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Bayesian Reliability Analysis with Imprecise Prior Probabilities

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Abstract

The Bayesian framework for statistical inference offers the possibility of taking expert opinions into account, and is therefore attractive in practical problems concerning reliability of technical systems. Probability is the only language in which uncertainty can be consistently expressed, and this requires the use of prior distributions for reporting expert opinions. In this paper an extension of the standard Bayesian approach based on the theory of imprecise probabilities and intervals of measures is developed. It is shown that this is necessary to take the nature of experts knowledge into account. The application of this approach in reliability theory is outlined.

The concept of imprecise probabilities allows us to accept a range of possible probabilities from an expert for events of interest and thus makes the elicitation of prior information simpler and clearer. The method also provides a consistent way for combining the opinions of several experts.

Key Words Bayesian inference, reliability theory, imprecise probabilities, intervals of measures, lower and upper envelopes, information, expert opinion, elicitation and combination.

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1. Introduction

Reliability analysis is concerned with the ability of a system to function satisfactorily under a given set of circumstances, and thus gives important information about the quality of a system, when to replace it, or when and how to maintain it. Many probabilistic models for the reliability of components and systems are available and can be used without too much difficulty whenever the parameters of the system are given or are deducible from the systems properties. The situation is more difficult when parameters must be estimated from data. The use of models is often limited by a lack of satisfactory data on which estimates of system parameters can be based. The reasons for the lack of data are numerous, sometimes there is no effective registration system, sometimes, the data are corrupted, and changes in design or operating conditions mean that historical data are no longer directly relevant.

The Bayesian approach to statistical inference (Lindley) provides a framework for dealing with expert opinions and prior knowledge together with historical or experimental data. Bayesian methods therefore offer solutions to some of the difficulties in reliability analysis (Martz and Waller; Sander and Badoux). However, there are also difficulties in using probability as language to express expert opinions (Walley). The problem has two aspects, the elicitation of expert knowledge, and the expression of that knowledge in the form of a prior distribution. Within the standard probability framework combined with the Bayesian approach it is difficult to deal with the uncertainty of an expert who says, for example "I think the chance of failure is between 25% and 35%". To deal with this uncertainty we can resort to hierarchical models, but they simply push the problem to another level. Sensi-
tivity analyses can also be used to examine the effect of the prior on the posterior and predictive distributions. In situations where there is plenty of good quality data available these options are likely to yield reasonable results because the evidence from the data weighs so much more heavily than the prior evidence.

A further difficulty arises when precise knowledge about some possible events exists besides ignorance about other events, this situation demands a prior exactly specified in only a part of the sample space.

A second area of difficulty is that of combining expert opinions and knowledge for use in decision making. The Kolmogorov axioms require exact probabilities, and do not provide a basis for reporting agreement or disagreement between experts. Within the Bayesian framework this can be solved by constructing the natural Bayesian weights or by using a hierarchical model in which the decision maker also has a prior for the experts (Cooke, Coolen, Mertens and Newby). However, again within the Bayesian framework at some level in the model a specified prior must be chosen.

Thus a calculus for consistent decision making is required within which the decision maker or expert does not need to enter the realm of probabilities in those cases where probability remains a mystery. Moreover, a calculus is required which is not restricted by the fact that most people cannot give a consistent estimation of a set of probabilities.
In section 2 of this paper, the concept of imprecise probabilities is briefly discussed in the context of reliability theory. In section 3 Bayesian analysis with imprecision, the elicitation and combination of experts' opinions, are discussed, and in section 4 two examples are given of application of the theory to reliability problems. Finally in section 5 the conclusions of this paper are summarized and some additional remarks are made.

2. Imprecise Probabilities and Reliability

In this section we show how the definitions of the measures of interest in reliability analysis can be extended in a consistent way using both imprecise probabilities expressed as imprecise prior distributions in a Bayesian analysis.

The Bayesian approach to statistical inference demands a prior distribution for the parameters of interest. The prior represents the information available before actual data come available. Walley\(^4\) presents a clear and extensive discussion about the major difficulties involved in indicating the amount of information, or lack of information, on which a prior distribution is based. When the prior information consists of opinions of several experts, it is important for the decision maker to report the level of disagreement among the experts. This is not possible with a single distribution.
To deal with the above shortcomings the concept of imprecise probabilities is introduced (Walley\textsuperscript{4}), where instead of one value for the probability $P(A)$ of an event $A$, an interval of values is chosen, bounded by lower and upper probabilities for $A$. This interval can be interpreted as a set of possible values for $P(A)$, between which no further distinction can be made given the information available. This implies that these bounds depend on you and your information, and therefore have a subjective nature.

This idea suggests that small intervals arise when there is plenty of relevant information, and larger intervals when there is a lack of information. Total lack of information about the probability of an event can be represented by assigning the values zero and one to the lower and upper probabilities for this event. Walley\textsuperscript{4} uses an interpretation related to the subjective nature of probability as advocated by De Finetti\textsuperscript{7}. In standard Bayesian theory the lower and upper probabilities for $A$ are equal. In this situation the probability for event $A$ is called precise.

Boole\textsuperscript{8} introduced an idea of imprecise probabilities, but little attention was paid to it for more than a century. More recently, Smith\textsuperscript{9} and Good\textsuperscript{10} contributed to the theory of imprecise probabilities; Dempster’s\textsuperscript{11} approach is also interesting, but can lead to unreasonable results in some cases (Walley\textsuperscript{4}; section 5.13). Walley and Fine\textsuperscript{12} and Wolfenson and Fine\textsuperscript{13} derive important results for the theory of imprecise probabilities, and Walley and Fine\textsuperscript{14} propose a generalization of the frequentist theory to include imprecision. When the imprecise probabilities refer to parameters the method of intervals of measures (DeRobertis and Hartigan\textsuperscript{15}) is especially useful, as in the Bayesian framework (Pericchi and Walley\textsuperscript{16}).
Although many authors have contributed to the theory of imprecise probabilities during the last decades, statistical models based on imprecision that can be applied easily in practice have received little attention. Adopting imprecise probabilities within the Bayesian framework offers the possibility of dealing with different levels of ignorance. The theory proposed by Berger\textsuperscript{17} (also Berger and Berliner\textsuperscript{18}) for robust Bayesian inference is similar to that discussed here, but there imprecision enters through sensitivity analyses, and not as a means of reporting important information itself.

The theory in this paper is fundamentally different to sensitivity analysis (Walley\textsuperscript{4}; section 5.9), and moreover different to the method of Meczarski and Zielinski\textsuperscript{19} who restrict their argument to a set precise prior distributions belonging to a parametric family. A review of theories for handling uncertainty is given by Walley\textsuperscript{20} who compares the theory discussed here with other theories for measuring uncertainty in expert systems, e.g. fuzzy sets.

For an event \( A \subseteq \Omega \) (\( \Omega \) the sample space), the lower and upper probabilities, \( \underline{\mathbb{P}}(A) \) and \( \overline{\mathbb{P}}(A) \) respectively, satisfy the basic axioms given by Smith\textsuperscript{9} (Wolfenson and Fine\textsuperscript{13}). Special cases of imprecise probabilities are lower and upper cumulative distribution functions (cdf) \( \underline{F}(x) \) and \( \overline{F}(x) \) for a real random variable \( X \). These are the lower and upper probabilities of the events \( X \leq x \) for \( x \in \mathbb{R} \), defined by \( \underline{F}(x) = \underline{\mathbb{P}}(X \leq x) \) and \( \overline{F}(x) = \overline{\mathbb{P}}(X \leq x) \).

Following DeRobertis and Hartigan\textsuperscript{15}, suppose there exist functions \( \ell(x) \) and \( u(x) \), \( 0 \leq \ell(x) \leq u(x) \) for all \( x \), such that all functions \( h \), \( \ell \leq h \leq u \), can, after normalization, be regarded as probability density functions (pdf) for \( X \). The functions \( \ell \) and \( u \) are called lower and upper density functions, and in this paper are assumed to be continuous and with non-zero and finite
integrals. If lower and upper densities are given, then the lower and upper cdf's are (for $x \in \mathbb{R}$)

$$\mathbb{F}(x) := \left( 1 + \int_{-\infty}^{x} \frac{u(\omega) \, d\omega}{t(\omega) \, d\omega} \right)^{-1} \quad \text{and} \quad \mathbb{F}(x) := \left( 1 + \int_{-\infty}^{x} \frac{\ell(\omega) \, d\omega}{u(\omega) \, d\omega} \right)^{-1},$$

which provide lower and upper envelopes (the lower and upper bounds with smallest distance of all lower and upper bounds) of all cdf's that can be constructed from densities (after normalization) that lie between $\ell$ and $u$ (Walley 4; section 4.6). The continuity of $\ell$ and $u$ ensures the continuity of these lower and upper cdf's. In the rest we use the term density to mean a function lying between $\ell$ and $u$ which after normalisation yields a probability density function (pdf).

The imprecision for an event $A$ is defined by $\Delta(A) = \mathbb{F}(A) - \mathbb{P}(A)$, so $0 \leq \Delta(A) \leq 1$. The extreme values, 0 and 1, occur only in the case of precision ($\mathbb{P}(A) = \mathbb{F}(A)$) or vacuous probabilities ($\mathbb{P}(A) = 0$ and $\mathbb{F}(A) = 1$). A simple measure for the amount of information about $A$ in $\mathbb{P}$ and $\mathbb{F}$ is: $I(A) = \Delta(A)^{-1} - 1$ (Walley 4; section 5.3.7). This is zero if and only if the probabilities are vacuous, and is infinite if and only if the probability for $A$ is precise.

In reliability theory, the reliability function, hazard rate and hazard function are important characterisations of probability distributions, with the hazard rate providing a natural description of a failure process. We define lower and upper hazard rates and hazard functions within the theory of imprecise probabilities. We restrict the discussion to a non-negative real random variable $\mathbb{T} \geq 0$ (time).
Definitions

Suppose continuous lower and upper densities \( l \) and \( u \) are specified for \( T \). The corresponding imprecise cdf's \( \underline{F} \) and \( \overline{F} \) are given by (1). Obvious definitions of the lower and upper Reliability functions are (\( t \geq 0 \))

\[
\underline{R}(t) := 1 - \underline{F}(t) \quad \text{and} \quad \overline{R}(t) := 1 - \overline{F}(t).
\] (2)

The lower and upper hazard rates are defined by

\[
\underline{h}(t) := \frac{\ell(t)}{\int_t^\infty u(x) \, dx} \quad \text{and} \quad \overline{h}(t) := \frac{u(t)}{\int_t^\infty \ell(x) \, dx}.
\] (3)

These imprecise hazard rates are the tightest lower and upper envelopes of the set of all hazard rates of pdf's that are derived from normalization of densities between \( l \) and \( u \), which is easily proved by taking a function \( q \) with \( \ell(t) \leq q(t) \leq u(t) \) for all \( t \geq 0 \) and \( C_q := \int_0^\infty q(x) \, dx \) the normalizing constant. The hazard rate corresponding to pdf \( q(t)/C_q \) is

\[
\underline{h}_q(t) = \frac{q(t)/C_q}{\int_t^\infty q(x)/C_q \, dx} = \frac{q(t)}{\int_t^\infty q(x) \, dx},
\]

and the fact that \( l \) and \( u \) are the lower and upper envelopes of all such \( q \) makes clear that the above defined \( \underline{h} \) and \( \overline{h} \) are the lower and upper envelopes of all such \( \underline{h}_q \).

The lower and upper Hazard functions are defined by

\[
\underline{H}(t) := -\ln(1-\underline{F}(t)) \quad \text{and} \quad \overline{H}(t) := -\ln(1-\overline{F}(t)).
\] (4)
It is again obvious that these are the lower and upper envelopes of all possible hazard functions relating to possible pdf's \( q(t)/C_q \) as above, since the hazard function for such a pdf is \( H_q(t) = -\ln(1-F_q(t)) \), with \( F_q \) the cdf corresponding to pdf \( q(t)/C_q \) and \( F \) and \( \bar{F} \) the lower and upper envelopes of all such \( F_q \).

Remark that \( h \) and \( H \) are not the first derivatives of \( H \) and \( H \), in fact \( h(t) \leq H'(t) \) and \( H(t) \geq H'(t) \) for all \( t \geq 0 \), with equality iff \( \ell(t) = u(t) \).

3. Bayesian Analysis and Imprecision

We concern ourselves here with a situation where the probabilistic model for the failure mechanism is known, and moreover is described by a failure pdf \( f_x(x|\theta) \). The form of the density is assumed to be known precisely, and the information that we obtain as prior opinions and as data enters the model through \( \theta \). Thus our model is \( f_x(x|\theta) \) and on the basis of data and prior opinion about \( \theta \) we wish to make probability statements about \( x \). Our prior knowledge is expressed imprecisely through imprecise lower and upper prior densities, \( \ell(\theta) \) and \( u(\theta) \), for \( \theta \). For simplicity, \( \theta \) and \( x \) are assumed to be real numbers, but the restriction is not necessary.

Given a vector \( x = (x_1, \ldots, x_n) \) of \( n \) independent observations, the likelihood function \( L(\theta|x) \) is determined and because \( f_x \) is of fixed form, so is the likelihood. Thus the updated versions of \( \ell(\theta) \) and \( u(\theta) \) are (Walley; section 4.6) \( \ell(\theta|x) := L(\theta|x)\ell(\theta) \) and \( u(\theta|x) := L(\theta|x)u(\theta) \), bounding (up to
a constant factor) all standard Bayesian posteriors related to prior pdf's $p$, that are normalized versions of functions between $l$ and $u$. It can be proved straightforwardly that the set of all pdf's that can be constructed by normalizing functions between these updated $l$ and $u$, is equal to the set of all posterior pdf's resulting from priors that follow by normalizing functions between the prior $l$ and $u$. The corresponding lower and upper posterior cdf's, $\theta(x)$ and $\bar{\theta}(x)$, are derived as in (1).

Consistent definitions (in the sense that the above $\ell(\theta|x)$ and $u(\theta|x)$ are consistent definitions of the posterior imprecise densities) of lower and upper predictive densities are:

$$\ell_x(x) := \int_{-\infty}^{\infty} f_x(x|\theta) \ell(\theta) d\theta \quad \text{and} \quad u_x(x) := \int_{-\infty}^{\infty} f_x(x|\theta) u(\theta) d\theta.$$  \hfill (5)

Again these are not pdf's. Corresponding lower and upper predictive cdf's for $X$ are derived as in (1). The predictive densities are updated by replacing $\ell(\theta)$ and $u(\theta)$ by $\ell(\theta|x)$ and $u(\theta|x)$.

In the classical Bayesian framework the essential concept is coherence (Lindley). An analogous role is played by the generalized coherence within this new framework, according to Walley (section 2.5). Coherence of the imprecise cdf's follows from the fact that they are non-decreasing functions with $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$ for all $\theta$ (analogously for the other defined cdf's, Walley; section 4.6).

For practical application of the theory statistical models are needed. In the first place guidelines for elicitation and combination of expert opinions are required, together with methods for assessment of imprecise prior
densities based on such subjective information (Walley\textsuperscript{4}; chapter 4; Coolen\textsuperscript{21}). Secondly, the idea of conjugate priors can be extended. In the usual approach conjugate priors simplify the description of the updating, and their parameters often also correspond to fictitious data. Coolen\textsuperscript{22,23} has proposed imprecise conjugate prior densities to allow these benefits to be incorporated in the approach through imprecise priors. In chapter 4 two examples using imprecise conjugate priors will be given. When these extensions are used in Bayesian decision theory, the expected loss for a particular decision is also imprecise. The standard Bayesian model with a precise prior yields a single value for the expected loss, in the present case one seems to have the problem of having to calculate all possible losses corresponding to all possible priors for each decision. Coolen\textsuperscript{24} shows that few calculations are required to derive the smallest interval containing all possible values of the expected loss corresponding to a decision. To determine an optimal decision the intervals for the expected losses have to be compared with the help of an extra criterion, for example a minimax rule. Thus while there is a little more to be done, the technique avoids an unsatisfactory aspect of classical Bayesian decision theory in which an expected loss corresponding to a possible decision can be calculated exactly, while the prior is based on little information (Walley\textsuperscript{4}; section 5.6).

Central to the theory is the concept of imprecision which relates to the value attributed to the available information on which subjective probability statements about an event are based. The imprecision changes as new data update the prior and predictive distributions. These changes in imprecision need to be considered and described by models which are reasonably in accord with intuition. Coolen\textsuperscript{25} in an analysis of Bernoulli experiments shows that quite simple models can easily respond correctly to cases as prior-data con-
flict (new data contradict prior information, Walley; section 5.4). Most statistical theories, Bayesian included, cannot deal with prior-data conflict. Also from this point of view the generalization of the classical precise concept of probabilities is necessary to deal correctly with all sorts of information.

4. Examples

We illustrate the possibilities of the concept by two simple examples that are of interest in the theory of reliability.

Example 1

In a reliability test the reliability is to be estimated by testing 10 items for a predetermined time $t$. The reliability at time $t$ is then $R(t)$ and we have a binomial situation with $n=10$ and $\pi=R(t)$. Thus our statements about $R(t)$ are based on statements about the parameter $\pi$ in the binomial distribution $\text{Bin}(10,\pi)$ (Coolen). Statements about future values of the number of failures in a time period $t$ are based on predictive distributions. In an experiment we observe that $x$ items survive the test, and $10-x$ have failed, thus the probability mass function for the number of failed items is just

$$P_X(X=x|\pi) = \binom{10}{x} \pi^x (1-\pi)^{10-x}$$

for $x\in\{0,1,\ldots,10\}$ and $0\leq \pi \leq 1$. 

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In analysing data derived from such a reliability test suppose that the imprecise densities for $\pi$ are derived from the beta family (Coolen\textsuperscript{21}):

$$\ell_\pi(p) = p(1-p)^{7}/B(2,3)$$

and

$$u_\pi(p) = \ell_\pi(p) + c_0 p^3 (1-p)^{5}/B(4,6), \text{ for } 0 \leq p \leq 1,$$

where we suppose $c_0 = 1$. If data $(n,\alpha)$ become available, where $n$ is the total number of data (assumed to be independent) and $\alpha$ the number of successes, then the likelihood is

$$L_X(X=\alpha|\pi) = \binom{n}{\alpha} \pi^\alpha (1-\pi)^{n-\alpha}$$

and the imprecise posterior densities for $\pi$ are:

$$\ell_\pi(p|n,\alpha) = p^{1+\alpha} (1-p)^{7+n-\alpha}/B(2,3) \quad \text{and}$$

$$u_\pi(p|n,\alpha) = \ell_\pi(p|n,\alpha) + c_n p^{3+\alpha} (1-p)^{5+n-\alpha}/B(4,6), \text{ with } c_n = (1+n/10)^{-1}.$$

This model is such that, according to Walley's measure of information, the value of prior subjective information to the decision maker is equivalent to the value of 10 observations. Table 1 and figures 1-6 (in the appendix) give the imprecise predictive cdf's and probabilities of $X$, corresponding to the imprecise priors and posteriors for two cases. The first posterior is based on data $(10,3)$, that seem to be reasonably in agreement with the prior information. The second posterior is based on data $(10,8)$, that are unlike-ly under the prior information. Thus in one case the data play a confirmatory role, and in the other there is conflict between the prior and the data. The conflict can be seen in the increase in the imprecision in the second case. Moreover, the value the decision maker adds to the total information decreases in the case of conflict. This effect is clear on looking at figures 1-6, representing the numbers of table 1.
Example 2

Suppose that we are interested in the length of intervals between breakdowns of a system, which follow a Poisson process, thus the intervals are independent and identically exponentially distributed. We consider the problem of determining a distribution for the length of the intervals between failures. To show how the idea of conjugate priors can be maintained we express the prior knowledge in terms of imprecise conjugate priors (Coolen21). In this case the distribution of the length, $X$, of intervals is the exponential

$$f_X(x|\lambda) = \lambda e^{-\lambda x}, \quad x \geq 0 \text{ and } \lambda > 0.$$ 

Since the density is exponential, the natural conjugates come from the gamma family. Thus suppose the imprecise prior densities for $\lambda$ are:

$$\ell_\lambda(l) = le^{-l}$$

and

$$u_\lambda(l) = \ell_\lambda(l) + c_0 \cdot 4le^{-2l}, \quad \text{with } c_0 = 1 \quad (l > 0).$$

Littlewood's26 software reliability model can be generalised in a similar way if we assume that the mixing distributions are given imprecisely in the gamma family.

If $n$ independent data $\{x_1, \ldots, x_n\}$ become available, sufficient statistics are $(n, t)$, with $t = \sum_{i=1}^n x_i$, and the imprecise posterior densities for $\lambda$ are:

$$\ell_\lambda(l|n, t) = (1+t) [(1+t)l]^{1+n} e^{-(1+t)l} / \Gamma(2+n)$$

and

$$u_\lambda(l|n, t) = \ell_\lambda(l|n, t) + c_n \cdot (2+t) [(2+t)l]^{1+n} e^{-(2+t)l} / \Gamma(2+n),$$

with $c_n = (1+n)^{-1}$. The value of $c_n$ is such that the value of the subjective prior information is equivalent to that of one observation. In figures 7
and 8 (appendix) the imprecise predictive reliability functions for $X$ and the corresponding hazard rates are given, together with the same functions for the imprecise posteriors derived after 2 observations summing to 2 have become available.

The same functions are given in figures 9 and 10 (appendix), but this time with posteriors derived from 2 observations summing to 5. This second case is an example of data that are in conflict with the prior thoughts, although very little value was added to the subjective opinions.

5. Conclusions and Additional Remarks

Thus, we have shown how the commonly used concepts in reliability theory can be extended in a sensible way and combined with prior knowledge through the use of imprecise probabilities. Little more is demanded than in a standard Bayesian approach, and in return some of the difficulties of the standard approach are avoided. In particular, the use of imprecise priors offers better ways of eliciting and expressing expert knowledge. At the same time, there is a calculus of imprecise probabilities\(^{21}\) which ensures that the expert opinions are consistently expressed and combined. The interpretation of results requires a little more of users since now all measures of interest are given as lying in a range between a lower and upper value. However, as we have seen the range can be an increasing or decreasing function of the amount of data; when the data are consistent with the preceding data and with the prior opinions, the ranges decrease; when the latest data are not consistent with the current state of knowledge, the range can increase.
Thus the intervals provide through the measures of imprecision an indication of changes in the object of study.

References

8. Boole, G. (1854) An investigation of the laws of thought on which are founded the mathematical theories of logic and probabilities (Macmillan, London; reprinted by Dover (1958)).


### Appendix

**Table 1 (example 1)**

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Figures 1-6 (example 1)

**Fig. 1:** Imprecise prior predictive CDFs

**Fig. 2:** Imprecise posterior predictive CDFs, data (10,3)
Fig. 3: imprecise posterior predictive CDFs, data (10,8)

Fig. 4: imprecise prior predictive probabilities
Fig. 5: imprecise posterior predictive probabilities, data (10,3)

Fig. 6: imprecise posterior predictive probabilities, data (10,8)
Figures 7-10 (example 2)

Fig. 7: imprecise predictive reliability functions

1: upper prior
2: lower prior
3: upper posterior
4: lower posterior

data (2,2)

Fig. 8: imprecise predictive hazard rates

1: upper prior
2: lower prior
3: upper posterior
4: lower posterior

data (2,2)
Fig. 9: Imprecise predictive reliability functions

1: upper prior
2: lower prior
3: upper posterior
4: lower posterior
Data (2,5)

Fig. 10: Imprecise predictive hazard rates

1: upper prior
2: lower prior
3: upper posterior
4: lower posterior
Data (2,5)
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