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MODELING, SIMULATION AND CONTROL OF A POWER LOOP TEST RIG FOR CONTINUOUSLY VARIABLE TRANSMISSIONS

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Short C.V. of the presenting author:  
Dr. P.A. Veenhuizen received his M.Sc. degree in experimental physics from the Free University of Amsterdam, The Netherlands in 1984. From 1988 he received his PhD degree from the University of Amsterdam on the subject of Electronic properties of Ce- and U-based compounds. In 1988 he joined SKF Engineering and Research Centre where his work focused on Electromagnetic detection techniques for material characterization and X-ray characterization of material fatigue. In 1995 he joined van Doorne’s Transmisie bv, where he was responsible for the realization of some advanced drive train projects. Since 2002 he is assistant professor in the CST group of the ME department of the Eindhoven University of Technology. His general interest is to link academic research to the (automotive) industry. He specializes in automotive transmission technology, with special focus on CVT’s and other force-closed components;
ABSTRACT

Durability testing of automotive transmission components under rated load conditions usually requires costly and large test rigs. In order to avoid this disadvantage the power loop test rig has been designed and realized. A design with two continuously variable transmission variators and one drive motor saves space and costs. An unconventional servo hydraulic actuation system is used to control clamping forces and transmission ratio. A physical model of the test rig is realized and a controller is designed which meets the requirements for durability testing. Experimental results show the validation of the model and functional measurements are presented.

Keywords: Modeling/ Control/ Durability Testing/ Servo Hydraulic/ Continuously Variable Transmission/ CVT
1 INTRODUCTION

Due to the combination of a smooth ride with relative high efficiency the continuously variable transmission (CVT) is regarded as a good solution to the demand for more comfort with low fuel consumption. However the development path of a high efficiency transmission involves extensive experimenting and testing. Conventional durability testing of automotive transmission components under load conditions involves high costs and occupation of useful space. The common setup with a drive motor and a brake motor requires two motors of the rated power, which are costly and space consuming. These two disadvantages were the main reason for Gear Chain Industrial BV (GCI) to start a joint development with the Eindhoven University of Technology of a Power Loop Test Rig (PLTR) for components of a CVT. The rig should not only allow for efficient durability testing but also for functional testing measurements like efficiencies, torque loss, slip behavior and steady state clamping forces. Both metal V-belts and V-chains can be tested. The PLTR consists of two CVT variators that are coupled in parallel. By creating a small ratio difference between both variators a torque is generated in the shafts. Since the mechanical power through one variator is transmitted to the other, a power loop is created. Due to the mechanical losses of the variators, power is dissipated and a small electric motor is needed to compensate for these losses. For durability testing the variator torque, motor speed and variator ratio should be kept constant by a control system. In this paper the layout of the test rig will be explained, followed by the control targets. Next a simulation model is presented which is used for controller design. The paper will proceed with control analysis. After this, test rig measurements and model validation results will be presented. Finally some conclusions will be drawn and recommendations are proposed.
2 POWER LOOP TEST RIG LAYOUT

In figure 1 the layout of the PLTR is shown. It consists of a drive motor and two variators coupled in parallel. The drive motor shaft is the primary side, the other the secondary. Variator A is mounted between the drive motor and variator B. Subscripts \( (1, 2, a, b) \) indicate the position of the object involved. Each variator is built into a modular beltbox that has an integrated manifold for the hydraulic system. The couplings connecting the beltboxes can be released without changing the position of the beltboxes. This enables quick (dis)-assembly of the test rig without the need for realigning the complete setup.

The bearings and belt are lubricated by a separate hydraulic circuit, which is fed by the lubrication pump \( (L_a, L_b) \). These circuits also feed the pressure pumps \( (P_a, P_b) \), which are used to control the pressure \( (p_{2a}, p_{2b}) \) in the secondary pulley cylinders of the variators. The primary pulley cylinders are pressurized by the ratio pumps \( (R_a, R_b) \), which control the flow between the primary and secondary cylinder. The pumps used are bi-directional external gear pumps with a displacement of 1.0 [cc/rev]. PWM controlled brushless 42 [V] DC servomotors are used to drive the pressure and ratio pumps.

![Figure 1: PLTR layout (pressure circuit in solid lines, lubrication circuit dashed)](image)

The hydraulic feed of the pulley cylinders is realized by an axial connection, which uses a sealed close clearance bushing to prevent excessive leakage. The design considerations are discussed in [7]. For the shaft connected to the motor the axial connection is not available and therefore a radial oil feed is designed. It consists of a chamber sealed with two rings in a groove on the shaft.

For control and measuring purposes the test rig is equipped with sensors for pressure \( (p_{1a}, p_{2a}, p_{1b}, p_{2b}) \), rotational speed \( (\omega_1, \omega_2) \), moveable pulley sheave position \( (x_{1b}, x_{2a}) \) and torque \( (T_1, T_2) \).
3 POWER LOOP TEST RIG CONTROL TARGETS

The PLTR is intended to test transmission elements like metal V-belts or V-chains. The current goal is to perform steady state durability tests, at fixed operating points. Controlled shifting between these operating points is necessary but no strict dynamic targets apply.

For durability testing a certain input torque, speed and transmission ratio are prescribed. The test rig is designed for a torque of 250 [Nm] both positive and negative torque. With an accuracy requirement of ±5.0 [%] this results in ±12.5 [Nm] fluctuation allowed. At 6000 [rpm] and ±1 [%] accuracy the input speed variation should be smaller than ±60 [rpm]. The transmission ratio, which is in the range of 0.43 [-] to 2.33 [-], can fluctuate up to respectively 0.01 [-] and 0.05 [-], due to the ±2 [%] accuracy requirement.

4 POWER LOOP TEST RIG MODEL

For control design purposes and to obtain more insight into the system behavior a model of the PLTR is made. This model consists of three parts. In figure 2 an overview of the model for one variator is given. The variator model, discussed in section 4.1 describes the behavior of one of the variators. The inputs for this model are the rotational speed ratio and the primary and secondary clamping forces $F_1$ and $F_2$. The outputs are a primary torque $T_{1A}$ and secondary torque $T_{2A}$ acting on the intermediate shafts of the PLTR. The hydraulic model is discussed in 4.3 and creates the clamping forces $F_1$ and $F_2$ for the variator. The speed of the ratio pump $\omega_{pump}$ and the torque of the pressure pump $T_{pump}$ are the inputs for this model. The mechanical model, discussed in section 4.2, couples two variators together and describes the dynamic behavior of the test rig assembly. Primary and secondary torque from both variators in combination with the drive motor torque $T_m$ result in a rotational speed ratio $r_{\omega}$ of the shafts.

### 4.1 VARIATOR MODEL

The variator model includes modeling of geometrical relations, shifting behavior, slip, torque loss and steady state clamping force balance. These sub-models are also shown in figure 2.

#### 4.1.1 Geometrical Model

The geometrical model describes the relation between the position of the moveable pulley sheaves and the variator ratio. First, a distinction should be made between the speed ratio $r_{\omega}$ and the geometrical ratio $r_g$. The speed ratio is defined as the ratio of the rotational shaft speeds:

$$r_{\omega} = \frac{\omega_2}{\omega_1} \quad (1)$$

Since in a variator some slip is present the speed ratio differs from the geometrical ratio. Neglecting spiral running of the V-belt, it can be calculated from the running radii of both pulleys:
The running radius $R_1$ can be calculated from a measurement of the axial position of the pulley sheave $x_1$, by means of equation (3)

$$R_1 = R_{1,\text{min}} + \frac{x_1}{2 \tan(\beta)} \tag{3}$$

Here, $R_{1,\text{min}}$ denotes the minimum running radius and the pulley sheave wedge angle is denoted by $\beta$. The secondary running radius $R_2$ can be calculated iteratively as indicated in equation (34) from Appendix A. With an approximation it results in a non-linear relation:

$$R_2 = R_1 - \frac{a \pi}{2} + \sqrt{\frac{a^2 \pi^2}{4} - 2a \pi R_1 - 2a^2 + aL} \tag{4}$$

### 4.1.2 Shifting Model

To describe the shifting behavior of the variator several models are available in the literature. According to [4] the model presented in [3] has the best resemblance with reality, except for very low rotational speeds. The primary pulley sheave can be seen as a mass-damper system on which the primary clamping force $F_1$ acts. Since the mass term is very much lower than the damping term in the equation describing shift dynamics it can be neglected. Hence the shifting behavior can be described with:

$$\dot{x}_1 = \frac{F_1 - F_1^*}{b_s} \tag{5}$$

Here $b_s$ represents a damping constant and $F_1^*$ is the clamping force which would result in a constant ratio. Calculation of this “balance” force is needed, and is described in section 4.1.5.

### 4.1.3 Slip Model

Conventional CVT modeling does not include slip since it is not an issue due to high over clamping. Since the PLTR is based on slip, due to a difference in ratio between both variators, slip modeling is obligatory. The variator slip $s$ is defined by:

$$s = 1 - \frac{r_0}{r_{0,0}} \tag{6}$$

The zero load speed ratio $r_{0,0}$ is defined as the speed ratio at zero secondary torque:

$$r_{0,0} = \left. \frac{\omega_2}{\omega_1} \right|_{T_2=0} \tag{7}$$

Since $r_{0,0}$ is not available when the variator transfers torque, it is assumed that the axial pulley position can be used to give a good estimate of $r_{0,0}$. Measurements reported in [1] have resulted in an empirical model to describe the variator torque transmission under slip conditions. It is based on the force balance between the axial clamping force and the input torque. The input torque $T_i$ acts on radius $R$ and results in a tangential force. It has to be counterbalanced by a friction force between belt and pulley sheaves, which is a function of
the pulley wedge angle $\beta$, and the axial clamping force $F_2$. The traction coefficient $\mu_{\text{eff}}$, which can be regarded as the effective friction coefficient is defined as:

$$\mu_{\text{eff}} = \frac{T_1 \cos(\beta)}{2F_2 R_1}$$  \hspace{1cm} (8)$$

In figure 3 results are shown for traction coefficient measurements. It turns out that it is mainly dependent on the slip value and the ratio. With these results it is possible to reconstruct the primary torque, when the ratio, slip and secondary clamping force are known.

4.1.4 Torque Loss

For realistic calculation of the secondary torque the torque loss has to be included. Variator torque loss has been measured on a single variator without load, since it has been shown that variator torque loss is largely input torque independent [2]. In figure 4 measured losses for a pushbelt are shown. The secondary torque is calculated with:

$$T_2 = \frac{(T_1 - T_{\text{loss}})}{r_{\omega 0}}$$  \hspace{1cm} (9)$$

4.1.5 Steady State Force Model

To describe the shifting behavior of the variator the steady state force $F_{1*}$ in equation (5) needs to be known. In Appendix A a simplified steady state v-belt or v-chain model is described to calculate the primary and secondary clamping forces as a function of the input torque $T_1$ and the zero load speed ratio $r_{\omega 0}$. In equation (36) of appendix A the calculation of the steady state primary clamping force $F_{1*}$ is described:

$$F_{1*} = f(T_1, r_{\omega 0}, F_{2,\text{actual}})$$  \hspace{1cm} (10)$$

It is a function of the input torque, the zero load ratio and the actual secondary clamping force. The dimensionless clamping force ratio $\zeta$ is defined as steady state clamping force ratio:

$$\zeta = \left(\frac{F_1}{F_2}\right)_{r_{\omega 0}=0} = \left(\frac{F_{1*}}{F_{2,\text{actual}}}ight)$$  \hspace{1cm} (11)$$
The steady state force ratio depends on the input torque, the zero load speed ratio and the torque ratio \( \tau \), which is defined by:

\[
\tau = \frac{T_1}{T_{1,\text{max}}}
\]  

(12)

Here \( T_{1,\text{max}} \) represents the maximum transmittable torque. The overclamping factor \( f_x \) indicates the amount of excess clamping that is used and may now be defined as:

\[
f_x = \frac{1}{\tau}
\]  

(13)

Results from model calculations are shown in figure 5 for the whole operating range, and in figure 6 for ratios low (0.43), medium (1.0) and overdrive (2.33).

The steady state force model is also used to predict the secondary clamping force for both variators in the PLTR. In equation (37) of appendix A the minimum clamping force is calculated, which is just sufficient to transfer the torque. Application of this clamping force is risky since there is no margin for disturbances. A large slip value at high clamping forces will ruin the contact surface between belt en pulley. To prevent excessive slip an over clamping factor as defined in equation (13) is used. For the variator used the secondary pressure setpoints are shown in figure 7 for both positive and negative maximum torque with an over clamping factor of 1.3 [-]. Clamping force setpoints for lower torque can be scaled linearly.

Figure 5: Steady state force ratio \( \zeta \) for complete operating range.

Figure 6: Steady state force ratio \( \zeta \) for ratio 0.43 (solid), 1.0 (dashed) and 2.33 (dash dotted).

Figure 7: Clamping force requirements for maximum torque with over clamping factor 1.3 [-].
4.2 MECHANICAL MODEL

The PLTR mechanical model consists of two variator models and two intermediate shafts as represented in figure 8. The primary intermediate shaft and motor inertia are lumped as inertia $J_1$. $J_2$ represents the secondary intermediate shaft’s inertia. Due to the ratio difference between the two variators a slip exists which results in a primary torque $T_{1A}$, $T_{1B}$ and secondary torque $T_{2A}$, $T_{2B}$ for respectively the primary and secondary shaft. These torques lead to primary and secondary shaft speeds according to:

$$J_1 \omega_1 = T_m + T_{1B} - T_{1A}$$  \hspace{1cm} (14)

$$J_2 \omega_2 = T_{2A} - T_{2B}$$  \hspace{1cm} (15)

With the shaft speeds, and equations (1), (7) and (6) the slip can be calculated which leads to the torques acting on the shafts.

![Figure 8: Free body model of the PLTR](image)

![Figure 9: Hydraulic actuation system overview](image)

4.3 HYDRAULIC MODEL

The model of the hydraulic system mentioned in section 2 is discussed in [6], and reproduced here in short. Figure 9 shows the relevant hydraulic layout for one variator. With the assumption of constant oil sump temperature isothermal oil compressibility is given by:

$$\dot{\rho} = \rho \kappa(p, \theta) \dot{p}$$  \hspace{1cm} (16)

The compressibility modulus $\kappa$, the primary cylinder pressure $p_1$ is linked to the density $\rho$, which is a function of the external flows from the ratio pump $q_{pump}$, the flow due to pulley displacement by shifting $q_{shift}$ and the leakage in the pulley cylinder $q_{leak}$:

$$\frac{1}{\kappa(V_{01} + A_1 x_1)} (q_{pump} - q_{leak1} - q_{shift1})$$  \hspace{1cm} (17)
The initial cylinder volume is represented by \( V_0 \), \( A_1 \) is the primary pulley cylinder area, and \( x_1 \) is the primary cylinder displacement. For the secondary system the same mass conservation law results in:

\[
\dot{p}_2 = \frac{1}{\kappa (V_0 + A_2 x_2)} (q_{pump} - q_{pump} - q_{\text{leak},2} - q_{\text{shift},2})
\]  

(18)

The primary cylinder displacement \( x_1 \) is calculated using (5) whereas the secondary cylinder displacement \( x_2 \) can be determined using the geometrical relations. The flow for pump \( i \) can be calculated with the rotational pump speed \( \omega_i \), the displacement volume \( C_y \) and the volumetric pump efficiency \( \eta_v \):

\[
q_i = \frac{\omega_i C_y}{2\pi} \eta_v
\]  

(19)

The leakage flow is assumed to be linearly dependent on the pressure in the cylinder:

\[
q_{\text{leak},i} = \frac{p_i}{R_{\text{leak},i}}
\]  

(20)

Here \( R_{\text{leak},i} \) represents the leakage resistance. Calculation of the flow due to shifting, \( q_{\text{shift}} \), is performed with:

\[
q_{\text{shift},i} = A_i \dot{x}_i
\]  

(21)

The pressure pump speed as used in equation (19) is calculated with the dynamics of the pressure pump:

\[
\dot{\omega}_{\text{pump}} = -\frac{1}{J_{\text{motorpump}}} \left(T_{\text{pump}} - b \omega_{\text{pump}} - \frac{C_y}{2\pi} (p_2 - p_0) \right) \eta_{\text{hm,ppump}}
\]  

(22)

\( J_{\text{motorpump}} \) represents the inertia of servomotor-pump combination, \( T_{\text{pump}} \) is the servomotor torque, \( b \) represents viscous damping in the pump, \( p_0 \) is the lubrication pressure, and the hydro mechanical efficiency of the pump is represented by \( \eta_{\text{hm,ppump}} \). The load torque for the ratio pump motor is described in a similar way:

\[
T_{\text{pump}} = \left\{ J_{\text{motorpump}} \dot{\omega}_{\text{pump}} + b \omega_{\text{pump}} + \frac{C_y}{2\pi} (p_2 - p_1) \right\} \frac{1}{\eta_{\text{hm,rpump}}}
\]

(23)

With (17) and (18) the primary and secondary pressure in the cylinders can be calculated. The resulting clamping forces are the input variables from the hydraulic model to the variator model. The PLTR model is complete and is used to better understand the behavior of the system. Controller design and tuning are simplified and will be discussed in the next section.
5 POWER LOOP TEST RIG CONTROL

The PLTR control consists of three layers as shown in the control overview in figure 10. The first control loop includes the rotational speed control in the motor amplifiers. The control of the hydraulic system from section 4.3 is the second layer. In [6] this control is described extensively. Finally on top there is a specific PLTR control layer. Since the variators are built modular so is the hydraulic actuation system. Each beltbox has its own speed and hydraulic control.

Figure 10: Overview of the complete control system of the power loop test rig.

The rotational speed control consists of a PI-feedback control, which can be adjusted in the motor amplifier. It is used to create a desired hydraulic flow, since according to (19) it is proportional to the rotational pump speed.

For the hydraulic control a decentralized control strategy is used. The ratio pump and the pressure pump have their own P-feedback control. To eliminate steady state ratio error of the PLTR, integration action is included in the Hydraulic Control of variator B. Interaction between both hydraulic control outputs \( p_2 \) and \( r_{\omega 0} \) is compensated by a model based interaction reduction. With equation (23) it is possible to predict the required secondary servomotor torque. Equation (17) prescribes the ratio pump servomotor speed for a certain ratio.

For controller design of the PLTR the model from section 4 is used. Simulation resulted in a strategy where variator B is controlled at a desired ratio \( r_{\omega 0,B,ref} \) and the desired torque \( T_{1,ref} \) is controlled by changing the desired ratio \( r_{\omega 0,A,ref} \) of variator A. In this way variator B is tested at a controlled operating point and efficiency can be measured.

The PLTR control consists of a pressure setpoint calculation and a torque control. The secondary clamping force setpoint from section 4.1.5 is used to predict the necessary clamping force for a demanded torque \( T_i \) and ratio \( r_{\omega 0} \) with an over clamping factor \( f_c \). To create the demanded primary torque a feedback PI-control is
used to adjust a desired ratio to the hydraulic control. The reference ratio $r_{\omega_0,\text{ref}}$ can be used as feedforward for variator A since only a small ratio difference is present between the variators.

For identification of the system frequency response measurements are performed. Nine operating points are considered: three geometrical ratios, low 0.5 [-], medium 1.0 [-] and overdrive 2.2 [-], and three torque levels are examined for negative $-100$ [Nm], zero 0 [Nm] and positive $100$ [Nm] torque. Since there is only little difference between the results obtained in different operating points, a typical one is shown in figure 11.

With the system identified the controller parameters are tuned to obtain stability within a bandwidth as high as possible. The achieved average bandwidth for the operating points equals 0.4 [Hz] for both inputs, which is shown in figure 11. For steady state testing this is sufficient, but for dynamic testing a higher bandwidth is needed. In figure 13 the response of both outputs for a step perturbation of both system inputs is shown. A disturbance signal is added to $r_{\omega_0,A,\text{ref}}$ and $\omega_{\text{pump},\text{ref},b}$ for respectively the torque and ratio control. None of the perturbations results in instability.

![Figure 11: Typical system transfer functions](image1.png)  ![Figure 12: Typical open loop transfer](image2.png)  

![Figure 13: Response on step perturbation for 3 operating points. Operating point 1: $T_B=100$ [Nm], $r_{\omega_0,B}=0.5$ [-]. Operating point 2: $T_B=0$ [Nm], $r_{\omega_0,B}=1$ [-]. Operating point 3: $T_B=100$ [Nm], $r_{\omega_0,B}=2.2$ [-].](image3.png)
EXPERIMENTAL RESULTS

To prove the validity of the simulation model a comparison is made between test rig measurements and simulation output. First a single unloaded variator is validated, with only the drive motor connected. As shown in figures 14 and 15 the model predicts the experimental results quite well.

For validation of the steady state force ratio model from section 4.1.5 the rig is modified to power loop mode, with both shafts connected. Variator B is set up for a certain fixed maximum torque $T_{1,max}$, with a resulting secondary pressure. The ratio of variator B is controlled at a constant ratio. The torque is applied by changing the ratio of variator A. The pressure setpoint $p_{2a}$ of variator A is changed with the applied torque, whereas the pressure setpoint $p_{2b}$ of variator B is held constant. In this way the torque $T_1$ is increased without increasing the $T_{1,max}$ value of variator B. The result is a measurement throughout the complete torque ratio $\tau$ as defined in equation (12). In figure 16 the measured steady state force ratio $\zeta$ is compared with the model for three ratios. For low (0.5) en medium (1) ratio, the model gives good results, for overdrive there is some difference. Since the model is a simplification not all forces are taken into account. For simulation purposes however the model has sufficient accuracy.

During the experiment to determine the steady state force ratio $\zeta$, variator B is going from over clamping mode to under clamping mode. With increasing torque ratio the slip also increases from microslip to macroslip regime. With the actual torque $T_1$ and clamping force $F_{2b}$ measured the effective traction...
coefficient $\mu_{\text{eff}}$ can be calculated according to equation (8). The results are shown in figure 17 for three ratio’s. Apparently medium (1) has best traction, followed by overdrive (2). The low (0.5) ratio has worst traction. The large traction coefficient difference between the operating points can be explained when equation (8) is examined. In this equation it is assumed that slip always occurs at the secondary pulley. Since for $r_{\text{eff}}<1$ slip will occur on the primary pulley it needs to be accounted for.

For durability testing a cycle should be run at several operating points. In between the operating points a controlled behavior is required, since no excessive slip should occur. The control targets mentioned in section 3 apply for steady state in particular. In figure 18 and 19 the results of a test cycle are shown. Both torque and ratio stay within the accuracy band, even for transient. Since primary speed is controlled by an external device and is not yet adjustable it is not taken into account.

Figure (18): Primary torque $T_i$ (solid) and accuracy band (dotted) during the cycle.

Figure (19): Ratio $r_{\text{eff},B}$ (solid) and accuracy band (dotted) during the cycle.

7 CONCLUSION

A simulation model is created for controller design and to gain insight in the behavior of an unusual setup. With the frequency response measurements a controller is designed which meets the requirements. If dynamic testing on drive cycles is required in the future, redesign of the control system is needed. The validity of the model is proven by the comparison with the experimental results. The steady state force ratio model predicts the behavior adequately; it can be used on simulation level instead of time-consuming measurement data.

The test rig can be used for durability testing with measurement of all relevant variables. Furthermore it is a complete test setup for experimenting. Steady state force ratios, slip, traction coefficients and efficiencies provide valuable information for improvements on the design and actuation of a CVT. In this way more efficient transmissions can be realized.
REFERENCES


APPENDIX A

In figure 21 the forces acting on a variator and the resulting angles are shown. The pull force in the chain is indicated with $W_1$ on the tensile part and $W_2$ on the slack part. On the rest angles $\nu_1$ and $\nu_2$ the forces remain respectively $W_1$ and $W_2$. On the creep angle $\alpha$ the pull forces $W_1$ and $W_2$ converge to each other. If one pulley contact is at slip limit there will be no rest angle at the pulley involved, but only a creep angle. According to Eytelwein the following relations apply:

$$\frac{W_1}{W_2} = e^{\frac{j\alpha}{\sin\beta}}; \quad W_1 - W_2 = \frac{T}{R_i} \tag{24}$$

Geometrical relations can relate the axial clamping forces $F_1$ and $F_2$, to the pull forces $W_1$ and $W_2$. In figure 20 the acting forces are shown for one pulley. In figure 20a the relation between the normal force $dN$ and axial force part of this force $dN_x$ on an infinitesimal small part of the chain is visualized:

$$dN_x = dN \cos \beta \tag{25}$$

From figure 20b the relation between the normal force $dN$ and the radial force $F_r$ can be derived:

$$dN = \frac{dF_r}{2 \sin \beta} \tag{26}$$

The radial force $dF_r$ in the chain is a function of the tension force $W$ along the angle $d\theta$ according to figure 20c:
\[ dF_r = 2W \sin \left( \frac{d \varphi}{2} \right) \approx Wd\varphi \]  

(27)

Substitution of (27) in (26), and (26) in (25)

\[ F_{ax} = \int_{\varphi}^{\gamma} W(\varphi) \frac{d \varphi}{2 \tan(\beta)} \]  

(28)

On the rest angle \( \nu \) the pull force \( W_{\nu}(\varphi) \) is constant, and on the creep angle \( \alpha \) the pull force \( W_{\alpha}(\varphi) \) can be described with Eytelwein:

\[ W_{\nu,1,2} = W_{1,2} \wedge W_{\alpha}(\varphi) = W_{2} e^{\frac{\mu \beta}{\sin(\beta)}} \]  

(29)

With equation (29) known, equation (28) results in a general expression for the axial clamping force:

\[ F_{1,2} = \int_{0}^{\alpha} \frac{W_{2}}{2 \tan(\beta)} e^{\frac{\mu \beta}{\sin(\beta)}} d \varphi + \int_{0}^{\nu} \frac{W_{1,2}}{2 \tan(\beta)} \nu_{1,2} \]  

(30)

Integrating the terms in (30) in combination with (24) leads to the following relations for positive torque:

\[ F_{1} = \left( \frac{\cos(\beta)}{2 \mu} + \frac{\mu}{e^{\frac{\mu}{\sin(\beta)}}} \right) \frac{\gamma_{1} - \alpha}{2 \tan(\beta)} \cdot T_{1} \cdot R_{1} ; F_{2} = \left( \frac{\cos(\beta)}{2 \mu} + \frac{1}{e^{\frac{\mu}{\sin(\beta)}}} \right) \frac{\gamma_{2} - \alpha}{2 \tan(\beta)} \cdot T_{1} \cdot R_{1} \]  

(31)

and for negative torque:

\[ F_{1} = \left( \frac{\cos(\beta)}{2 \mu} + \frac{1}{e^{\frac{\mu}{\sin(\beta)}}} \right) \frac{\gamma_{1} - \alpha}{2 \tan(\beta)} \cdot T_{1} \cdot R_{1} ; F_{2} = \left( \frac{\cos(\beta)}{2 \mu} + \frac{\mu}{e^{\frac{\mu}{\sin(\beta)}}} \right) \frac{\gamma_{2} - \alpha}{2 \tan(\beta)} \cdot T_{1} \cdot R_{1} \]  

(32)

These equations are in general form, it has to be remarked that at slip limit some simplifications are possible. Since at slip limit the pulley with the smallest running radius becomes critical, substitution of \( \nu_{a} \) or \( \nu_{b} \) for \( \mu \) can be made for respectively \( r_{w0}<1 \) and \( r_{w0}>1 \). With this substitution the second term of the critical clamping force becomes zero since no over clamping is present for the pulley at slip limit. The substituted form of the formula is hereafter named \( F_{1,min} \) and \( F_{2,min} \) for respectively primary pulley critical \( (r_{w0}<1) \) or secondary pulley critical \( (r_{w0}>1) \).
For the use of equations (31) and (32) the creep angle is needed. As shown in figure 21 the following relations apply:

\[
\gamma_1 = \pi - 2\delta \\
\gamma_2 = \pi + 2\delta
\]  

(33)

The angle \(\delta\), which is ratio dependent, is needed for calculation. From the following relations it can be calculated iteratively:

\[
\sin(\delta) = \frac{R_2 - R_1}{a} \\
L = 2\delta(R_2 - R_1) + \pi(R_2 + R_1) + 2\alpha \cos(\delta) \\
\gamma_2 - \gamma_1 = 4\delta
\]  

(34)

With the angle \(\delta\) known the wrapped angles \(\gamma_1\) and \(\gamma_2\) can be calculated. When the value of the actual secondary clamping force \(F_{2,\text{actual}}\) is substituted for \(F_2\) in equation (31) or (32), the creep angle \(\alpha\) can be calculated iteratively:

\[
\alpha = f(F_{2,\text{actual}}, T_1, r_{\text{w}}, \omega, \delta)
\]  

(35)

The steady state primary clamping force, which balances this secondary force, can now be calculated with (31) or (32):

\[
F_1^* = f(\alpha, T_1, r_{\text{w}}, \omega, \delta) = f(T_1, r_{\text{w}}, \omega, F_{2,\text{actual}})
\]  

(36)

The minimal secondary clamping force \(F_{2,\text{min}}\) is calculated using the simplified equations (31) and (32) for slip limit operation, where the creep angle is equal to one of the wrapped angles:

\[
F_{2,\text{min}} = f(T_1, r_{\text{w}}, \omega, \delta)
\]  

(37)