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F.W. Steutel and J.G.F. Thiemann

1. INTRODUCTION

This paper developed from the following simple problem. Let $K_1(\mu), \ldots, K_n(\mu)$ be i.i.d. having a Poisson distribution with mean $\mu$. Let $K_{1:n} \leq \cdots \leq K_{n:n}$ denote the corresponding order statistics. Problem: determine $E K_{j:n}$ and $\text{var} K_{j:n}$.

It turns out that for $n=2$ the answer can be given explicitly; we give the solution for $K_{2:2}$ (see [7] for details):

$$EK_{2:2} = \mu + \mu e^{-2\mu} (I_0(2\mu) + I_1(2\mu)),
\text{var} K_{2:2} = \mu - \mu^2 e^{-4\mu} (I_0(2\mu) + I_1(2\mu))^2 + \mu e^{-2\mu} I_0(2\mu),$$

where $I_j$ denotes a modified Bessel function of order $j$.

For $n > 2$ the problem seems intractable, so we look for approximations for large $\mu$.

When approximating the Poisson distribution by a normal one, there is always the problem of continuity corrections. We try to avoid that by first considering a continuous variant of the Poisson distribution, and that is where the Gamma process comes in.

Let $Z(t), t > 0$, be a Gamma process with unit mean, i.e. $Z$ is a process with stationary independent increments and $Z(1)$ has an exponential distribution with mean one. Now let $T(\mu)$ be the exceedance time of level $\mu$, i.e.

$$T(\mu) = \inf \{t > 0 \mid Z(t) > \mu\}.$$

Then

$$\{T(\mu) \leq t\} = \{Z(t) > \mu\} \text{ a.s.}$$

and so

$$P(T(\mu) \leq t) = \int_\mu^\infty \frac{x^{t-1} e^{-x}}{\Gamma(t)} \, dx \quad (t > 0).$$

Now $T(\mu)$ has a continuous distribution function which is increasing on $(0, \infty)$. Moreover, for
integer arguments it coincides with the Poisson distribution function with mean $\mu$, since from (3) we easily deduce

\begin{equation}
P(T(\mu) \leq k) = \sum_{i=0}^{k-1} \frac{\mu^i}{i!} e^{-\mu} \quad (k = 1, 2, \cdots).
\end{equation}

For future reference we introduce some notation. For each real number $x$ let $\lfloor x \rfloor$ be its integer part, i.e. the largest integer not exceeding $x$, and put $\{x\} := x - \lfloor x \rfloor$, the fractional part of $x$. The property of $T$ mentioned above can then be states as: $[T(\mu)]$ has a Poisson distribution with mean $\mu$. As a result, this paper contains some new results for exceedance times in Gamma processes and an approximate solution of the above-mentioned problem about order statistics.

Before discussing the special case of the Gamma process we consider exceedance times of constant levels for more general processes.

2. EXCEEDANCE TIMES IN PROCESSES WITH STATIONARY, INDEPENDENT INCREMENTS

In this section $Z(t), t > 0$, is a non-negative process with stationary, independent increments, scaled such that $Z(0) = 0$ and $EZ(1) = 1$. For each $\mu > 0$ we define the exceedance time $T(\mu)$ by (1), as before and we shall be interested in the behaviour of $[T(\mu)]$ and $(T(\mu))$ for large values of $\mu$ (see Fig. 1, which is slightly misleading since the paths of $Z$ are not continuous)

The random variable $T(\mu)$ is a.s. finite: $P(T(\mu) > n) \leq P(Z(n) \leq \mu) \to 0 \quad (n \to \infty)$.

Since the process $Z$ is continuous in probability and non-decreasing, it is at each point a.s. continuous. From this one can deduce that, for each $t > 0$ and for all but at most countably many $\mu > 0$, the sets

$$\{\mu < Z(t)\}, \{\mu \leq Z(t)\}, \{T(\mu) < t\} \text{ and } \{T(\mu) \leq t\}$$

are a.s. equal.
We are now ready to prove the following result.

**Theorem 1.** When $Z(1)$ is non-lattice, then

\[(5) \lim_{\mu \to \infty} P\left(\{T(\mu)\} < u\right) = u \quad (0 < u < 1) .\]

**Proof:** For each $u \in (0, 1)$ we have a.s.

\[
\{\{T(\mu)\} < u\} = \bigcup_{k=0}^{\infty} \{k \leq T(\mu) < k + u\} = \bigcup_{k=0}^{\infty} \{Z(k) \leq \mu < Z(k + u)\} .
\]

So

\[
P\left(\{T(\mu)\} < u\right) = \sum_{k=0}^{\infty} \int_{\mu}^{\infty} P(Z(u) > \mu - x) \, dF_{Z(u)}(x) ,
\]

where $F_Z$ denotes the distribution function of $Z$. An appeal to the key renewal theorem (see [4]) easily yields

\[
\lim_{\mu \to \infty} P\left(\{T(\mu)\} < u\right) = \frac{EZ(u)}{EZ(1)} = u .
\]

\[\square\]

**Remark 1.** If one writes $Z(1) = Z(1 - u) + Z(u)$ then Theorem 1 is a special case of a well-known result for alternating renewal processes.

**Remark 2.** Theorem 1 also holds without the non-lattice condition. The proof for the lattice case can be given along similar lines.

Laplace transformation with respect to $\mu$ is an efficient tool for obtaining asymptotic results for $\mu \to \infty$. In view of this, as a preparation for the special case in the next section we give a few lemmas. For this purpose we need some more notation. Since $Z(1)$ has an infinitely divisible distribution we have, for $t, s > 0$,

\[
Ee^{-sZ(1)} = \phi(s)^t = e^{-t\psi(s)} ,
\]

with

\[(6) \phi(s) := Ee^{-sZ(1)} \quad \text{and} \quad \psi(s) = -\log \phi(s) .\]

We shall use the following simple fact: if $F$ is a distribution function on $[0, \infty)$, then
\( e^{-\tau} F(x) = s \int_0^\infty e^{-\tau} F(x) dx \quad (s > 0) \).

**Lemma 1.**

\[
\int_0^\infty e^{-\tau\mu} F_{T(\omega)}(t) d\mu = s^{-1}(1 - e^{-\tau \psi(s)}) \quad (s, t > 0).
\]

**Proof:** Since \( \{\mu < Z(t)\} \subset \{\mu \leq Z(t)\} \) we have, for all but at most countably many \( \mu \)'s, \( F_{T(\omega)}(t) = 1 - F_Z(\mu) \). So

\[
\int_0^\infty e^{-\tau\mu} F_{T(\omega)}(t) d\mu = \int_0^\infty e^{-\tau\mu}(1 - F_Z(\mu)) d\mu =
\]

\[
= s^{-1} - s^{-1} \int_0^\infty e^{-\tau\mu} dF_Z(\mu) = s^{-1} - s^{-1} E e^{-\tau Z(t)} =
\]

\[
= s^{-1}(1 - e^{-\tau \psi(s)}).
\]

**Lemma 2.**

\[
\int_0^\infty e^{-\tau\mu} e^{-\tau T(\omega)} d\mu = \frac{\psi(s)}{s(\tau + \psi(s))} \quad (s, \tau > 0).
\]

**Proof:** By (7) we have

\[ E e^{-\tau T(\omega)} = \tau \int_0^\infty e^{-\tau u} F_{T(\omega)}(t) dt. \]

Laplace transformation with respect to \( \mu \), an appeal to Lemma 1 and a change of order of integration, give the required result.

**Lemma 3.**

\[
\int_0^\infty e^{-\tau\mu} E(T(\mu)^k) d\mu = \frac{k!}{s^k \psi(s)^k} \quad (s > 0, k \in \mathbb{N}).
\]
Proof: We get the result by differentiating \( k \) times both sides of the equality in Lemma 2 and letting \( \tau \downarrow 0 \). 

Since \( T(\mu) \) is non-decreasing in \( \mu \), as follows from the definition of \( T \), Lemma 3 implies that \( T(\mu) \) has finite moments of all orders.

**Lemma 4.** For each \( u \in (0, 1) \) the function \( \mu \mapsto F_{\{T(\mu)\}}(u) \) is the difference of two increasing functions, and

\[
\int_0^\infty e^{-st} F_{\{T(\mu)\}}(u) \, d\mu = \frac{1-\phi(s)^u}{s(1-\phi(s))} \quad (s > 0) .
\]

Proof: Let \( u \in (0, 1) \). Then, for all but at most countably many \( \mu \)'s, by (6) we have

\[
F_{\{T(\mu)\}}(u) = \sum_{k=0}^\infty P\left(k \leq T(\mu) \leq k + u\right) = \sum_{k=0}^\infty [F_{T(\mu)}(k + u) - F_{T(\mu)}(k)] .
\]

Hence, by Lemma 1,

\[
\int_0^\infty e^{-st} F_{\{T(\mu)\}}(u) \, d\mu = \sum_{k=0}^\infty [s^{-1}(1 - e^{-(k+u)s}) - s^{-1}(1 - e^{-ks})] = \\
= \frac{1 - e^{-\psi(s)}}{s (1 - e^{-\psi(s)})} = \frac{1-\phi(s)^u}{s(1-\phi(s))} .
\]

Furthermore, for all \( \mu > 0 \), we have

\[
F_{\{T(\mu)\}}(u) = \sum_{k=0}^\infty P\left(k \leq T(\mu) \leq k + u\right) = \sum_{k=0}^\infty P\left(T(\mu) \geq k\right) - \sum_{k=0}^\infty P\left(T(\mu) > k + u\right) ,
\]

where convergence of both series follows from majorization by the integral

\[
\int_0^\infty P\left(T(\mu) > x\right) \, dx = ET(\mu) < \infty .
\]

So \( F_{\{T(\mu)\}}(u) \) can be written as the difference of two increasing functions of \( \mu \). 

In the Lemmas 1-4 we have Laplace transforms of functions that are monotone or equal to the difference of two monotone functions and, hence, the inversion theorem for the Laplace transformation is in force (see e.g. [10] §7.3). When the random variables \( Z(t) \) (\( t > 0 \)) have continuous distributions then these functions are continuous and therefore they can be obtained by application of the inversion theorem to their Laplace transforms. We use Lemmas 1,3 and 4 to obtain asymptotic results as \( \mu \to \infty \).
Theorem 2.

i) \[ \lim_{\mu \to \infty} \frac{E(T(\mu)^k / \mu^k = 1 \quad (k = 1, 2, \cdots) \]

ii) \[ \lim_{M \to \infty} M^{-1} \int_0^M F(T(\mu))(u) \, d\mu = u \quad (0 < u < 1) \]

Proof: We apply a Tauberian theorem (see Feller, vol. 2 Th. XIII.5.1) to the Laplace transforms in the Lemmas 1, 3 and 4.

For the function \( F_T(\mu)(1) \) we have

\[ \lim_{M \to \infty} \int_0^M F_T(\mu)(1) \, d\mu = \int_0^\infty P(\mu < Z(1)) \, d\mu = EZ(1) = 1 \]

So for its Laplace transform, given by Lemma 1, we have, by the Tauberian theorem,

\[ \lim_{s \downarrow 0} s^{-1}(1 - e^{-\psi(s)}) = 1, \] which implies \( \lim_{s \downarrow 0} s^{-1} \psi(s) = 1. \]

Next we consider the Laplace transform \( f(s) \) of \( E(T(\mu)^k) \). By Lemma 3 and the result just obtained we have \( \lim_{s \downarrow 0} s^{k+1} f(s) = \lim_{s \downarrow 0} k! s^k \psi(s)^{-k} = k! \). From this i) follows by the Tauberian theorem.

Finally, for the Laplace transform \( g(s) \) of \( F(T(\mu))(u) \) by Lemma 4 we have \( \lim_{s \downarrow 0} s g(s) = u \), which implies ii) by the Tauberian theorem.

To obtain Theorem 1, which is stronger than ii), a more powerful Tauberian theorem would be needed. For the special case of the Gamma process much sharper results will be obtained from the Lemmas 3 and 4.

3. THE GAMMA PROCESS

This is the particular case we are most interested in. Now we have (cf. (6))

\[ \phi(s) = \frac{1}{1+s} \quad \text{and} \quad \psi(s) = \log (1 + s). \]

Moreover, the random variables \( Z(t) \) \((t > 0)\) have continuous distributions, so the functions occurring in the Lemmas 3 and 4 can be obtained by applying the inversion formula to their Laplace transforms. This will give us quite sharp versions of the Theorems 1 and 2.
Theorem 1'.

\[ 0 < F_{(T_0)}(u) - u < (\pi u e^{-\mu})^{-1} \quad (0 < u < 1, 0 < \mu). \]

Proof: Let \( u \in (0, 1) \). For the Laplace transform \( f \) of \( F_{(T_0)}(u) \) we have, by Lemma 4,

\[ f(s) = \frac{1 - \phi(s)^u}{s(1 - \phi(s))} = s^{-2}[1 + s - (1 + s)^{1-u}] \]

Therefore \( f(s) = O(1/s^{1-1}) \) \( (1/s \to \infty) \), \( f \) has a pole in 0 and a branch point in -1. Consequently, in the version formula

\[ F_{(T_0)}(u) = \frac{1}{2\pi i} \lim_{b \to \infty} \int_{a-ib}^{a+ib} e^{\mu s} f(s) \, ds \]

we can modify the path of integration as shown in Fig. 2. Since the residue of the integrand in 0 equals \( u \), we get

\[ F_{(T_0)}(u) = u - \frac{1}{2\pi i} \int_{-\infty}^{-1} e^{\mu s} s^{-2}[1 + s]^{1-u} e^{-\pi i (1-u)} \, ds + \]

\[ - \frac{1}{2\pi i} \int_{-\infty}^{-1} e^{\mu s} s^{-2}[1 + s]^{1-u} e^{\pi i (1-u)} \, ds = \]

![Figure 2](image-url)
\[ u = \frac{1}{\pi} \int_{-\infty}^{1} e^{\mu s} \frac{1}{1 + s} \sin(\pi(1 - u)) \, ds. \]

So

\[ 0 < F(T(\mu)) (u) - u < \frac{1}{\pi} \int_{-\infty}^{1} e^{\mu s} \, ds = \frac{1}{\mu} \, e^{-\mu}. \]

Theorem 2'. For all \( k \in \mathbb{N} \) and \( \mu > 0 \),

\[ \mathbb{E} (T(\mu)^k) / k! = \sum_{l=0}^{k} c_{-l} \mu^l l! \leq (\pi^{k+1} \mu e^{\mu})^{-1}, \]

where the coefficients \( c_{-l} \) are defined by

\[ (\log(1 + s))^{-k} = \sum_{l=-k}^{\infty} c_{-l} s^l \quad (1 \leq s < 1). \]

Proof: Let \( k \in \mathbb{N} \). By Lemma 3 the Laplace transform of \( \mathbb{E} (T(\mu)^k) / k! \) is \( s^{-1} [\log(1 + s)]^{-k} \).

Exactly the same procedure as used in the proof of Theorem 2' yields

\[ \mathbb{E} (T(\mu)^k) = \sum_{l=0}^{k} c_{-l} \mu^l l! + \]

\[ + \frac{1}{2\pi i} \int_{-\infty}^{-1} e^{\mu s} s^{-1} [\log |1 + s| - \pi i]^{-k} \, ds + \]

\[ + \frac{1}{2\pi i} \int_{1}^{\infty} e^{\mu s} s^{-1} [\log |1 + s| - \pi i]^{-k} \, ds, \]

where again the first term on the right is the residu in 0.

Now both integrals are easily seen to be bounded in absolute value by

\[ \frac{1}{2\pi} \int_{-\infty}^{-1} e^{\mu s} s^{-k} \, ds = \frac{1}{2} (\pi^{k+1} \mu e^{\mu})^{-1}, \]

hence the result.

The following corollaries are important for our purposes. Corollary 2 is rather surprising and shows a behaviour that is similar to that observed in [6] for \( Y, \{ Y/e \} \) and \( \{ Y/e \} = \{ Y/e \} \) as \( e \downarrow 0 \).

We recall that \( \{ T(\mu) \} \) has a Poisson distribution with mean \( \mu \), and we refer to Theorem 1' as well as to Theorem 2'.
Corollary 1.

\[
ET(\mu) = \mu + \frac{1}{2} + O(e^{-\mu}) ,
\]

\[
\text{var } T(\mu) = \mu - \frac{1}{12} + O(e^{-\mu}) \quad (\mu \rightarrow \infty).
\]

Corollary 2.

\[
\text{cov}(T(\mu), \{T(\mu)\}) = O(e^{-\mu}) ,
\]

\[
\text{cov}([T(\mu)], \{T(\mu)\}) = -\frac{1}{12} + O(e^{-\mu}) \quad (\mu \rightarrow \infty).
\]

From Corollary 2 it follows that in

\[
K_{ijL} = T_{ijL} - U ,
\]

the random variables in the right-hand side are practically uncorrelated (for an interesting case where \(X \) and \(\{X\} \) are independent we refer to [9]). In combination with Theorem 1' it follows that, as far as the first two moments are concerned, \(K(\mu)\) is quite well approximated by

\[
K(\mu) = T(\mu) - U ,
\]

where \(T(\mu)\) and \(U\) are independent and \(U\) is uniformly distributed on \((0, 1)\).

4. APPROXIMATING \(K_{i,jL}(\mu)\)

In this section we derive approximations for \(EK_{i,jL}(\mu)\) and \(\text{var } K_{i,jL}(\mu)\) for large \(\mu\). Starting from (8) and using a normal approximation for \(T_{ijL}\).

Let \(\mu > 0\). Since the distribution function \(F\) of \(T(\mu)\) is continuous and increasing one has

\[
T(\mu) = F^{-1}(U) ,
\]

for any random variable \(U\) that is uniformly distributed on \((0, 1)\). In particular we have

\[
T(\mu) = F^{-1}(\Phi(X)) ,
\]

where \(X\) is a standard normal random variable and \(\Phi\) its distribution function. The right-hand side of this equality depends on \(\mu\) via \(F\), and we make this dependence explicit by writing

\[
T(\mu) = G(X, \mu) .
\]

As both \(F\) and \(\Phi\) are increasing functions, \(G\) is increasing in its first argument and therefore the relation (9) implies a similar relation for the order statistics corresponding to \(T(\mu)\) and \(X\):
For the function \( G \) we have the following result, which we give without its straight-forward but rather lengthy proof.

**Lemma 5.** Let \( q \) be defined by

\[
q(x, \mu) = \mu + x \mu^{\frac{1}{6}} + \frac{x^2+2}{6} - \frac{x^3+2x}{72} \mu^{-\frac{1}{2}}.
\]

Then for \( G \) as defined by (9) one has

\[
G(x, \mu) = q(x, \mu) + r(x, \mu)
\]

with

\[
|r(x, \mu)| \leq C \frac{x^{4+1}}{\mu} \quad \text{for} \quad |x| \leq \mu^{ \frac{1}{6}}
\]

and \( \mu \) sufficiently large, where \( C \) is a constant.

The expansion in (11) is very similar to an expansion given by Riordan [8] without error term, and is related to Edgeworth expansions.

It is helpful for the intuition to combine (10) and (11) to

\[
T_{j,n}(\mu) = \mu + X_{j,n} \mu^{\frac{1}{6}} + \frac{1}{6} (X_{j,n}^2 + 2) - \frac{1}{72} (X_{j,n}^3 - 2X_{j,n}) \mu^{-\frac{1}{2}} + \frac{1}{2} (X_{j,n}^4 + 1) \mu^{-1}
\]

on \( \{X_{j,n} \leq \mu^{ \frac{1}{6}} \} \).

In order to relate the expectation and variance of \( T_{j,n}(\mu) \) to those of \( X_{j,n} \) we need one further estimate on \( G(x, \mu) \), which enables us to obtain bounds on the tails of the corresponding integrals.

**Lemma 6.** For \( x \) and \( \mu \) such that \( 2 \leq \mu^{ \frac{1}{6}} \leq |x| \) we have

\[
G(x, \mu) \leq x^{14} e^{x^2/(2\mu)} \quad \text{and} \quad q(x, \mu) \leq 3x^6.
\]

**Proof:** Let \( t \) be such that \( F_T(t) = \Phi(x) \). We shall prove that \( t \leq 2\mu^2 |x| e^{x^2/(2\mu)} \) and so we may suppose \( 2\mu^2 < t \).
Now let \( k \in \mathbb{N} \) be such that
\[
\mu^2 - 1 \leq k \leq \frac{t}{2} < k + 1
\]
then
\[
1 - F_{T}(y) \leq 1 - F_{T}(2k) = \int_{0}^{\frac{\mu^{2k-1}}{\Gamma(2k)}} e^{-y} dy \leq \frac{\mu^{2k}}{(2k)!} \leq \frac{\mu^{2(k+1)}}{(k+1)!} = (\frac{\mu^{2}}{k+1})^{k} \leq (\frac{\mu^{2}}{k+1})^{k+1} \leq (\frac{2\mu^{2}}{t})^{k}
\]
and
\[
1 - \Phi(x) \geq 1 - \Phi(|x|) \geq (2\pi)^{-\frac{1}{2}} (1 - x^{-2}) |x|^{-1} e^{-x^2/2} \geq |x|^{-1} e^{-x^2/2}
\]
hence
\[
t \leq 2\mu^2 |x|^{1 \mu} e^{x^2/(2\mu)}.
\]
Since \( t = G(x, \mu) \), from (13) and \( 2 \leq \mu^6 \leq |x| \) the estimates for \( G \) and \( q \) are easily deduced. \( \Box \)

Lemmas 5 and 6 imply

Theorem 3.
\[
ET_{j,n}(\mu) = \mu + \mu^{\frac{1}{2}} EX_{j,n} + \frac{1}{6} (EX_{j,n}^2 + 2) + O(\mu^{-\frac{1}{2}}) \quad (\mu \to \infty),
\]
\[
\text{var} T_{j,n}(\mu) = \mu \text{ var } X_{j,n} + \frac{1}{3} \mu^{\frac{1}{2}} \text{ cov}(X_{j,n}, X_{j,n}^2) + O(1) \quad (\mu \to \infty).
\]

Finally, we need to go from the random variable \( T_{j,n}(\mu) \) to \( [T_{j,n}(\mu)] \) \( d \) \( K_{j,n}(\mu) \). This is done in the following lemma.

Lemma 7. For each \( j, n \in \mathbb{N} \) with \( j \leq n \) and for each \( r > 0 \)
\[
EK_{j,n}(\mu) = ET_{j,n}(\mu) - \frac{1}{2} + O(\mu^{-r}) \quad (\mu \to \infty)
\]
\[
\text{var} K_{j,n}(\mu) = \text{var} T_{j,n}(\mu) + \frac{1}{12} + O(\mu^{-r}) \quad (\mu \to \infty).
\]

Proof: Let \( j, n \in \mathbb{N} \) with \( j \leq n \). Then
(14) \[ P(T_{j;n} > t) - P(T > t) = Q(I(\mu, t)) , \]

where \( I(\mu, t) := P(T > t) \) and the polynomial \( Q \) is defined by

\[ Q(x) = \sum_{i=0}^{j-1} \left( \begin{array}{c} n \\ i \end{array} \right) (1-x)^i x^{n-i} - x . \]

Integration of (14) gives

(15) \[ ET_{j;n}(\mu) - ET(\mu) = \int_{0}^{\infty} Q(I(\mu, t)) \, dt . \]

In a similar way we obtain

(16) \[ E[T_{j;n}(\mu)] - E[T(\mu)] = \sum_{k=1}^{\infty} Q(I(\mu, k)) . \]

Now, since \( Q(0) = Q(1) = 0 \), the polynomial \( Q(x) \) contains a factor \( x(1-x) \), and it follows from an Euler-McLaurin-type result in [2] that the difference of the right-hand side of (15) and (16) is \( O(\mu^{-r}) \) \( (\mu \to \infty) \) for every \( r > 0 \). By Corollary 1 the result on \( EK_{j;n}(\mu) \) can now be obtained

For the result on \( \text{var} \, K_{j;n} \) a similar proof applies.

Now Theorem 3 leads to the result we started out to obtain:

**Theorem 4.** For each \( j, n \in \mathbb{N} \) with \( j \leq n \) and for \( \mu \to \infty \)

\[ EK_{j;n}(\mu) = \mu + \mu^\frac{1}{3} EX_{j;n} + \frac{1}{6}(EX_{j;n}^2 - 1) + O(\mu^{-\frac{1}{3}}) \]

\[ \text{var} \, K_{j;n}(\mu) = \mu \text{ var } X_{j;n} + \frac{1}{3} \mu^\frac{1}{3} \text{ cov}(X_{j;n}, X_{j;n}^2) + O(1) . \]

Theorem 4 has been proved in [1] by more laborious, purely analytic methods. The present proof is a bit simpler, and formulas like (12) provide some more insight. Tables and asymptotic formulas for moments of \( X_{j;n} \) needed to apply Theorem 4 can be found in Harter [5]. It turns out that the estimates are quite accurate even for moderate values of \( \mu \) and fairly large values of \( n \). The first terms (without error term) are easily obtained from the central limit theorem; the accuracy is considerably increased by the extra terms, but at a cost. Though the order terms are not uniform in \( n \), they can be shown to be quite good as long as \( n \) does not increase faster than polynomially in \( \mu \).

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<td>Reversed self-decomposability</td>
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<td>The singular zero-sum differential game with stability using $H_{\infty}$ control theory</td>
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<td>M 89-10</td>
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<td>L.J.G. Langenhoff</td>
<td>An analytical theory of multi-echelon production/distribution systems</td>
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<td>A.H.W. Geerts</td>
<td>The Algebraic Riccati Equation and Singular Optimal Control</td>
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<td>M 89-12</td>
<td>May</td>
<td>D.A. Overdijk</td>
<td>De geometrie van de kroonwieloverbrenging</td>
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<td>M 89-13</td>
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<td>I.J.B.F. Adan</td>
<td>Analysis of the shortest queue problem</td>
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<td>A.A. Stoorvogel</td>
<td>The singular $H_{\infty}$ control problem with dynamic measurement feedback</td>
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<td>P.C. Schuur</td>
<td>On the asymptotic convergence of the simulated annealing algorithm in the presence of a parameter dependent penalization</td>
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<td>A priori results in linear-quadratic optimal control theory (extended version)</td>
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<td>An approximation for the response time of an open CP-disk system</td>
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<td>On randomness of random number generators</td>
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<td>Synchronously Parallel: Boltzmann Machines: a Mathematical Model</td>
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