq \vdash / xs : \#us' \leq N \cdot w' - j)$

where "$\sqsubseteq$" denotes the prefix order:

$p \sqsubseteq q \equiv p \in \text{inits} \cdot q$, $p \sqcap q \equiv p \sqsubseteq q \land p \neq q$. 
The universal quantification in (OSj) is chosen because
- \( us \) and \( w \) in the setting (S) are arbitrarily chosen such that \( z \not\in T \). It is desirable to have a condition that is independent of that choice.
- (OSj) is a property of the \( Q \)-segmentation \( xs \) alone (even optimality is not used).

The established "thinning out" may be formulated as:

(8) Lemma. Let \( xsOPx \). In each of the following two cases:

- \( L0 \) : \( Q \) is overlap closed and \( xs \) satisfies \( OS0 \)
- \( L1 \) : \( Q \) is postfix closed and \( xs \) satisfies \( OS1 \)

the quantification in (6) may be thinned out to

\[ N.x\# [a] = (\{us, vs : us\# vs = xs \land Q.(\#/vs)\# [a] : \#us + 1\}. \]

Proof.

\[ N.x\# [a] \]
\[ = \{ (6), Lf \text{ hence restriction to } z \in T \} \]
\[ (\{w, z : z \in T \land w\# z = x \land Q.z\# [a] : N.w + 1\} \]
\[ = \{ z \in T \equiv (\{us, vs : us\# vs = xs : z = \#/vs \}; \text{calc} \} \]
\[ (\{us, vs : us\# vs = xs : \]
\[ (\{w, z : z = \#/vs \land w\# z = x \land Q.z\# [a] : N.w + 1\}) \]
\[ = \{ \#/xs = x \text{ and } w\# z = (\#/(us/\# z \equiv w = \#/us) \]
\[ (\{us, vs : us\# vs = xs \land Q.(\#/vs)\# [a] : N.(\#/us) + 1\} \]
\[ = \{ xsOPx \land us \subseteq xs, (3) \} \]
\[ (\{us, vs : us\# vs = xs \land Q.(\#/vs)\# [a] : \#us + 1\} \]

Lemma (8) only guarantees efficiency improvement if the (OSj) property is an invariant in the (successive) construction of optimal segmentations. This will be addressed in the next section.

Construction of an optimal segmentation

In the following blueprint for the calculation of an optimal segmentation for \( X \in \alpha^* \), only the invariance of \( I2 \) is left to be proved:

\[ I0 \] \( x\# x' = X \)
\[ I1 \] \( xsOPx \)
\[ I2 \] \( xs \) satisfies \( OSj \)
\begin{verbatim}
x',x',zs := [], X, [] \{I\}
; do x' \neq []
    \rightarrow a := \text{hd} x'
    \quad ; S \{(ys, zs) is a witness for
    \quad \begin{align*}
    (\perp (us, vs) : us \uplus vs &= xs \land Q.(\perp /vs) \uplus [a] : \#us + 1)}
    \quad \end{align*}
    \quad ; xs := ys \uplus [/(zs) \uplus [a]] \{I| x := x \uplus [a] \land I2!} \}
    ; x, x' := x \uplus [a], \text{tl} x' \{I\}
\od \{ I \land x = X, \text{hence } xs \mathcal{OP} X \}
\end{verbatim}

In order to prove the invariance of \(I2\), assume
\begin{enumerate}
  \item \(\perp / (ys \uplus [q]) = (\perp / xs) \uplus [a]\) \{where \(q = (\perp / zs) \uplus [a]\)}
  \item \(ys \subseteq xs\) \{(ys, zs) is a witness\}
  \item \(N.\perp / xs = \#xs\) \{I1 \land \text{def. } N\}
\end{enumerate}
then \(ys \uplus [q] \) satisfies \(\text{OSj}\)
\[
= \{ \text{def OSj} \}
= \{ \text{Aus, } w : us \subseteq ys \uplus [q] \land \perp /w \subseteq \perp / (ys \uplus [q]) : \#us \leq N.w - j \}
= \{ \text{(i); } \subseteq \}
= \{ \text{Aus, } w : us \subseteq ys \land \perp /w \subseteq \perp / xs : \#us \leq N.w - j \}
\leftarrow \{ \text{(ii); } \text{split off } w = \perp / xs ; \perp /w \subseteq \perp / xs \Rightarrow us \subseteq xs \}
= \{ \text{Aus, } w : us \subseteq xs \land \perp /w \subseteq \perp / xs : \#us \leq N.w - j \}
\land \{ \text{Aus : } us \subseteq xs : \#us \leq N.\perp / xs - j \}
= \{ \text{def OSj}; (iii) and } j \in \{0, 1\}\}
x \text{satisfies } OSj (\land \text{true})
\end{verbatim}

Note that \(OS1\) is an invariant for the construction in both cases, \(Q\) is overlap closed and \(Q\) is postfix closed.

For the construction of \(S\) in case \(Q\) is overlap closed I don't see a better solution than just checking all splittings of \(xs\). However, in case \(Q\) is postfix closed, things are a lot more attractive: since
\[
\neg Q.q \Rightarrow \neg Q.p \perp q
\]
S boils down to a linear search:

\[
ys, zs, q := xs, [], [a] \\
\{ys \# zs = xs \land Q.q \land q = (\# / zs) \# [a]\} \\
\text{do } \text{ ys } \neq [] \text{ cand } Q.(last.ys) \# q \\
\rightarrow ys, zs, q := \text{front.ys} , [last.ys] \# zs , (last.ys) \# q \\
\text{od}
\]

S can easily be mixed with the assignment to zs. [Identify ys and zs, forget about zs in the above].

The complete algorithm is linear (modulo Q-calculations) which is evident from the variant function

\[
2 \star \# X - 2 \star \# x + \# xs.
\]

For completeness sake: the algorithm, in case Q is postfix closed, is:

\[
x, x', xs := [], X, [] \\
\text{do } x' \neq [] \\
\rightarrow a := \text{hd}.x' ; q := [a] \\
\text{do } xs \neq [] \text{ cand } Q.(last.xs) \# q \\
\rightarrow xs, q := \text{front.xs} , (last.xs) \# q \\
\text{od} \\
\text{do } x, x', xs := x \# [a], \text{tl}.x', xs \# [q] \\
\text{od}
\]

**Greedy Q-segmentations**

Interpretation of the strongest OS condition (OS1) leads to some feeling of greediness.

The definition of (left-) greediness for Q-segmentations (see [B]):

\[
(9) \quad \text{Greedy.}[] \\
\text{Greedy.}[x] \# zs \equiv \text{Greedy.xs} \land x = ([z : z \subseteq x \# (\# / zs) \land Q.z : z])
\]

The following lemma shows that the construction in the former section is a construction for the greedy Q-segmentation:
Lemma. Let $xs$ be a $Q$-segmentation with $Q.\vdash xs \equiv \#xs \leq 1$. Then $xs$ satisfies $OS1 \Rightarrow \text{Greedy} .xs$.

Proof. By induction on $\#xs$. The base-case, $\#xs \leq 1$, is trivial.

Suppose $\#xs \geq 1$. Then for $Q$-segmentation $[x] \vdash xs$:

$[x] \vdash xs$ satisfies $OS1$

$\Rightarrow \{\text{domain restriction}\}$

$(Axs, w : [x] \subseteq us \subseteq [x] \vdash xs \land \vdash us \subseteq w \subseteq [x] \vdash (\vdash xs) : \#us < N.w)$

$= \{\text{dummy change for } us, w\}$

$(Axs, w : us \subseteq xs \land \vdash us \subseteq w \subseteq xs : \#us + 1 < N.x \vdash w)$

$\Rightarrow \{Q.x, so N.x \vdash w \leq 1 + N.w; \text{ def } OS1\}$

$xs$ satisfies $OS1$

$\Rightarrow \{\text{Ind. hyp.}\}$

Greedy.xs

and

$[x] \vdash xs$ satisfies $OS1$

$\Rightarrow \{\text{instantiate } us := [x]; \#xs \geq 1\}$

$(Aw : x \subseteq w \subseteq x \vdash (\vdash xs) : 1 < N.w)$

$\Rightarrow \{1 < N.w \Rightarrow 1 \neq N.w; w \neq [] \Rightarrow (1 = N.w \equiv Q.w)\}$

$(Aw : x \subseteq w \subseteq x \vdash (\vdash xs) : \neg Q.w)$

$= \{\#([x] \vdash xs) > 1 \Rightarrow \neg Q.x \vdash (\vdash xs); Q.x\}$

$x = (\downarrow w : w \subseteq x \vdash (\vdash xs) \land Q.w : w)$

Afterthought and acknowledgements

The derivation of the requirements on $Q$ and the corresponding algorithms were what I was after. However, also the solutions themselves are interesting: The shape of the "postfix-closed" version is very familiar. It has a striking resemblance with the algorithms for

- the maximal pre- and postfix of a string [(W0)].
- the largest rectangle under a histogram [(W1)].
- the maximal one-sided extreme segment (and the like).
A common root for all these problems would be very interesting. I don't mean simply the use of a stack that is apparent in these examples, but a general recognition strategy and a theorem that converts the recognition (almost) immediately into an algorithm.

The problem and the challenge to derive the solution resulted from discussions in the algorithmics working group at the Rijks Universiteit van Utrecht. Hans Zantema gave a functional solution using a direct proof that greedy is optimal. The solution presented here inspired Maarten Fokkinga to give a full account of promotion possibilities for an optimal segmentation problem, leading to a kind of "taxonomy" of their solution schemes ([F]). Oege de Moor presented a Bird-Meertens derivation in Ameland ([M]).
References


\* \* LNCS 875 (1995) \* \*