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Citation for published version (APA):

Speetjens, M. F. M., Clercx, H. J. H., & Heijst, van, G. J. F. (2006). Merger of coherent structures in time-periodic viscous flows. *Chaos*, 16(4), 043104-1/8. [043104]. <https://doi.org/10.1063/1.2355656>

DOI:

[10.1063/1.2355656](https://doi.org/10.1063/1.2355656)

Document status and date:

Published: 01/01/2006

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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- The final published version features the final layout of the paper including the volume, issue and page numbers.

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Merger of coherent structures in time-periodic viscous flows

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(Received 19 January 2006; accepted 23 August 2006; published online 11 October 2006)

Inertia-induced changes in transport properties of an incompressible viscous time-periodic flow are studied in terms of the topological properties of volume-preserving maps. In the noninertial limit, the flow admits one constant of motion and thus relates to a so-called one-action map. However, the invariant surfaces corresponding to the constant of motion are topologically equivalent to spheres rather than the common case of tori. This has fundamental ramifications for the effect of inertia and leads to a new kind of response scenario: resonance-induced merger of coherent structures. © 2006 American Institute of Physics. [DOI: 10.1063/1.2355656]

Passive tracer advection in time-periodic viscous flows serves as a model problem for laminar mixing in industry. Such flows admit analysis in terms of volume-preserving maps classified by the number of constants of motion (“actions”).¹ An important aspect for mixing applications is the response of coherent structures that occur in one-action (invariant surfaces) and two-action (invariant curves) maps to perturbations. These coherent structures form transport barriers to unrestricted tracer motion and their destruction (by some perturbation) is an essential condition for the attainment of efficient mixing. Here the response of such structures to weak (inertial) perturbations is considered.

I. INTRODUCTION

Perturbation of two-action maps causes coalescence of invariant curves into surfaces and thus strictly transforms them into one-action maps.¹ Under specific conditions these invariant surfaces may have local defects that enable global advection via the mechanism of resonance-induced dispersion.^{1–3} These are the two essential response scenarios of two-action maps known to date. Perturbation of one-action maps hitherto appear examined only for invariant surfaces topologically equivalent to tori and gives the “classical” Hamiltonian scenario: KAM-like survival of nonresonant tori; Poincaré-Birkhoff-type break-up of resonant tori.¹ However, despite their practical relevance, response scenarios of one-action maps with invariant surfaces other than tori received little attention thus far. Studies on such one-action maps thus far performed primarily concern the dynamics within invariant surfaces rather than the response to perturbations and, moreover, involve highly idealized configurations.^{4,5} Investigations on the response of the one-action state to perturbations appear nonexistent to date, though. This motivates the study hereafter on the response of an experimentally realizable one-action map involving in-

variant surfaces topologically equivalent to spheres to perturbation by inertial effects.

Considered is the tracer advection in the time-periodic viscous flow set up in the nondimensional square cylinder $\mathcal{D}: [r, \theta, z] = [0, 1] \times [0, 2\pi] \times [-1, 1]$ by a two-step forcing protocol.⁶ The first and second forcing steps involve steady translation of the bottom wall in positive x and y direction, respectively, with fixed nondimensional displacement⁷ $D = VT_f/2L = 5$ (with V , T_f , and L , respectively, translation velocity, period time and side length of the corresponding dimensional problem). This forcing leads under the present highly viscous conditions to stepwise steady flow. (Essentially unsteady effects involved with changeover between forcing steps are negligible under the present viscous flow conditions.) Figure 1 provides a schematic of the two-step time-periodic forcing protocol.⁸

The motion of passive tracers is described by the kinematic equation

$$\frac{dx}{dt} = \mathbf{v}(\mathbf{x}, t) \Rightarrow \mathbf{x}_{k+1} = \Phi(\mathbf{x}_k), \quad (1)$$

with $\mathbf{x}_k$ the position of a tracer released at $\mathbf{x}_0$ after k periods and Φ the volume-preserving map corresponding to one pe-

FIG. 1. Schematic of the two-step time-periodic forcing protocol. The first and second forcing steps involve steady translation of the bottom wall in the x and y direction, respectively, over a distance D .

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riod of the incompressible time-periodic flow $\mathbf{v}(\mathbf{x}, t)$. Note $T=2D$ is the corresponding nondimensional period time. Such maps are alternatively denoted Liouvillian maps and have volume-preservation as the most fundamental property.¹

The time-periodic flow $\mathbf{v}(\mathbf{x}, t)$ on symmetry grounds admits expression entirely in terms of the steady flow $\mathbf{u}(\mathbf{x})$ during the first step (translation of the bottom wall in the x direction) of each forcing cycle. Thus the time-periodic flow becomes

$$\mathbf{v}(x, y, z, t) = \begin{cases} \mathbf{u}(x, y, z) & \text{for } (k-1)T \leq t \leq (k-1/2)T \text{ (first step),} \\ \mathbf{u}(y, -x, z) & \text{for } (k-1/2)T < t < kT \text{ (second step),} \end{cases} \quad (2)$$

with $k=[1, 2, \dots]$ the sequence of forcing cycles, advancing the steady flow $\mathbf{u}$ as the basic flow field.⁶ This flow is governed by the nondimensional steady Navier-Stokes equations

$$\text{Re } \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (3)$$

with p the pressure and $\mathbf{u}$ vanishing on all boundaries, except for the normalized x -wise translation velocity U_x of the bottom wall. The problem is characterized entirely by the Reynolds number $\text{Re} = VL/\nu$, with ν the kinematic viscosity. The analysis hereafter thus involves two flow fields: the steady flow field $\mathbf{u}$ ("basic flow"), governed by (3), and the time-periodic flow field $\mathbf{v}$ ("time-periodic flow"), governed by (2).

Resolution of the nonlinear flow model (3) is performed by numerical treatment with the spectral scheme proposed in Speetjens and Clercx.⁹ This scheme has been specifically developed for numerical resolution of the Navier-Stokes equations in the laminar flow regime and yields a highly accurate approximation to the basic flow $\mathbf{u}$. [The numerical algorithm imposes $U_x = (r^2 - 1)^2$ instead of $U_x = 1$ in order to eliminate the discontinuity in the boundary condition for u_x that would otherwise occur at the bottom rim of the cylinder ($r=1$), where the nonmoving cylinder wall and translating bottom

FIG. 2. Basic flow $\mathbf{u}$ (first forcing step of the time-periodic flow $\mathbf{v}$) in the noninertial limit: closed streamlines (solid) encircling a stagnation line in the plane $x=0$ (heavy).

wall meet. This smoothing (U_x dropping off to zero with r approaching unity) is essential for attainment of spectral convergence of the algorithm and may be done without loss of generality.] Typical departures from the incompressibility constraint are $\nabla \cdot \mathbf{u} \sim \mathcal{O}(10^{-12})$ and thus imply that volume preservation, the most fundamental property of Liouvillian maps (see above), is satisfied to a very high degree. Numerical integration of the kinematic equation (1) is carried out with an explicit third-order Taylor-Galerkin scheme using velocity-field interpolation based upon the spectral expansion of $\mathbf{u}$. This ensures an equal level of approximation of $\mathbf{u}$ at any position $\mathbf{x}$. Tests reveal that this numerical scheme significantly outperforms more conventional schemes in the particular context of tracer dynamics in volume-preserving maps; in particular its strong retention of the incompressibil-

FIG. 3. Axisymmetric constant of motion F_1 of the noninertial time-periodic flow $\mathbf{v}$. Panel a gives the concentric circular isopleths of f_1 in the rz plane; panel b gives the corresponding surfaces of revolution that define the axisymmetric invariant surfaces of F_1 . The facing quarters of the invariant surfaces are omitted so as to expose their concentric arrangement. The center of the invariant surfaces (dot) corresponds with a stagnation point on the z axis.

ity constraint transpires as crucial for faithful representations of long-term tracer dynamics in volume-preserving maps.⁹

The noninertial limit of the basic steady flow field $\mathbf{u}$ is of the form¹⁰

$$\begin{aligned} u_r(r, \theta, z) &= \tilde{u}_r(r, z) \cos \theta, & u_\theta(r, \theta, z) &= \tilde{u}_\theta(r, z) \sin \theta, \\ u_z(r, \theta, z) &= \tilde{u}_z(r, z) \cos \theta, \end{aligned} \quad (4)$$

with (r, θ, z) the cylindrical frame of reference. Its topology (Fig. 2) consists of closed streamlines (solid) centered upon a stagnation line that sits in the plane $x=0$ (heavy). The streamlines are self-symmetric about $x=0$ and form symmetric pairs about $y=0$. (Note the noninertial flow field during the second forcing step identifies with that shown in Fig. 2 upon a rightward quarter turn around the cylinder axis.) The closed streamlines imply two constants of motion $F(x)$ (governed by $\mathbf{u} \cdot \nabla F = 0$) and thus classify the noninertial basic flow as a two-action map.¹¹ Relations (4) imply the form $F(x) = f(r, z)g(\theta)$ that, following substitution into the governing equation $\mathbf{u} \cdot \nabla F = 0$ and separation of variables, gives

$$\tilde{u}_r \frac{\partial f}{\partial r} + \tilde{u}_z \frac{\partial f}{\partial z} + \frac{\lambda \tilde{u}_\theta}{r} f = 0, \quad \sin \theta \frac{\partial g}{\partial \theta} - \lambda \cos \theta g = 0, \quad (5)$$

with λ the separation constant. Valid eigenfunctions are $g(\theta) = A$ ($\lambda=0$) and $g(\theta) = A \sin \theta$ ($\lambda=1$), with A an arbitrary constant, say $A=1$. This gives two distinct constants of motion, namely

$$F_1(r, z) = f_1(r, z), \quad F_2(r, \theta, z) = f_2(r, z) \sin \theta, \quad (6)$$

that are axisymmetric (F_1) and antisymmetric about $y=0$ (F_2). Functions $f_1(r, z)$ and $f_2(r, z)$ are determined by the leading equation in (5) for $\lambda=0$ and $\lambda=1$, respectively. The latter results through $-\tilde{u}_\theta/r = \nabla' \cdot \mathbf{u}'$ in

$$\mathbf{u}' \cdot \nabla' f_1 = 0, \quad \mathbf{u}' \cdot \nabla' (\ln f_2) = \nabla' \cdot \mathbf{u}', \quad (7)$$

with $\mathbf{u}' = (\tilde{u}_r, \tilde{u}_z)$ and $\nabla' = (\partial/\partial r, \partial/\partial z)$. These expressions may be recast as

$$\frac{\tilde{u}_r}{f_2} = \frac{\partial f_1}{\partial z}, \quad \frac{\tilde{u}_z}{f_2} = -\frac{\partial f_1}{\partial r}, \quad \nabla' \cdot \left(\frac{\mathbf{u}'}{f_2} \right) = 0, \quad (8)$$

exposing f_1 as a stream function of the rescaled velocity field $\mathbf{u}^* = \mathbf{u}'/f_2$ in the (r, z) plane. [Reconciliation of the nonseparable flow field $\mathbf{u}'$, i.e., $\nabla \cdot \mathbf{u}' \neq 0$, with the stream-function formulation requires rescaling $\mathbf{u}^* = \mathbf{u}'/c$ such that $\nabla \cdot \mathbf{u}^* = 0$. From $\nabla \cdot \mathbf{u}^* = \nabla \cdot (\mathbf{u}'/c) = 0$ it readily follows that the scaling factor c identifies with f_2 .] The flow topology is invariant to the rescaling, meaning streamlines of $\mathbf{u}^*$ and $\mathbf{u}'$ are identical and coincide with isopleths of f_1 . Furthermore, property $\nabla' \cdot \mathbf{u}' \neq 0$ implies that f_2 changes along streamlines corresponding with $\mathbf{u}'$, meaning f_1 and f_2 are different functions and thus parameterize two distinct (intersecting) families of invariant curves in the (r, z) plane. This, in turn, implies that F_1 and F_2 parameterize two noncoinciding families of invariant surfaces that intersect at the streamlines of the noninertial basic flow.⁶

The axisymmetric constant of motion F_1 is invariant to changes in translation direction of the bottom wall and thus is a constant of motion also for the noninertial time-periodic

FIG. 4. Basic topological makeup of the noninertial time-periodic flow $\mathbf{v}$: invariant surfaces of constant of motion F_1 (surfaces) and period-1 line in the plane $y=-x$ (curve). The solid and heavy sections on the period-1 line are hyperbolic and elliptic segments, respectively; the dot is the parabolic stagnation point upon which the invariant surfaces are centered; the stars are the parabolic period-1 points that bound the elliptic segments of the period-1 line.

flow $\mathbf{v}$. Its invariant surfaces correspond with surfaces of revolution of the stream function f_1 in the rz plane and comprise a family of concentric axisymmetric invariant surfaces that are topologically equivalent to spheres (Fig. 3). The constant of motion F_2 is destroyed by changes in translation direction and is consequently no longer present in the time-periodic case. Thus the noninertial time-periodic flow $\mathbf{v}$ corresponds to a one-action map (F_1) with invariant surfaces other than tori [Fig. 3(b)]. A second topological feature of the noninertial flow $\mathbf{v}$ relevant in the present context is a period-1 line in the plane $y=-x$.⁶ Figure 4 gives the basic topological make-up of the noninertial time-periodic flow $\mathbf{v}$: spherical invariant surfaces of constant of motion F_1 (surfaces) and the period-1 line in the plane $y=-x$ (curve). The normal and heavy sections on the period-1 line are hyperbolic and elliptic segments, respectively; the dot is the parabolic stagnation point upon which the invariant surfaces are centered; the stars are the parabolic period-1 points that bound the elliptic segments. (Note that the outer parabolic points coincide with the boundary.) For $\text{Re}=0$ tracers are restricted to the invariant surfaces parameterized by F_1 and consequently the motion is essentially 2D. Tracer motion within invariant surfaces is typically chaotic, however. Figure 5(a) demonstrates this 2D intrasurface chaos with the Poincaré section (10 000 forcing periods) of a single tracer. Figure 5(b) shows the corresponding projection in the rz plane; the projection coincides with a single curve and thus demonstrates that the tracer, despite its chaotic motion, is indeed confined on a single invariant surface. The arrows in Fig. 5(a) indicate the two highly localized elliptic islands that are centered on the intersections of the invariant surface with the two elliptic segments of the period-1 line (Fig. 4). Figure

FIG. 5. 2D intrasurface chaotic motion on invariant surfaces of constant motion F_1 in the noninertial time-periodic flow $\mathbf{v}$. Shown is the Poincaré section (10 000 forcing periods) of a single tracer in perspective view (panel a) and the corresponding projection in the rz plane (panel b). The star represents the initial position of the tracer. Panel c gives a close-up of the left elliptic island of the pair of elliptic islands (indicated by arrow in panel a) corresponding with the elliptic segments of the period-1 line.

5(c) gives a close-up of the left elliptic island from the same viewpoint as adopted in panel a.

The noninertial time-periodic flow $\mathbf{v}$ is the baseline flow in terms of which to examine the response of one-action maps with invariant surfaces other than tori to weak (inertial) perturbations. This response is, by virtue of relation (2), intimately related to the response of the basic flow $\mathbf{u}$. The following provides an extensive investigation on this matter.

II. RESULTS AND DISCUSSION

Inertia introduces centrifugal forces to the (steady) basic flow $\mathbf{u}$ that destroy the symmetry about $x=0$ and thus lead to nonclosed streamlines. Figure 6(a) demonstrates this for a

streamline confined to the plane of symmetry $y=0$ at $\text{Re}=10$; the streamline (solid) steadily spirals outward, thus progressively distancing itself from its closed noninertial counterpart (heavy), in the course of time. A measure for the nonclosedness (and, inherently, the departure from the noninertial baseline) may be found in the separation δ of the nonclosed streamline from the corresponding closed streamline in the noninertial limit. Figure 6(b) gives the separation δ for the streamline shown in Fig. 6(a) with increasing displacement D of the bottom wall. The progression reveals that δ to good approximation grows linearly with increasing D , implying a (nearly) constant growth rate $\dot{\delta}=d\delta/dD$. This linear relation is found throughout the considered range 0

FIG. 6. Inertial effects in the basic flow. Panel a demonstrates the outward-spiraling motion along a streamline in the plane of symmetry $y=0$ at $Re=10$ induced by centrifugal forces. This results in a nonclosed streamline (solid) that progressively distances itself from its closed noninertial counterpart (heavy). Panel b gives the corresponding separation δ of the inertial from the noninertial streamline with increasing displacement D of the bottom wall. Panel c shows the growth rate $\dot{\delta}=d\delta/dD$ of the separation as a function of Re .

$\leq Re \leq 10$, advancing the growth rate $\dot{\delta}$ as an appropriate measure for the effect of inertia upon the flow field. Figure 6(c) displays $\dot{\delta}$ versus Re , exposing a monotonic increase of the growth rate, signifying a progressive departure from the noninertial baseline $Re=0$ ($\dot{\delta}=0$) with increasing Re .

For small nonzero Re the nonclosed streamlines coalesce into two families of concentric tori, symmetrically arranged about the preserved symmetry plane $y=0$, and the topology of the basic flow transforms from a two-action state ($Re=0$) to a one-action state ($Re>0$).¹ (Note the phenomenon of resonance-induced dispersion mentioned before does not occur here.) Figure 7 demonstrates this for $Re=10$. Panel (a) shows a typical nonclosed streamline wound around a torus. Panel (b) shows the intersections of five tori on one side of the symmetry plane $y=0$ with the plane $x=0.12$. The topological transition happens for any nonzero Re , meaning the noninertial limit is a singular state.¹ Thus the behavior of the basic flow demonstrated for $Re=10$ is representative for that found throughout the considered range $0 \leq Re \leq 10$. Within this range changes brought on by variation of Re are, consistent with the findings attached to Fig. 6, entirely quantitative.

The inertia-induced symmetry-breaking in the (steady) basic flow $\mathbf{u}$ leads in the time-periodic flow $\mathbf{v}$ to drifting of tracers transverse to their corresponding invariant surfaces. Thus tracers occupy a shell of finite thickness that is centered upon the invariant surface of the time-periodic flow $\mathbf{v}$ in the noninertial limit and gradually expands as time progresses. However, for the small departures from the noninertial limit

considered here, the expansion rate involves time scales several orders of magnitude larger than the period time of the flow, meaning tracers remain restricted to thin shells of approximately constant thickness for prolonged periods of time. Formation of such a long-lived shell is demonstrated in Fig. 8 with the Poincaré section of a single tracer for 10 000 forcing periods and $Re=0.1$.

The above behavior suggests that for small departures from the noninertial limit, constant of motion F_1 is approximately an adiabatic invariant. The analogy is somewhat loose in the sense that tracers are not strictly confined to invariant surfaces of F_1 , but are allowed limited freedom of movement in their direct proximity. Thin shells such as that shown in Fig. 8 (“adiabatic shells”) are the corresponding family of coherent structures. These shells expand continuously in time and thus restrict the lifespan T_A of the adiabatic state to finite periods in time. This limited durability is inherent in adiabatic approximations.¹² An estimate for the lifespan T_A may be found through the shell thickness δ_s . Investigation of the case $Re=0.1$ reveals $\delta_s \sim \mathcal{O}(10^{-2})$ for relative time spans $T_A/T \sim \mathcal{O}(10\,000)$, meaning expansion is insignificant and thus implying that adiabatic shells remain intact for the period of time considered (Fig. 8). Beyond this time span, expansion slowly yet gradually increases beyond $\delta_s \sim \mathcal{O}(10^{-2})$, resulting in significant expansion that effectively causes shells to dissolve and the adiabatic state to break down. For higher Re , on the other hand, shells almost immediately exhibit notable expansion, meaning that an adiabatic state is never reached. Thus adiabatic states are

FIG. 7. Basic flow $\mathbf{u}$ under inertial conditions ($\text{Re}=10$). Panel a shows a typical nonclosed streamline wound around a torus. Panel b shows the intersections of five tori on one side of the symmetry plane $y=0$ with the plane $x=0.12$.

restricted to $\text{Re} \leq \mathcal{O}(0.1)$ and have a typical lifespan $T_A/T \sim \mathcal{O}(10\,000)$ within this range.

A second family of adiabatic structures emanates from the islands centered upon the elliptic segments of the period-1 line in the noninertial limit. These islands consist of closed orbits encircling the period-1 line and coincide with a given invariant surface of F_1 (Fig. 5). The closed orbits imply a local constant of motion $H(\eta, \zeta)$ within the local intra-surface frame of reference (η, ζ) . (Note the map is area-preserving in the proximity of the islands.⁴) Hence, the tracer dynamics are locally governed by two constants of motion, namely $F_1(r, z)$ and $H(\eta, \zeta)$, implying that Φ is locally a two-action map. Perturbation effectuates coalescence of the closed orbits into tubes that are parameterized by an adiabatic invariant¹ and transforms the local two-action map into a local one-action map in essentially the same way as the basic flow.¹³ Thus two families of concentric tubes, each centered upon an elliptic segment of the period-1 line, are created. The perturbed map Φ in consequence accommodates (at least) three families of coherent structures, viz., the

FIG. 8. Long-lived thin shell centered upon an axisymmetric invariant surface (not shown) of the time-periodic flow $\mathbf{v}$ in the noninertial limit, as visualized with the Poincaré section of a single tracer for 10 000 periods. Shown is the projection in the rz plane. The star represents the initial position (same as in Fig. 5) of the tracer.

adiabatic shells corresponding with F_1 and the two families of concentric tubes corresponding with the two elliptic segments of the period-1 line. However, these three families, rather than coexist, appear to merge into one family of coherent structures. Two adiabatic shells of F_1 connect through local “openings” via one tube on each elliptic segment of the period-1 line and form one adiabatic structure. This is shown schematically in a cross-sectional view in Fig. 9: two tubes (shaded), each corresponding with one elliptic segment (heavy) of the period-1 line (solid) and two adiabatic shells (circles) corresponding with F_1 (panel a) merge into one adiabatic structure (panel b) via attachment of the tubes to the shells. Figure 10 shows an actual adiabatic structure thus formed in perspective view (panel a) and its intersection with the plane $y=-x$ (panel b), together with the period-1 line (curve). (Note the adiabatic structures are topologically equivalent to a torus.¹⁴) The merger occurs such that fully closed adiabatic structures are formed that enclose one another. This arrangement is demonstrated in Fig. 11 for the structure of Fig. 10 and a second structure. Note the outer shell of the latter virtually coincides with the boundary.

Tubes transport tracers from inner to outer shells within time spans $T_i/T \sim \mathcal{O}(1000)$, with T_i the residence time of tracers in tubes and T the period time of the forcing. Typical residence times on the inner shell prior to extraction are several orders of magnitude higher. The formation of adiabatic structures occurs for any arbitrarily small $\text{Re} > 0$, meaning the noninertial limit of the time-periodic flow $\mathbf{v}$ is, similar to that of the basic flow $\mathbf{u}$, a singular state.¹

The cause for the merger must be sought in the local breakdown of the averaging principle, which underlies the formation of adiabatic structures,¹⁵ at the ends of the tubes. This breakdown leads to local “openings” in the adiabatic shells that enable switching of tracers between tubes and shells, thus effectively connecting them. Key to the formation of an adiabatic shell is that, relative to the corresponding

FIG. 9. Schematic of the merger of the three families of coherent structures into one family of adiabatic structures. Panel a gives the individual coherent structures in cross-sectional view: two tubes (shaded), each corresponding with one elliptic segment (heavy) of the period-1 line (solid), and two adiabatic shells (circles) corresponding with F_1 . Panel b gives the adiabatic structure formed by merger of these individual structures: inner and outer shells connected via the two tubes.

invariant surface of F_1 , tangential tracer motion significantly exceeds transverse tracer motion. This essential condition is violated in the proximity of the parabolic period-1 points (stars in Fig. 4) that bound both ends of the elliptic segments of the period-1 line, where fluid parcels deform (shear) yet perform no (appreciable) net motion. These parabolic points are resonances,¹ that is, material points in the proximity of which fluid motion is greatly reduced (or ceases altogether) to such an extent that the averaging principle fails and the adiabatic shells develop local defects.¹⁶ This creates the abovementioned “openings” that facilitate the merger of adiabatic shells and tubes. Moreover, this implies that chaotic motion in an invariant surface, which is inherently nonresonant, is essential to its survival as an adiabatic shell.

Resonances are hitherto known only to cause (local) breakdown of coherent structures in one- and two-action maps (see introductory paragraph). However, for the one-action map studied here resonances cause merger of structures. This is a fundamental difference that sets the behavior found here apart from that known to date and thus suggests an essentially new response scenario: resonance-induced merger. The underlying mechanisms are not yet fully under-

FIG. 10. Inertia-induced merger of two adiabatic shells and two elliptic tubes, centered upon the period-1 line (curve), into one adiabatic structure. The structure is shown in perspective view (panel a) and its intersection with the plane $y=-x$ (panel b).

stood, though. However, key is the topological equivalence of the invariant surfaces to spheres. First, this invalidates the Hamiltonian response scenarios for one-action maps, which hinge explicitly on the assumption that both intrasurface coordinates are periodic. This condition is fulfilled for tori yet violated for spheres. Second, this causes resonances to emerge as isolated points rather than as curves (one-action maps with tori) or surfaces (two-action maps). The topological equivalence to spheres namely means that the invariant surfaces typically accommodate one (or more) isolated periodic points that, through connectivity of fluid parcels, each join with adjacent points into one (or more) periodic lines intersecting the invariant surfaces.^{4,5} (Note that in the present configuration the period-1 line is implied by symmetry properties of the flow.⁶) Nondegenerate periodic lines thus formed are typically partitioned into elliptic and hyperbolic segments that connect through isolated parabolic points (Fig. 4). This implies resonances (parabolic points), if existent,

FIG. 11. Arrangement of adiabatic structures. Shown are the intersections of two structures with the plane $y=-x$.

emerge as isolated points on periodic lines. These fundamental topological differences between the “classical” toroidal invariant surfaces and the present spherical invariant surfaces are believed essential for the occurrence of merger rather

FIG. 12. Breakdown of the adiabatic state for higher departures from the noninertial baseline. Shown are the (remnants of) the adiabatic structures in Fig. 11 (black) and that of an additional adiabatic structure (grey) for $\text{Re}=1$. (Intersections of the structures with the plane $y=-x$.)

than (local) disintegration of coherent structures in the time-periodic flow $\mathbf{v}$.

The adiabatic state of the time-periodic flow $\mathbf{v}$ exists only in a narrow Re range and disintegration already sets in for departures from the noninertial limit slightly beyond that range (see before). Figure 12 demonstrates this for $\text{Re}=1$. Shown are the (remnants of) the adiabatic structures in Fig. 11 (black) and that of an additional adiabatic structure (grey). The transverse expansion of the shells has increased dramatically to such an extent that (local) merger into chaotic regions begins to occur, thus signifying the breakdown of the adiabatic state. The range of existence of the adiabatic state of the time-periodic flow $\mathbf{v}$ is significantly more narrow than that of the basic flow $\mathbf{u}$ (concentric tori in Fig. 7). The former is restricted to $\text{Re} \leq \mathcal{O}(0.1)$; the latter survives perturbations up to $\text{Re} \sim \mathcal{O}(10)$. This suggests the adiabatic state of $\mathbf{v}$ is very vulnerable to disturbances (e.g. nonideal conditions) and may thus be encountered in reality only in low- Re flows (e.g., microfluidics or highly viscous flows).

ACKNOWLEDGMENTS

M.F.M.S. gratefully acknowledges financial support from the Dutch Foundation for Fundamental Research on Matter (FOM). We furthermore wish to extend our gratitude to Professor V. V. Meleshko (Kiev National Taras Shevchenko University, Kiev, Ukraine) and Professor V. V. Sharko (Institute of Mathematics, National Academy of Sciences of Ukraine, Kiev, Ukraine) for their expert advice on various topological issues.

¹J. H. E. Cartwright, M. Feingold and O. Piro, J. Fluid Mech. **316**, 259 (1996).

²J. H. E. Cartwright, M. Feingold and O. Piro, Phys. Rev. Lett. **75**, 3669 (1995).

³I. Mezić, Physica D **154**, 51 (2001).

⁴A. Gómez and J. D. Meiss, Chaos **12**, 289 (2002).

⁵P. Mullowney, K. Julien and J. D. Meiss, SIAM J. Appl. Dyn. Syst. **4**, 159 (2005).

⁶M. F. M. Speetjens, H. J. H. Clercx and G. J. F. van Heijst, J. Fluid Mech. **514**, 77 (2004).

⁷This is a typical value that may be changed without loss of generality of the discussion here.

⁸In Ref. 6, the time-periodic flow corresponds with forcing protocol A.

⁹M. F. M. Speetjens and H. J. H. Clercx, Int. J. Comput. Fluid Dyn. **19**, 191 (2005).

¹⁰V. S. Malyuga, V. V. Meleshko, M. F. M. Speetjens, H. J. H. Clercx and G. J. F. van Heijst, Proc. R. Soc. London, Ser. A **458**, 1867 (2002).

¹¹K. Bajer, Chaos, Solitons Fractals **4**, 895 (1994).

¹²E. Ott, Phys. Rev. Lett. **42**, 1628 (1979).

¹³The tubes are complete; the phenomenon of resonance-induced dispersion mentioned in the introductory paragraph does not occur here.

¹⁴The incomplete inner shell and the two connecting tubes (“inner section”) as well as the incomplete outer shell (“outer section”) are topologically equivalent to a tube [Fig. 9(b)]. Both structures connect at the circles that bound the “openings” in the incomplete outer shell and thus form one closed structure that is topologically equivalent to a torus. The outer side of the torus thus created corresponds with the outer side of the inner section and the inner side of the outer section.

¹⁵V. I. Arnol’d, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).

¹⁶This reasoning may suggest that elliptic and hyperbolic periodic points are resonances as well. However, in contrast with parabolic points, elliptic and hyperbolic points have significant tangential tracer motion in their direct proximity that upholds the averaging principle.