Reverse self-decomposability

Citation for published version (APA):
Universiteit Eindhoven.

Document status and date:
Published: 01/01/1989

Publisher Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
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Reverse self-decomposability

B.G. Hansen
REVERSE SELF-DECOMPOSABILITY

by

Bjørn Gårn Hansen
Eindhoven University of Technology

ABSTRACT

In this paper we consider the analog of multiple self-decomposability for characteristic functions \( \phi \) satisfying \( \phi(t) = \phi^{c^a}(c^{-1}t) \phi_c(t) \). We obtain results analogous to those known for multiple self-decomposable characteristic functions.

1. INTRODUCTION AND PRELIMINARIES

A characteristic function \( \phi \) of a random variable on \( \mathbb{R} = (-\infty, \infty) \) is said to be self-decomposable if for every \( c \in (0, 1) \) there exists a characteristic function \( \phi_c \) such that

\[
\phi(t) = \phi(ct) \phi_c(t),
\]

(cf. Lukacs (1970)). The set of all self-decomposable characteristic functions is denoted by \( L \). Urbanik (1973) considered self-decomposable characteristic functions \( \phi \), where \( \phi_c \) also has a decomposability property. He defined, inductively, the sets \( L_n \), \( n \in \mathbb{N}_0 := \{0, 1, \ldots\} \), of \( n \)-times-self-decomposable characteristic functions by \( L_0 := L \) and \( L_n := \{ \phi \mid \phi \in L_{n-1} \text{ and } \phi_c \in L_{n-1} \text{ for all } c \in (0, 1) \} \). O'Connor (1979) and (1981), Jurek (1989) and Hansen (1988) generalized the notion of self-decomposability. An infinitely divisible characteristic function \( \phi \) is said to belong to the set \( U_\alpha((c)_{c \in (0, 1)}) \) if \( \phi \) satisfies some differentiability conditions and

\[
\phi(t) = \phi^{c^a}(ct) \phi_c(t),
\]

for all \( c \in (0, 1) \). Multiple self-decomposability in the sense of (1.2) has been studied by Hansen (1989). In this paper we consider characteristic functions \( \phi \) satisfying

\[
\phi(t) = \phi^{c^a}(c^{-1}t) \phi_c(t),
\]

for all \( c \in (0, 1) \). Infinitely divisible characteristic functions satisfying (1.3) and some
differentiability condition are said to belong to the set $U_a((c^{-1})_{c\in(0,1)})$ and are called reverse $\alpha$-self-decomposable. Analogous to Urbanik (1973) we introduce the sets $U^n_a((c^{-1})_{c\in(0,1)})$ of $n$-times reverse $\alpha$-self-decomposable characteristic functions. We show that a random variable has its characteristic function in $U^n_a((c^{-1})_{c\in(0,1)})$ if and only if it is the weak limit of a sequence of partial sums of a uan triangular array of $n$-times reverse $\alpha$-unimodal random variables. The canonical form for such characteristic functions is established and it is shown that the Lévy spectral function of $\phi$ is $n$-times reverse $\alpha$-unimodal (cf. Section 2). We also characterize the sets $\cup_{a\in\mathbb{R}} U^n_a((c^{-1})_{c\in(0,1)})$ and $\cap_{n\in\mathbb{N}_0} U^n_a((c^{-1})_{c\in(0,1)})$. We conclude this section with three theorems. For the proof of the first two theorems we refer to Loève (1977) and of the third to Lukacs (1970).

**THEOREM 1.1.** A function $\phi$ is an infinitely divisible characteristic function if and only if it can be written in the form

$$
\ln \phi(t) = ita_{\phi} - \frac{1}{2} \sigma_{\phi}^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} k(t,x) \, dM(x),
$$

where $a_{\phi} \in \mathbb{R}, \sigma_{\phi}^2 \in \mathbb{R}_+$, $k(t,x) = e^{itx} - 1 - itx (1 + x^2)^{-1}$, and such that the function $M$ (called the Lévy spectral function) satisfies

(i) $M(x)$ is non-decreasing on $(-\infty,0)$ and $(0,\infty)$;

(ii) $M(-\infty) = M(\infty) = 0$;

(iii) The integrals $\int_{-1}^{0} x^2 \, dM(x)$ and $\int_{0}^{1} x^2 \, dM(x)$ are finite.

The representation is unique.

An infinitely divisible characteristic function $\phi$ is uniquely determined by the triple $[a_{\phi}, \sigma_{\phi}^2, M]$ in Theorem 1.1. We therefore write $\phi = [a_{\phi}, \sigma_{\phi}^2, M]$.

**THEOREM 1.2.** Let $X$ be a random variable with characteristic function $\phi$ and let $(X_{k,l})$, $k = 1,2,\ldots,l$, $l \in \mathbb{N}_+: = \{1,2,\ldots\}$ be a uan triangular array of random variables with distribution functions $(F_{k,l})$, $k = 1,2,\ldots,l$, $l \in \mathbb{N}_+$. There exists a sequence $(a_l)$ such that

$$
\sum_{k=1}^{l} X_{k,l} + a_l \xrightarrow{w} X \quad \text{as } l \to \infty,
$$

if and only if

...
(i) there exists a function $M$ satisfying (i) and (iii) of Theorem 1.1 such that

$$M_l := \sum_{k=1}^{l} F_{k,l} \to M \text{ as } l \to \infty$$

outside every neighbourhood of the origin.

(ii) $\lim_{l \to \infty} \lim_{I \to 0} \sum_{k=1}^{l} \int_{|x| \leq I} x^2 dF_{k,l}(x) - \left( \int_{|x| \leq I} x dF_{k,l}(x) \right)^2 = \sigma_{\phi}^2$.

Necessarily $\phi$ is infinitely divisible with $\phi = [a_{\phi}, \sigma_{\phi}^2, M]$ for some $a_{\phi} \in \mathbb{R}$.

**Theorem 1.3.** A function $\phi$ is the characteristic function of a stable distribution if and only if $\phi$ is either normal or $\phi$ can be written in the form

$$\ln \phi(t) = ita_{\phi} - c \ln |t| \delta (1 + i \beta \text{sgn}(t) w(\ln |t|, \delta))$$

where $c \geq 0$, $1|\beta| \leq 1$, $\delta \in (0,2)$ and $a_{\phi} \in \mathbb{R}$. The function $w(\ln |t|, \delta)$ is given by

$$w(\ln |t|, \delta) = \begin{cases} \tan(\pi \delta/2) & \text{if } \delta \neq 1 \\ -(2/\pi) \ln |t| & \text{if } \delta = 1 \end{cases}$$

Equivalently, $\phi$ is the characteristic function of a stable distribution if and only if $\phi$ is infinitely divisible and either (cf. Theorem 1.1) $\sigma_{\phi}^2 > 0$ and $M(x) \equiv 0$ or $\sigma_{\phi}^2 = 0$ and $M(x) = C_\delta |x|^\delta$ for $x < 0$ and $M(x) = -C_\delta |x|^{-\delta}$ for $x > 0$. The parameters satisfy $\delta \in (0,2)$, $C_- \geq 0$, $C_+ \geq 0$ and $C_- + C_+ \geq 0$. The parameter $\delta$ is called the exponent of stability of $\phi$ and we write $\phi_{\text{st}}(\delta)$.

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2. **REVERSE $\alpha$-UNIMODALITY**

A random variable $X$ with distribution function $F$ and density $f$ is said to be unimodal, with mode at $x_0$ (not necessarily unique), if $f(x)$ is non-decreasing for $x < x_0$ and non-increasing for $x > x_0$. We will throughout this paper assume that $x_0 = 0$, i.e., if a function is said to be unimodal (or (n-times reverse) $\alpha$-unimodal) it is understood that its mode is at the origin. Khintchine (1938) showed that $X$ is unimodal (at zero) if $X = UY$, with $U$ and $Y$ independent and $U$ uniformly distributed on $(0,1)$. Olshen and Savage (1970) generalized this concept; a random variable is said to be $\alpha$-unimodal (at zero) if it is of the form $U^{1/\alpha}Y$, with $U$ and $Y$ independent, $U$ uniformly distributed on $(0,1)$ and $\alpha > 0$. If $Y$ has distribution function $G$, then

$$f(x) = \alpha x^{\alpha-1} \int_{x}^{\infty} v^{-\alpha} dG(v), \ x \in \mathbb{R}_+,$$

$$f(x) = \alpha x^{\alpha-1} \int_{-\infty}^{x} v^{-\alpha} dG(v), \ x \in \mathbb{R}_-.$$
This result corresponds to Corollary 2, p. 28 in Olshen and Savage (1970). Hence, f is \( \alpha \)-unimodal if and only if \( |x|^{1-\alpha} f(x) \) is non-decreasing on \((-\infty,0)\) and non-increasing on \((0,\infty)\). We will be interested in the reverse notion of \( \alpha \)-unimodality, that is in functions \( f \) where \( |x|^{1+\alpha} f(x) \) is non-increasing on \((-\infty,0)\) and non-decreasing on \((0,\infty)\). Let \( \alpha > 0 \). A random variable \( X \) is said to be reverse \( \alpha \)-unimodal (at zero) if it is of the form \( U^{-1/\alpha} Y \), with \( U \) and \( Y \) independent and \( U \) uniformly distributed on \((0,1)\). If \( Y \) has distribution function \( G \) and \( X \) density \( f \), then

\[
f(x) = \alpha x^{-\alpha-1} \int_0^x v^\alpha dG(v), \quad x \in \mathbb{R}_+,
\]

or equivalently

\[
F(x) = \alpha \int_0^x G(v) v^\alpha dv.
\]

By iteration we see that a random variable \( X \) is of the form \( X = U_1^{-1/\alpha} \cdots U_{n+1}^{-1/\alpha} Y \), with \( U_i, i = 1, 2, ..., n+1 \), mutually independent, independent of \( Y \) and all uniformly distributed on \((0,1)\) if and only if (here we use \( \mathbb{R}_- := (-\infty,0)\))

\[
f(x) = \alpha^{n+1} (n!)^{-1} x^{-\alpha-1} \int_0^x (\ln(x/v))^n v^\alpha dG(v), \quad x \in \mathbb{R}_+, \tag{2.1a}
\]

\[
f(x) = \alpha^{n+1} (n!)^{-1} |x|^{-\alpha-1} \int_0^x (\ln(x/v))^n |v|^\alpha dG(v), \quad x \in \mathbb{R}_-, \tag{2.1b}
\]

or equivalently

\[
F(x) = \alpha^{n+1} (n!)^{-1} \int_0^1 G(v) (\ln v)^n v^{-\alpha-1} dv.
\]

We will use the notion of reverse \( \alpha \)-unimodality in connection with Lévy spectral functions, so a more general definition is needed.

**Definition 2.1.** A function \( f \) is said to be \( n \)-times reverse \( \alpha \)-unimodal and belong to the set \( \mathcal{R}_\alpha^n \) for some \( \alpha \in \mathbb{R} \) and some \( n \in \mathbb{N}_0 \), if there exists constants \( C_+, C_- \in \mathbb{R}_+ \) and a right continuous function \( N \), non-decreasing on \((-\infty, 0)\) and \((0, \infty)\) such that

\[
f(x) = \begin{cases} 
  x^{-\alpha-1} \left( \int_0^x (\ln(x/v))^n v^\alpha dN(v) + C_+ \right) & x > 0, \\
  0 & x = 0, \\
  |x|^{-\alpha-1} \left( \int_x^0 (\ln(x/v))^n |v|^\alpha dN(v) + C_- \right) & x < 0.
\end{cases} \tag{2.2}
\]
and such that the integrals converge for every \( x \in \mathbb{R} \).

It is clear that \( R^n_\alpha \) form an increasing sequence of sets in \( \alpha \) and a decreasing sequence of sets in \( n \). Let \( F \) be absolutely continuous with Radon-Nikodym derivative \( F' = f \), let \( \alpha \neq 0 \) and let \( xy > 0 \). It can be shown that (2.2) is equivalent to

\[
F(x) - \lambda = \int_0^1 N(vx) (\ln v^{-1})^{n} v^{x - 1} C_{\text{sign}(x)} |x|^{-\alpha},
\]

for some \( \lambda \in \mathbb{R} \). We will henceforth assume, without loss of generality, that \( \lambda = 0 \). In the rest of this section we study the functions in \( R^n_\alpha \) closer, in particular the distribution functions in \( R^n_\alpha \) and we also give some properties of characteristic functions having Lévy spectral functions \( M \) with Radon-Nikodym derivative \( M' \in R^n_\alpha \). The following theorem is an immediate consequence of Definition 2.1.

**Theorem 2.2.** The set \( R^n_\alpha \) is closed under limits.

It can be verified that \( N \) in (2.2) is, up to additive constants, unique. Let \( T_{\epsilon} \) be a linear operator, acting on sets, and defined by \( T_{\epsilon}B = \{ cb \mid b \in B \} \), for any Borel set \( B \). Also, denote the Raydon-Nikodym derivative of the function \( F \) (or \( M \)) by \( F' \) (or \( M' \)). We first prove a lemma which shows that a function is in \( R^n_\alpha \) if and only if it satisfies some inequalities.

**Lemma 2.3.** Let \( \alpha \in \mathbb{R}, \beta_\epsilon \) be the set of Borel sets on \( \mathbb{R} \setminus (-\epsilon, \epsilon) \), for any \( \epsilon > 0 \) and let \( c(k) = (c_1, c_2, \ldots, c_k), \ k \in \mathbb{N}_+ \). Define, inductively, the functions \( F_{c(k)} \) by \( F_{c(0)} = F \) and

\[
F_{c(k)}(x) := F_{c(k-1)}(x) - c_k \cdot F_{c(k-1)}(c_k x), \ k \in \mathbb{N}_+,
\]

for all \( c_i \in (0, 1), 1 \leq i \leq k \). \( F \) is absolutely continuous on \( \mathbb{R} \setminus \{0\} \) and \( F' \in R^n_\alpha \) if and only if for all \( c \in (0, 1) \) and all \( c_k \in (0, 1), 1 \leq k \leq n \)

\[
F_{c(n)}(B) \geq c^\alpha F_{c(n)}(T_{\epsilon}B) \text{ and } F_{c(n)}(B) \geq 0 \text{ for all } B \in \beta_\epsilon.
\]

**Proof.** Let \( \alpha > 0 \) and let \( F_{c(n)} \) satisfy (2.5). Fix \( \epsilon > 0 \) and let \( B = (a, b), 0 < \epsilon \leq a < b \). Then (2.5) is equivalent to

\[
F_{c(n)}(b) - F_{c(n)}(a) \leq c^{-\alpha} (F_{c(n)}(b/c) - F_{c(n)}(a/c)).
\]

Let \( w(x) := F_{c(n)}(x^{-1/\alpha}) \). Let \( x = (b/c)^{-\alpha}, y = (a/c)^{-\alpha}, x' = b^{-\alpha} \) and \( y' = a^{-\alpha} \). It follows that

\[
\frac{w(y') - w(x')}{y' - x'} \leq \frac{w(y) - w(x)}{y - x}.
\]
Hence $w$ is convex. By Theorem A, p. 4, Roberts and Varberg (1973), $w$ and hence also $F_{c(n)}$ is absolutely continuous on $(e, \infty)$. Observe that

$$\int_{B} F'_{c(n)}(x) \, dx = F_{c(n)}(B) \geq c^{\alpha} F_{c(n)}(T_{c}B) = c^{\alpha+1} \int_{B} F'_{c(n)}(cx) \, dx ,$$

(2.6)

with $B = (a, b)$. Differentiating (2.6) with respect to $a$ and multiplying both sides by $a^{1+\alpha}$, we see that $a^{1+\alpha} F'_{c(n)}(a)$ is non-decreasing on $\mathbb{R}_{+}$. Similarly for $b < a \leq e < 0$ we obtain that $a^{1+\alpha} F'_{c(n)}(a)$ is non-increasing on $\mathbb{R}_{-}$. Hence $F_{c(n)}' \in R^{0}_{\alpha}$ and so $F_{c(n)}$ is of the form (2.3). Suppose that $n > 0$. Since $F_{c(n)}(B) \geq 0$, we see from (2.4) that $F_{c(n-1)}$ satisfies (2.5). By (2.4) $F_{c(n-1)}$ is non-decreasing on $\mathbb{R} \setminus \{0\}$ and so $F'_{c(n-1)}$ exists and $F'_{c(n-1)} \in R^{0}_{\alpha}$. For $x > 0$

$$N_{n}(x) := \lim_{c_{n} \uparrow 1} (1-c_{n})^{-1} F_{c(n)}(x) = \frac{\partial}{\partial c_{n}} c_{n} \left. F_{c(n-1)}(c_{n}x) \right|_{c_{n}=1}$$

$$= \alpha F_{c(n-1)}(x) + x F_{c(n-1)}(x) = x^{1-\alpha} \frac{\partial}{\partial x} x^{\alpha} F_{c(n-1)}(x),$$

From (2.3) it follows that $x^{\alpha} F_{c(n)}(x) \to -\alpha^{-1} C_{+}$ as $x \downarrow 0$, and so

$$F_{c(n)}(x) = x^{-\alpha} \left[ \int_{0}^{x} N_{n}(v) v^{\alpha-1} \, dv + C_{+} \right] = \int_{0}^{1} N_{n}(vx) v^{\alpha-1} \, dv + C_{+} x^{-\alpha} .$$

Similarly for $x < 0$. By Theorem 2.2, $N'_{n} \in R^{0}_{\alpha}$ and so $F'_{c(n-1)} \in R^{1}_{\alpha}$. Proceeding similarly we see that $F'_{c(0)} := F' \in R^{n}_{\alpha}$. The proof for $\alpha \leq 0$ is almost identical and therefore omitted.

Conversely, suppose $F$ is absolutely continuous with $F' \in R^{n}_{\alpha}$. Then (cf. (2.3))

$$F(x) = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} N(v \ldots v_{n}x)(v_{1} \ldots v_{n})^{\alpha-1} \, dv_{n} \ldots dv_{1} - \alpha^{-1} C_{\text{sign}}(x) \, |x|^{-\alpha} .$$

Obviously $F_{c(n)}(B) \geq 0$ and

$$F_{c(n)}(x) = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} N(v \ldots v_{n}x)(v_{1} \ldots v_{n})^{\alpha-1} \, dv_{n} \ldots dv_{1} - \alpha^{-1} C_{\text{sign}}(x) \, |x|^{-\alpha}$$

$$= \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} N_{c(1)}(v \ldots v_{n-1}x)(v_{1} \ldots v_{n-1})^{\alpha-1} \, dv_{n-1} \ldots dv_{1}$$

$$- \alpha^{-1} C_{\text{sign}}(x) \, |x|^{-\alpha} ,$$

where

$$N_{c(1)}(x) := \int_{0}^{1} N(v_{n}x) v_{n}^{\alpha-1} \, dv_{n} .$$

Proceeding similarly $n-2$ times we have that
\[ F_{c(n)}(x) = \int_{0}^{1} N_{c(n)}(v_1 x) v_1^{\alpha-1} dv_1 - \alpha^{-1} C_{\text{sign}(x)} |x|^{-\alpha}. \]

Hence
\[
F_{c(n)}(B) = \int_{0}^{1} N_{c(n)}(T_{v_1} B) v_1^{\alpha-1} dv_1 - \alpha^{-1} \int_{B} C_{\text{sign}(x)} |x|^{-\alpha} dx
\]
\[
= c^\alpha \int_{0}^{1} N_{c(n)}(T_{uc} B) u^{\alpha-1} du - c^\alpha \alpha^{-1} \int_{T_{uc} B} C_{\text{sign}(x)} |x|^{-\alpha} dx
\]
\[
\geq c^\alpha \int_{0}^{1} N_{c(n)}(T_{uc} B) u^{\alpha-1} du - c^\alpha \alpha^{-1} \int_{T_{uc} B} C_{\text{sign}(x)} |x|^{-\alpha} dx
\]
\[
= c^\alpha F_{c(n)}(T_{uc} B).
\]

**Remark 2.4.** It can be shown (cf. the proof of Lemma 2.3) that a distribution function has a density in \( R_{c}^0 \) if and only if \( \alpha > 0 \) and for any \( c \in (0,1) \) there exists a distribution function \( F_c \) such that
\[
F(x) = c^\alpha F(cx) + (1 - c^\alpha) F_c(x).
\] (2.7)

Let \( X \) and \( X' \) have distribution function \( F \), \( X_c \) have distribution function \( F_c \) and define the random variable \( Y_c \) by
\[
\mathbb{P}(Y_c = 1) = c^\alpha = 1 - \mathbb{P}(Y_c = 0).
\]

Then (2.7) is equivalent to
\[
X = Y_c c^{-1} X' + (1 - Y_c) X_c,
\]
with \( X, X', X_c \) and \( Y_c \) all independent. Let \( DF \) denote the set of all distribution functions. Define the sets \( F_{\alpha}^n \) for all \( c \in (0,1) \) by \( F_{\alpha}^{-1} := DF \) and
\[
F_{\alpha}^n := \{ F \in F_{\alpha}^{-1} | F \text{ satisfies (2.7) with } F_c \in F_{\alpha}^{-1} \}, n \in \mathbb{N}_0.
\]

Proceeding as in the proof of Lemma 2.3 and using the above (see also the proof of Theorem 3.4) it can be verified that the following statements are equivalent.

(i) \( F \in F_{\alpha}^{n}((c^{-1})_{c \in (0,1)}) \);

(ii) There exists a unique distribution function \( G \) such that
\[
F(x) = \alpha^{n+1} (n!)^{-1} \int_{0}^{1} G(xv) (ln v^{-1})^{n} v^{\alpha-1} dv;
\]
(iii) \( F \) is absolutely continuous and \( F' \in R^n \).

It is well known that the sum of two unimodal random variables need not be unimodal (cf. for example Olshen and Savage (1970)). With the help of Theorem 1.2 we characterize in the following theorem the random variables which can be obtained as the weak limit of a sequence of partial sums of \( n \)-times reverse \( \alpha \)-unimodal random variables.

**Theorem 2.5.** Let \( \alpha > 0 \). There exists a uan triangular array \((X_{k,1}), k = 1, 2, ..., l, l \in \mathbb{N}_+\), with \( X_{k,1} \) \( n \)-times reverse \( \alpha \)-unimodal (i.e., its distribution function \( F_{k,1} \) has a density \( f_{k,1} \) of the form (2.1) for some distribution function \( G_{k,1} \) and a sequence \((a_l)\) such that

\[
\sum_{k=1}^{l} X_{k,1} + a_l \to X \text{ as } l \to \infty.
\]

if and only if \( X \) is a random variable with characteristic function \( \phi = [a_\varphi, \sigma^2_\varphi, M] \) where \( M \) is absolutely continuous and \( M' \in R^n \). Necessarily \( \sum_{k=1}^{l} G_{k,1} \to N \) as \( l \to \infty \) outside every neighbourhood of the origin with \( M' \) and \( N \) related by (2.2) with \( f \) replaced by \( M' \).

**Proof.** The necessity follows from Theorem 1.2 and Theorem 2.2. Suppose \( \phi = [a_\varphi, \sigma^2_\varphi, M] \), with \( M' \in R^n \). We consider three cases.

**Case 1.** Let \( M(0+) = -\infty \) and \( M(0-) = \infty \). Let \((b_l)\) and \((c_l)\) be two sequences with \( b_l \downarrow 0, c_l \uparrow 0 \) as \( l \to \infty \) and such that

\[
-M(b_l) = \frac{1}{2} l = M(c_l).
\]

Also let \( F_l(x) = 1 + M(x) l^{-1} \) if \( x \geq b_l \), \( F_l(x) = M(x) l^{-1} \) if \( x \leq c_l \) and \( F_l(x) = M(c_l) l^{-1} \) if \( x \in (c_l, b_l) \). Hence \( F_l \) is a distribution function with an \( n \)-times reverse \( \alpha \)-unimodal density, i.e., it has a density almost everywhere of the form (2.1) for some distribution function \( G_l \). Note that for any Borel set \( B \), bounded away from the origin, there exists an \( l_0 \) such that for all \( l > l_0 \), \( IF_l(B) = M(B) \) and that for all \( l \in \mathbb{N}_+ \), \( IF_l(B) \leq M(B) \). Let \( X_{k,1} \) have distribution function \( F_l \) for each \( 1 \leq k \leq l \). Necessarily \((X_{k,1}), k = 1, 2, ..., l, l \in \mathbb{N}_+ \) is uan and

\[
0 \leq l \left( \int_{|x| \leq \infty} x dF_l(x) \right)^2 \leq l \left( \int_{|x| \leq \infty} x^2 dF_l(x) \right) \leq \int_{|x| \leq \infty} x^2 dM(x).
\]

By Theorem 1.2, \( \sum_{k=1}^{l} X_{k,1} \to X \) as \( l \to \infty \), with \( \sigma_\varphi = 0 \). Since \( IF_l(B) = M(B) \) for sufficiently large \( l \) we have that \( IF_l'(x) = M'(x) \) for sufficiently large \( l \) and hence by the uniqueness of the measure \( N \) in (2.2), \( IG_l(B) = N(B) \) for sufficiently large \( l \). Thus \( N \) is the weak limit of \( l G_l \) as \( l \to \infty \) outside every neighbourhood of the origin.
CASE II. Let $M(0+) > -\infty$ and $M(0-) < \infty$. Assume without loss of generality that $M(\mathbb{R}) = 1$. Let $F_i(B) := l^{-1}M(B) + (1-l^{-1})1_0(B)$, where $1_0(B) = 0$ if $0 \notin B$ and $1_0(B) = 1$ if $0 \in B$. The rest of the proof is as in case I.

CASE III. If $M(0+) > -\infty$ and $M(0-) = \infty$ or $M(0+) = -\infty$ and $M(0-) < \infty$, then by combining the approaches in cases I and II we can prove the theorem.

From Theorem 2.5 we see that $\sum_{k=1}^{l} G_{k,l} \to N$ as $l \to \infty$, with $M'$ and $N$ related by (2.2) where $f$ is replaced by $M'$. In view of Theorem 1.2 we can expect that $N$ is a Lévy spectral function. We prove this in the following lemma.

**Lemma 2.6.** Let $\alpha > 0$ and let $M$ be such that $M' \in R_0^\alpha$, i.e.,

$$M'(x) = \begin{cases} 
  x^{-\alpha-1} \left( \int_0^x (\ln(x/v))^\alpha dN(v) + C_+ \right) & x > 0 \\
  l x^{-\alpha-1} \left( \int_x^0 (\ln(x/v))^\alpha dN(v) + C_- \right) & x < 0 
\end{cases} \quad (2.8)$$

with the integrals converging for every $x \neq 0$. Then $M$ is a Lévy spectral function if and only if $N$ is a Lévy spectral function with $C_+ = C_- = 0$ for $\alpha \geq 2$.

**Proof.** Let $n = 0$. First let $M$ be a Lévy spectral function satisfying (2.8) with $x > 0$. From (2.8) it follows that for $\alpha \neq 2$

$$\int_0^1 u^2 dM(u) = \int_0^1 \int_0^x x^{-\alpha-1} v^\alpha dN(v) dx + C_+ \int_0^1 x^{-\alpha-1} dx$$

$$= (2-\alpha)^{-1} \left[ (\int_0^1 v^\alpha dN(v) - \delta^{2-\alpha} \int_0^1 \delta^\alpha dN(v) \right]^x_0$$

$$- \int_0^1 \nu^2 dN(v) + C_+ \delta^{2-\alpha}. \quad (2.9)$$

Since both integrals on the right hand side of the first equality are positive, both must converge as $\delta \downarrow 0$. Hence $C_+ = 0$ if $\alpha \geq 2$. Let $\alpha > 2$. Since the second and third integrals on the right hand side of (2.9) are positive, both must converge as $\delta \downarrow 0$. Hence $N$ satisfies condition (iii) of Theorem 1.1. Let $\alpha \leq 2$. Since $|x|^\alpha \geq |x|^2$ for $|x| \leq 1$, $\int_0^1 x^2 dN(x) < \infty$ and so condition (iii) of Theorem 1.1 is also met in this case. From (2.3) with $n = 0$ it follows that

$$M(x) = \int_0^1 N(vx) v^{\alpha-1} dv - \alpha^{-1} C_+ x^{-\alpha}. \quad (2.10)$$
Since \( M(\infty) = 0 \) necessarily \( |N(\infty)| < \infty \), and so we may take \( N(\infty) = 0 \). Hence \( N \) satisfies condition (ii) of Theorem 1.1. Similarly for \( x < 0 \).

Conversely, if \( N \) satisfies (ii) and (iii) of Theorem 1.1, then by (2.10), \( M \) must satisfy (ii) and since the second integral on the right hand side of (2.9) is negative, \( M \) also satisfies (iii) of Theorem 1.1.

The lemma is now easily proved for \( n > 0 \) by induction.

In Theorem 2.5 and Lemma 2.6 we use explicitly that \( \alpha > 0 \). Infact, if \( M \) is a \( \text{Lévy} \) spectral function, \( M' \in R_0^\alpha \) with \( \alpha \leq 0 \) and \( u > v > 0 \), then

\[
M(u) - M(v) = \int_v^u M'(x) \, dx \geq vM'(v) (\ln(u/v)).
\]

Hence \( M(\infty) = \infty \) which is a contradiction. So \( \alpha \) must be positive. Since \( \phi \) is infinitely divisible, \( \phi \) has no real zeros (cf. Lukacs (1970)) and hence \( \ln \phi(t) \) is a well defined function for \( t \in \mathbb{R} \). We now prove a lemma which gives the canonical form of infinitely divisible characteristic functions with \( M' \in R_0^\alpha \) (compare with Remark 2.4).

**Lemma 2.7.** Let \( \alpha > 0 \). If \( \phi \) is an infinitely divisible characteristic function with an absolutely continuous \( \text{Lévy} \) spectral function \( M \) where \( M' \in R_0^\alpha \), then there exists an infinitely divisible characteristic function \( \gamma \), a stable characteristic function \( \phi_{\text{STABLE}(\alpha)}(t) \), possibly degenerate, and constants \( a, \sigma \in \mathbb{R} \) such that

\[
\ln \phi(t) = iat - \frac{1}{2}\sigma^2 t^2 + \int_0^1 \ln \gamma(v^{-1}t) (\ln v^{-1})^n v^{\alpha-1} dv + \ln \phi_{\text{STABLE}(\alpha)}(t),
\]

with \( \phi_{\text{STABLE}(\alpha)}(t) \equiv 1 \) if \( \alpha > 2 \).

**Proof.** Suppose that \( M' \in R_0^\alpha \). By Theorems 1.1 and 1.3 and Lemma 2.6 we have

\[
\int k(t,x) \, dM(x) = \int k(t,x) \left( \int_0^x (\ln x/v)^n (v/x)^\alpha x^{-1} \, dN(v) \, dx \right)
+ \int k(t,x) C_{\text{sign}(x)} x^{-\alpha-1} \, dx
= \int_0^1 k(t,v/u) (\ln u^{-1})^n u^{\alpha-1} du \, dN(x) + \ln \phi_{\text{STABLE}(\alpha)}(t)
\]

We consider two cases.

**Case 1.** Let \( \alpha > 1 \). In this case

\[
\int \int_{\mathbb{R}\setminus\{0\}} [k(t,v/u) - k(t/u,v)] (\ln u^{-1})^n u^{\alpha-1} du \, dN(v) := ia't
\]
converges and the lemma is proved.

**CASE II.** Let $\alpha \in (0, 1]$. Since the integral in (2.8) converges,

\[
\int_{\mathbb{R}\setminus\{0\}} (\ln u)^n u^{\alpha - 1} (v/u) / (1 + (v/u)^2) \, du \, dN(v) := a' < \infty; \\
\int_{\mathbb{R}\setminus\{0\}} x / (1 + x^2) \, dN(x) := a'' < \infty,
\]

and so

\[
\int_{\mathbb{R}\setminus\{0\}} k(t, x) \, dM(x) = \int_{\mathbb{R}\setminus\{0\}} [ \int_{\mathbb{R}\setminus\{0\}} k(t/u, x) \, dM(x) + ia''t/u ] \, u^{\alpha - 1} \, du .
\]

Theorem 2.5 shows that the set of infinitely divisible characteristic functions with $M' \in \mathbb{R}_A^\infty$ is the solution of a central limit problem. This central limit problem provided the motivation for this work. In the next section we study this set closer. Our starting point is however different than that of Theorem 2.5, but similar to that used by for example Lukacs (1970) to define the set of self-decomposable characteristic functions.

### 3. REVERSE SELF-DECOMPOSABILITY

Let $ID$ denote the set of all infinitely divisible characteristic functions. We begin with two definitions.

**Definition 3.1.** The characteristic function $\phi$ is said to belong to $U_{\alpha}((c^{-1})_{c \in (0, 1)})$, for some $\alpha \in \mathbb{R}$, if $\phi$ is infinitely divisible, $\phi'(t)$ exists for $t \neq 0$, $t \phi'(t) \to 0$ as $t \to 0$ and for every $c \in (0, 1)$ there exists a characteristic function $\phi_c$ such that

\[
\phi(t) = \phi^c(c^{-1} t) \phi_c(t), \quad t \in \mathbb{R}.
\]

**Definition 3.2.** Define the sets $U^n_{\alpha}((c^{-1})_{c \in (0, 1)})$, $n \in \mathbb{N}_0$, (or $U^n_{\alpha}$ for short), inductively by $U^n_{\alpha}((c^{-1})_{c \in (0, 1)}) := U_{\alpha}((c^{-1})_{c \in (0, 1)})$ and

\[
U^n_{\alpha} := \{ \phi \in U^{n-1}_{\alpha} \mid \text{for all } c \in (0, 1), \phi_c \in U^{n-1}_{\alpha} \}, n \in \mathbb{N}_+. 
\]
It is evident from Definition 3.2 that \( U^n_\alpha ((c^{-1})_{c\in(0,1)}) \), \( n \in \mathbb{N}_0 \), form a decreasing sequence of sets for each fixed \( \alpha \), i.e.,

\[
U_\alpha := U^0_\alpha \supseteq U^1_\alpha \supseteq \ldots \supseteq U^n_\alpha := \bigcap_{n \in \mathbb{N}_0} U^n_\alpha .
\]

We will need the following lemma, which gives a canonical representation of \( \phi_c \) in Definition 3.1.

**Lemma 3.3.** Let \( \alpha > 0 \) and let \( \phi = [a_\phi, \sigma_\phi^2, M] \). If \( \phi \in U_\alpha ((c^{-1})_{c\in(0,1)}) \) then there exists a unique infinitely divisible characteristic function \( \gamma \) such that

\[
\ln\phi_c(t) = \int \frac{1}{c} \ln \gamma(v^{-1}t) v^{\alpha-1} dv , \quad (3.2)
\]

and so \( \phi_c \in ID \) for every \( c \in (0,1) \).

**Proof.** Suppose \( \phi \in U_\alpha ((c^{-1})_{c\in(0,1)}) \). For any \( v \in (0,1) \) and any \( r > 0 \) let \( (c_l) \) be an increasing sequence with \( c_l \uparrow 1 \) as \( l \to \infty \) and such that \( v^{\alpha-1} r(1-c_l)^{-1} \in \mathbb{N}_+ \). By (3.1)

\[
v^{\alpha-1} \ln \gamma_{r,l}(v^{-1}t) := \ln \phi_{c_l}(v^{-1}t) v^{\alpha-1} r(1-c_l)^{-1} \]

is the logarithm of a characteristic function. Letting \( l \to \infty \) in (3.3) yields

\[
v^{\alpha-1} \ln \gamma_{r,l}(v^{-1}t) = r \frac{\partial}{\partial v} v^{\alpha-1} \ln \phi(v^{-1}t) = \alpha v^{\alpha-1} \ln \phi(v^{-1}t) - r v^{\alpha-2} \frac{\phi'(v^{-1}t)}{\phi(v^{-1}t)} .
\]

\( \phi \) is continuous at zero with \( \phi(0) = 1 \) and by assumption \( t\phi'(t) \to 0 \) as \( t \to 0 \), so \( \gamma_{r,1} \) is continuous at zero with \( \gamma_{1}(0) = 1 \) and hence by the continuity theorem for characteristic functions, \( \gamma_{r,1}^{\alpha} \) is a characteristic function for every \( r > 0 \) and every \( v \in (0,1) \), and thus \( \gamma := \gamma_1 \) is an infinitely divisible characteristic function. Integrating over \( (c,1) \) in \( v \) and rearranging we get

\[
\ln \phi_c(t) := \ln \phi(t) - c^\alpha \ln \phi(c^{-1}t) = \int \frac{1}{c} \ln \gamma(v^{-1}t) v^{\alpha-1} dv .
\]

Hence \( \phi_c \) is of the form (3.2) and it is infinitely divisible. \( \square \)

We are now ready to prove the main theorem of this section (compare with Remark 2.4).

**Theorem 3.4.** Let \( \alpha > 0 \), let \( n \in \mathbb{N}_0 \) and let \( \phi = [a_\phi, \sigma_\phi^2, M] \). The following statements are equivalent.
(i) \( \phi \in U^\alpha_a ((c^{-1})_{c \in (0,1)}) \)

(ii) There exists a unique infinitely divisible characteristic function \( \gamma \) and a stable, possibly degenerate, characteristic function \( \phi_{\text{STABLE}(\alpha)} \) such that

\[
\ln \phi(t) = \int_0^1 \ln \gamma(v^{-1} t) (\ln v^{-1})^\alpha v^{\alpha-1} dv + \ln \phi_{\text{STABLE}(\alpha)}(t),
\]

with \( \phi_{\text{STABLE}(\alpha)}(t) = 1 \) if \( \alpha > 2 \).

(iii) \( \sigma = 0 \) if \( \alpha < 2 \) and \( M \) is absolutely continuous with \( M' \in R^\alpha_a \).

PROOF. Lemma 2.7 proves (iii) \( \Rightarrow \) (ii). That (ii) \( \Rightarrow \) (i) is proved as in the first part of the converse part of the proof of Lemma 2.3. Suppose (i) holds. By Lemma 3.3, \( \phi_c \in ID \).

Let \( c(k) := (c_1, ..., c_k) \), \( k \in \mathbb{N}_+ \) and define the characteristic functions \( \phi_c(k) \) inductively, by \( \phi_c(0) := \phi \) and

\[
\ln \phi_c(k)(t) := \ln \phi_c(k-1)(t) - c_k^\alpha \ln \phi_c(k-1)(c_k^{-1} t).
\]

If \( M_c(k) \) is the Lévy spectral function of \( \phi_c(k) \), then \( M_c(k) \) is defined by (2.4) with \( F \) replaced by \( M \). By Definition 3.2, \( \phi_c(n) \in U^0_a ((c^{-1})_{c \in (0,1)}) \), i.e.,

\[
\ln \phi_c(t) := \ln \phi_c(n)(t) - c^\alpha \ln \phi_c(n)(c^{-1} t), \ c \in (0,1),
\]

for some \( \phi_c \in ID \). Let \( M_c \) be the Lévy spectral function of \( \phi_c \). Then

\[
M_c(x) = M_c(n)(x) - c^\alpha M_c(n)(cx)
\]

(cf. Lukacs (1970), p. 163), and so \( M_c(n) \) satisfies the inequality in Lemma 2.3. By Lemma 2.3, \( M' \in R^\alpha_a \).

If \( \phi \) is stable with exponent \( \delta \), then \( \phi \) has a Lévy spectral function \( M \) with \( M' \in R^\delta_a \) and hence \( M \) is such that \( M' \in R^\alpha_a \) for all \( \alpha \geq \delta \). We thus have the following corollary to Theorem 3.4 (iii).

**COROLLARY 1.** \( \phi \) is stable with exponent \( \delta \), then \( \phi \in U^\alpha_a ((c^{-1})_{c \in (0,1)}) \) for every \( \alpha \geq \delta \).

**COROLLARY 2.** If \( \alpha \leq 0 \) then \( U^\alpha_a ((c^{-1})_{c \in (0,1)}) \) contains only degenerate characteristic functions.

Since \( \phi \) is given by (3.4) and \( \phi_c \) by (3.2) it follows from (3.1) with \( c^\alpha = n^{-1} \) that

**COROLLARY 3.** If \( \phi \in U^\alpha_a ((c^{-1})_{c \in (0,1)}) \), \( \alpha \in (0, 2] \), then

\[
\lim_{n \to \infty} n^{-1} \ln \phi(n^{1/\alpha} t) \to \ln \phi_{\text{STABLE}(\alpha)}(t).
\]
COROLLARY 4. \( U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \) is a multiplication semigroup, closed under limits.

In the next theorem we prove a relationship between \( \phi \) and \( \gamma \) in (3.4) when \( n = 0 \).

**THEOREM 3.5.** Let \( \phi \) and \( \gamma \) be related by

\[
\ln \phi(t) = \int_0^1 \ln \gamma(v^{-1} t) v^{\alpha - 1} \, dv + \ln \Phi_{\text{STABLE}}(\alpha)(t),
\]

and let \( n \in \mathbb{N}_+ \). \( \phi \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \) if and only if \( \gamma \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \) and \( \gamma \) has no stable component with exponent \( \alpha \).

**PROOF.** Let \( \phi \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \), i.e., let \( \phi \) satisfy (3.1) with \( \phi_c \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \) for every \( c \in (0,1) \). From (3.2) we have that

\[
\ln \gamma(t) = \lim_{c\uparrow 1} (1 - c)^{-1} \ln \phi_c(t).
\]

Also from Lemma 3.3, \( \phi_c \) has no stable component with exponent \( \alpha \) and so neither does \( \gamma \). Since \( \phi_c \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \subseteq \text{ID} \), we have that \( \phi_c^{v(1-c)^{-1}} \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \). By Corollary 4 to Theorem 3.4, \( U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \) is closed under limits and hence \( \gamma \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \).

Conversely, let \( \gamma \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \) (with no stable component with exponent \( \alpha \)). Observe that

\[
\ln \phi_c(t) = \ln \phi(t) - c^\alpha \ln \phi(c^{-1} t)
\]

\[
= \int_0^1 \left[ \ln \gamma(v^{-1} t) - c^\alpha \ln \gamma(v^{-1} c^{-1} t) \right] v^{\alpha - 1} \, dv
\]

\[
= \int_0^1 \ln \gamma_c(v^{-1} t) v^{\alpha - 1} \, dv.
\]

Obviously, if \( \gamma \) can be decomposed \( n \) times in this fashion, then so can \( \phi \). Hence \( \phi \in U^n_{\alpha}((c^{-1})_{c\in(0,1)}) \).

We can also prove the equivalence between (i) and (ii) in Theorem 3.4, by applying induction to Theorem 3.5. In the next remark we give necessary and sufficient conditions for the convergence of the integral in (3.4).

**REMARK 3.6.** Let \( M \) and \( N \) be the Lévy spectral functions of \( \phi \) and \( \gamma \), respectively, in (3.4). From the proof of Theorem 3.4, \( M \) and \( N \) are related by (2.8). Hence,
(i) if $\alpha > 2$, then the integral in (3.4) converges for all infinitely divisible characteristic functions $\gamma$;

(ii) if $\alpha = 2$, then the integral in (3.4) converges if and only if $\gamma$ has no normal component;

(iii) if $\alpha \in (1,2)$, then the integral in (3.4) converges if and only if $\gamma$ has no normal component and

$$\int_{-1}^{0} x^\alpha \ln(x^{-1})^n \, dN(x) < \infty \quad \text{and} \quad \int_{-1}^{0} |x|^\alpha \ln(|x|^{-1})^n \, dN(x) < \infty ;$$

(iv) if $\alpha \in (0,1]$, then the integral in (3.4) converges if and only if $\gamma$ has no normal or degenerate component and

$$\int_{0}^{1} x^\alpha \ln(x^{-1})^n \, dN(x) < \infty \quad \text{and} \quad \int_{-1}^{0} |x|^\alpha \ln(|x|^{-1})^n \, dN(x) < \infty .$$

We now characterize the set of completely reverse $\alpha$-self-decomposable characteristic functions: $U^\alpha_{\alpha}((c^{-1})_{c \in (0,1)}) := \cap_{\alpha \in \mathbb{R}} U^\alpha_{\alpha}((c^{-1})_{c \in (0,1)})$.

**Theorem 3.7.** Let $\alpha > 0$. $\phi \in U^\alpha_{\alpha}((c^{-1})_{c \in (0,1)}) := \cap_{\alpha \in \mathbb{R}} U^\alpha_{\alpha}((c^{-1})_{c \in (0,1)})$ if and only if there exists a distribution function $G$ such that

$$\ln \phi(t) = \int_{(0,\min(2,\alpha)]} \ln \phi_{\text{STABLE}(\alpha)}(t) \, dG(a) .$$

**Proof.** Let $LSF_{\alpha}^n$ denote the set of Lévy spectral functions of characteristic functions $\phi \in U^\alpha_{\alpha}((c^{-1})_{c \in (0,1)})$ and let $K = \{ M \mid M \in LSF_{\alpha}^n \text{ and } M(1) = -1 \text{ and } M(-1) = 1 \}$. By (2.3) with $n = 0$ it follows that $|x|^\alpha M(x)$ is monotone and converges to zero as $x \to 0$. Hence $|M(x)| \in [0,1]$ for all $x \notin (-1,1)$ and

$$0 \leq c^\alpha M(c) \geq -1 , \quad 0 \leq c^\alpha M(-c) \leq 1 ,$$

for any $c \in (0,1]$. It can now be checked that $K$ is compact, convex and spans $L_{\alpha\alpha}^\alpha$. By Choquet’s theorem (cf. Choquet (1960)) every point of $K$ is the barycenter of a probability measure concentrated on the set of extreme points of $K$.

Let $M \in K$ and suppose that $M$ has no stable component with exponent $\alpha$ (i.e., $C_+ = C_- = 0$ in (2.2) and (2.3)). For each $c \in (0,1)$ let

$$f(c) = -c^\alpha M(c) \leq -M(1) = 1 .$$

By the monotonicity of $|x|^\alpha M(x)$ there exists an $\varepsilon \in (0,1]$ such that $f(c) < 1$ for all $c \in (0,\varepsilon)$. For any $x > 0$ and any $c \in (0,\varepsilon)$, let
\[ M_{1,c}(x) = f(c)^{-1} c^\alpha M(cx) \quad (3.7) \]

Obviously \( M_{1,c}, M_{2,c} \in K \). Hence

\[ M(x) = f(c) M_{1,c}(x) + (1 - f(c)) M_{2,c}(x). \]

If \( M_\varepsilon \) is an extreme point of \( K \) and \( M_\varepsilon \) has no stable component with exponent \( \alpha \), then necessarily \( M_\varepsilon = M_{1,c} = M_{2,c} \) for all \( c \in (0, \varepsilon) \). By (3.7)

\[ M_\varepsilon(x) = f(c)^{-1} c^\alpha M_\varepsilon(cx), \]

for all \( c \in (0, \varepsilon) \). Necessarily \( f(c_1 c_2) = f(c_1) f(c_2) \) and hence \( f(c) = c^a \), for some \( a > 0 \). Similarly for \( x < 0 \). It now follows that \( M_\varepsilon(x) = C \| x \|^a \) and so (cf. Theorem 1.3), \( M_\varepsilon \) is the Lévy spectral function of a stable characteristic function with exponent \( \alpha - a \). Obviously, if \( \phi = [\sigma, \sigma_0] \in \mathcal{N}_{(\alpha)} ((c^{-1})_{c \in (0,1)}), \alpha < 2 \), then \( \sigma_\phi = 0 \). Hence, by applying Choquet’s theorem, we see that \( \phi \) is of the form (3.6). The converse is easily verified.

If \( \alpha(1) > \alpha(2) \) and \( \phi \in U_{\alpha(2)}((c^{-1})_{c \in (0,1)}), \) then (cf. (3.1))

\[ \phi(t) = \phi^{\alpha(1)}(c^{-1} t) \phi^{\alpha(2)} - \alpha(1) (c^{-1} t) \phi(c(t)). \]

Since \( \phi \) is infinitely divisible, \( \phi^{\alpha(2)} - \alpha(1) \) is a characteristic function and hence \( \phi \in U_{\alpha(1)}((c^{-1})_{c \in (0,1)}). \) Let \( \Gamma_\alpha(u) \) denote the distribution function of a gamma \( (\alpha, n+1) \) distributed random variable. Theorem 3.4 (ii) implies for any \( \phi \in \mathcal{D}, \)

\[ \phi_\alpha(t) := \exp \left( \int_0^1 \ln \phi(v^{-1} t) \alpha^{n+1} (\ln v^{-1})^n (n!)^{-1} v^{\alpha-1} dv \right) \]

\[ = \exp \left( \int_0^\infty \ln \phi(e^{-t}) d\Gamma_\alpha(u). \right) \]

is in \( U_\alpha((c^{-1})_{c \in (0,1)})) \) for all \( \alpha > 2 \), with the integral converging by Remark 3.6. Obviously \( \Gamma_\alpha(u) \) tends to a distribution function with total mass at zero, and so by Helly’s second theorem \( \phi_\alpha \to \phi \) as \( \alpha \to \infty \). Hence

\[ \cup_{\alpha \in \mathbb{R}} U_\alpha((c^{-1})_{c \in (0,1)}) = \mathcal{D}. \]

We collect these results in the following theorem.

**THEOREM 3.8.** The sets \( U_\alpha((c^{-1})_{c \in (0,1)}) \) are multiplication semigroups, closed under limits and provide a classification of \( \mathcal{D}, \) i.e.,

(i) if \( \alpha(2) < \alpha(1) \) then \( U_{\alpha(2)}((c^{-1})_{c \in (0,1)}) \subseteq U_{\alpha(1)}((c^{-1})_{c \in (0,1)}) \) for \( n \in \mathbb{N} \cup \{\infty\}; \)

\[ \]
(ii) if $n \in \mathbb{N}_0$ then $\bigcup_{\alpha \in \mathbb{R}} U_{\alpha}^n ((c^{-1})_{c \in (0,1)}) = \text{ID}$;
References


