Solution to Problem 74-17: A stability problem

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Here $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel's functions of the first and second kind, respectively, and $\nu = \sqrt{as}$, where s is the Laplace transform parameter, a is a constant and $x \neq y \neq z$.

The problem arose in a study of the chemical processes occurring by diffusion and an exchange or tunneling mechanism, and involved the solution of

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial r^2} - \alpha c e^{-\beta r},$$

where α , β and D are constants.

Problem 76-4, Geometric Probability, by IWAO SUGAI (Applied Physics Laboratory, Johns Hopkins University).

Two points are chosen at random, uniformly with respect to area, one each from the two plane regions $0 \leq x^2 + y^2 \leq a^2$ and $(a-b)^2 \leq x^2 + y^2 \leq a^2$, respectively. Find the probability P that the distance between the two points is at most b ($0 < b < a$).

This problem may have application in ASW sonar buoy deployment, triangulation tactics of disabled vehicles for automatic monitors [1], mutual visibility for satellite communication [2], and rocket re-entry impact distribution.

REFERENCES

- [1] S. RITER, W. B. JONES, JR. AND H. DOZIER, *Speeding the deployment of emergency vehicles*, IEEE Spectrum, 10 (1973), pp. 56-62.
- [2] I. SUGAI, *Probability of isotropic link connectivity using comsats in elliptic orbits*, Proc. IEEE (Correspondence), 53 (1965), pp. 541-542.

Problem 76-5, An Arithmetic Conjecture*, by D. J. NEWMAN (Yeshiva University).

To determine positive integers $a_1, a_2, \dots, a_n$ such that $S_n \equiv \sum_{i=1}^n 1/a_i < 1$ and S_n is a maximum, it is conjectured that at each choice one picks the smallest integer still satisfying the inequality constraint. For example, for $n = 4$, one would choose

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43}.$$

Problem 76-6, An n -th Order Linear Differential Equation, by M. S. KLAMKIN (University of Waterloo).

Solve the differential equation

$$[x^{2n}(D - a/x)^n - k^n]y = 0.$$

SOLUTIONS

Problem 74-17, A Stability Problem, by O. BOTTEMA (University of Delft, the Netherlands).

Unit masses are fixed at each of the 2^n vertices of a hypercube in E^n ($n \geq 1$). A test particle which can move freely in space is attracted by the unit masses according to Newton's inverse square law of gravitation. The center of the cube

is obviously a position of equilibrium for the test particle. Is this a position of stable or unstable equilibrium?

Solution by O. P. LOSSERS (Technological University, Eindhoven, the Netherlands).

We consider the more general case where the test particle is attracted with a force proportional to the k th power of the inverse of the distance. Here k is an integer satisfying $k \geq 2$. Then the total force acting on the test particle is proportional to

$$\mathbf{F}(\mathbf{x}) = - \sum_{i=1}^N (\mathbf{x} - \mathbf{a}_i) |\mathbf{x} - \mathbf{a}_i|^{-k-1},$$

where $N = 2^n$, and $\mathbf{a}_1, \dots, \mathbf{a}_N$ are the vertices of the unit cube. We have $\mathbf{F}(\mathbf{x}) = \nabla \varphi(\mathbf{x})$, where the potential φ is given by

$$\varphi(\mathbf{x}) = (k-1)^{-1} \sum_{i=1}^N |\mathbf{x} - \mathbf{a}_i|^{-(k-1)}.$$

The origin is stable if and only if φ has a local maximum at $\mathbf{x} = \mathbf{0}$. We calculate the Hessian $(\varphi_{x_i x_j})$, that is, the functional matrix $\mathbf{F}_x(\mathbf{x}) = (\partial F_i / \partial x_j)$ at $\mathbf{x} = \mathbf{0}$. If $\mathbf{f}(\mathbf{x}) = (\mathbf{x} - \mathbf{a}) |\mathbf{x} - \mathbf{a}|^{-k-1}$, then

$$\mathbf{f}_x(\mathbf{x}) = |\mathbf{x} - \mathbf{a}|^{-k-1} \mathbf{I} - (k+1) |\mathbf{x} - \mathbf{a}|^{-k-3} (\mathbf{x} - \mathbf{a})(\mathbf{x} - \mathbf{a})^T.$$

Consequently, using $|a_i| = \sqrt{n}$, $\sum_{i=1}^N a_i a_i^T = NI$ (as is easily verified), we obtain

$$\begin{aligned} \mathbf{F}_x(\mathbf{0}) &= - \sum_{i=1}^N |\mathbf{a}_i|^{-k-3} \{ |\mathbf{a}_i|^2 \mathbf{I} - (k+1) \mathbf{a}_i \mathbf{a}_i^T \} \\ &= -n^{-(k+3)/2} N(n-k-1) \mathbf{I}. \end{aligned}$$

If $n > k+1$, then $\mathbf{F}_x(\mathbf{0})$ is negative definite, hence the origin is stable. If $n < k+1$, then $\mathbf{F}_x(\mathbf{0})$ is positive definite, which implies the instability of the origin. If $n = k+1$, the potential φ is a harmonic function, $\Delta \varphi = 0$. It is well known that a harmonic function does not have maxima or minima. Hence $\mathbf{0}$ is unstable.

Also solved by D. J. BORDELON (Naval Underwater Systems Center, Newport, R.I.), N. FUNAYAMA (Yamagata University, Japan), R. MANOHAR and G. L. SAINI (University of Saskatchewan, Canada), H. B. ROSENSTOCK (Naval Research Laboratory, Washington, D.C.) and the proposer.

Problem 74-18, Constrained Minimization of an Integral Functional, by MICHAEL H. MOORE (Vector Research, Inc.).

Let $N(a, b)$ denote the set of all nonnegative functions n on R^1 with support $[-a, a]$ for which

$$\int_{-a}^a n(t) dt = b.$$