Linear feedback : an algebraic approach

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LINEAR FEEDBACK—AN ALGEBRAIC APPROACH*

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Abstract. The algebraic theory of linear input-output maps is reexamined with the objective of accommodating the concept of (state) feedback in this theory. The concepts of extended and restricted linear i/o maps (and linear i/s maps) are introduced and investigated. It is shown how “fraction representations” of transfer matrices arise naturally in this new theoretical framework.

Conditions are given for when the change caused to a linear input-output map by an (open loop) “cascade compensator” can also be accomplished by utilization of (closed loop) state feedback. In particular, it is shown that the change caused to a linear input-output map by cascading (composing) it with an input space isomorphism, can also be effected by feedback, provided the input space isomorphism is “bicausal”, i.e. it does not change the causal structure of the input-output map. Further detailed characterizations of feedback are also given especially in connection with the newly introduced concepts of degree chain and degree list.

1. Introduction. Probably one of the most important contributions to linear systems theory since the introduction of the concepts of controllability and observability, has been the discovery by R. E. Kalman (1965) (see also Kalman (1968) and Kalman et al. (1969, Chap. 10)) that the theory of linear systems can be naturally accommodated in classical module theory. This observation led to a completely satisfactory theory of realization, i.e. the theory that links (external) input-output descriptions with (internal) state space descriptions of systems, the most recent complete discussion of which can be found in Eilenberg (1974, Chap. 16).

Yet, despite the power of the module theoretic approach in attacking the realization problem, there seemed to be no apparent contact between this theory and even some of the most elementary control theoretic questions of linear systems especially insofar as the concept of feedback is concerned.

Two, completely unrelated, approaches were used to study feedback problems: One is the so called “geometric” approach, forwarded and promoted by Wonham and Morse (see e.g. Wonham (1974)), which has been successfully applied to solve such problems as decoupling, regulator design, design of model following systems, and investigating feedback invariant structures, (see. e.g. Wonham and Morse (1970), (1972), Morse and Wonham (1970), (1971), Morse (1973), (1975), Wonham and Pearson (1974) and Wonham (1973)). The second approach, which was widely used and was developed mainly by Rosenbrock (1970) and by Wolovich (see, e.g., Wolovich (1974)), used polynomial matrix techniques for the study of a variety of control theoretic questions. This latter approach, whose primary power derives from the surprising usefulness of fraction

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representations of transfer function matrices, seemed to be especially successful in providing convenient and quite powerful computational algorithms with a capability of yielding various abstract results. Also, fraction representations of transfer matrices provide a convenient vehicle for studying such problems as minimal realization (see e.g. Heymann (1972), Forney (1975) and Fuhrmann (1976)) and feedback invariants as in Heymann (1972).

Probably the most striking paradox in this state of affairs is the fact that, historically, the module theoretic approach to linear systems seemed to support the prevailing viewpoint that transfer function matrices in the form $H(z)/\psi(z)$ ($\psi(z)$ a polynomial) are *the* natural (and theoretically sound) concrete representations of linear input-output maps (see Kalman et al. (1969, Chap. 10)). The representation of transfer function matrices as matrix fractions seemed therefore to be nothing more than a useful technical trick. This discrepancy has been recognized notably by Eckberg (1974) and by Fuhrmann (1976) who attempted to reconcile the two representations within the module theoretic framework. Yet, both of these attempts provide a rather ad-hoc accommodation. Specifically, while both Eckberg and Fuhrmann make a very sound case for viewing transfer matrices as matrix fractions, no successful contact is made with the theory of input-output maps. More importantly, there is no satisfactory contact with feedback theory. While in Fuhrmann (1976) there is no attempt in this direction, in Eckberg (1974) the treatment of feedback is not very successful in that it fails in exhibiting module theory as a powerful or even a convenient framework for dealing with the feedback concept altogether.

The main purpose of the present paper is to reexamine the module theoretic setting of linear input-output maps with the explicit objective of accommodating the concept of "state feedback" within this framework.

In the theory of realization, the concept "canonical" (equivalently reachable and observable) plays a very central and fundamental role in that it defines what is essentially a unique state space. Yet, the property of being a *canonical* realization is *not* invariant under feedback (i.e. a canonical state space can be modified by state feedback to become noncanonical and vice versa). Since the input-output map defines uniquely (or essentially uniquely) only a canonical state space, it is clear why the concept of state feedback somehow seems incompatible with the "classical" module theoretic setting of linear input-output maps. It is easily seen however that reachability *is* invariant under feedback. The importance of the matrix fraction representation for the input-output maps derives from the fact that the representation essentially fixes a reachable state space. Specifically, the representation determines uniquely a reachable realization. As a consequence, the concept of feedback (and especially its effects) can be studied at the level of input-output maps without going through the process of constructing state space descriptions first. Hence, in the study of feedback from an input-output point of view, the process of (concrete) *realization* can essentially be bypassed.

It has been technically well known for quite some time that the modification caused to the input-output map of a linear system by state feedback can be accomplished by cascading the system with a linear device (sometimes called a compensator). Conversely, the effect on a linear input-output map by cascading it with certain types of linear "devices" can sometimes also be accomplished by a

feedback implementation. Yet this converse problem is much less well understood even on a purely “technical” (rather than theoretical) level.

In this paper we give necessary and sufficient conditions (in a module theoretic framework) for a cascade “compensator” to be realizable by a feedback implementation. To formulate these conditions certain revisions in the way an input-output map is viewed are necessary in order to accommodate feedback. We shall adopt here a point of view that was already taken previously in the unpublished notes of Wyman (1972) (see also Sontag (1976)).

The paper is organized as follows: In § 2 the concepts of extended and restricted linear i/o maps are introduced and their relation is investigated (the latter concept coinciding with the standard linear input-output map defined in Kalman et al. (Chap. 10)). In § 3 the concepts of abstract realization and semirealization are discussed and the concept of linear i/s map associated with a (reachable) semirealization is introduced and investigated. Theorems 3.5 and 3.9 give characterizations of linear i/s maps. In § 4 the results of § 3 are specialized to the case where the input and output spaces are finite dimensional and in particular for the case in which the realization or semirealization (i.e. the associated state space) is finite dimensional. Specifically, it is shown that with every reachable realization there is associated a concrete representation of the extended linear i/o map in terms of a (matrix) fraction. This establishes the naturalness of fraction representations and explains their usefulness in the study of feedback. In a similar manner it is shown how linear i/s maps are concretely represented as matrix fractions in association with reachable semirealizations of linear i/o maps. In § 5 the concept of feedback is abstractly introduced and its relation with linear i/s maps is investigated. The notion of a *bicausal* isomorphism (in the extended input space) is defined and investigated. It is seen that every feedback transformation can be implemented in “open loop” through a bicausal isomorphism of the input space. Conversely, conditions are found for the implementability of a bicausal isomorphism of the input space as a feedback transformation (Theorems 5.7 and 5.10). The section is concluded with Theorem 5.13, which states essentially that if the linear i/o map is rational (“rational” being appropriately defined) then every bicausal isomorphism of the input space can be implemented as feedback in some finite dimensional reachable (although not necessarily observable) state space. This result has the intuitive interpretation that the change caused to a linear i/o map by (externally) modifying its input structure can also be accomplished by (internally) implementing feedback provided the external input change does not alter the causal structure of the linear i/o map and is reversible. The paper is concluded in § 6, where the study of feedback is further expanded. In particular, it is noted that feedback can be investigated by studying the structure of the kernel of a (restricted) linear i/s map. Since this kernel is a submodule of the input module, the study of feedback is generalized by studying the structure of an essentially general submodule. In this connection, the concept of the *degree chain* of a submodule is introduced and investigated. The main result of § 6 is Theorem 6.10, which gives a complete characterization of “feedback equivalent” submodules. This reestablishes from a new viewpoint the central role of certain feedback invariants which have been introduced previously. In addition, it is shown how certain previously known (but not well understood) facts find a natural

accommodation within this new theoretical framework (in particular Forney's concept of minimal basis and Wolovich's concept of column properness).

Throughout this paper it will be assumed that the reader is familiar with the now classical module theory of linear systems as can be found for example in Kalman (1968), and Kalman et al. (1969, Chap. 10).

2. Extended and restricted linear i/o maps. We shall begin by introducing some notation. Throughout the paper K will denote a field and U and Y will denote K -linear spaces. The space U will be referred to as the *input value space* and Y as the *output value space* (of an underlying dynamical system Σ). We shall make finite dimensionality assumptions on U and Y only when explicitly stated.

We let $\mathbb{Z}$ denote the set of integers. If S is a K -linear space (in particular U and Y), we consider the set of all sequences $s = (s_t)_{t \in \mathbb{Z}} = (\cdots, s_{-1}, s_0, s_1, \cdots)$ possessing the following properties: (i) $s_t \in S$ for all $t \in \mathbb{Z}$, and (ii) there exists $t_0 \in \mathbb{Z}$ such that $s_t = 0$ for all $t < t_0$. These sequences will be identified with (formal) *S-Laurent series* in z^{-1} , i.e. series of the form

$$(2.1) \quad s = \sum_{t=t_0}^{\infty} s_t z^{-t}.$$

We shall denote the set of S -Laurent series by $S((z^{-1}))$ or alternatively by ΛS . It is then well known that the set $\Lambda K = K((z^{-1}))$ of (scalar) K -Laurent series is a field with convolution as scalar multiplication and the obvious (coefficientwise) addition. Also, with convolution as scalar multiplication and the usual addition, ΛS is a ΛK -linear space. This, in particular, implies that ΛS is also K -linear and also a $K[z]$ -module (where $K[z]$ is the ring of polynomials in z). For an element $s \in \Lambda S$ given by (2.1), the *order* of s is defined by

$$(2.2) \quad \text{ord } s := \begin{cases} \min \{t \in \mathbb{Z} | s_t \neq 0\} & \text{if } s \neq 0, \\ \infty & \text{if } s = 0. \end{cases}$$

Furthermore, for an element $s \in \Lambda S$, multiplication by z results in a shift of the sequence (s_t) to the left, that is

$$(s_t)_{t \in \mathbb{Z}} \mapsto z \cdot (s_t)_{t \in \mathbb{Z}} = (s_{t+1})_{t \in \mathbb{Z}}.$$

We now introduce the *extended input space* ΛU , and the *extended output space* ΛY . If a map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ is K -linear, we say that $\bar{f}$ is *time invariant* provided it satisfies

$$\bar{f}(z \cdot w) = z \bar{f}(w) \quad \text{for all } w \in \Lambda U.$$

The following elementary but important result then follows (see also Wyman (1972)):

THEOREM 2.3. *A K -linear map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ is time invariant if and only if it is ΛK -linear.*

A K -linear map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ is called *causal* if $\text{ord } \bar{f}(w) \geq \text{ord } w$ and *strictly causal* if $\text{ord } \bar{f}(w) > \text{ord } w$ for all $w \in \Lambda U$. We now introduce the following

DEFINITION 2.4. A map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ is called an *extended linear input-output map* (or *extended linear i/o map*) if it is K -linear, strictly causal, and time invariant.

Theorem 2.3 provides an algebraic characterization of time invariance. In order to complete the characterization of extended linear i/o maps we now turn to the question of causality. Let us denote by L the K -linear space of all K -linear maps $U \rightarrow Y$ and consider the space ΛL of all L -Laurent series. This space can be identified with the space of ΛK -linear maps $\Lambda U \rightarrow \Lambda Y$ as follows: We define the K -linear maps

$$\bar{i}_u: U \rightarrow \Lambda U: u \mapsto u \quad (\text{canonical injection}),$$

$$\bar{p}_k: \Lambda Y \rightarrow Y: \sum y_t z^{-t} \mapsto y_k,$$

and for every ΛK -linear map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ and every $k \in \mathbb{Z}$ we define the K -linear map $A_k: U \rightarrow Y$ by

$$(2.5) \quad A_k := A_k(\bar{f}) := \bar{p}_k \circ \bar{f} \circ \bar{i}_u.$$

The L -Laurent series associated with the map $\bar{f}$ is then given by

$$(2.6) \quad Z_{\bar{f}}(z^{-1}) := \sum A_t(\bar{f}) z^{-t}$$

and is called the *impulse response* or the *transfer function* of $\bar{f}$. Now, if $Z = \sum A_t z^{-t}$ is any element of ΛL we define the action of Z on $w = \sum u_t z^{-t} \in \Lambda U$ by

$$(2.7) \quad Z w := \sum_t \sum_k (A_k u_{t-k}) z^{-t}.$$

If $\bar{f}: \Lambda U \rightarrow \Lambda Y$ is defined as the map whose action is given by (2.7), it is then easily verified that

$$Z_{\bar{f}}(z^{-1}) = Z.$$

We now have the following immediate characterization of causality: *The map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ is causal if and only if $A_k(\bar{f}) = 0$ for $k < 0$ and is strictly causal if and only if $A_k(\bar{f}) = 0$ for $k \leq 0$.* The following is then proved:

THEOREM 2.8. *A map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ is an extended linear i/o map if and only if it is ΛK -linear and $A_k(\bar{f}) = 0$ for all $k \leq 0$.*

For a linear space S we denote by $S[z]$ or alternatively by ΩS the set of all polynomials $\sum_{k=0}^N s_k z^k$ with coefficients s_k in S . Obviously, ΩS is a subset of ΛS and it is easily seen that it is also a K -linear subspace and even a $K[z]$ -submodule of ΛS . It is not, however, a ΛK -linear subspace.

In the development of the module theoretic treatment of linear systems, one of Kalman's primary objectives was to provide a rigorous framework for relating linear input-output maps to state space descriptions, (i.e. the problem of realization). In his treatment Kalman considered the following "experimental" setup (see e.g. Kalman (1968) and Kalman et al. (1969, Chap. 10)): All inputs terminate (i.e. become identically zero) at $t = 0$ and the outputs are observed only for $t \geq 1$. This special set of inputs ΩU will henceforth be called the *restricted input space*, and elements $w \in \Omega U$ will be called *restricted inputs*. Since in this setting outputs are observed only for $t \geq 1$, it follows that two outputs are indistinguishable if their difference is in ΩY . Thus, if we introduce the notation

$$(2.9) \quad \Gamma S := \Lambda S / \Omega S$$

for any K -linear space S , then according to the foregoing we define ΓY to be the *restricted output space*. Let $\Lambda_+ Y$ denote the subset of ΛY consisting of all elements of the form $\sum_{t=1}^{\infty} y_t z^{-t}$. It is readily verified that there is a bijective correspondence between ΓY and $\Lambda_+ Y$ with the property that with every $\gamma \in \Gamma Y$ is associated a unique $\tilde{\gamma} \in \Lambda_+ Y$ satisfying $\pi \tilde{\gamma} = \gamma$ where $\pi: \Lambda Y \rightarrow \Gamma Y$ is the canonical projection. Furthermore, ΓY induces a unique module structure on $\Lambda_+ Y$ by the requirement that the above bijection become a $K[z]$ -module homomorphism. We shall henceforth identify elements of ΓY with those of $\Lambda_+ Y$ and refer to the latter as *restricted outputs*. (It is precisely the module $\Lambda_+ Y$ which was defined in Kalman et al. (1969, Chap. 10) as the output module of a linear system.)

DEFINITION 2.10. A map $f: \Omega U \rightarrow \Gamma Y$ is called a *restricted linear input-output map* (or *restricted linear i/o map*) if it is a $K[z]$ -module homomorphism.

Remark 2.11. In the above definition of restricted linear i/o maps the properties of causality and time invariance are automatically built in. Wyman (1972) called the same a *Kalman linear i/o map* and it is, in fact, precisely the input-output map derived in Kalman's work (see e.g. Kalman et al. (1969, Chap. 10)). We shall later make extensive use both of extended and of restricted linear i/o maps.

In addition to the extended and restricted i/o maps, we introduce the concept of *linear i/o value map*

$$(2.12) \quad f: \Omega U \rightarrow Y$$

as follows: If $\tilde{f}$ is an extended linear i/o map, we define f (associated with it) by

$$(2.13) \quad f(w) := \bar{p}_1 \circ \tilde{f}(w); \quad w \in \Omega U.$$

Alternatively, if $\tilde{f}$ is a restricted linear i/o map, we construct the i/o value map f by

$$(2.14) \quad f := p_1 \circ \tilde{f}$$

where (with ΓY identified with $\Lambda_+ Y$)

$$(2.15) \quad p_1: \Gamma Y \rightarrow Y: \sum_{t=1}^{\infty} y_t z^{-t} \mapsto y_1.$$

Conversely, if $f: \Omega U \rightarrow Y$ is any K -linear map, we can regard it as a linear i/o map by associating with it the maps $\tilde{f}$ and $\bar{f}$ which are constructed from f using the conditions of time invariance and causality. In particular, we have

$$(2.16) \quad \tilde{f}(w) = \sum_{t \geq 0} f(z^t w) z^{-t-1}$$

and

$$(2.17) \quad \bar{f}(w) = \sum_{t \in \mathbb{Z}} f(\mathcal{S}(z^t w)) z^{-t-1}$$

where the *truncation operator* $\mathcal{S}: \Lambda U \rightarrow \Omega U$ is defined by

$$(2.18) \quad \mathcal{S}(\sum u_t z^{-t}) := \sum_{t \leq 0} u_t z^{-t}.$$

(Compare also Kalman and Hautus (1972, formulas (2) and (4)).) It is easily verified that the maps $\tilde{f}$ and $\bar{f}$ as defined in (2.16) and (2.17) are, respectively, a restricted and an extended linear i/o map and that the formulas (2.13) and (2.14) hold. The relation between $\bar{f}$, $\tilde{f}$ and f is indicated in the following commutative diagram:

$$\begin{array}{ccc}
 \Lambda U & \xrightarrow{\bar{f}} & \Lambda Y \\
 \uparrow i & & \downarrow \pi \\
 \Omega U & \xrightarrow{\tilde{f}} & \Gamma Y \\
 \uparrow i & & \downarrow p_1 \\
 \Omega U & \xrightarrow{f} & Y
 \end{array}
 \begin{array}{c}
 \\
 \\
 \bar{p}_1
 \end{array}$$

where i is the identity map, j is the canonical injection, and π is the canonical projection. The maps f , p_1 and $\bar{p}_1$ are K -linear, the maps j , π and $\tilde{f}$ are $K[z]$ -homomorphisms and $\bar{f}$ is a ΛK -linear map. The maps $\tilde{f}$ and $\bar{f}$ are called the *restricted and extended i/o maps associated with f* .

3. Linear i/s maps. Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a restricted linear i/o map. By an *abstract realization* of $\tilde{f}$ we refer to a triple (X, g, h) where X is a $K[z]$ -module and $g: \Omega U \rightarrow X$ and $h: X \rightarrow \Gamma Y$ are $K[z]$ -module homomorphisms such that the diagram

$$\begin{array}{ccc}
 \Omega U & \xrightarrow{\tilde{f}} & \Gamma Y \\
 g \searrow & & \nearrow h \\
 & X &
 \end{array}$$

commutes. For the system theoretic interpretation of an abstract realization, the reader is referred to Kalman (1968) and Kalman et al. (1969, Chap. 10). The module X is called the *state space* (and is sometimes regarded only as a K -linear space).

If (X, g, h) is a given abstract realization of a restricted linear i/o map $\tilde{f}$, one can construct from it a concrete realization of $\tilde{f}$ (see Kalman et al. (1969, Chap. 10)). Such a concrete realization is uniquely determined by the abstract realization (X, g, h) and we shall henceforth call (X, g, h) simply a *realization* of $\tilde{f}$. In keeping with standard systems terminology we then call a realization *reachable* if g is surjective and *observable* if h is injective. A realization is called *canonical* if it is both reachable and observable.

Consider now a restricted linear i/o map $\tilde{f}: \Omega U \rightarrow \Gamma Y$. Let X be a $K[z]$ -module and let $g: \Omega U \rightarrow X$ be a $K[z]$ -module homomorphism. The pair (X, g) will be called a *semirealization* of $\tilde{f}$ if it can be extended to a realization (X, g, h) with some $K[z]$ -homomorphism $h: X \rightarrow \Gamma Y$. (X, g) will be called a *reachable semirealization* if g is surjective, and *canonical* if every extension realization (X, g, h) of (X, g) is canonical. The following characterization of reachable semirealizations is easily verified:

THEOREM 3.1. *Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a restricted linear i/o map, let X be a $K[z]$ -module, and let $g: \Omega U \rightarrow X$ be a surjective $K[z]$ -homomorphism. Then*

- (i) *(X, g) is a (reachable) semirealization of $\tilde{f}$ if and only if $\ker g \subset \ker \tilde{f}$.*
- (ii) *If (X, g) is a reachable semirealization of $\tilde{f}$ there is a unique h such that (X, g, h) is a realization of $\tilde{f}$.*
- (iii) *(X, g) is a canonical semirealization of $\tilde{f}$ if and only if $\ker g = \ker \tilde{f}$.*

Consider now a $K[z]$ -homomorphism $g: \Omega U \rightarrow X$ where X is a $K[z]$ -module. If we refer only to the K -linear structure of X and regard g as a K -linear map then, as in (2.16) and (2.17), we can construct the restricted and extended linear i/o maps

$$(3.2) \quad \tilde{g}: \Omega U \rightarrow \Gamma X,$$

$$(3.3) \quad \bar{g}: \Lambda U \rightarrow \Lambda X$$

associated with g . In view of the fact that X is a $K[z]$ -module and not just a K -linear space and g is a $K[z]$ -homomorphism and not just a K -linear map, the maps $\tilde{g}$ and $\bar{g}$ have properties that distinguish them from ordinary i/o maps. For this reason we will call $\tilde{g}$ and $\bar{g}$, respectively, *a restricted* and *an extended linear i/s map* (where i/s stands for input-state). More generally, a restricted linear i/o map $\tilde{f}: \Omega U \rightarrow \Gamma Y$ and the corresponding extended i/o map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ are called *i/s maps* if Y can be endowed with a $K[z]$ -module structure, compatible with its K -vector space structure, such that the associated i/o value map

$$f := \bar{p}_1 \circ \bar{f} \circ j = p_1 \circ \tilde{f}$$

is a $K[z]$ -module homomorphism. It is easily verified that if $\tilde{f}$ is a restricted linear i/s map and f the associated value map then

$$(3.4) \quad \ker \tilde{f} = \ker f.$$

Indeed, for $w \in \Omega U$, $\tilde{f}(w) = 0$ if and only if $f(z^t w) = 0$ for all $t \geq 0$ (see (2.16)). However, in view of the fact that f is a $K[z]$ -homomorphism, the latter is equivalent to $f(w) = 0$ since $f(z^t w) = z^t f(w)$.

Assume now that $\tilde{f}: \Omega U \rightarrow \Gamma Y$ is a (restricted) linear i/s map and that its associated value map f is surjective. Then (3.4) and Theorem 3.1 imply that (Y, f) is a reachable (in fact even a canonical) semirealization of $\tilde{f}$. We shall henceforth call a restricted linear i/s map $\tilde{f}$ *reachable* whenever the pair (Y, f) is a reachable semirealization of $\tilde{f}$ or, equivalently, whenever the associated i/o value map f is surjective. (The name i/s was adopted precisely for the reason that in these special i/o maps the output value space qualifies as state space.)

Assume next that $\tilde{f}: \Omega U \rightarrow \Gamma Y$ is a linear i/o map such that its associated i/o value map f is surjective and that (3.4) holds. Then Y is isomorphic as a K -linear space to $\Omega U / \ker \tilde{f}$ which is a $K[z]$ -module. This isomorphism induces a compatible $K[z]$ -module structure on Y and it is easily seen that f is then a $K[z]$ -homomorphism. We summarize the situation in the following

THEOREM 3.5. *Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a restricted linear i/o map such that the associated i/o value map $f = p_1 \circ \tilde{f}: \Omega U \rightarrow Y$ is surjective. Then $\tilde{f}$ is an i/s map if and only if (3.4) holds.*

If $H: X \rightarrow Y$ is a K -linear map then it induces in a natural way a ΛK -linear map

$$(3.6) \quad H: \Lambda X \rightarrow \Lambda Y: \sum x_i z^{-i} \mapsto \sum (Hx_i) z^{-i}$$

and also a $K[z]$ -homomorphism

$$(3.7) \quad H: \Gamma X \rightarrow \Gamma Y: \sum_{i \geq 1} x_i z^{-i} \mapsto \sum_{i \geq 1} (Hx_i) z^{-i}$$

where we have identified $\Lambda_+ X$ with ΓX and $\Lambda_+ Y$ with ΓY . The maps defined by (3.6) and (3.7) are called *static* maps and it will be convenient to denote them by the same symbol H . We now have the following

THEOREM 3.8. *Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a restricted linear i/o map, let $g: \Omega U \rightarrow X$ be a $K[z]$ -homomorphism and let $\tilde{g}$ be the restricted linear i/s map associated with g .*

(i) *If (X, g) is a semirealization of $\tilde{f}$, then there exists a K -linear map $H: X \rightarrow Y$ such that $\tilde{f} = H \cdot \tilde{g}$ (correspondingly $\tilde{f} = H \cdot \tilde{g}$).*

(ii) *Conversely, if g is surjective and there exists $H: X \rightarrow Y$ such that $\tilde{f} = H \cdot \tilde{g}$, then (X, g) is a semirealization of $\tilde{f}$.*

Proof. (i) Let $h: X \rightarrow \Gamma Y$ be a $K[z]$ -homomorphism such that (X, g, h) is a realization of $\tilde{f}$, and define

$$H: X \rightarrow Y: x \mapsto p_1 \circ h(x),$$

where p_1 is defined in (2.15). Then, if $f := p_1 \circ \tilde{f}$, we have

$$f = p_1 \circ \tilde{f} = p_1 \circ h \circ g = H \circ g$$

and consequently for $w \in \Omega U$ we have by (2.16) and (3.7)

$$\begin{aligned} \tilde{f}(w) &= \sum_{i \geq 0} f(z^i w) z^{-i-1} = \sum_{i \geq 0} H \circ g(z^i w) z^{-i-1} \\ &= H \circ \sum_{i \geq 0} g(z^i w) z^{-i-1} = H \cdot \tilde{g}, \end{aligned}$$

and the proof of (i) is complete.

(ii) The existence of H such that $\tilde{f} = H \cdot \tilde{g}$ implies that $\ker g = \ker \tilde{g} \subset \ker \tilde{f}$. Hence, since g is surjective the result follows from Theorem 3.1. $\square$

The following result gives another characterization of the linear i/s maps amongst all linear i/o maps.

THEOREM 3.9. *Let $\tilde{g}: \Omega U \rightarrow \Gamma Y$ be a restricted linear i/o map. Then $\tilde{g}$ is a reachable restricted linear i/s map if and only if the following condition holds: for every restricted linear i/o map $\tilde{f}: \Omega U \rightarrow \Gamma S$ satisfying $\ker \tilde{g} \subset \ker \tilde{f}$ (where S is a K -linear space), there exists a unique K -linear map $H: Y \rightarrow S$ such that $\tilde{f} = H \cdot \tilde{g}$.*

Proof. Assume that $\tilde{g}$ is a reachable restricted linear i/s map and that $\tilde{f}$ satisfies $\ker \tilde{g} \subset \ker \tilde{f}$. Then the existence of H such that $\tilde{f} = H \cdot \tilde{g}$ follows from Theorems 3.1 and 3.8. Also, $f = H \cdot g$ where $f = p_1 \circ \tilde{f}$ and $g = p_1 \circ \tilde{g}$, and from the surjectivity of g (the reachability of $\tilde{g}$) it follows that H is unique.

Conversely, assume the condition holds. Define $S := \Omega U / \ker \tilde{g}$ and let $f: \Omega U \rightarrow S$ be the canonical projection. Let $\tilde{f}: \Omega U \rightarrow \Gamma S$ be the restricted linear i/o map associated with f . Then $\tilde{f}$ is an i/s map and $\ker \tilde{f} = \ker f = \ker \tilde{g}$. It follows that

there exists a unique K -linear map $H: Y \rightarrow S$ such that $\tilde{f} = H \cdot \tilde{g}$ or, equivalently, $f = H \cdot g$. Hence $\ker g \subset \ker f = \ker \tilde{g}$ and, since $\ker \tilde{g} \subset \ker g$ is obvious, it follows that $\ker \tilde{g} = \ker g$. From the uniqueness of H we have that g is surjective and the result follows from Theorem 3.5. $\square$

We conclude this section with the following result (due to Wyman (1972)) which summarizes in a transparent way our current point of view.

THEOREM 3.10. *Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a restricted linear i/o map. Then every reachable realization (X, g, h) of $\tilde{f}$ induces a unique commutative diagram*

4. Finite dimensionality and matrix representations. Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a restricted linear i/o map. Two reachable realizations (X_1, g_1, h_1) and (X_2, g_2, h_2) of $\tilde{f}$ are called *isomorphic* if there exists a $K[z]$ -isomorphism $\alpha: X_1 \rightarrow X_2$ such that the diagram

commutes. Let $\Delta \subset \ker \tilde{f}$ be a submodule. There is associated with Δ (uniquely) a reachable realization $(X_\Delta, g_\Delta, h_\Delta)$ of $\tilde{f}$ where $X_\Delta := \Omega U / \Delta$, $g_\Delta: \Omega U \rightarrow X_\Delta$ is the canonical projection and $h_\Delta: X_\Delta \rightarrow \Gamma Y$ is defined via Theorem 3.1(ii). Suppose now that (X, g, h) is any reachable realization of $\tilde{f}$ and let $\Delta = \ker g$. It is easily verified that (X, g, h) is then isomorphic to $(X_\Delta, g_\Delta, h_\Delta)$. It follows that the class of reachable realizations of $\tilde{f}$ is in one-one correspondence (modulo isomorphisms) with the class of submodules of $\ker \tilde{f}$. Specifically, with every submodule $\Delta \subset \ker \tilde{f}$ is associated a state space $\Omega U / \Delta$, and the structural properties of all reachable

realizations (X, g, h) for which $\Delta = \ker g$ are essentially completely determined by Δ . This fact will be particularly useful in our study of feedback where it will turn out to be easier to deal with (the submodule) Δ than with (the quotient module) $\Omega U/\Delta$.

So far we have imposed no finite dimensionality conditions on either U , X or Y . In this section we shall specialize our results to the finite dimensional case and shall henceforth assume that U and Y are finite dimensional linear spaces. More specifically, we let

$$U = K^m, \quad Y = K^p$$

so that $\Omega U = K^m[z]$ and $\Gamma Y = K^p((z^{-1}))/K^p[z]$.

Thus, in this case ΩU is a free $K[z]$ -module and has a basis of m elements. Since $K[z]$ is a principal ideal domain, every submodule Δ of ΩU has at most m free generators and their number d , which we call rank Δ , is independent of their choice (see e.g. Hartley and Hawkes (1970, Thm. 7.8)).

Our main interest is in the representation of finite dimensional linear systems, and we are concerned with the case in which for a submodule $\Delta \subset \Omega U$, $\Omega U/\Delta$ is a finite dimensional K -linear space, or equivalently, a torsion module (see e.g. Lang (1965, p. 388)). Thus we shall make use of the following standard but important result (the proof of which can be found in Fuhrmann (1976)).

THEOREM 4.1. *Let $\Delta \subset \Omega U$ be a submodule. Then $\Omega U/\Delta$ is a torsion module (or equivalently a finite dimensional K -linear space) if and only if rank $\Delta = m$.*

Let $\Delta \subset \Omega U$ be a submodule of rank m and let $d_1, \dots, d_m$ be a basis for Δ . Define the $K[z]$ -homomorphism $D: \Omega U \rightarrow \Delta$ by $De_i = d_i$, $i = 1, \dots, m$, where $e_1, \dots, e_m$ denotes the natural basis in K^m , (as well as in ΩU). We can view D also as an $m \times m$ polynomial matrix (i.e. a matrix with entries in $K[z]$) by regarding $d_i \in K^m[z]$ as the i th column of D . Conversely, if $D = D(z)$ is an $l \times m$ polynomial matrix, we can regard D as a $K[z]$ -homomorphism $K^m[z] \rightarrow K^l[z]: e_i \mapsto d_i$, $i = 1, \dots, m$, where $d_i = d_i(z) \in K^l[z]$ is the i th column of D . If in particular $l = m$, then $d_1, \dots, d_m$ are elements of $\Omega U = K^m[z]$ and are thus generators of a submodule $\Delta \subset \Omega U$ defined by

$$\Delta = D\Omega U := \{Dw \mid w \in \Omega U\}.$$

Clearly rank $\Delta = \text{rank } D$ (rank D being the matrix rank of D), and rank $\Delta = m$ if and only if D is nonsingular (i.e. $0 \neq \det D \in K[z]$).

Consider now the $K[z]$ -homomorphism $\Omega U \rightarrow \Omega U$ defined by an $m \times m$ nonsingular polynomial matrix D . It is easily verified that there exists a unique ΛK -linear map $\bar{D}: \Lambda U \rightarrow \Lambda U$ such that the diagram

$$\begin{array}{ccc} \Lambda U & \xrightarrow{\bar{D}} & \Lambda U \\ \uparrow j & & \uparrow j \\ \Omega U & \xrightarrow{D} & \Omega U \end{array}$$

commutes, where j , as usual, denotes the canonical injection. It is readily noted that the transfer function of $\bar{D}$ is given by the (polynomial) matrix D . [We shall

refer to a ΛK -linear map whose transfer function is a polynomial matrix as a *polynomial map*.] Since D is nonsingular, it is obviously invertible over ΛK , and thus $\bar{D}$ is an invertible map. We shall henceforth denote both the maps D and $\bar{D}$ and their associated polynomial matrix by the single symbol D . The meaning will always be clear from the context.

An $m \times m$ polynomial matrix R is called *unimodular* if its determinant is a nonzero constant, i.e. if its inverse is also a polynomial matrix. The following theorem whose elementary proof is omitted and the corresponding immediate corollary will be useful:

THEOREM 4.2. *Let $\Delta, \Delta^* \subset \Omega U$ be submodules given by $\Delta = D\Omega U$ and $\Delta^* = D^*\Omega U$ where D and D^* are nonsingular. Then $\Delta^* \subset \Delta$ if and only if there exists a polynomial matrix R such that $D^* = DR$.*

COROLLARY 4.3. *Under the conditions of Theorem 4.2, $\Delta^* = \Delta$ if and only if there exists a unimodular matrix R such that $D^* = DR$.*

We now turn to the study of representations of linear i/o maps (and their realizations). Let $\bar{f}: \Lambda U \rightarrow \Lambda Y$ be an extended linear i/o map, let $\tilde{f}$ be the associated restricted linear i/o map and let (X, g, h) be a reachable realization of $\tilde{f}$ with X a finite dimensional K -vector space (i.e. $\text{rank ker } g = m$). Let $\ker g = \Delta = D\Omega U$ and define the map

$$(4.4) \quad N: \Lambda U \rightarrow \Lambda Y: w \mapsto \bar{f}(Dw).$$

Since N is a composite of two ΛK -linear maps, it is clearly also ΛK -linear. Let

$$Z_N(z) = \sum N_t z^{-t}$$

denote the transfer function of N . We observe that Z_N is a polynomial matrix (i.e. $N_t = 0$ for all $t > 0$). Indeed, for $w \in \Omega U$

$$\pi(Nw) = \pi \circ \bar{f}(Dw) = \tilde{f}(Dw) = 0,$$

with the last equality holding since $Dw \in \ker g \subset \ker \tilde{f}$. It follows that $Nw \in K^p[z]$. Since $w \in \Omega U$ was chosen arbitrarily, we see (compare (2.7)) that

$$Z_N(z) = \sum_{t \leq 0} N_t z^{-t}$$

as claimed. We shall now (just as we have done for D) identify (notationally) Z_N with N so that N will denote both the map and its associated polynomial matrix. In view of the relation $Nw = \bar{f}(Dw)$ for every $w \in \Lambda U$, and the fact that D is invertible we obtain the following expression:

$$\bar{f}(w) = ND^{-1}w \quad (w \in \Lambda U).$$

Thus, the transfer function of $\bar{f}$ has the representation

$$(4.5) \quad Z_{\bar{f}}(z) = \sum_{t=1}^{\infty} A_t z^{-t} = N(z)D^{-1}(z).$$

A ΛK -linear map $\bar{f}: \Lambda U \rightarrow \Lambda Y$ and its L -Laurent series $Z_{\bar{f}}(z)$ are called *rational* if and only if there exists a (nonsingular) polynomial matrix $D(z)$ such

that $Z_{\tilde{f}}D$ is a polynomial matrix. Thus we have proved the necessity part of the following

THEOREM 4.6. *A restricted linear i/o map $\tilde{f}: \Omega U \rightarrow \Gamma Y$ has a reachable realization with finite dimensional state space if and only if $\tilde{f}$ is rational.*

Proof (sufficiency). If $\tilde{f}$ is rational, there exist polynomial matrices N and D such that $\tilde{f} = ND^{-1}$. Then $(X_{\Delta}, g_{\Delta}, h_{\Delta})$, where $\Delta = D\Omega U$, is a reachable realization of $\tilde{f}$ with finite dimensional state space. $\square$

Two matrices N and D are called *right coprime* if every $m \times m$ polynomial matrix R satisfying $N = N_1R$ and $D = D_1R$ for some polynomial matrices N_1 and D_1 , is necessarily unimodular.

THEOREM 4.7. *Let (X, g, h) be a reachable realization of a restricted linear i/o map $\tilde{f}$. Let $\ker g = \Delta = D\Omega U$ and assume D is nonsingular. Let $N = Z_{\tilde{f}}D$. Then (X, g, h) is observable (i.e. $\ker \tilde{f} = \ker g$) if and only if N and D are right coprime.*

Proof. Let D_1 be chosen such that $\ker \tilde{f} = D_1\Omega U$. Then $D\Omega U = \ker g \subset \ker \tilde{f} = D_1\Omega U$, and by Theorem 4.2 there exists a polynomial matrix R such that $D = D_1R$. D_1 is then nonsingular and if N_1 is defined such that $N_1D_1^{-1} = Z_{\tilde{f}}$, it is readily seen that $N = N_1R$. If we assume that N and D are right coprime, then the matrix R must be unimodular and by Corollary 4.3 $\ker g = \ker \tilde{f}$ and observability holds. Conversely, let $N = N_1R$ and $D = D_1R$. Then R is nonsingular and $Z_{\tilde{f}} = ND^{-1} = N_1D_1^{-1}$. It follows that $D_1\Omega U \subset \ker \tilde{f}$ and we have

$$\ker g = D\Omega U \subset D_1\Omega U \subset \ker \tilde{f}.$$

If (X, g, h) is observable, then $\ker g = \ker \tilde{f}$ so that $D\Omega U = D_1\Omega U$ and by Corollary 4.3 R is unimodular, whence N and D are right coprime. $\square$

Remark 4.8. The representation $Z_{\tilde{f}} = ND^{-1}$ of a transfer function is widely known as a *fraction representation*. We have just seen how fraction representations arise naturally from reachable (abstract) realizations (X, g, h) of an input-output map. In fact, there is essentially a one-one correspondence between the class of all finite dimensional reachable realizations of $\tilde{f}$ and the class of fraction representations. It is important to observe that by writing $Z_{\tilde{f}} = ND^{-1}$ we essentially have (modulo isomorphism) a specific reachable realization in mind. The observability along with the associated right coprimeness is essentially of no consequence in our study of feedback as we shall see later.

Let $X = K^n$. Assume X is also endowed with a module structure and let $g: \Omega U \rightarrow X$ be a surjective $K[z]$ -homomorphism. Let $\tilde{g}: \Omega U \rightarrow \Gamma X$ be the restricted (reachable) linear i/s map associated with g . Then (X, g) is a canonical semirealization of $\tilde{g}$ and there exists $h: X \rightarrow \Gamma X$ such that (X, g, h) is a (canonical) realization of $\tilde{g}$. According to the foregoing we can write

$$(4.9) \quad Z_{\tilde{g}} = SD^{-1},$$

where $Z_{\tilde{g}}$ is the transfer function of $\tilde{g}$, with D being a (nonsingular) polynomial matrix such that $\ker \tilde{g} = D\Omega U$, and S being an $n \times m$ polynomial map. In view of the observability of (X, g, h) it follows from Theorem 4.7 that D and S are right coprime, and moreover, $\deg \det D = n$.

If $Z_{\tilde{f}}$ is the transfer function of an extended linear i/o map $\tilde{f}$, then the strict causality of $\tilde{f}$ is equivalent to $Z_{\tilde{f}}$ being a *strictly proper* rational matrix; that is, each entry of $Z_{\tilde{f}}$ is a fraction of polynomials with the degree of the denominator higher

than the degree of the numerator. We now specialize Theorem 3.9 to the finite dimensional case.

THEOREM 4.10. *Let S and D be right coprime polynomial matrices with D nonsingular, and assume that $Z := SD^{-1}$ is strictly proper (or equivalently strictly causal). Then Z is the transfer function of a reachable extended linear i/s map $\tilde{g}$ if and only if for every polynomial matrix N satisfying the condition ND^{-1} is strictly proper, there exists a unique constant matrix H (with entries in K) such that $N = HS$.*

Proof. Assume that $Z = Z_{\tilde{g}}$ where $\tilde{g}$ is a reachable extended linear i/s map. By the coprimeness of S and D we have $\ker \tilde{g} = D\Omega U$ and if $\tilde{f}_N$ is the restricted linear i/o map associated with $Z_N = ND^{-1}$ (ND^{-1} strictly proper), then $\ker \tilde{g} = D\Omega U \subset \ker \tilde{f}_N$. By Theorem 3.9 there exists a unique $H: X \rightarrow Y$ such that $\tilde{f}_N = H \cdot \tilde{g}$ so that if H also denotes the corresponding matrix we have $N = (ND^{-1})D = (HSD^{-1})D = HS$. Conversely, let $\tilde{f}$ be a restricted linear i/o map with $\ker \tilde{g} \subset \ker \tilde{f}$. Then $D\Omega U \subset \ker \tilde{f}$ and there exists a polynomial matrix N such that $\tilde{f} = ND^{-1}$ ($\tilde{f}$ denoting the extended linear i/o map associated with $\tilde{f}$). By hypothesis there exists a unique constant matrix H such that $N = HS$, and consequently $\tilde{f} = H \cdot \tilde{g}$ and by Theorem 3.9 the proof is complete. $\square$

COROLLARY 4.11. *Let D be a nonsingular $m \times m$ polynomial matrix and let $\det D$ have degree n . The set $\hat{X}$ of m -row polynomial vectors $v(z)$ such that $v(z)D^{-1}(z)$ is strictly proper, is an n dimensional vector space over K . If S is a polynomial matrix such that SD^{-1} is strictly proper, then SD^{-1} is the transfer function of a reachable linear i/s map if and only if the rows of S form a basis of $\hat{X}$.*

Proof. That $\hat{X}$ is a K -vector space is obvious. Suppose SD^{-1} is the transfer function of a linear i/s map. If $v(z) \in \hat{X}$ is any vector, then by Theorem 4.10 there exists a (row) matrix H such that $v = HS$ and thus, the rows of S generate $\hat{X}$. The uniqueness condition of H implies that the rows of S are linearly independent (over K) and thus form a basis for $\hat{X}$. Conversely, if the rows of S form a basis for $\hat{X}$ then SD^{-1} is a strictly proper matrix and for each matrix N whose rows are elements of $\hat{X}$ there exists a unique K -matrix H such that $N = HS$ whence, by Theorem 4.10, SD^{-1} is the transfer function of a linear i/s map. Since the number of rows in S is equal to n , it follows that $\dim \hat{X} = n$. $\square$

5. Feedback. Consider a restricted linear i/o map $\tilde{f}: \Omega U \rightarrow \Gamma Y$, let (X, g) be a reachable semirealization of $\tilde{f}$ and let $\tilde{g}: \Lambda U \rightarrow \Lambda X$ be the extended linear i/s map associated with g . Suppose we modify $\tilde{g}$ in the following way. For each $w \in \Lambda U$ instead of letting $\tilde{g}$ act on w we let $\tilde{g}$ act on $\varphi := V \cdot w - L \cdot \xi$, where $V: U \rightarrow U$ and $L: X \rightarrow U$ are K -linear (static) maps with V invertible, and where ξ is such that $\xi = \tilde{g}(\varphi)$. Then w , φ , and ξ are related by the equations

$$(5.1) \quad \varphi = V \cdot w - L \cdot \xi; \quad \xi = \tilde{g}(\varphi),$$

which schematically can be described by the following *feedback* block diagram:

In line with the above block diagram we call the pair (L, V) a *feedback pair*. The map L will be called a *feedback map* and V will be called a *(static) input transformation*.

From (5.1) we can eliminate ξ , to obtain

$$(5.2) \quad (I + L \cdot \bar{g}) \cdot \varphi = V \cdot w.$$

Since $L \cdot \bar{g}$ is strictly causal, it is easily verified that $\text{ord } (I + L\bar{g})\varphi = \text{ord } \varphi$ for each $\varphi \in \Lambda U$. This implies that $\ker (I + L\bar{g}) = 0$, from which it follows that $I + L\bar{g}$ is invertible. Thus, (5.2) has a unique solution for φ given by

$$\varphi = \bar{l} \cdot w,$$

where

$$(5.3) \quad \bar{l} := \bar{l}_{L,V} := (I + L\bar{g})^{-1} V.$$

Upon substituting for φ in the second formula of (5.1), we obtain the expression

$$(5.4) \quad \xi = \bar{f}_{L,V}(w),$$

where

$$(5.5) \quad \bar{f}_{L,V} := \bar{g} \circ \bar{l}.$$

The schematic interpretation of (5.5) is given by the block diagram

where $\bar{l}: \Lambda U \rightarrow \Lambda U$ is regarded as a *dynamic input transformation*. It is readily noted (see also Lemma 5.6 below) that the map $\bar{l}$ has the following properties:

- (i) $\bar{l}$ is an invertible ΛK -linear map.
- (ii) Both $\bar{l}$ and $\bar{l}^{-1}$ are causal maps.

We shall henceforth call a map $\bar{l}: \Lambda U \rightarrow \Lambda U$ which satisfies both (i) and (ii) a *bicausal isomorphism* on ΛU .

It is obvious that $\bar{f}_{L,V}$ is an extended linear i/o map. This follows immediately from the fact that the composite of ΛK -linear maps is ΛK -linear, and that the composite of a causal map with a strictly causal one is strictly causal. In fact, it will be seen later that $\bar{f}_{L,V}$ is even a reachable linear i/s map.

We have seen that if we can construct $\bar{f}_{L,V}$ from $\bar{g}$ by feedback (as in our first interpretation) then we can also construct it by cascading $\bar{g}$ with a bicausal isomorphism $\bar{l} = \bar{l}_{L,V}$ (which is an “open loop” construction). We shall now turn to the more difficult question: when can a ΛK -linear map $\bar{l}: \Lambda U \rightarrow \Lambda U$ be expressed as in (5.3) for some L and V . If this is the case we call $\bar{l}$ a *feedback transformation* (corresponding to (L, V)).

LEMMA 5.6. *Let $\bar{l}: \Lambda U \rightarrow \Lambda U$ be a causal ΛK -linear map and let $Z_{\bar{l}}(z^{-1}) = \sum_{k=0}^{\infty} A_k(\bar{l})z^{-k}$ be its transfer function. Then $\bar{l}$ has a causal inverse if and only if $A_0(\bar{l}): U \rightarrow U$ is invertible, in which case $A_0(\bar{l}^{-1}) = (A_0(\bar{l}))^{-1}$.*

The easy proof of Lemma 5.6 is by direct calculation and is omitted.

THEOREM 5.7. *Let $\tilde{g}: \Lambda U \rightarrow \Lambda X$ be a reachable extended linear i/s map. If $\bar{l}: \Lambda U \rightarrow \Lambda U$ is a ΛK -linear map, then there exists a feedback pair (L, V) such that $\bar{l} = \bar{l}_{L,V}$ (as in (5.3)) if and only if*

- (i) *$\bar{l}$ is a bicausal isomorphism,*
- (ii) *for $w \in \Omega U$, $\tilde{g}(w) \in \Omega X$ implies $\bar{l}^{-1}(w) \in \Omega U$ (equivalently, $\bar{l}^{-1}(\ker \tilde{g}) \subset \Omega U$).*

Proof. Assume first that $\bar{l} = \bar{l}_{L,V}$. The bicausality of $\bar{l}$ has been noted before so that (i) holds. Also, since $\bar{l}^{-1} = V^{-1} + V^{-1}L\tilde{g}$ it is readily seen that if $w \in \Omega U$, then $\tilde{g}(w) \in \Omega X$ implies that $\bar{l}^{-1}(w) \in \Omega U$ so that (ii) also holds.

Conversely, assume $\bar{l}$ satisfies (i) and (ii). Let $V: U \rightarrow U$ be defined by

$$V := A_0(\bar{l}).$$

By (i) and Lemma 5.6, V is invertible and $V^{-1} = A_0(\bar{l}^{-1})$. Hence, the map $\bar{l}^{-1} - V^{-1}$ (where V^{-1} is regarded as a static map) is strictly causal and

$$(5.8) \quad \tilde{f} := \bar{l}^{-1} - V^{-1}$$

is an extended linear i/o map $\Lambda U \rightarrow \Lambda U$. We claim that $\ker \tilde{g} \subset \ker \tilde{f}$ (where $\tilde{f}$ is the restricted linear i/o map associated with $\tilde{f}$). Indeed $\tilde{g}(w) = 0$ implies $\tilde{g}(w) \in \Omega X$ and by (ii) $\bar{l}^{-1}(w) \in \Omega U$. By (5.8) it then follows that $\tilde{f}(w) \in \Omega U$ so that $\tilde{f}(w) = 0$. Upon employing Theorem 3.9, we conclude the existence of a K -linear map $H: X \rightarrow U$ such that $\tilde{f} = H \cdot \tilde{g}$ (or equivalently $\bar{F} = H \cdot \bar{g}$). Letting $L: X \rightarrow U$ be defined by $L = VH$ we obtain upon substituting into (5.8)

$$\bar{l}^{-1} = V^{-1} + V^{-1}L\tilde{g}$$

and the proof is complete. $\square$

COROLLARY 5.9. *Let $\tilde{g}: \Lambda U \rightarrow \Lambda X$ be a reachable extended linear i/s map. Then for every feedback pair (L, V) the map $\tilde{f}_{L,V}$ defined by (5.5) is a reachable extended linear i/s map.*

Proof. We shall prove the corollary by showing that $\tilde{f}_{L,V}$ satisfies Theorem 3.9. Let $\tilde{f}: \Omega U \rightarrow \Gamma S$ be any restricted linear i/o map such that $\ker \tilde{f}_{L,V} \subset \ker \tilde{f}$. Let $\bar{f}$ be the extended linear i/o map associated with $\tilde{f}$ and define $\bar{f}_1: \Lambda U \rightarrow \Lambda S$ by

$$\bar{f}_1: \Lambda U \rightarrow \Lambda S: w \mapsto \bar{f}\bar{l}^{-1}(w).$$

If we can show that $\ker \tilde{g} \subset \ker \bar{f}_1$, then by Theorem 3.9 there exists a unique $H: X \rightarrow S$ such that $\bar{f}_1 = H \cdot \tilde{g}$ (or equivalently $\bar{f}_1 = H \cdot \bar{g}$), whence $\bar{f} = \bar{f}_1 \circ \bar{l} = H \cdot \bar{g} \circ \bar{l} = H \cdot \tilde{f}_{L,V}$ (or equivalently $\tilde{f} = H \cdot \tilde{f}_{L,V}$) showing that $\tilde{f}_{L,V}$ satisfies Theorem 3.9.

Hence, the proof will be complete upon showing that $\ker \tilde{g} \subset \ker \bar{f}_1$. If $w \in \Omega U$ and $\tilde{g}(w) = 0$ then $\tilde{g}(w) \in \Omega X$ and by Theorem 5.7(ii) $w_1 := \bar{l}^{-1}(w) \in \Omega U$. It follows that

$$\tilde{f}_{L,V}(w_1) = \tilde{g} \circ \bar{l}(w_1) = \tilde{g} \circ \bar{l} \circ \bar{l}^{-1}(w) = \tilde{g}(w) \in \Omega X$$

and consequently $\tilde{f}_{L,V}(w_1) = 0$ from which $\tilde{f}(w_1) = 0$ by assumption. Thus, $\bar{f}(w_1) \in \Omega S$ and

$$\bar{f}_1(w) = \bar{f} \circ \bar{l}^{-1}(w) = \bar{f}(w_1) \in \Omega S,$$

so that $\bar{f}_1(w) = 0$ and $\ker \tilde{g} \subset \ker \bar{f}_1$. $\square$

We shall now specialize our results to the finite dimensional case and assume, just as in § 4, that $U = K^m$ and $Y = K^p$. Moreover, if $\tilde{f}: \Omega U \rightarrow \Gamma Y$ is a restricted linear i/o map, we shall assume that $\tilde{f}$ is rational, i.e. that it has a realization with finite dimensional state space.

First we have the following specialization of Theorem 5.7.

THEOREM 5.10. *Let $\tilde{g}: \Lambda U \rightarrow \Lambda X$ be a reachable extended finite dimensional (i.e. $X = K^n$) linear i/s map, and let $D\Omega U = \ker \tilde{g}$. If $\bar{l}: \Lambda U \rightarrow \Lambda U$ is a ΛK -linear map, there exists a pair (L, V) such that $\bar{l} = \bar{l}_{L,V}$ (see (5.3)) if and only if $\bar{l}$ is a bicausal isomorphism and $Z_{\bar{l}}^{-1}D$ is a polynomial matrix (where $Z_{\bar{l}}$ denotes the transfer function of $\bar{l}$).*

Remark 5.11. The condition that $Z_{\bar{l}}^{-1}D$ is a polynomial matrix is obvious upon examination of (5.3). In the finite dimensional case we have $Z_{\tilde{g}} = SD^{-1}$ (compare Theorem 4.10), and consequently (5.3) implies that

$$Z_{\bar{l}}^{-1}D = V^{-1}(I + LSD^{-1})D = V^{-1}(D + LS).$$

It is interesting to observe that if $\bar{f}_{\bar{l}}$ is given by $\tilde{g} \circ \bar{l}$ (where $\bar{l} = \bar{l}_{L,V}$), then $Z_{\bar{f}_{\bar{l}}} = S(D + LS)^{-1}V$ and consequently

$$(5.12) \quad \ker \tilde{f}_{\bar{l}} = V^{-1}(D + LS)\Omega U.$$

Alternatively, (5.12) can also be written in the form

$$\ker \tilde{f}_{\bar{l}} = V^{-1}(D + Q)\Omega U,$$

where Q is an arbitrary polynomial matrix such that QD^{-1} is strictly proper (the matrix L being of course uniquely specified through Theorem 4.10). The interesting point in this alternate formulation is the fact that we have eliminated L from explicit consideration.

It is also interesting to make explicit the possible i/o maps (or transfer functions) obtainable by feedback. If $\tilde{f} = ND^{-1}$ where it is understood that the factorization defines a realization of $\tilde{f}$, then the transfer functions obtainable using feedback are given by

$$\tilde{f}_{L,V} = N(D + Q)^{-1}V,$$

where Q is any polynomial matrix such that QD^{-1} is strictly proper (L being, as before, specified through Theorem 4.10).

The reader should notice the similarity of the situation with the single input single output case.

Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a rational restricted linear i/o map, and let $\bar{l}: \Lambda U \rightarrow \Lambda U$ be a rational causal ΛK -linear map. Let (X, g, h) be a reachable finite dimensional ($X = K^n$) realization of $\tilde{f}$ and let $\tilde{g}$ be the extended linear i/s map associated with $\tilde{f}$. Theorem 5.10 then states that there exists a feedback pair (L, V) such that $\bar{l} = (I + L\tilde{g})^{-1}V$ provided $\bar{l}$ is a bicausal isomorphism and $Z_{\bar{l}}^{-1}D$ is a polynomial matrix, where $Z_{\bar{l}}$ is the transfer function of $\bar{l}$, and D is a polynomial matrix satisfying $D\Omega U = \ker \tilde{g}$. While the bicausality of $\bar{l}$ is clearly a necessary condition for its feedback implementation in a given realization, it is not sufficient. We shall conclude this section by showing that there always exists a reachable realization in which $\bar{l}$ can be implemented by feedback if and only if $\bar{l}$ is a rational bicausal isomorphism.

THEOREM 5.13. *Let $\tilde{f}: \Omega U \rightarrow \Gamma Y$ be a rational restricted linear i/o map and let $\bar{l}: \Lambda U \rightarrow \Lambda U$ be a rational causal ΛK -linear map. A necessary and sufficient condition for the existence of a finite dimensional reachable realization (X, g, h) of $\tilde{f}$ such that*

$$Z_{\bar{l}} = (I + LSD^{-1})^{-1}V$$

for some feedback pair (L, V) (where SD^{-1} is the transfer function of the linear i/s map associated with g) is that $\bar{l}$ is a bicausal isomorphism.

Proof. The necessity of the condition is obvious from Theorem 5.10. We will prove sufficiency. If $\tilde{f}$ is rational, then $Z_{\tilde{f}} = ND^{-1}$ for polynomial matrices N and D and $D\Omega U \subset \ker \tilde{f}$. Also if $\bar{l}$ is a bicausal isomorphism, then it can be expressed as $\bar{l} = (I + PQ^{-1})^{-1}V$ (with PQ^{-1} strictly proper). Let $R := \det Q \cdot I$. Then $Z_{\tilde{f}} = NR(DR)^{-1}$ and consider the realization $(X_{\Delta}, g_{\Delta}, h_{\Delta})$ where $\Delta = DR\Omega U$. Clearly $Z_{\bar{l}}^{-1}DR = V^{-1}(I + PQ^{-1})D \cdot \det Q \cdot I$ is a polynomial matrix satisfying the condition of Theorem 5.10. $\square$

Remark 5.14. Theorem 5.13 gives a complete characterization of those dynamical compensators which can be implemented by “state feedback” in a (possibly unobservable) reachable state-space. This characterization problem received some attention in the literature and the reader is referred to Wolovich (1974).

6. Chains and invariants. Throughout this section we shall assume that $U = K^m$ and $Y = K^p$. Let $\Delta \subset \Omega U$ be a submodule of rank m and let (X_{Δ}, g_{Δ}) be the (finite dimensional) reachable semirealization given by $X_{\Delta} := \Omega U / \Delta$ and $g_{\Delta}: \Omega U \rightarrow X_{\Delta}$ the canonical projection. Let $\tilde{g} = \tilde{g}_{\Delta}$ be the (restricted) linear i/s map associated with g_{Δ} . Then $\ker \tilde{g} = \Delta$ and, as we have seen at the beginning of § 4, all reachable realizations and hence also all reachable linear i/s maps whose kernel is Δ are isomorphic. Thus, modulo state space isomorphism, $\tilde{g}$ is uniquely determined by Δ . If feedback $\bar{l}$ is applied to $\tilde{g}$, then $\tilde{g}$ is transformed into a new reachable linear i/s map $\tilde{f}_{\bar{l}}$ (see (5.5)) and correspondingly the submodule Δ is transformed into a submodule $\Delta(\bar{l}) := \ker \tilde{f}_{\bar{l}}$ (see (5.12)). Two submodules Δ and Δ' of ΩU of rank m will be called *feedback equivalent* if there exists a feedback $\bar{l}$ such that if $\Delta = \ker \tilde{g}$, then $\Delta' = \ker \tilde{f}_{\bar{l}}$. Clearly feedback equivalence is an equivalence relation and as we have just seen, each feedback equivalence class of submodules characterizes a class of feedback equivalent linear i/s maps. In the present section we study this equivalence relation of submodules.

For $w \in \Omega U$ define the (polynomial) degree of w by $\deg w := -\text{ord } w$. Let Δ and Δ' be submodules of ΩU . A $K[z]$ -homomorphism $q: \Delta \rightarrow \Delta'$ is called *degree preserving* if for each $w \in \Delta$, $\deg q(w) = \deg w$.

LEMMA 6.1. *Two submodules Δ and Δ' of ΩU of rank m are feedback equivalent if and only if there exists a degree preserving $K[z]$ -isomorphism $q: \Delta \rightarrow \Delta'$.*

Proof. If Δ and Δ' are feedback equivalent then $\Delta = \ker \tilde{g}$ and $\Delta' = \ker \tilde{f}_{\bar{l}}$ where $\tilde{f}_{\bar{l}} = \tilde{g} \circ \bar{l}$ for a feedback transformation $\bar{l}$. Let q denote the restriction of $\bar{l}^{-1}$ to Δ . We see that q maps Δ into Δ' . Indeed, if $w \in \Delta$, then $\tilde{g}(w) \in \Omega X$, and by Theorem 5.7(ii) we also have $v := q(w) = \bar{l}^{-1}(w) \in \Omega U$. Thus $\tilde{f}_{\bar{l}}(v) = \tilde{g} \circ \bar{l} \circ \bar{l}^{-1}(w) = \tilde{g}(w) \in \Omega X$, whence $\tilde{f}_{\bar{l}}(v) = 0$ and $q(w) = v \in \Delta'$. From the bicausality of $\bar{l}$, it is clear that $\text{ord } \bar{l}^{-1}(w) = \text{ord } w$ for all $w \in \Lambda U$ and consequently the homomorphism $q: \Delta \rightarrow \Delta': w \mapsto \bar{l}^{-1}(w)$ is a degree preserving isomorphism.

Conversely, assume that Δ and Δ' are submodules of ΩU of rank m and that a degree preserving $q: \Delta \rightarrow \Delta'$ exists. Let $d_1, \dots, d_m$ be a basis for Δ and denote $d'_i := q(d_i)$, $i = 1, \dots, m$. Let D and D' be the polynomial matrices $D := [d_1, \dots, d_m]$ and $D' := [d'_1, \dots, d'_m]$. Then surely D is nonsingular and for each $w \in \Delta$ we have $q(w) = D'D^{-1}w$. By formally regarding the matrix $D'D^{-1}$ as a transfer function, we see that q extends uniquely to a ΛK -linear map

$$\bar{q}: \Lambda U \rightarrow \Lambda U; \quad Z_{\bar{q}} := D'D^{-1}.$$

We will complete the proof by showing that $\bar{q}$ is a bicausal isomorphism and that the map $\bar{l} := \bar{q}^{-1}$ is the desired feedback transformation such that $\Delta' = \Delta(\bar{l})$. First note that $\bar{q}$ (and hence also $\bar{l}$) is a bicausal isomorphism if and only if

$$(6.2) \quad \text{ord } \bar{q}(w) = \text{ord } w$$

for all $w \in \Lambda U$. Recall that (6.2) holds for $w \in \Delta$ by assumption. Since Δ is of rank m (and hence $\Omega U/\Delta$ is a torsion module) there exists a polynomial ψ such that $\psi w \in \Delta$ for all $w \in \Omega U$. Thus $\text{ord } \bar{q}(\psi w) = \text{ord } \psi w$ for all $w \in \Omega U$ and it follows that (6.2) also holds for all $w \in \Omega U$. Next, for $w \in \Lambda U$ let k be an integer such that $w_1 := \mathcal{S}(z^k w) \neq 0$. Then $w_1 \in \Omega U$ and $\text{ord } \bar{q}(w_1) = \text{ord } w_1 = \text{ord } z^k w$. Since for k sufficiently large $\text{ord } \bar{q}(w_1) = \text{ord } \bar{q}(z^k w)$, it follows that (6.2) holds for all $w \in \Lambda U$ and $\bar{q}$ is a bicausal isomorphism. Finally, to see that $\bar{l} = \bar{q}^{-1}$ is the desired feedback transformation we need to show (see Theorem 5.10) that $Z_{\bar{l}}^{-1}D$ is a polynomial matrix. Indeed this is true since $Z_{\bar{l}}^{-1}D = Z_{\bar{q}}^{-1}D = D'D^{-1}D = D'$ and the proof is complete. $\square$

In order to apply Lemma 6.1 in checking as to whether two submodules Δ and Δ' are feedback equivalent one has to verify the existence (or nonexistence) of a degree preserving homomorphism $q: \Delta \rightarrow \Delta'$. To this end we introduce the concept of *degree chain*.

DEFINITION 6.3. Let Δ be a submodule of ΩU and for each $i = 0, 1, 2, \dots$, let $\Delta_i \subset \Delta$ be the submodule generated by the elements of Δ whose polynomial degree is less than or equal to i . Let $\nu_i := \nu_i(\Delta) := \text{rank } \Delta_i$. The sequence $(\Delta_i)_{i=0}^{\infty}$ of submodules is called the *degree chain* of Δ and the sequent $(\nu_i)_{i=0}^{\infty}$ is called the *degree list*.

It follows immediately that $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \dots \subset \Delta$, and $\nu_0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu := \text{rank } \Delta$. Since ΩU is Noetherian, it satisfies the ascending chain condition and we have $\Delta_r = \Delta$ for some finite r . Hence

$$\Delta_0 \subset \Delta_1 \subset \dots \subset \Delta_r = \Delta, \quad 0 \leq \nu_0 \leq \nu_1 \leq \dots \leq \nu_r = \nu.$$

We also introduce the following notation. Let $a \neq 0$ be a polynomial or a polynomial vector. We denote by $\hat{a}$ the leading coefficient (vector) of a . That is, if $a = \sum_{i=0}^k a_i z^i$ and $a_k \neq 0$ then $\hat{a} = a_k$. Also, if $a = 0$ we define $\hat{a} := 0$. Recall that if $d_1, \dots, d_k$ are elements of a $K[z]$ -module which generate a submodule M , they are called *free generators of M* , or simply *free* if they are $K[z]$ -linearly independent. We now introduce the following

DEFINITION 6.4. Let $d_1, \dots, d_k$ be elements of a finitely generated $K[z]$ -module (specifically of ΩU). Then $d_1, \dots, d_k$ are called *properly free* if $\hat{d}_1, \dots, \hat{d}_k$ are K -linearly independent.

LEMMA 6.5. *If $d_1, \dots, d_k$ are properly free then they are free.*

Proof. We prove the lemma by negation. If $d_1, \dots, d_k$ are not free, there exist polynomials $\alpha_1, \dots, \alpha_k$, not all zero, such that $\sum_{i=1}^k \alpha_i d_i = 0$. Let $r := \max \deg(\alpha_i d_i)$ and define

$$\varepsilon_i := \begin{cases} 0 & \text{if } \deg(\alpha_i d_i) < r, \\ 1 & \text{if } \deg(\alpha_i d_i) = r. \end{cases}$$

Adding the terms of degree r in $\sum \alpha_i d_i$ yields

$$\sum_{i=1}^k \varepsilon_i \hat{\alpha}_i \hat{d}_i = 0,$$

implying that $\hat{d}_1, \dots, \hat{d}_k$ are K -linearly dependent since not all $\varepsilon_i \hat{\alpha}_i$ are zero. $\square$

DEFINITION 6.6. Let Δ be a free $K[z]$ -module. A basis $d_1, \dots, d_k$ of Δ is called *proper* if $d_1, \dots, d_k$ are properly free.

The following is an instrumental result.

LEMMA 6.7. *Let Δ be a submodule of ΩU of rank $\nu(>0)$. Then there exists a proper basis $d_1, \dots, d_\nu$ of Δ such that*

$$(6.8) \quad \deg d_j = i \quad \text{for } \nu_{i-1} < j \leq \nu_i \text{ and } i = 0, 1, 2, \dots$$

where $(\nu_i)_{i=0}^\infty$ is the degree list of Δ and $\nu_{-1} := 0$.

Proof. First observe that if $k \geq 0$ is the first integer such that $\nu_k \neq 0$, we can choose $0 \neq d_1 \in \Delta_k$ such that $\deg d_1 = k$. Then d_1 is clearly properly free and satisfies (6.8). We proceed stepwise and assume that for $l > 0$, $d_1, \dots, d_l$ are properly free elements of Δ satisfying (6.8) and let S denote the submodule generated by $d_1, \dots, d_l$. If $l \neq \nu$ (and hence $S \neq \Delta$), then there exists an integer t such that

$$\Delta_{t-1} \subset S \subsetneq \Delta_t.$$

Thus, there exists an element $d_{l+1} \in \Delta_t$ such that $d_{l+1} \notin S$ and $\deg d_{l+1} = t$. Obviously $d_1, \dots, d_{l+1}$ satisfies (6.8) and we complete the proof by showing that this set is also properly free. Assume to the contrary that $\hat{d}_{l+1}$ is a K -linear combination of $\hat{d}_1, \dots, \hat{d}_l$:

$$\hat{d}_{l+1} = \sum_{i=1}^l \lambda_i \hat{d}_i.$$

Consider the element

$$p := d_{l+1} - \sum_{i=1}^l \lambda_i z^{\delta_i} d_i$$

where $\delta_i := t - \deg d_i$, $i = 1, \dots, l$. Obviously $p \in \Delta$ and it is easily verified that $\deg p \leq t - 1$. Thus $p \in \Delta_{t-1} \subset S$ and we conclude that $d_{l+1} \in S$, a contradiction to our assumption. $\square$

COROLLARY 6.9. *Let $\Delta \subset \Omega U$ be a submodule of rank ν and let $(\Delta_i)_{i=0}^\infty$ and $(\nu_i)_{i=0}^\infty$ be the degree chain and degree list of Δ respectively. There exists a proper basis $d_1, \dots, d_\nu$ of Δ such that for each i satisfying $\nu_i \neq 0$ the set $d_1, \dots, d_{\nu_i}$ is a basis for Δ_i .*

Proof. The basis constructed in Lemma 6.7 satisfies the desired property. $\square$

DEFINITION 6.10. Two submodules Δ and Δ' of ΩU are called *chain isomorphic* if there exists a $K[z]$ -isomorphism $q: \Delta \rightarrow \Delta'$ such that $q(\Delta_i) = \Delta'_i$ for $i = 0, 1, 2, \dots$.

The following is the main result of the present section.

THEOREM 6.11. *Let Δ and Δ' be two submodules of ΩU of rank m . Then the following statements are equivalent:*

- (i) Δ and Δ' are feedback equivalent.
- (ii) There exists a degree preserving $K[z]$ -homomorphism $q: \Delta \rightarrow \Delta'$.
- (iii) Δ and Δ' are chain isomorphic.
- (iv) $\nu_i(\Delta) = \nu_i(\Delta')$ for $i = 0, 1, 2, \dots$.

Proof. The equivalence of (i) and (ii) follows from Lemma 6.1. We prove the remaining implications.

(ii) $\Rightarrow$ (iii). If $q: \Delta \rightarrow \Delta'$ is a degree preserving $K[z]$ -homomorphism then $q(\Delta_i) \subset \Delta'_i$. Indeed, if $w \in \Delta_i$, then $w = \sum \alpha_j w_j$ with $\alpha_j \in K[z]$, and $w_j \in \Omega U$ satisfying $\deg w_j \leq i$. Hence $q(w) = \sum \alpha_j q(w_j) \in \Delta'_i$ since $\deg q(w_j) = \deg w_j \leq i$. Similarly we have that $q^{-1}(\Delta'_i) \subset \Delta_i$ and hence $q(\Delta_i) = \Delta'_i$.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (ii). Suppose $\nu_i(\Delta) = \nu_i(\Delta') = \nu_i$ for all $i = 0, 1, 2, \dots$. Let $d_1, \dots, d_m$ and $d'_1, \dots, d'_m$ be bases of Δ and Δ' respectively as in Lemma 6.7 (see also Corollary 6.9). Define the $K[z]$ -homomorphism $q: \Delta \rightarrow \Delta'$ by $d_i \mapsto d'_i$ for $i = 1, \dots, m$. It is readily noted that this q is degree preserving. $\square$

Remark 6.12. For a module $\Delta \subset \Omega U$ of rank m , a sequence of submodules which is essentially equivalent to our degree chain has previously been introduced by Eckberg (1974) in his study of so called canonical matrices. Eckberg's motivation for the introduction of this chain was the construction of a certain unique "canonical" basis for Δ (see also Remark 6.13 below). It is interesting to observe that this chain was completely ignored in Eckberg's study of feedback and he apparently never recognized its feedback invariance properties.

Remark 6.13. As we have already mentioned earlier, fraction representations of the form ND^{-1} were used successfully over the past several years for the study of a variety of technical problems associated with feedback. While it is usually required (unnecessarily) that N and D be right coprime it was also recognized correctly that it is useful to select D in a special form. Specifically, it was first recognized by Wolovich that the matrix D should be taken to be "column proper", i.e. that the sum of the degrees of the columns of D equal the degree of the determinant of D (see Wolovich (1974)). Similar requirements were subsequently made by others (see e.g. Heymann (1972, Chap. 6) and Popov (1970), (1972)). The requirement of column properness has been introduced upon the technical observation that certain "canonical" forms arise naturally provided the matrix D is chosen to be column proper. However, there was no deep understanding of the reasons for this fact since everything was "technique" oriented and motivated. Eckberg (1974) tried to formulate the column properness in his module theoretic study and had moderate success. Forney (1975) developed a theory of *minimal bases* for rational vector spaces for the purpose of giving respectability to what is essentially also nothing else but column properness. The drawback in these approaches is that they depend on defining at the outset (and without any prior motivation) certain "canonical" structures. Yet, it is easily

verified that the column properness property is essentially the same as the property that the columns of D are proper basis for the submodule $\Delta = D\Omega U$. In view of Theorem 6.11 it thus becomes clear that if D is chosen to be column proper, then input isomorphisms are easily tested for the degree preserving property. This explains (in a natural way) the usefulness of this construction for the study of state feedback and it is precisely this important fact, that remained obscure in previous investigations.

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