A generalized mean-variance metric of route choice model under travel time uncertainty

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A generalized mean-variance metric of route choice model under travel time uncertainty

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ABSTRACT
Route choice modeling under travel time uncertainty is essential for analyzing travelers’ mobility patterns. This paper investigates the impact of travel time uncertainty on route choice behavior in user equilibrium models based on a generalized mean-variance metric (GMV). This model can capture the influence of risk attitudes and schedule unpunctuality on route choice using a generalization of expected travel time, variance, and expected early or late arrival penalties, of which travelers are assumed to minimize the GMV of trips considering a certain on-time arrival probability. This paper establishes the properties of GMV and formulates the GMV-based static user equilibria as a variational inequality (VI) problem, for which the existence and uniqueness of the solutions are also analyzed. An effective traffic assignment algorithm without path enumeration is developed to solve the proposed user equilibrium problem. Numerical examples are conducted to demonstrate the properties of the proposed model.

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Travel time uncertainty; route choice; generalized mean-variance; user equilibrium

1. Introduction
It is crucial to understand the route choice behavior of travelers for mobility pattern analysis. Much research has concentrated on route choice modeling to incorporate various contexts, for example, travel time uncertainty (Lo, Luo, and Siu 2006), within-day dynamics (Bellei, Gentile, and Papola 2005), day-to-day dynamics (Bie and Lo 2010; Liu et al. 2020), bounded rationality (Di and Liu 2016), multi-modal (Zhang et al. 2011; Liao 2016; Li et al. 2018), and activity-travel chains in multi-state supernetworks (Liao, Arentze, and Timmermans 2010, 2011, 2012, 2013; Liu et al. 2015). Particularly, route choice under uncertainty receives increasing attention (Liao, Rasouli, and Timmermans 2014; Rasouli and Timmermans 2014). Understanding the complex relations between travel time uncertainty and route choice is crucial for predicting mobility patterns.

Travel time uncertainty can be accounted for from two different perspectives: supply...
degradations and travel demand fluctuations. Supply degradations fall within the category of exogenous sources and usually cause non-recurrent congestion, while demand fluctuations are regarded as endogenous factors and always lead to recurrent congestion (Lo, Luo, and Siu 2006; Li, Lam, and Sumalee 2008; Chen and Zhou 2010; Li, Lam, and Wong 2011). Being the sources of travel time uncertainty, demand and supply aspects interact and affect travelers’ route choice behavior significantly (Bates et al. 2001; Lam, Shao, and Sumalee 2008; Wang, Ehrgott, and Chen 2014).

To capture the effects of travel time uncertainty on travelers’ route choice behavior, travel time reliability (TTR) has been extensively studied. For example, Tilahun and Levinson (2010) used a computer-administered stated preference survey to estimate the value of TTR and explored the tradeoffs that travelers make for route choice. Woodard et al. (2017) stated that travel time variability strongly affects the desirability of routes in the road network. Moreover, various empirical studies (Li, Hensher, and Rose 2010; Sweet and Chen 2011; Li and Hensher 2013) have made a convincing proposition that TTR plays a key role in travelers’ route choice behavior.

To quantify travel time variability, several metrics have been proposed in traffic equilibrium models, including travel time budget (TTB), mean-excess travel time (METT), perceived travel time, and multi-attribute utility functions. All these metrics are derived from two different approaches, namely, the mean-variance approach and the scheduling approach. Jackson and Jucker (1982) proposed the mean-variance metric as the weighted sum of the mean and variance of travel time. Aligned with this effort, Lo, Luo, and Siu (2006) factored travelers’ different risk attitudes according to their on-time arrival probabilities using the concept of TTB for degradable transport networks. As another extension, Chen and Zhou (2010) took the conditional expectation of travel time beyond TTB into consideration and suggested a form of METT that combines a buffer time with the tardy time. Based on TTB, METT, and several other models, Tan, Yang, and Guo (2014) examined the Pareto efficiency of traffic equilibria. Mean-standard deviation indifference curves were introduced to geometrically analyze the risk-taking behavior of travelers. Assuming that travelers want to minimize the mean and standard deviation of travel time, Wang, Ehrgott, and Chen (2014) proposed a general TTR bi-objective user equilibrium (UE) model and proved that the model encompasses the single-objective of the TTB-UE model (Lo, Luo, and Siu 2006) and the late arrival penalized UE model (Watling 2006).

Based on Vickrey (1969), Small (1982) first proposed the classic schedule delay concept, which has been extended to several cases. For instance, Noland and Small (1995) analyzed the effects of uncertain travel time and derived the optimized expected utility function for both a uniform and an exponential distribution of travel time. To allow for time-dependent travel time distributions, Bates et al. (2001) proposed another form for the expected utility function and argued that the mean-variance approach and scheduling approach are approximated for a wide range of distributions. Based on the ‘schedule delay’ paradigm, Watling (2006) defined a new disutility function by adding a schedule delay term to the expected travel cost and developed a late arrival penalized UE model. Fosgerau and Karlström (2010) proved the equivalence of both approaches and derived that the preference parameters in the mean-variance approach depend on the parameters in the scheduling approach. Li and Hensher (2013) introduced a rank-dependent utility theory model and proposed an attribute-specific extension, where maximizing expected utility is a special case. In addition, several approaches based on alternative choice-making mechanisms, such as
prospect theory and regret theory (Chorus 2012; Li and Huang 2017), were developed based on TTR.

To accommodate a variety of route risk attitudes, we propose a generalized mean-variance (GMV) metric for route choice under travel time uncertainty. GMV uses a form of ‘generalized cost’ structure with individual preferences for the associated terms. It is capable of ensuring a preferable on-time arrival probability and capturing the influence of two mutually exclusive schedule delays on travelers’ route choice. We show three special forms of GMV and prove the continuity and monotonicity, which were only assumed in the previous studies. Due to the non-additivity of GMV, two dominance conditions are developed for searching the reliable shortest path. Moreover, a GMV-based user equilibrium (GMVUE) problem is formulated as a variational inequality (VI) problem and solved by an effective traffic assignment algorithm without path enumeration. Note that we focus on the model framework in this study and leave the empirical analysis in a separate study.

The remainder of this paper is organized as follows. Section 2 provides the preliminary knowledge of route choice under travel time uncertainty. Section 3 introduces the GMV metric and proposes the corresponding UE model. The properties of GMV are also presented and analyzed. Section 4 develops a GMV-based traffic assignment algorithm for solving the GMVUE problem. Numerical examples are given in Section 5. Finally, conclusions and recommendations for future work are provided.

The following abbreviations are used in the paper:

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full name</th>
<th>Abbreviation</th>
<th>Full name</th>
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</thead>
<tbody>
<tr>
<td>GMV</td>
<td>generalized mean-variance</td>
<td>CDF</td>
<td>cumulative distribution function</td>
</tr>
<tr>
<td>TTR</td>
<td>travel time reliability</td>
<td>SD</td>
<td>standard deviation</td>
</tr>
<tr>
<td>TTB</td>
<td>travel time budget</td>
<td>BPR</td>
<td>Bureau of Public Road</td>
</tr>
<tr>
<td>METT</td>
<td>mean-excess travel time</td>
<td>M-V</td>
<td>mean-variance</td>
</tr>
<tr>
<td>MTT</td>
<td>mean travel time</td>
<td>M-GMV</td>
<td>mean-GMV</td>
</tr>
<tr>
<td>MLTT</td>
<td>mean-less travel time</td>
<td>MSA</td>
<td>method of successive average</td>
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2. Preliminaries

This section provides some preliminary knowledge of route choice under travel time uncertainty in a transport network \( G(N, A) \), where \( N \) and \( A \) are the sets of nodes and links respectively.

2.1. Link and path travel time distribution

According to Lo, Luo, and Siu (2006), the relationship between traffic flow and travel time due to supply degradations is established by the Bureau of Public Road (BPR) link performance function

\[
T_I(v_l, C_l) = t_0^l \left[ 1 + \beta \left( \frac{v_l}{C_l} \right)^\delta \right], I \in A
\] (1)
where $T_l$ is the random travel time of link $l$; $v_l$, $c_l$ and $t^0_l$ are the corresponding traffic flow, random link capacity after degradation, and free-flow travel time respectively; $\beta$ and $\delta$ are deterministic parameters.

By assuming that $c_l$ is independent of $v_l$ and follows a uniform distribution, Lo, Luo, and Siu (2006) analytically derived the mean $\mu_l$ and standard deviation (SD) $\sigma_l$ of $T_l$ as follows

$$
\mu_l = E(T_l) = t^0_l + \beta v_l t^0_l c_l^{-\delta} c_l^{-\delta} (1 - \theta_l)(1 - \delta), \forall l \in A
$$

$$
\sigma_l = \sqrt{Var(T_l)} = \sqrt{\beta^2 (t^0_l)^2 v_l^2 \left\{ \frac{1 - \theta_l^{1-\delta}}{c_l^{2\delta} (1 - \theta_l)(1 - 2\delta)} - \left[ \frac{1 - \theta_l^{1-\delta}}{c_l^{2\delta} (1 - \theta_l)(1 - \delta)} \right]^2 \right\}}, \forall l \in A
$$

where $c_l^\delta$ and $\theta_l c_l^\delta$ are the upper and lower bounds of the uniform distribution respectively.

By assuming that link travel times are statistically independent, the travel time $T_p$ of path $p$ is the sum of related link travel times along $p$, and the mean and SD of $T_p$ can be represented respectively as

$$
\mu_p = E(T_p) = \sum_{l \in A} \mu_l x_{lp}, \quad \forall p \in P^{rs}
$$

$$
\sigma_p = \sqrt{Var(T_p)} = \sqrt{\sum_{l \in A} \sigma_l^2 x_{lp}}, \quad \forall p \in P^{rs}
$$

where $x_{lp}$ is a 0–1 variable regarding the link-path incidence relationship. $x_{lp} = 1$ denotes that link $l$ is on path $p$, and $x_{lp} = 0$ otherwise. $P^{rs}$ is the set of paths between origin $r$ and destination $s$.

Under the above assumptions, when those independent link travel times are added, the path travel time tends to be normally distributed according to the Central Limit Theorem even if the link travel times are not. Thus, we have

$$
T_p \sim N(\mu_p, \sigma_p^2), \quad \forall p \in P^{rs}
$$

The Central Limit Theorem is applicable when a path contains many links. The assumptions of mutually independent and normally distributed link travel times or disutilities offer an alternative (Yin, Lam, and Ieda 2004; Fu and Lam 2014), which leads to the normal distribution of path travel times or disutilities. To relax the assumptions, for example, Seshadri and Srinivasan (2017) relaxed the independence assumption in a robust traffic assignment model. Still, these assumptions are widely adopted due to the simplicity and analytic properties for route choice, network design, and land use modeling (Li et al. 2008; Chen et al. 2013; Tan, Yang, and Guo 2014; Liao, Rasouli, and Timmermans 2014; de Jong and Bliemer 2015; Sun, Cheng, and Ma 2018; Chen, Li, and Lam 2018)

2.2. Travel time budget and mean-excess travel time

Lo, Luo, and Siu (2006) introduced the concept of travel time budget (TTB) to relate travel time variability due to stochastic link capacity variations to travelers’ risk-averse route
choice behavior as

$$\xi_p(\alpha) = E(T_p) + \gamma(\alpha)\sqrt{\text{Var}(T_p)} = \mu_p + \gamma(\alpha)\sigma_p, \quad \forall p \in \mathcal{P}^{rs}$$  \hspace{1cm} (7)

where $\xi_p(\alpha)$ is the TTB of path $p$ required to ensure on-time arrival at confidence level $\alpha$, and $\gamma(\alpha)$ is a parameter for describing the requirement of punctual arrival. A larger value of $\alpha$ corresponds to a larger $\gamma(\alpha)$. The value of $\xi_p(\alpha)$ can be expressed in relation to on-time arrival probability $\alpha$:

$$P(T_p \leq \xi_p(\alpha)) = G(\xi_p(\alpha)) = \alpha, \quad \forall p \in \mathcal{P}^{rs}$$  \hspace{1cm} (8)

where $G(\cdot)$ denotes the cumulative distribution function (CDF) of $T_p$. Let $\Phi(\cdot)$ be the CDF of the standard normal distribution. Substituting Eq. (7) into Eq. (8) gives

$$\gamma(\alpha) = \Phi^{-1}(\alpha)$$  \hspace{1cm} (9)

Alternatively, Shao, Lam, and Tam (2006) assumed that the travel time variations are deduced from the daily demand variations, which follow a normal distribution. Based on the Central Limit Theorem, they concluded that the path travel time follows a normal distribution and derived the formulation of TTB. Both formulations were developed by applying the Central Limit Theorem with the assumption of independent link travel times.

To capture the unreliability aspects of travel time variability, Chen and Zhou (2010) considered the tardy time and formulated the mean-excess travel time (METT) as the following equation.

$$\eta_p(\alpha) = \xi_p(\alpha) + E(T_p - \xi_p(\alpha)|T_p \geq \xi_p(\alpha)), \quad \forall p \in \mathcal{P}^{rs}$$  \hspace{1cm} (10)

### 3. Formulation

This section first formulates a generalized mean-variance (GMV) metric and analyzes its properties. Next, the corresponding UE model (GMVUE) and VI formulation are proposed. To keep consistency, we use the notations above with the same definitions unless explained otherwise.

#### 3.1. Generalized mean-variance metric

Under travel time uncertainty, mean travel time, and travel time variance are two important components affecting travelers’ choices. Travelers with different preferable on-time arrival probabilities have different attitudes toward travel time variability. For example, risk-averse travelers with a large on-time arrival probability perceive that travel time uncertainty will lead to a high penalty, and they may pre-assign a larger travel time for their trips. However, the early and late trips are undesirable but unavoidable in reality. To quantify travel time variability, a generalized mean-variance (GMV) metric is expressed as

$$u^{rs}_p = \omega_1 \cdot \mu_p + [\omega_2(\alpha) \cdot E(T_p - \xi_p(\alpha))^+ \downarrow \omega_3(\alpha) \cdot E(T_p - \xi_p(\alpha))^+] + \omega_4(\alpha) \cdot \sigma_p$$  \hspace{1cm} (11)

where $(T_p - \xi_p(\alpha))^+$ is the early arrival time defined as $\max(0, \xi_p(\alpha) - T_p)$; $(T_p - \xi_p(\alpha))^-$ is the late arrival time defined as $\max(0, T_p - \xi_p(\alpha))$; $\downarrow$ is an operator to denote that one and
only one component of the two sides is effective; \( \omega_1, \omega_2(\alpha), \omega_3(\alpha) \) and \( \omega_4(\alpha) \) are collective weight coefficients, and \( \omega_i(\alpha)(i = 2, 3, 4) \) denotes a preference parameter related to \( \alpha \).

Imposing a constraint \( \omega_2(\alpha) \cdot \omega_3(\alpha) = 0 \), Eq. (11) is reduced to

\[
\mathbf{u}_p^{rs} = \omega_1 \cdot \mu_p + \omega_2(\alpha) \cdot E(T_p - \xi_p(\alpha))^+ + \omega_3(\alpha) \cdot E(T_p - \xi_p(\alpha))^- + \omega_4(\alpha) \cdot \sigma_p
\]  

(12)

Setting aside the weight coefficients, the first term in Eq. (12) is the expectation of path travel time. It reflects the value of the average travel time within a long time frame. The second term is the expected travel cost related to early arrival, which can be seen as the opportunity cost of interrupting the prior trip. The third term is the expected travel cost of being late, seen as the opportunity cost of interrupting the current trip. The last term is the safety margin, which captures the sensitivity to path travel time dispersion.

Regarding the weight coefficients, \( \omega_1 \) and \( \omega_4(\alpha) \) are used to capture the degree of importance of mean travel time and variance to GMV. \( \omega_4(\alpha) \) is set equal \( \omega_1 \gamma(\alpha) \) in this paper to illustrate travelers’ different risk attitudes toward travel time uncertainty unless otherwise specified. \( \omega_2(\alpha)(\leq 0) \) and \( \omega_3(\alpha)(\geq 0) \) are parameters to indicate the degrees of attitude toward the early and late arrivals.

Taken together, Lo, Luo, and Siu (2006) used the sum of the first and fourth terms represents the TTB to captures the ‘reliability aspect’ (i.e. travelers arrive at the destination with a travel time less than or equal to the travel time budget). However, travelers may still arrive late with a probability \( (1 - \alpha) \), as shown by the red area in Figure 1. Therefore, Chen and Zhou (2010) introduced an additional term \( \omega_3(\alpha) \cdot E(T_p - \xi_p(\alpha))^+ \) to represent an additional safety margin, which is the mean late arrival time beyond the TTB. The proposed METT is the conditional expectation of the late trips (red area) and used to capture the estimation of travel time for risk-averse travelers. Alternatively, the second term \( \omega_2(\alpha) \cdot E(T_p - \xi_p(\alpha))^- \) can be seen as the opportunity cost and used to hedge against early arrival. MLTT is the conditional expectation of the early trips (green area) for risk-prone travelers. Note that early and late arrivals are mutually exclusive, the operator \( \bot \) in Eq. (11) and the condition \( \omega_2(\alpha) \cdot \omega_3(\alpha) = 0 \) of Eq. (12) are used to represent either early or late arrival. The statements and extensions are expressed by the following remark.

**Remark:** GMV is a generalized mean-variance metric for route choice under travel time uncertainty:
(i) GMV is equivalent to the expected/mean travel time (MTT) (Hall 1986) when travelers only concentrate on expected travel time, i.e. $\omega_1 = 1$ and $\omega_2(\alpha) = \omega_3(\alpha) = \omega_4(\alpha) = 0$.

(ii) GMV is equivalent to TTB (Lo, Luo, and Siu 2006) when travelers factor expected travel time and variance into their route choice decision, i.e. $\omega_1 = 1$, $\omega_2(\alpha) = \omega_3(\alpha) = 0$ and $\omega_4(\alpha) = \gamma(\alpha)$.

(iii) GMV is equivalent to METT (Chen and Zhou 2010) when travelers factor the reliable aspect of travel time variability (defined by TTB) and the unreliable aspect with the proportion of $(1/(1 - \alpha))$ into route choice decision, i.e. $\omega_1 = 1$, $\omega_2(\alpha) = 0$, $\omega_3(\alpha) = (1/(1 - \alpha))$ and $\omega_4(\alpha) = \gamma(\alpha)$.

(iv) GMV is equivalent to MLTT when travelers factor TTB and the early arrival with the proportion of $-(1/\alpha)$ into route choice decision, i.e. $\omega_1 = 1$, $\omega_2(\alpha) = -(1/\alpha)$, $\omega_3(\alpha) = 0$ and $\omega_4(\alpha) = \gamma(\alpha)$.

**Proof:** See Appendix 1.

As illustrated above, MTT, TTB, MLTT, and METT are three special cases of GMV. Since TTB and METT were developed by applying the Central Limit Theorem, this study uses the same theorem.

### 3.2. Properties of GMV

Understanding the relationship between GMV and the value components, such as on-time arrival probability, expected travel time, and SD, is important for evaluating the path alternatives. Following the widely assumed condition that path travel times are normally distributed, continuity and monotonicity of GMV are obtained below.

**Proposition 1 (Continuity and monotonicity):**

(a) GMV (Eq. (12)) is continuous with $\alpha$.

(b) TTB, METT, and MLTT, three different forms of GMV, are monotonically increasing with $\alpha$.

**Proof:** See Appendix 2.

Although Proposition 1 is proved by assuming normally distributed path travel times, it can be easily obtained that the property of continuity is guaranteed with any other continuous distributions. For the monotonicity, as depicted by Figure 2, MLTT, TTB, and METT are increasing with the increase of on-time arrival probability. The late arrival coefficient $\omega_3(\alpha)$ of METT has a steep increase, which leads to that the value of METT (green curve) increases rapidly when $\alpha$ approaches 1. Whereas, the value of MLTT (blue curve) increases rapidly when $\alpha$ is very small.

**Corollary 1:** GMV is non-additive, i.e. the path GMV is not necessarily the sum of the associated link GMVs.

**Proof:** Based on Remark (ii, iii), TTB and METT are two special cases of GMV. GMV is non-additive because of the non-additivity of TTB and METT.
The non-additivity of GMV leads to the violation of Bellman’s Principle of Optimality (Bellman 1958) and disallows the application of classical shortest pathfinding algorithms to search for the minimal GMV path. Dominance-based methods (Chen et al. 2013) provide a straightforward way to overcome the non-additive property for solving the reliable shortest path problem. Inspired by their work, the following GMV-based dominance definitions and conditions are proposed.

**Definition 1:** Let $p^{ij}_1 = p^r_i \oplus p^{ij}$ and $p^{ij}_2 = p^j_2 \oplus p^{ij}$ be two paths from node $r$ to $j$ with the same sub-path $p^{ij}$; $p^{ij}_1$ GMV-based dominates $p^{ij}_2$ (denoted by $p^{ij}_1 \succ p^{ij}_2$) if and only if $u^{ij}_1 < u^{ij}_2$ for any path $p^{ij} \in P^{ij}$ and any node $j \in N$, where $\oplus$ is a path concatenation operator.

**Definition 2:** A path $p^{ij}_1 \in P^{ij}$ is a GMV-based non-dominated path, if and only if $p^{ij}_1$ is not dominated by any other path $p^{ij}_2 \in P^{ij}$.

Based on Definition 1 and 2, the GMV-based principle of optimality is presented as follows.

**Corollary 2:** A sub-path of any GMV-based non-dominated path must be a GMV-based non-dominated path itself.

**Proof:** ‘Reduction to absurdity’ is applied to prove this corollary. Suppose $p^{ij}_2$ is a sub-path of a GMV-based non-dominated path $p^{ij}_2 = p^{ij}_1 \oplus p^{ij}$ and $p^{ij}_1 \succ p^{ij}_2$. Let $p^{ih}_1 = p^{ij}_1 \oplus p^{ih}$ and $p^{ih}_2 = p^{ij}_2 \oplus p^{ih}$, then $u^{ih}_1 < u^{ih}_2$ for any path $p^{ih} \in P^{ih}$ and any node $h \in N$ according to Definition 1. It is reasonable to assume that $p^{ih} = p^{ij} \oplus p^{ih}$, where $p^{ih}$ denotes any path from node $j$ to $h$. Thus, $p^{ih}_1 = (p^{ij}_1 \oplus p^{ij}) \oplus p^{ih}$, $p^{ih}_2 = (p^{ij}_2 \oplus p^{ij}) \oplus p^{ih}$, and $u^{ih}_1 < u^{ih}_2$ for any path $p^{ih} \in P^{ih}$ and any node $h \in N$. In other words, $(p^{ij}_1 \oplus p^{ij}) \succ (p^{ij}_2 \oplus p^{ij}) = p^{ij}_2$. This contradicts the precondition that $p^{ij}_2$ is a GMV-based non-dominated path.

**Corollary 3:** The path with the minimal GMV is a GMV-based non-dominated path.
This corollary, combined with Corollary 2, can be used to find the path with the minimal GMV. Starting from an origin, the GMV-based non-dominated sub-paths are stored and extended until the destination is reached. To determine GMV-based non-dominated paths in a transport network under uncertainty, mean-variance (M-V) dominance and mean-GMV (M-GMV) dominance are proposed as follows.

**Proposition 2 (M-V dominance):** Given $\alpha$ and two different paths, $p_1^i$ and $p_2^i$, of path set $P^i$, $p_1^i > p_2^i$ if $p_1^i$ and $p_2^i$ satisfy either

(a) $\mu_1^i \leq \mu_2^i$ and $z\sigma_1^i < z\sigma_2^i$ or
(b) $\mu_1^i < \mu_2^i$ and $z\sigma_1^i \leq z\sigma_2^i$

where $z = \omega_1\gamma(\alpha) + \omega_2(\alpha)\gamma(\alpha) - \omega_3(\alpha)(1 - \alpha)\gamma(\alpha) + \frac{\omega_2(\alpha) + \omega_3(\alpha)}{\sqrt{2\pi}}e^{-\left(\frac{\gamma(\alpha)}{\sqrt{2}}\right)^2}$.

**Proof:** See Appendix 3.

**Proposition 3:** (M-GMV dominance): Given $\alpha$ and two different paths $p_1^i$, $p_2^i \in P^i$, $p_1^i > p_2^i$ if $p_1^i$ and $p_2^i$ satisfy $\mu_1^i \leq \mu_2^i$ and $u_1^i < u_2^i$.

**Proof:** See Appendix 3.

Note that the M-V dominance identified in Proposition 3 is different from the counterpart in Chen et al. (2013), in which $z$ is equal to $\gamma(\alpha)$. In addition, the M-GMV dominance is more effective for identifying the GMV-based non-dominated paths as stated by Proposition 4.

**Proposition 4:** Given $p_1^i$ and $p_2^i$, if $p_1^i$ M-V dominates $p_2^i$, then $p_1^i$ M-GMV dominates $p_2^i$.

**Proof:** When $\mu_1^i \leq \mu_2^i$ and $z\sigma_1^i < z\sigma_2^i$, we have $u_1^i - u_2^i = \omega_1(\mu_1^i - \mu_2^i) + (z\sigma_1^i - z\sigma_2^i) < 0$. When $\mu_1^i < \mu_2^i$ and $z\sigma_1^i \leq z\sigma_2^i$, we have $u_1^i - u_2^i = \omega_1(\mu_1^i - \mu_2^i) + (z\sigma_1^i - z\sigma_2^i) < 0$. Therefore, $p_1^i$ M-GMV dominates $p_2^i$ according to Proposition 3.

Based on this proposition, some GMV-based dominated paths that are not identified under the M-V dominance condition can be discarded when searching the minimal GMV path. This conclusion contributes to speeding up the path search process.

### 3.3. Illustrative example

Figure 3 depicts a network with one OD (1-6) to illustrate the different outcomes of TTB, METT, and MLTT. All link travel times are assumed to be normally distributed and independent with each other. The means and standard deviations are attached to the respective links. Suppose all travelers are risk-averse and use the same confidence level of on-time arrival probability $\alpha = 0.8$.

According to Remark (ii-iv), TTB, METT and MLTT are obtained by setting weight coefficients as $[1, 0, 0, \gamma(\alpha)]$, $[1, 0, 1/(1 - \alpha), \gamma(\alpha)]$ and $[1, -1/\alpha, 0, \gamma(\alpha)]$ respectively. Figure 4 provides the comparison results, where the x-axis represents the three route choice metrics, and the y-axis represents the corresponding values. It is postulated that travelers search
Figure 3. A simple network for illustrating different route choice metrics.

Figure 4. The results of different route choice metrics.

for paths with optimal values according to certain metrics. As shown, different route choice metrics lead to different optimal paths. For example, to avoid late arrival, travelers would add a safety margin to ensure their predetermined on-time arrival probability, and choose path 3 with the minimal TTB (11.85). Besides the mean-variance of path travel time, travelers may budget their travel costs based on the expected travel delay cost, for which METT includes the expected excess delay beyond the TTB. Thus, travelers would switch to path 1 to decrease the expected excess delay. When travelers use MLTT as the metric, based on the last group of bars in Figure 4, path 1 is no longer the optimal choice, and travelers would switch to path 4. Similar to TTB and METT, MLTT is non-additive. For example, considering path 3 (consisting of links (1, 5) and (5, 6)), the MLTTs on links (1, 5) and (5, 6) are 2.53 and 2.97 respectively according to Eq. (12), but path 3 has a larger MLTT, i.e. 5.82 (greater than the sum of 2.53 and 2.97).

3.4. Path-based user equilibrium

The path-based approaches have become more common recently in networks with non-additive link travel costs. In combination, the column generation technique (Leventhal, Nemhauser, and Trotter 1973; Wang et al. 2019) makes it possible to address large-scale networks. This study proposes a path-based user equilibrium model based on GMV. It is assumed that travelers aim to minimize GMV to accomplish their trips in traffic networks under uncertainty, and the GMV-based UE (GMVUE) is reached after long-term adaptations. The flow pattern at equilibrium is stated as: for any OD pair, all the used paths have equal
GMV, while the unused paths have equal or higher GMVs. Formally, the conditions can be expressed by a set of complementarity conditions:

$$f_{rs}^{ps}[u_{rs}^{ps} - \pi_{rs}] = 0, \quad \forall p \in P^{rs}, rs \in RS$$

(13)

where $u_{rs}^{ps}$ is the GMV incurred by travelers departing from $r$ and using path $p$ to reach destination $s$, $\pi_{rs}$ the minimal GMV of OD pair $r$-$s$, and $f_{rs}^{ps}$ the traffic flow on $p$.

The demand of OD pair $r$-$s$, $q_{rs}$, is assumed fixed in this study. Let $f$ denotes the path flow vector $(\ldots, f_{rs}^{ps}, \ldots)^T$ and $u(f)$ the GMV vector. The GMVUE problem is to find $f$ such that Eq. (13) and the following demand conservation and non-negativity constraints are satisfied.

$$\sum_p f_{rs}^{ps} = q_{rs}, \quad \forall p \in P^{rs}, rs \in RS$$

(14)

$$f \geq 0$$

(15)

The flow conservation Eq. (14) ensures that the demand of any OD pair is equal to the sum of the flows on all paths of the same OD pair. The GMVUE problem Eqs. (13)-(15) can be formulated as a finite-dimensional variational inequality problem $VI(f, \Omega)$ to find a vector $f^*$ such that

$$(f - f^*)^T u(f^*) \geq 0, \forall f \in \Omega$$

(16)

$$\Omega = \{f|\Lambda f = q, f \geq 0\}$$

(17)

The superscript ‘*’ refers to a solution of $f$ that fulfills the GMVUE conditions, $q$ denotes the demand vector, and $\Lambda$ denotes an OD-path incidence matrix. The equivalence between the GMV equilibrium problem and $VI(f, \Omega)$ is established as follows.

**Proposition 5:** According to Chen and Zhou (2010), given that $u(f)$ is non-negative, the solution of $VI(f, \Omega)$ is equivalent to the equilibrium solution of the GMVUE problem.

The existence of solutions to $VI(f, \Omega)$ requires that $u(f)$ is a continuous function of $f$, and $\Omega$ is a compact closed convex set. For the GMVUE problem, the second requirement is satisfied for the linear demand constraints and non-negativity constraints depicted in Eq. (17). Note that the schedule delay $(T_p - \xi_p(\alpha))^-$ and $(T_p - \xi_p(\alpha))^+$ are random variables, which are discontinuous at several points. However, we find below that the GMV formulation is continuous with the link traffic flows.

**Proposition 6:** The GMV established in Eq. (12) is continuous with link flows.

**Proof:** See Appendix 4.

Since link flow is the sum of all path flows using this link, the continuity of GMV to path flows is guaranteed with the incorporation of Proposition 6. Thus, there exists at least one solution to $VI(f, \Omega)$. The uniqueness requires that the Jacobian matrix of $u(f)$ is positive definite, which, however, cannot be guaranteed.
4. Solution algorithm

In this section, a solution algorithm to the GMVUE problem is proposed. To address a real transportation network under travel time uncertainty, the algorithm integrates a GMV-based shortest path algorithm, a column generation scheme, and the method of successive average (MSA). Although the MSA with the predetermined sequence of step size may suffer slow convergence, it has been widely used in traffic assignment problems (Fu and Lam 2014; Levin et al. 2015) due to its simplicity and the forced convergence property.

**GMVSP:** Search for a path with the minimal GMV

Step 1. Initialization
Create a path $p^0_{rt}$ from $r$ to itself, and set $\mu_{rt}^0 = 0, (\sigma_{rt}^0)^2 = 0, \text{ and } u_{rt}^0 = 0$. Add $p^0_{rt}$ into label-vector $P^0$ and a list of candidate labels $SE$.

Step 2. Label selection
Take label $p^0_{ri} \in P^0$ at node $i$ from $SE$ in a FIFO order. If $SE = \emptyset$, then go to Step 4.

Step 3. Path extension
For every outgoing link $a$ of chosen node $i$ ($j$ denotes a successor node of $i$):

- Step 3.1. Generate a new label $p^0_{rj} \in P^0$. Set $\mu_{rj}^0 = \mu_{ri}^0 + \mu_a, (\sigma_{rj}^0)^2 = (\sigma_{ri}^0)^2 + \sigma_a^2$ and $u_{rj}^0 = \omega_1 u_{ri}^0 + z \sigma_a^0$.  

- Step 3.2. If $p^0_{rj} \in P^0$ is acyclic, then go to Step 3.3; otherwise, scan the next link.

- Step 3.3. If $p^0_{rj}$ is an M-GMV non-dominated path, insert $p^0_{rj}$ into $P^0$ and $SE$, and remove all paths M-GMV dominated by $p^0_{rj}$ from $P^0$ and $SE$.

End for.
Return to Step 2.

Step 4. Determine the GMV-based shortest path $p^0_{rs}$ and stop.

As illustrated in subsection 3.3, GMV is non-additive since the GMV of a path is not necessarily the sum of the GMVs of the associated links. To overcome this difficulty, a bi-criteria label-correcting method (Liao, Rasouli, and Timmermans 2014) is adopted to find a reliable path with the minimal GMV. The algorithm for solving the GMV-based shortest path problem is hereafter referred to as GMVSP, and the detailed steps are described above.

Accordingly, the GMV-based traffic assignment algorithm has two loops (Figure 5). The outer loop is for updating the path sets (left-hand side of Figure 5). For each outer loop iteration $n$, the GMVSP algorithm is adopted to generate GMV-based shortest paths for each OD pair and to update the path set using the column generation technique. Next, MSA is used to assign traffic flows on the updated paths, which resides in the inner loop to solve the GMVUE model.

A gap function is defined to measure the convergence of MSA:

$$\text{Gap}(f^\tau) = \frac{[u(f^\tau) - \pi(f^\tau)]^T f^\tau}{\pi(f^\tau)^T f^\tau}$$

where $\pi(f^\tau) = (\ldots, \pi^\tau(f^\tau), \ldots)$ is the minimal GMV of all OD pairs. Note that if the UE conditions are satisfied, the above gap function is less than a predefined convergence tolerance $\varepsilon$ ($\varepsilon > 0$). The detailed steps are given by the GMVUE algorithm.

In the solution algorithm, path extensions and flow assignments are the most time-consuming components for large-scale networks. For the path searches, the run-time complexity of GMVSP with the Fibonacci heap is $O(|A||P| + |N|\log(|N|))$, where $|A|$ and $|N|$ are the numbers of network links and nodes respectively, and $|P|$ is the maximum number of
non-dominated paths associated with a node. The value of $|P|$ is smaller using M-GMV dominance condition than the M-V dominance condition according to Proposition 4. Regarding the flow updating, it should be noted that MSA is performed twice flow assignments (Step 3.2 and 3.3) at each iteration.

**GMVUE algorithm:**
Determine the flow patterns at equilibrium

**Step 0. Initialization**
Given $\alpha$ and $\varepsilon$, set $n = 0$ and $f_n = 0$. For OD pair $r$-$s$, let $P_n^0 = \emptyset$ be the initial set of used paths.

**Step 1. (Column generation) Update path set for each OD pair**
For each OD pair $r$-$s$
Call GMVSP to search the minimal GMV path $p_n^1$. If $p_n^1 \notin P_n^0$, then set $P_n^0 = P_n^0 \cup \{p_n^1\}$, $f_n^1 = 0$ and $f_n = [f_n; f_n^1]$. End for.

**Step 2. (Stopping criterion) If $P_n^0 = P_{n-1}^0$, then stop; otherwise, continue.**

**Step 3. (MSA) Update path flow**

**Step 3.0. Initialization.** Set inner loop iteration index $\tau = 1$ and feasible path flow vector $f^\tau = f_n$.

**Step 3.1. Update the GMV vector $u^\tau(f^\tau)$ for each OD pair $r$-$s$**

**Step 3.2. Perform all-or-nothing assignment on the basis of path GMV $u^\tau(f^\tau)$, yielding auxiliary path flows $(\tilde{f}^\tau)_r^s$, for each OD pair $r$-$s$.**

**Step 3.3. For each OD pair $r$-$s$, calculate new path flows $(f^{\tau+1})_r^s = (f^\tau)_r^s + [(\tilde{f}^\tau)_r^s - (f^\tau)_r^s]/\tau$.**

**Step 3.4. Check the stopping criterion of MSA.** If $\text{Gap}(f^\tau) < \varepsilon$, set $f_n = f^\tau$ and $n = n + 1$, and go to Step 1; otherwise, set $\tau = \tau + 1$, and go to Step 3.1.

**Figure 5.** Flowchart of GMV-based traffic assignment algorithm.

5. Numerical examples

This section presents two numerical examples to illustrate GMVUE. The first example is adopted for illustrating detailed results. The second example presents the convergence results in a relatively large traffic network. The solution algorithm is run on a personal computer with an Intel(R) Core(TM) i7-6700 3.40 GHz CPU and 8.00 GB RAM. The link performance uses BPR function with $\beta = 1$, $\delta = 4$ on all the links. As a new special form of GMV, MLTT is focused on in this section and the corresponding weight coefficients equal $[1, -1/\alpha, 0, \gamma(\alpha)]$. Moreover, we set $\varepsilon = 10^{-5}$ and $\alpha = 0.9$ unless otherwise explained.
Figure 6. The test network.

5.1. Example 1: six-node network

The small-scale test network (Figure 6) is adopted from Shao, Lam, and Tam (2006), which has two OD pairs, six nodes, seven links, and four paths. The link number, capacity, and degradable parameter are shown near the links. The demands for OD pairs 1–3, 2–4 are 15 and 40 units respectively. Although the uniqueness of the solutions to the VI problem cannot be guaranteed, it is found the flow patterns at the equilibrium states are stable with multiple random start points. The steady state is reached after 0.19 s of computation time on average. Based on the above setting, Figure 7 shows the convergence curves of the MSA. As seen, the path MLTTs and flows fluctuate greatly at the first 15 iterations and converge to a steady state gradually. The fluctuations during the convergence course is a common issue in MSA applications (Carey and Ge 2012). At the steady state, travelers of OD pair 1–3 are concentrated on path 1 and disfavor path 2 (flow curve is coincident with the x-axis) due to a higher MLTT of path 2. For OD pair 2-4, paths 3 and 4 possess the same MLTT (25.07), and the traffic flows on both paths are 31.11 and 8.89 respectively. These results are consistent with the GMVUE conditions and constraints in Eqs. (13)-(15).

To illustrate the impact of on-time arrival probability on traffic flow, Figure 8 presents the equilibrium results of OD pair 1–3 under different OD demands and different $\alpha$, of which (a)-(b) and (c)-(d) correspond to 17 and 40 units of demand respectively (both are set arbitrarily for illustration purpose). As shown in Figure 8 (a), all travelers choose path 1 when $\alpha < 0.6$. If $\alpha$ is increased over 0.6, several flows on path 1 switch to path 2. When there are 40 units of demand, traffic flows are assigned to paths 1 and 2 under different $\alpha$. When $\alpha$ is relatively small (less than 0.4), there are more travelers choosing path 1 to avoid the penalty caused by travel delays. When $\alpha$ is larger than 0.4, the traffic flow on path 2 is greater than that on path 1. These curves depicted in Figure 8 are consistent with the GMVUE conditions.

MLTT, TTB, and METT, as three special cases of GMV, take both expected travel time and travel time variance into consideration. Figure 9 shows the different equilibrium flows on path 1 under these metrics when the demand of OD pair 1–3 is equal to 17 and 40 units respectively, where the left-hand side red dashed line denotes the total demand and the right-hand side denotes the half demand for reference purpose. Most travelers with high on-time arrival probability will choose paths with small mean and variance of path travel time when the demand is small. Since the mean and variance of travel time in path 1 are smaller than those of path 2, travel flows concentrate on path 1 under MLTT, TTB, and METT. When the demand increases to 40, traffic congestion occurs. Some travelers on path 1 shift...
Figure 7. Convergence curves of the path flows and GMVs. (a) Path flow and (b) Path MLTT.

to path 2 to avoid high penalties due to congestion. Fewer travelers choose path 1 under these criteria. The results are consistent with Eq. (2) and (3), which indicates that traffic congestion results in large mean and variance of path travel time.

5.2. Example 2: Anaheim network

This example uses a real network in the City of Anaheim (USA) to illustrate the effectiveness and present the sensitivity analysis of the proposed GMV-based traffic assignment algorithm. This network consists of 416 nodes, 914 links, and 1406 OD pairs. The network topology, link capacities, free-flow travel times, and original OD demands are obtained from http://www.bgu.ac.il/bargera/tntp/. The demands are enlarged two times from the original demands to produce congestion effects. To analyze the uncertainty of this network, $\theta_l$ is obtained by linear projecting the length of link $l$ to the interval [0.5, 0.9].

After new paths are generated, considerable flows shift from the existing paths to the new ones at the first a few inner iterations due to small $\tau$. Hence, we exclude the first five
Figure 8. Equilibrium results under different on-time arrival probabilities. (a) path flow under 17 units of demand (b) path MLTT under 17 units of demand (c) path flow under 40 units of demand and (d) path MLTT under 40 units of demand.

Figure 9. Travel flows on path 1 of OD pair 1–3 under different route choice metrics.
inner iterations after new paths are generated, and the convergence curve is depicted in Figure 10. As shown, the curve consists of several fluctuations. When the result of the current outer iteration (see Figure 5) approaches the equilibrium solution, new paths have similar MLTTs as those used paths. As shown, fewer iterations are needed to achieve the equilibrium state. This conclusion is demonstrated by the decreasing distances between two adjacent peaks. Overall, the gap decreases to a small value within a few iterations and then move downwards slowly due to the nature of MSA.

To compare the route choice outcomes of MLTT, TTB, and METT, we first sort the values of MLTT ascendingly and show the MLTT values of 1000 OD pairs in red (Figure 11 (a)). Correspondingly, we show the values of TTB and METT of the same OD pairs in orange and blue respectively. Regarding the flow comparison, we find the common paths of each OD pair under different route choice metrics and calculate the proportions of the path flows to corresponding OD demands. For a clear presentation, the travel flow proportions of only 100 OD pairs are depicted in Figure 11 (b). It can be seen that at the equilibrium state, the travel costs and flows of the three metrics are different across OD pairs. Note that TTB is bounded by MLTT and METT despite the fluctuations of TTB and METT in Figure 11 (a). This result is consistent with the definitions of the metrics.

For further comparisons, the distributions of the numbers of paths per OD pair under different criteria are plotted in Figure 12. It shows that the number of paths of the OD pairs is primarily concentrated in the first four groups, with very few OD pairs having more than seven paths at equilibrium. In particular, with the MLTT criterion, about 65% of OD pairs use no more than three paths, while the percentage is around 55% for TTB. Although the numbers of the identified paths may also be affected by the congestion level, the results indicate that the column generation scheme takes effects for identifying the relevant paths for traffic assignment. Also, the analyses further confirm that different criteria result in different route choice results.

The degree of supply uncertainty could be reduced by intelligent transportation system applications, for example, emerging traffic management measures and operations strategies. An example is that the lower bounds of the uniform distributions of capacities are increased due to the deployment of connected vehicles. To demonstrate the influence of these changes on travel time reliability (TTR), $\alpha$ is adopted as the criterion to quantify TTR.
and set as 0.6 initially, and then $\theta_l(\forall l)$ is increased by 1%, 5% and 10% respectively. Figure 13 shows the on-time probability $\alpha$ of the paths after the capacity improvement. As shown, the values of $\alpha$ of most paths increase, indicating that the TTR of travelers is improved. Moreover, the rightward shift of the histograms illustrates that TTR gains more improvement with a higher $\theta_l$. Taking Figure 13 (b) for example, although $\theta_l$ increases by only 5%, the improvement of TTR is significant: 81.8% of the paths have TTR within the range [0.7, 0.9]. It is also found that there are a few paths with TTR lower than 0.6. This outcome is caused by the fact that the generated paths for a minority of OD pairs are different and also
have different means and variances. Based on the numerical results, we can conclude that capacity improvements in the transport network improve TTR.

6. Conclusions and future research

This paper proposed a generalized mean-variance (GMV) metric for route choice under travel time uncertainty and developed a GMV-based user equilibrium model (GMVUE) in a transport network. Instead of focusing solely on expected travel time, the GMVUE model is capable of factoring travel time variance, early arrival, or late arrival into route choice considerations. As illustrated, GMV has a more generalized form than several currently widely used metrics, such as MTT, TTB, and METT. This paper also analyzed some properties of GMV, including continuity and non-additivity. Continuity is satisfied without the assumption of normal and independent distributions. To overcome non-additivity, GMV-based dominance definitions and conditions were established and used to search the reliable shortest paths. The GMVUE model was formulated as a variational inequality (VI) problem. The existence and uniqueness of the solutions to the VI problem were also discussed.
With the incorporation of a bi-criteria label-correcting algorithm, MSA, and column generation technique, an effective traffic assignment algorithm without path enumeration was developed to solve the GMVUE model for real networks. As illustrated in the numerical examples, different weight coefficients result in different GMV forms and traffic flow assignment schemes.

On the basis of GMVUE, several extensions are worthy of exploring in future studies. First, path travel times in this study were assumed to follow normal distributions. However, this assumption is unlikely or even logically impossible to hold in reality. Many other travel time distributions, such as the lognormal or truncated normal distribution, should be investigated for considering the asymmetry of the distributions. Second, this study only incorporated the link capacity degradation into consideration for travel time reliability. However, the uncertainty of travel times exists on both the supply and demand sides. Thus, GMVUE should also be extended by considering demand and supply fluctuations simultaneously. Lastly, route choice with the GMV metric can be applied for multimodal mobility, emerging mobility services (Li et al. 2018), and also the combination with bounded rationality (Wang et al. 2019). We will address these issues in our future work.

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References


Appendices

Appendix 1. Proof of Remark

**Proof:** While the correctness of Remark (i–ii) is obvious, Remark (iii) and (iv) need some manipulations to prove. Recalling the definition of GMV, we obtain

\[
u^*_{P} = \mu_p + \frac{1}{1-\alpha} E(T_P - \xi_p(\alpha))^+ + \gamma(\alpha)\sigma_P \quad (A.1)
\]

Based on the definition of TTB in Eq. (7), Eq. (A.1) is rewritten as

\[
u^*_{P} = \xi_p(\alpha) + \frac{1}{1-\alpha} E(T_P - \xi_p(\alpha))^+ \quad (A.2)
\]

The second term of the right-hand side of Eq. (A.2) can be derived as follows

\[
11 - \alpha E(T_P - \xi_p(\alpha))^+ = 11 - \alpha \int_{\xi_p(\alpha)}^{b_p} (T - \xi_p(\alpha)) \ast g(T_P) dT_P
\]

\[
= \int_{\xi_p(\alpha)}^{b_p} (T - \xi_p(\alpha)) \ast g(T_P) Pr(T_P \geq \xi_p(\alpha)) dT_P
\]

\[
= E(T_P - \xi_p(\alpha) | T_P \geq \xi_p(\alpha)) \quad (A.3)
\]

where \(g(T_P)\) denotes the probability density function of the \(T_P\). Substituting Eq. (A.3) into Eq. (A.2), we obtain

\[
u^*_{P} = \xi_p(\alpha) + E(T_P - \xi_p(\alpha) | T_P \geq \xi_p(\alpha)) = E(T_P | T_P \geq \xi_p(\alpha)) \quad (A.4)
\]

Eq. (A.4) is consistent with the definition of METT proposed by Chen and Zhou (2010). The proof of remark (iv) is similar and thus omitted here.

Appendix 2. Proof of Proposition 1

**Proof of continuity:** Path travel time \(T_P\) is a random variable, so is the late schedule delay \((T_P - \xi_p(\alpha))^+\) denoted by \(\tilde{T}_P\) given by

\[
\tilde{T}_P = (T_P - \xi_p(\alpha))^+ = \begin{cases} 
0, & T_P < \xi_p(\alpha) \\
T_P - \xi_p(\alpha), & T_P \geq \xi_p(\alpha)
\end{cases} \quad (A.5)
\]

As \(T_P\) follows a normal distribution, the CDF of \(\tilde{T}_P\) is given by

\[
Pr(\tilde{T}_P \leq y) = \begin{cases} 
0, & y < 0 \\
G(y + \xi_p), & y \geq 0
\end{cases}
\]

where \(\xi_p(\alpha)\) is abbreviated as \(\xi_p\) for convenience. The expectation of \(\tilde{T}_P\) is calculated by

\[
E(\tilde{T}_P) = 0 \ast Pr(\tilde{T}_P = 0) + \int_{\xi_p}^{+\infty} (T_P - \xi_p) \ast g(T_P) dT_P = \int_{\xi_p}^{+\infty} (T_P - \xi_p) \ast g(T_P) dT_P \quad (A.6)
\]

where \(g(T_P)\) denotes the probability density function of \(T_P\). Similarly, we can get the expectation of early schedule delay below

\[
E(T_P - \xi_p(\alpha))^- = \int_{-\infty}^{\xi_p} (\xi_p - T_P) \ast g(T_P) dT_P \quad (A.8)
\]
According to Eq. (12), we have

\[
\upsilon_p^{\text{rs}} = \omega_1 \mu_p + \omega_2(\alpha) \int_{-\infty}^{\xi_p} (\xi_p - T_p) \cdot g(T_p) \, dT_p + \omega_3(\alpha) \int_{\xi_p}^{+\infty} (T_p - \xi_p) \cdot g(T_p) \, dT_p + \omega_1 \gamma(\alpha) \sigma_p
\]

\[
= \omega_1 \xi_p + \omega_2(\alpha) \alpha \xi_p - \omega_2(\alpha) \int_{-\infty}^{\xi_p} T_p \cdot g(T_p) \, dT_p + \omega_3(\alpha) \int_{\xi_p}^{+\infty} T_p \cdot g(T_p) \, dT_p - \omega_3(\alpha)(1 - \alpha) \xi_p
\]

(A.9)

The second line of Eq. (A.9) starts with the definition of TTB as shown in Eq. (7). Through integral manipulations, the fourth term on the right-hand side of Eq. (A.9) is rewritten as

\[
\omega_3(\alpha) \int_{\xi_p}^{+\infty} T_p \cdot g(T_p) \, dT_p = \omega_3(\alpha) \mu_p (1 - \alpha) + \frac{\omega_3(\alpha) \sigma_p}{\sqrt{2\pi}} e^{-\left(\frac{\xi_p - \mu_p}{\sigma_p}\right)^2}
\]

(A.10)

Similarly, the third term on the right-hand side of Eq. (A.9) is represented as

\[
- \omega_2(\alpha) \int_{-\infty}^{\xi_p} T_p \cdot g(T_p) \, dT_p = - \omega_2(\alpha) \mu_p \alpha + \frac{\omega_2(\alpha) \sigma_p}{\sqrt{2\pi}} e^{-\left(\frac{\xi_p - \mu_p}{\sigma_p}\right)^2}
\]

(A.11)

By combining Eqs. (A.9)–(A.11), \( \upsilon_p^{\text{rs}} \) is reduced to

\[
\upsilon_p^{\text{rs}} = \omega_1 \xi_p + \omega_2(\alpha) \alpha (\xi_p - \mu_p) + \omega_3(\alpha)(1 - \alpha)(\mu_p - \xi_p) + \frac{(\omega_2(\alpha) + \omega_3(\alpha)) \sigma_p}{\sqrt{2\pi}} e^{-\left(\frac{\xi_p - \mu_p}{\sigma_p}\right)^2}
\]

(A.12)

Substituting Eq. (7) into Eq. (A.12), we obtain

\[
\upsilon_p^{\text{rs}} = \omega_1 [\mu_p + \gamma(\alpha) \sigma_p] + \omega_2(\alpha) \alpha \gamma(\alpha) \sigma_p - \omega_3(\alpha)(1 - \alpha) \gamma(\alpha) \sigma_p + \frac{(\omega_2(\alpha) + \omega_3(\alpha)) \sigma_p}{\sqrt{2\pi}} e^{-\left(\frac{\xi_p - \mu_p}{\sigma_p}\right)^2}
\]

(A.13)

Recalling the definition of \( \gamma(\alpha) \) shown in Eq. (9), \( \gamma(\alpha) \) is continuous with \( \alpha \); thus, the GMV \( \upsilon_p^{\text{rs}} \) calculated by Eq. (A.13) is also continuous with \( \alpha \).

**Proof of monotonicity:** Based on the relation between \( \gamma(\alpha) \) and \( \alpha \) shown in Eq. (9), \( \alpha \) is represented as

\[
\alpha = \Phi(\gamma(\alpha)) = \int_{-\infty}^{\gamma(\alpha)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx
\]

(A.14)

For TTB, the first order of derivatives of \( \upsilon_p^{\text{rs}} \) with respect to \( \gamma(\alpha) \) are

\[
(\upsilon_p^{\text{rs}})' = \sigma_p 
\]

(A.15)

Recalling remark (iii) and (iv) in subsection 3.1, coefficients \( \omega_2(\alpha) \) and \( \omega_3(\alpha) \) of GMV are \( \alpha \)-related. The mathematical form of the first order derivative of METT and MLTT with respect to \( \gamma(\alpha) \) can be derived from the following equations respectively.

\[
(\upsilon_p^{\text{rs}})' = \frac{\sigma_p \cdot \Phi'(\gamma(\alpha))}{\sqrt{2\pi} [1 - \Phi(\gamma(\alpha))]^2} \int_{\gamma(\alpha)}^{+\infty} (x - \gamma(\alpha)) e^{-\frac{x^2}{2}} \, dx
\]

(A.16)

\[
(\upsilon_p^{\text{rs}})' = \frac{\sigma_p \cdot \Phi'(\gamma(\alpha)) + \Phi(\gamma(\alpha))}{\sqrt{2\pi} \cdot \Phi'(\gamma(\alpha))} e^{-\frac{\gamma^2(\alpha)}{2}}
\]

(A.17)

It is obvious that the values of Eqs. (A.15)–(A.17) are positive. Therefore, TTB, METT, and MLTT are monotonically increasing with \( \gamma(\alpha) \). As \( \gamma(\alpha) \) is monotonically increasing with \( \alpha \), it is concluded that TTB, METT, and MLTT are monotonically increasing with \( \alpha \).
Appendix 3. Proof of Proposition 2 and Proposition 3

Proof of Proposition 2: Let $p_1^r = p_1^r \oplus p_2^r$ and $p_2^r = p_2^r \oplus p_2^r$ be two paths from node $r$ to $j$ with the same sub-path $p_1^r$, the deviation between $u_1^r$ and $u_2^r$ is

$$f_1^r(p_1^r) = u_1^r - u_2^r = \omega_1 (\mu_1^r - \mu_2^r) + \frac{z}{2} \left( \sqrt{\frac{\omega_1^r}{2} + \frac{\omega_1^r}{2}} - \sqrt{\frac{\omega_1^r}{2} + \frac{\omega_1^r}{2}} \right) \quad (A.18)$$

The derivative of the above equation with respect to $(\sigma_p)^2$ can be formulated as follows

$$(f_1^r)'(p_1^r) = \frac{df_1^r(p_1^r)}{d(\sigma_p)^2} = \frac{2}{z} \frac{1}{1 + (\sigma_p)^2} \quad (A.19)$$

When $\sigma_1^r < \sigma_2^r$, we have $(f_1^r)'(p_1^r) > 0$. Note that $(\sigma_p)^2 \in (0, +\infty)$, thus $f_1^r(p_1^r) < \mu_1^r - \mu_2^r \leq 0$. The last inequality is followed by $\mu_1^r \leq \mu_2^r$. This completes the proof of case (a) according to Definition 1, and case (b) can be concluded similarly.

Proof of Proposition 3: When $z\sigma_1^r < z\sigma_2^r$ and $\mu_1^r \leq \mu_2^r$, we have $(f_1^r)'(p_1^r) > 0$ according to Eq. (A.19). Thus, $f_1^r(p_1^r) < \omega_1 (\mu_1^r - \mu_2^r) \leq 0$. When $z\sigma_1^r > z\sigma_2^r$ and $u_1^r < u_2^r$, it can be obtained that $f_1^r(p_1^r) < 0$. Thus, $f_1^r(p_1^r) < u_1^r - u_2^r < 0$ when $z\sigma_1^r = z\sigma_2^r$ and $u_1^r < u_2^r$, we have $f_1^r(p_1^r) = 0$. Thus, $f_1^r(p_2^r) = u_1^r - u_2^r < 0$. Therefore $p_1^r > p_2^r$ if $p_1^r$ and $p_2^r$ satisfy $\mu_1^r \leq \mu_2^r$ and $u_1^r < u_2^r$.

Appendix 4. Proof of Proposition 6

Proof: Given $\alpha$ and $v_l$, the value of TTB $\xi_p(\alpha)$ is a constant, which can be calculated by Eqs. (1)–(5), (7). According to Eq. (A.12), the GMV can be simplified as

$$u_p^s = \omega_1 \xi_p + \omega_2 (\alpha)(\xi_p - \mu_p) + \omega_3 (\alpha)(1 - \alpha)(\mu_p - \xi_p) + \frac{(\omega_2(\alpha) + \omega_3(\alpha))\sigma_p}{\sqrt{2\pi}} e^{-\frac{(v_p - \psi_p)^2}{\sqrt{2\pi}} \sigma_p^2} \quad (A.20)$$

For any path $p$, the continuity of $u_p^s$ is conditional on the continuity of $\xi_p$, $\mu_p$, and $\sigma_p$ because the weight coefficients and on-time arrival probability are constants. Referring to Eqs. (2)–(3), it is obvious that $\mu_1$ and $\sigma_1$ are continuous with link traffic flows $v_l$. Hence, $\mu_p$ and $\sigma_p$ are continuous (Eqs. (4), (5)). In addition, as shown in Eq. (7), TTB is continuous since it is a weighted sum of $\mu_p$ and $\sigma_p$.

Let $\epsilon$ be the smallest possible positive real number, and we set the link traffic flows as $v_l = \max\{\epsilon, v_l\}$ for calculating GMV. This modification ensures $v_l > 0$, and therefore a positive $\sigma_p$, as shown in Eqs. (3), (5). Moreover, $\sigma_p$ hardly changes if $\epsilon$ is sufficiently small. With this pretreatment, the denominator in Eq. (A.20) is larger than zero.

Therefore, GMV is continuous with the link traffic flows.