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Optimal Leadtimes Planning in a Serial Production Systems

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Abstract

Consider an N stage serial production line where the processing times of orders may be random. Since the carrying costs increases from stage to stage, the standard production procedure, that is, to determine a total leadtime for the entire order by taking an appropriate percentile of the distribution of total processing time and then release the order immediately from stage to stage during the process, may not be optimal since it ignores inventory carrying costs. This article studies a per stage planned leadtime dispatching policy for such systems. The order will not be released immediately to the next workstation prior to a predetermined delivery time, or planned leadtime. The vector of planned leadtimes at workstations is to be determined by trading off expected holding costs at all stages and expected penalty costs for exceeding the total planned leadtime. We show that the optimal vector of planned leadtimes may be obtained efficiently by solving an equivalent serial inventory model of the type considered in Clark and Scarf (1960), and as a corollary, we conclude that the planned leadtime vector satisfies a generalized news boy problem.

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1. Introduction

Realistic manufacturing processes exhibit uncertainties due to the involvement of unreliable machines, fluctuating yield, and unsteady environmental conditions. The processing times of an order may be random at some or all stages of the line. A commonly used production procedure is to determine a total leadtime for the entire order by taking an appropriate percentile of the distribution of total processing time, and to forward the order as soon as possible from stage to stage during the process. While this procedure is easy to implement and control, it ignores inventory carrying costs. If the cost rate per unit of time of carrying the order increases as the order moves from stage to stage (as is usually the case, due to additional value being added), we may be better off specifying a planned leadtime per stage, and not to release the order from a given stage prior to the planned leadtime. On the other hand, setting a planned leadtime at each stage may also reduce the variability of the total production leadtime and eventually the system variability.

Significance of leadtime variability effect on the system performance was studied by Grasso and Taylor (1984) through simulation. The problem is thus to determine the length of planned leadtimes at each workstation by trading off expected holding costs at all stages and expected penalty costs for exceeding the total planned leadtime. In this paper, we show that the optimal vector of planned leadtimes may be obtained efficiently by solving an equivalent serial inventory model of the similar type considered in Clark and Scarf (1960).

While a lot of work has been done in the areas of production planning, scheduling and inventory, relative few of them is very close to the specific problems of this type. Most of the existing research on this problem are limited to simulation or single stage production system. Whybark and Williams (1976) used simulation to compare the effectiveness between keeping safety time and safety stock in their studying of MRP systems. They found that safety time is a more efficient and effective cushion than safety stock against leadtime variability for MRP systems. Week (1981) studied a single stage system with tardiness cost from which he concluded that the single stage system is actually a simple newsboy problem.

Some interesting research on two or three stage system was done by Yano(1987a, b, c). Yano (1987a) studied a serial two stage production system. The author assumed that the actual procurement processing times at workstations follow some known random variables. The author then formulated the problem of optimizing leadtime (safety time) into a mathematical programming model and discussed the solution structure and optimal condition of a two stage serial production system. To confirm the complexity of the problem, the author also

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demonstrated the non-convexity of this type of problem for a two stage system in the paper. Although some of the results in the paper need to be revised, some results of the paper do provide some insight on solving this type of problem generally. Yano (1987b, c) studied the similar problem for a three-station two-level distribution-type networks system and a three-station two-level assembly system with stochastic processing time. First order conditions were investigated in the papers based on the discussion for the two stage serial production system in the previous paper. An algorithm was developed for a two level assembly system which gives the optimal planned leadtime with the objective of minimizing the total holding and tardiness costs. Heuristic policies were also developed for the distribution system.

The purpose of this article is to study a general N-stage serial production system as defined in Yano (1987a). We assume that a batch of products needs to be processed at an N-stage serial production system. The process time of the batch at each stage is assumed to be a known random variable. We consider two types of costs in the model, (a) inventory holding cost that is caused by "earlier" completion of the batch than scheduled time. This cost may occur at every workstation. (b) penalty cost, or tardiness cost for missing the due date. Tardiness cost only occurs for finished product at the final workstation. The objective is to minimize the total costs. Since the process time at each work station will not be affected by the dispatching policy, there will be no difference whether we consider holding costs during process or not in our model.

The arrangement of the paper is as follows. We describe and formulate the system in the next section. In section 3, we show the equivalency between our production leadtime problem and the well known inventory system. Section 4 discusses the some solution structure of the problem and provides an algorithm to solve the leadtime problem approximately. A three station example is given to demonstrate the computation aspect of the algorithm.

2. Model description and formulation

The problem of determining optimal planned leadtime in a serial system can be described as follows: Assume that we have an N-stage serial production system in which a fixed batch of products needs to be processed. The actual processing time at each workstation is assumed to follow a random variable with known distribution. Random variables at different workstations may follow arbitrary distributions and are assumed statistically independent. Since each stage in the production adds a value to the batch, a non-decreasing linear inventory holding cost is assumed to occur at each stage if the batch is not delivered immediately to the next stage after it

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finishes processing (by crushing the workstations, we can easily extend to arbitrary holding cost rate case). A delivery schedule is determined when a customer makes an order. A missing of promised delivery on the due date will be penalized. The holding and penalty costs are assumed proportional to the time of delay.

Assume that x_i be the decision variable of planned leadtime for stage i, for $i = 1, 2,...,$ N. The dispatching rule at each stage is that a batch is dispatched at the planned dispatching time, that is, $x_N + x_{N-1} +...+ x_i$ (if the starting process time is 0) at stage i if the batch is completed earlier or on time at this stage. Otherwise, the "tardy" batch will be dispatched immediately after it finishing processing. The system can be described by the following figure 1.
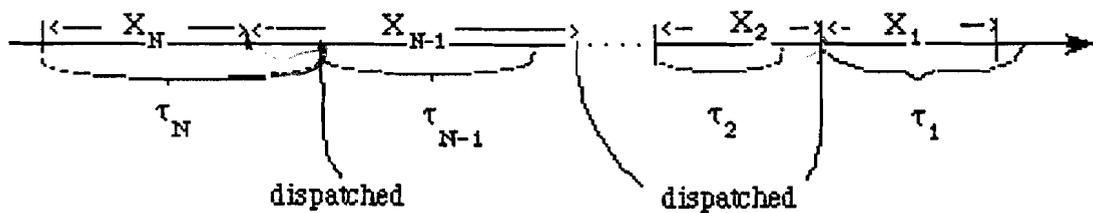
Figure 1

As shown in the figure 1, we will always assume that the batch begins processing at stage N, then (N-1), (N-2), ..., and processes are finished at stage 1. Process starting time is assumed to be zero.

The objective is to determine the leadtime variable x_i, i = 1, 2, ...N, at each stage so that the sum of the inventory holding costs, caused by "earlier" completion at each stage, and the penalty cost, caused by the tardiness of the finished products, is minimized. Let

h_i = the inventory holding cost rate at stage i.

p = penalty cost rate for missing the due date

x = $(x_1, x_2, ..., x_N)$ = the planned leadtime vector.

$E(\xi)$ = expectation of random variable ξ.

$[c]^+$ = $\text{Max}\{c, 0\}$, and $[\xi]^- = \text{Max}\{-c, 0\}$, for any real value c.

Let $S_i = \sum_{k=1}^{i} x_k$ be the planned finish time at workstation i and $G_i(x)$ be the difference of time between the planned finish time of the order at workstation i and the actual finish time at this workstation, then, given the dispatching rule described above, we have

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$$G_i(x) = \min\{(x_i - \tau_i), (x_i - \tau_i) + (x_{i+1} - \tau_{i+1}), \dots, (x_i - \tau_i) + (x_{i+1} - \tau_{i+1}) + \dots + (x_N - \tau_N)\}. \tag{1}$$

The objective is thus to find a non-negative vector x that minimizes:

$$\text{Min } \Phi(x) = \sum_{i=1}^{N} h_i E(G_i(x)^+) + p E(G_i(x)^-) \tag{2}$$

Alternatively, we can also assume that carrying costs are incurred while the order is being processed as well. In that case, it is reasonable to assume that the holding cost with rate h_i applies from the time the order has completed stage i until it completes stage i+1. This results in the objective

$$\text{Min } \Phi'(x) = \sum_{i=1}^{N} h_i E[G_i(x)^+] + \sum_{i=2}^{N} h_i E(\tau_{i-1}) + p E[G_i(x)^-] \tag{3}$$

Since two objectives (2) and (3) differ by a constant and therefore have the same optimal solution(s). We thus confine ourselves to $\Phi'(x)$, without loss of generality.

The above mathematical programming formulation is not easy to solve. It is shown by Yano (1986) that it is a non convex problem even for a two workstation system. Although it is possible to analyze the problem directly from basic Kuhn-Tucker condition, it is complex, tedious and less intuitive. We shall show in the next section that the above planned leadtime problem (3) equivalents to an inventory system that was studied by Clark and Scarf (1960). As a result, the problem of finding an optimal leadtime policy is can be reduced to solve N one variable convex sub problems in the alternative system structure.

3. An equivalency between the production leadtime model and inventory model

Consider now a periodical review N stages serial inventory system, with stage 1 receiving stock from stage 2, 2 from 3 etc. and stage N from an outside supplies (denoted by stage N+1). stage N+1 is assumed to have infinite stock. A constant transportation leadtime L_i is required from stages i+1 to i. Demand is assumed to occur only at stage 1 according to a known stationary process. Denote the cumulative demand during leadtime L_i as τ_i. Unfilled demand is backordered. For the convenience of analysis, we assume that demand arrival and delivery occurs at the right beginning of each period and inventory at each stage is calculated immediately after the delivery and arrival of the stock. Let

$I_i(t)$ = the on hand inventory level at stage i at time t.

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$IE_i(t)$ = the echelon inventory at stage i at time t, which equals inventory on-hand at stage i plus inventories in-transit to or on-hand at stage j, j=i-1, i-2, ..., 1, at time t after demand arrival and all deliveries (at all stages) at time t.

$B(t)$ = the backorder quantity at stage 1 at time t.

$IL_i(t)$ = $IE_i(t)$ - $B(t)$ = the echelon inventory level at stage i at time t.

$IP_i(t)$ = echelon inventory position at stage i at time t, it equals $IL_i(t)$ plus the inventories in-transition to stage i at period t from stage i+1.

h_i = holding cost rate for every unit in stock at stage i or in transit to stage i-1.

p = backlogging rate.

It is well know from Clark and Scarf (1960) that the steady state total holding and backlogging costs are minimized by adopting a simple order up-to policy with parameters (S_1, S_2, ..., S_N): at any period and for all stage i = 1, 2, ..., N, the echelon i inventory position is increased to S_i if supplies are available at stage i+1, otherwise all available stock at stage i+1 is shipped to stage i and the rest of them will be delivered later when there is stock available. Clearly, $S_i \geq S_{i-1}$. The optimal replenish policy implies that echelon inventory position, $IP_i(t)$, at stage i at time t equals S_i when echelon inventory level $IL_{i+1}(t)$ at stage i+1 is greater then S_i, or echelon inventory level $IL_{i+1}(t)$ if $IL_{i+1}(t)$ is less than S_i. That is

$$IP_i(t) = \min\{S_i, IL_{i+1}(t)\}. \tag{4}$$

Further, from the definition of $IL_i(t)$ and $IP_i(t)$, we can easily show they satisfy the following relationships:

$$IL_i(t+L_i) = IP_i(t) - \tau_i, \qquad i = 1, 2, ..., N; \ t \geq 0. \tag{5}$$

Combining (4) and (5) yields

$$\begin{aligned} IL_i(t) &= \min\{S_i - \tau_i, IL_{i+1}(t-L_i) - \tau_i\} \\ &= \min\{S_i - \tau_i, S_{i+1} - (\tau_i + \tau_{i+1}), ..., S_N - (\tau_i + \tau_{i+1} + ... + \tau_N)\}. \end{aligned} \tag{6}$$

By definition of echelon inventory level, for i ≥ 2, the on hand inventory level, $I_i(t) = [IL_i(t) - S_{i-1}]^+$ and hence, from (6) we have

$$\begin{aligned} I_i(t) &= [\min\{S_i - S_{i-1} - \tau_i, S_{i+1} - S_{i-1} - (\tau_i + \tau_{i-1}), ..., S_N - S_{i-1} - (\tau_i + \tau_{i+1} + ... + \tau_N)\}]^+ \\ &= [\min\{x_i - \tau_i, x_{i+1} - \tau_{i+1} + x_i - \tau_i, ..., \sum_{k=i+1}^{N}(x_k - \tau_k)\}]^+. \end{aligned} \tag{7}$$

where $x_i = (S_i - S_{i-1}) \geq 0$, for $i = 2, 3, ..., N$, and $x_1 = S_1$.

When $i = 1$, it is clear that

$$I_1(t) + \text{stock in-transition} = (IL_1(t))^+$$
$$= [\min \{x_1 - \tau_1, x_1 - \tau_1 + x_2 - \tau_2, ..., \sum_{k=1}^{N} (x_k - \tau_k)\}]^+. \tag{8}$$

The backorder quantity is thereby

$$(IL_1(t))^- = [\min \{x_1 - \tau_1, x_1 - \tau_1 + x_2 - \tau_2, ..., \sum_{k=1}^{N} (x_k - \tau_k)\}]^- \tag{9}$$

<u>Proposition 1:</u>   The vector x^* is the unique optimal solution of the planned leadtime problem if it is optimal for the equivalent serial inventory model.

<u>Proof:</u> The expected inventory in transition from stage $i+1$ to stage i is $E\tau_i$ by an application of Little's law. The objective in the serial inventory system is thus

$$\min \{ \sum_{k=2}^{N} [h_k E(I_k(t)) + E\tau_k] + h_1 E(IL_1(t)^+) + pE(IL_1(t)^-) \} \tag{10}$$

Compare (1) and (7)-(9), we conclude that the planned leadtime problem (3) and the serial inventory problem (10) are identical if τ_i, $i = 1, 2,..., N$, the processing time in the leadtime planning model and leadtime time demand of stage i in inventory model, are identical. It follows from Federgruen and Zipkin (1984 a, b) that the latter has a unique solution.

In existing inventory model, we usually assume that demand process follows stationary process. Therefore the corresponding leadtime demand at each stage is originated from a common distribution in the inventory model while the processing times for different stages in leadtime planning problem are usually independent, arbitrary random variables. However, equivalency in Proposition 1 is still true for arbitrary random variable τ_i, since relation (5), (6) and (7) still hold for arbitrary leadtime demand random variable τ_i if the order up to policy is still optimal for the inventory model.

From the above discussion, we know that the optimal solution of planned leadtime problem can be obtained through an optimal (s, S) policy as long as the corresponding inventory system has an optimal (s, S) policy. To construct an inventory system with above leadtime demand pattern, let us consider the same N-stage serial inventory system as we considered before except that customer "demands" occur not only for final product but also for subassembled products in intermediate stages. The quantity of subassembled product "demand"

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at stage i is such that the total demand at this stage during its leadtime L_i equals τ_i, which is independent of τ_j, for $j \neq i$, This is achievable by assuming that the subassembled "demands" can sometimes be suppliers. Further we assume that the "demands" for subassembled product at each stage have a higher priority than the demands from the down steam stages and there are no backorder penalty costs if the "demands" can not be fulfilled immediately. The results of Clark and Scarf (1960) can easily be extended to such a scenario. Thus we know that (s, S) policy is still optimal for such an inventory system.

For the purpose of self contain and to develop an algorithm for the leadtime model, we provide a brief discussion of optimality in the following section for a two stage inventory system described above. This discussion is very similar to those in the article by Langenhoff and Zijm (1990).

4: Optimal solution structure and an algorithm

The purpose of this section is to provide an approximated algorithm of solving the leadtime planning vector. The first part of this section is to give an average cost analysis for a system has demand pattern described in the last part of section 3. Consider a two stage inventory system with such a demand pattern. If the echelon inventory position at stage 2 is ordered up to an level y_2 at period t, it will affect the decision of ordering only after period $t + L_2$ since the delivery leadtime L_2 involved and the expected quantity in transition from stage 2 to stage 1 is $E(\tau_2)$ (by an easy application of Little's law), which has no impact to the optimal policy. The important costs are only physical inventories and backlog. The physical inventory level at the stage 2 at period $t+L_2$ equals $(y_2 - \tau_2 - y_1)^+$ when the echelon inventory position at stage 1 is ordered up to a level y_1 at period $t + L_2$. In such a circumstance, the physical inventory position of stage 1 at period $t+L_1+L_2$ equals $y_1 - \tau_1$, if $y_2 - \tau_2 - y_1 \geq 0$, or, $y_2 - \tau_2 - \tau_1$, if $y_2 - \tau_2 - y_1 < 0$. Clearly, $y_2 \geq y_1$. Define

$$\hat{L}_1(y) = \begin{cases} h_1 y & \text{if } y \geq 0 \\ -py & \text{if } y < 0 \end{cases}. \tag{11}$$

If its inventory position is ordered up to a level y, the expected cost at stage 1 becomes

$$\hat{D}^{(1)}(y) = E(\hat{L}_1(y - \tau_1)).$$

Thus expected one period total cost $C^{(2)}(y_1,y_2)$ in the two stage system is (regardless the distribution of leadtime demand in the workstations)

$$C^{(2)}(y_1,y_2) = E[\hat{D}^{(1)}(y_1)|y_2-y_1\geq\tau_2]+E[\hat{D}^{(1)}(y_2-\tau_2)|y_2-y_1<\tau_2]+h_2E[(y_2-\tau_2-y_1)^+]+h_2E(\tau_1)$$

$$= E[\hat{D}^{(1)}(y_1)+h_2(y_2-\tau_2-y_1) \,|y_2-y_1\geq\tau_2] + E[\hat{D}^{(1)}(y_2-\tau_2) \,|\, y_2-y_1<\tau_2] + h_2E(\tau_1)$$

$$= E[\hat{D}^{(1)}(y_1)+h_2(y_2-\tau_2-y_1)] + E[\hat{D}^{(1)}(y_2-\tau_2)-\hat{D}^{(1)}(y_1)+h_2(y_2-\tau_2-y_1)\,|\,y_2-y_1<\tau_2]+h_2E(\tau_1)$$

$$= E[\hat{L}_1(y_1-\tau_1)-h_2(y_1-\tau_1)]$$

$$+ E[h_2(y_2-\tau_2)] + E\{ E[\hat{L}_1(y_2-\tau_2-\tau_1)-h_2(y_2-\tau_2-\tau_1)] - E[\hat{L}_1(y_1-\tau_1)-h_2(y_1-\tau_1)] \,|\, y_2-y_1<\tau_2 \}$$

Let $L_1(y) = \hat{L}_1(y) - h_2y$, and $D^{(1)}(y) = E[L_1(y-\tau_1)]$, then

$$C^{(2)}(y_1,y_2) = D^{(1)}(y_1) + E[h_2(y_2-\tau_2)] + E\{ D^{(1)}(y_2-\tau_2) - D^{(1)}(y_1) \,|\, y_2-y_1< \tau_2\}$$

From the above derivation, we can obtain the following proposition,

Proposition 2: The expected one period cost function, $D^{(2)}(y_1,y_2)$, of a two stage inventory system with a general demand pattern can be written as

$$C^{(2)}(y_1,y_2) = D_1(y_1) + D_2(y_1,y_2). \tag{12}$$

Where,

$$D_1(y_1) = D^{(1)}(y_1) = E(L_1(y_1-\tau_1)), \tag{13}$$

$$D_2(y_1,y_2) = E(h_2(y_2-\tau_2)) + E[D_1(y_2-\tau_2) - D_1(y_1)|\tau_2 \geq y_2-y_1]. \tag{14}$$

The above result can easily be extended to N-stage inventory system and we have the total expected one period cost

$$C^{(N)}(y_1,y_2) = D_1(y_1) + D_2(y_1,y_2) + ... + D_N(y_1,y_2,...,y_N) \tag{15}$$

where

$$D_1(y_1) = E(L_1(y_1-\tau_1))$$

$$D_i(y_1,y_2) = (h_i-h_{i+1})E(y_i-\tau_i)+E[D_{i-1}(y_1,y_2,...,y_{i-2},y_i-\tau_i)-D_{i-1}(y_1,y_2,...,y_{i-1})|\tau_i\geq y_i-y_{i-1}]. \tag{16}$$

It is easy to show that $D_k(y_1,y_2,...,y_k)$ is a convex function of y_k given y_i, $i < k$. From Clark and Scarf (1960) or Langenhoff and Zijm (1990), the above separable property of expected one period cost guarantees that a (s, S) policy is optimal.

So far, we have demonstrated that the planned leadtime problem has a optimal (s,S) policy with virtually no assumption on distributions of process time random variables. Therefore, existing algorithms for solving optimal policy of an inventory system, such as Federgruen and Zipkin (1984 a, b), Rosling (1989) and Zheng and Federgruen (1991), can be used to solve the leadtime planning problem. Consider the following algorithm

Step 1: $i=1$
Step 2:   Given $S_1, S_2, ..., S_{i-1}$, solve S_i that minimize $D_i(S_1, S_2, ..., S_{i-1}, y_i)$.
              and let $S_k = \min \{S_k, S_i\}$,   for $k = 1, 2, ..., i-1$.
step 3:   if $i < N$, $i = i+1$, go to step 2.
              otherwise,   stop.

This algorithm is initially developed by Langenhoff and Zijm (1990). It is important to emphasize that it obtains the optimal solution by solving exactly N one variable equations. On the other hand, if we look at the problem from the aspect of leadtime planning, some interesting results can be obtained. We can show

Proposition 3:   Let $S_1, S_2, ..., S_{i-1}$ be the solution obtained from the above algorithm, we have

$$\frac{dD_i(S_1, S_2, ..., S_{i-1}, y)}{dy} = (p+h_1)P_r\{\tau_1 \leq S_1, ..., \tau_1+...+\tau_{i-1} \leq S_{i-1}, \tau_1+...+\tau_i \leq y\} - (p+h_{i+1}), \quad (17)$$

for $i = 2, 3, ..., N$, and when $i = 1$,

$$\frac{dD_1(y)}{dy} = \frac{dE(L_1(y-\tau_1))}{dy} = (p+h_1)P_r\{\tau_1 \leq y\}] - (p + h_2). \quad (18)$$

where $h_{N+1} = 0$.

The proof of this result is pure calculus and is given in the appendix.

Since planned leadtime $x_i = S_i - S_{i+1}$, we know to solve S_i is equivalent to solve x_i from the following equation for known $x_k, k = 1, 2, ..., i-1$.

$$P_r\{x_1 - \tau_1 \geq 0, (x_1 - \tau_1) + (x_2 - \tau_2) \geq 0, ..., (x_1 - \tau_1) + ... + (x_i - \tau_i) \geq 0\} = \frac{p+h_{i+1}}{p+h_1} \quad (19)$$

From (1), the left hand side of the equation (19) is the probability of earliness of an i-stage system, (from stage i to 1) with processing time τ_i at stage i. The probability of earliness is also an increasing function of decision variable x_i. Especially, since $h_{N+1} = 0$ when $i = N$, (19) becomes a generalized news boy problem. Define

$T_i = (\tau_i - x_i)^+,$

$T_j = (\tau_j - x_j + T_{j+1})^+, \qquad 2 \le j \le i-1$

and

$G_{1i}(x) = (x_1 - \tau_1 - T_j)^+.$

Then the left hand of (19) can easily be rewritten into $P_r\{G_{1i}(x) > 0\}$.

Assume that we have obtain $x_1 \ge 0$, $x_2 \ge 0$,...,$x_{i-1} \ge 0$, from the algorithm, then x_i can be solved by a simple bisection method if the probability on the left side of (19) can be estimated easily. An approximation by mixture of Erlang distributions through the first two moments is able to provide us an effective and simple solution to solve the problem (19). In the approximation of mixture of Erlang distributions, for given $x_1 \ge 0$, $x_2 \ge 0$,...,$x_{i-1} \ge 0$, and x_i, we calculate the first two moments of $G_{1i}(x)$ by the following procedure

Step 1: $\qquad E(T_{i+1}) = 0, \; Var(T_{i+1}) = 0.$

Step 2: $\qquad$ For known $E(T_{k+1})$ and $Var(T_{k+1})$, $k \le i$,

$\quad$ If $\sum\limits_{j=k}^{i} x_j < 0$, let $\qquad E(T_k) = E(T_{k+1}) + E(\tau_k)$, $Var(T_{i+1}) = Var(T_{k+1}) + Var(\tau_k)$

$\quad$ If $\sum\limits_{j=k}^{i} x_j \ge 0$ and $\sum\limits_{j=k+1}^{i} x_j < 0$, then let

$$E(T_k) = E[T_{k+1} + \tau_k - \sum\limits_{j=k}^{i} x_j]^+ , \quad Var(T_{i+1}) = Var[T_{k+1} + \tau_k - \sum\limits_{j=k}^{i} x_j]^+$$

$\quad$ If $\sum\limits_{j=k+1}^{i} x_j \ge 0$, then, $E(T_k) = E[T_{k+1} + \tau_k - x_k]^+$, $Var(T_{i+1}) = Var[T_{k+1} + \tau_k - x_k]^+$

$\qquad$ Fitting a mixture of Erlang distributions to random variable $T_{k+1} + \tau_k$ by using the moments obtained above.

Step 3: $\qquad$ Compute $P_r(G_{1i}(x) > 0) = P_r(T_2 + \tau_1 < x_1)$

Combining the above approximation and the algorithm discussed before, we have a quite robust method to solve the leadtime planning model.

The follow example gives the exact and approximated solutions for a three workstation system with exponential processing time random variables. The result from mixture of Erlang distributions also works very well.

<u>Example:</u> Consider a three stage serial system with the following parameters and processing time distributions:

$$h_3 = \frac{1}{2}, h_2 = 1, h_1 = 2, \text{ and } p = 2.$$

τ_i = exponential distribution with mean 1, for i = 1, 2, 3.

Then the optimal leadtime vector can be obtained by solving

For i = 1, $\quad \frac{p+h_2}{p+h_1} = P_1^1(x) = P(x_1-\tau_1 \geq 0)$, that is, $1 - e^{-x_1} = \frac{3}{4}$, we have, $x_1 = \log 4$.

For i = 2, $\quad \frac{p+h_3}{p+h_1} = P_1^2(x) = P(x_1-\tau_1 \geq 0, x_1-\tau_1+x_2-\tau_2 \geq 0)$, which, after substitute x,

satisfies $\quad \frac{5}{8} = \frac{3}{4} - \frac{1}{4} \log(4) e^{-x_2}$, so $x_2 = \log(4\log 2)$.

For i = 3, $\quad \frac{p}{p+h_1} = P_1^3(x) = P(x_1-\tau_1 \geq 0, x_1-\tau_1+x_2-\tau_2 \geq 0, x_1-\tau_1+x_2-\tau_2+x_3-\tau_3 \geq 0)$,

which can be re-expressed as,

$$\frac{1}{2} = \frac{5}{8} - \frac{1}{8} \log(8\log 2)e^{-x_3}, \text{ so } x_3 = \log(\log(8\log 2))$$

Thus the optimal leadtime vector for the problem is

$$x^* = (x_1^*, x_2^*, x_3^*) = (\log 4, \log(4\log 2), \log(\log(8\log 2))) = (1.39, 1.02, 0.54).$$

By using the mixture of Erlang distributions approximation, we have the solution (1.39, 1.02, 0.53).

<u>Reference</u>

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Appendix

Proof of Proposition 3: From the definition of $L_1(y-\tau_1)$, we have

$$\frac{dD_1(y)}{dy} = \begin{cases} (p+h_1)P_r\{\tau_1 \leq y\}] - (p+h_2) & \text{if } y \geq 0 \\ - (p+h_2) & \text{if } y < 0 \end{cases}$$

$$= (p+h_1)P_r\{\tau_1 \leq y\}] - (p + h_2), \quad \text{since } \tau_1 \geq 0.$$

Assume that it is true for $k = 1, 2, ..., i$, then, from (16) we have

$$\frac{dD_i(S_1, ...,S_i, y)}{dy} = h_{i+1}-h_{i+2} + \int_{y-S_i}^{+\infty} \frac{dD_i(S_1, ..., S_{i-1}, y-t)}{dy}dF_i(t) \qquad (A1)$$

where $F_i(t)$ is the distribution function of τ_i.

$$= h_{i+1}-h_{i+2}+ \int_{y-S_i} (p+h_1)P_r\{\tau_1 \leq S_1,...,\tau_1+...+\tau_{i-1} \leq S_i, \tau_1+...+\tau_i \leq y-t\} - (p+h_{i+1})dF_i(t).$$

when $y-S_i < 0$.

$$= h_{i+1} - h_{i+2} + \int_0^{} (p+h_1)P_r\{\tau_1 \leq S_1,\ldots,\tau_1+\ldots+\tau_{i-1} \leq S_i, \ \tau_1+\ldots+\tau_i \leq y-t\}dF_i(t) - (p+h_{i+1})$$

$$= \int_0^y (p+h_1)P_r\{\tau_1 \leq S_1,\ldots,\tau_1+\ldots+\tau_{i-1} \leq S_{i-1}, \ \tau_1+\ldots+\tau_i \leq y-t\}dF_i(t) - (p+h_{i+2})$$

$$= (p+h_1)P_r\{\tau_1 \leq S_1,\ldots,\tau_1+\ldots+\tau_{i-1} \leq S_{i-1}, \ \tau_1+\ldots+\tau_i+\tau_{i+1} \leq y\}dF_i(t) - (p+h_{i+2}).$$

since $S_i > y$, it is equivalent to

$$= (p+h_1)P_r\{\tau_1 \leq S_1,\ldots,\tau_1+\ldots+\tau_i \leq S_i, \ \tau_1+\ldots+\tau_i+\tau_{i+1} \leq y\}dF_i(t) - (p+h_{i+2}).$$

When $y - S_i \geq 0$, after integral by parts and use the result $\dfrac{dD_i(S_1, \ldots, S_{i-1}, S_i)}{dy} = 0$, (A1) yields

$$= (p+h_1)\int_{y-S_i}^{+\infty} P_r(\tau_{i+1} \leq t)dP_r\{\tau_1 \leq S_1,\ldots,\tau_1+\ldots+\tau_i \leq S_i, \ \tau_1+\ldots+\tau_i+\tau_{i+1} \leq y-t\} - (p+h_{i+2})$$

let $t' = t - y$

$$= (p+h_1)\int_0^{S_i} P_r(\tau_{i+1} \leq y-t')dP_r\{\tau_1 \leq S_1,\ldots,\tau_1+\ldots+\tau_i \leq S_i, \ \tau_1+\ldots+\tau_i+\tau_{i+1} \leq t'\} - (p+h_{i+2})$$

$$= (p+h_1)P_r\{\tau_1 \leq S_1,\ldots,\tau_1+\ldots+\tau_i \leq S_i, \ \tau_1+\ldots+\tau_i+\tau_{i+1} \leq y\} - (p+h_{i+2}).$$

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