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Refinements of the Nash equilibrium concept for bimatrix games

by

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Eindhoven, May 1981

The Netherlands
REFINEMENTS OF THE NASH EQUILIBRIUM CONCEPT FOR

BIMATRIX GAMES

by

E.E.C. van Damme

Abstract
In the literature several refinements of the Nash equilibrium concept have been introduced. Among these there are: the essential equilibrium point [16], the quasi-strong equilibrium point [2], the perfect equilibrium point [15], and the proper equilibrium point [11]. In this paper bimatrix games are considered. For this class of games a new refinement of the equilibrium concept, called nondegenerate equilibrium point, is introduced. It is proved that nondegenerate equilibrium points are essential, quasi-strong, proper and perfect. Furthermore, it is shown that for almost all bimatrix games all equilibrium points are nondegenerate.

1. Introduction
It is well-known that for noncooperative n-person non-zero sum games in normal form not all Nash equilibrium points are equally suited to be chosen as the solution. The reason for this fact is, that (in general) some equilibrium points are more stable than others. Therefore, in the literature various refinements of the Nash equilibrium concept have been introduced. Each of these refinements requires the equilibrium point to satisfy a particular stability condition.
Among these refinements there are:
- the essential equilibrium point, introduced by Wu Wen-tsün and Jiang Jia-he [16],
- the quasi-strong equilibrium point, introduced by Harsanyi [2],
- the perfect equilibrium point, introduced by Selten [15], and
- the proper equilibrium point, introduced by Myerson [11].

Loosely speaking we may say that for a general n-person game $G$ in normal form
- an equilibrium point $\pi$ is essential if each game near to $G$ has an equilibrium point near to $\pi$,
- an equilibrium point is quasi-strong if each player uses each pure strategy, which is a best reply against the combination of strategies played by the other players, with a positive probability,
- an equilibrium point is perfect if each player plays a strategy, which is a best reply against small perturbations of the combination of strategies, played by the other players, and
- an equilibrium point is proper if each player plays a strategy, which is a best reply against particular small perturbations of the combination of strategies, played by the other players.

In this paper, we consider bimatrix games (noncooperative 2-person general sum games) and derive relations between the refinements introduced above for this special class of games. The results we obtain are related to the results obtained by Jansen ([6] and [7]). However, he only considered essential equilibrium points (which he calls stable equilibrium points) and quasi-strong equilibrium points (which he calls regular equilibrium points). In this paper we restate some of his results. Other results of him are provided with a different proof.

The organisation of the paper is as follows.
In section 2 the formal definitions of the refinements introduced above are given for the special case of a bimatrix game.
In section 3 isolated equilibrium points are introduced (as equilibrium points which are such that there are no equilibrium points close to them) and a characterization of these equilibria is derived. Next, nondegenerate equilibrium points are introduced. An equilibrium point is nondegenerate if it is both isolated and quasi-strong. A useful characterization of nondegenerate equilibrium points is derived.
In section 4 we prove that nondegenerate equilibrium points are essential and that nondegenerate equilibrium points are proper (and hence also perfect). Unfortunately, not all bimatrix games possess nondegenerate equilibrium points. However, we prove that for almost all bimatrix games all equilibrium points are nondegenerate. More precisely, we show that the set of all m-by-n bimatrix games, for which at least one equilibrium point is degenerate, is a closed set, which has Lebesgue measure zero, within the class of all m-by-n bimatrix games.

Finally, in section 5, we derive a characterization of perfect equilibrium points of bimatrix games. Furthermore, three characterizations of non-degenerate equilibrium points are derived.

2. Preliminaries

An m-by-n bimatrix game consists of two m-by-n matrices

\[ A = [a_{ij}]_{i=1}^{m}{}_{j=1}^{n} \quad \text{and} \quad B = [b_{ij}]_{i=1}^{m}{}_{j=1}^{n} , \]

where \( a_{ij} \) (resp. \( b_{ij} \)) represents the payoff for player 1 (resp. player 2) when player 1 uses his pure strategy \( i \) and player 2 uses his pure strategy \( j \).

\( G_{m \times n} \) is the set of all m-by-n bimatrix games.

If \( (A,B), (A',B') \in G_{m \times n} \) we define the distance between \( (A,B) \) and \( (A',B') \) by:

\[
d((A,B);(A',B')) = \max \left\{ |a_{ij} - a'_{ij}|, |b_{ij} - b'_{ij}| ; \quad i \in \{1,\ldots,m\}, \quad j \in \{1,\ldots,n\} \right\} .
\]

Furthermore, we define for \( \varepsilon > 0 \) and \( (A,B) \in G_{m \times n} \):

\[
B_\varepsilon(A,B) = \{(A',B') \in G_{m \times n} ; \quad d((A,B);(A',B')) < \varepsilon \}.
\]

In this paper vectors will always be column vectors. If \( x \) is a vector, then \( x^T \) is the transpose of \( x \) and \( x_i \) is the \( i \)th coordinate of this vector.

For natural numbers \( k \) we define \( \mathbb{N}_k := \{1,2,\ldots,k\} \) and
If \((A, B)\) is an \(m\)-by-\(n\) bimatrix game, then \(S^m(S^n)\) is the set of all mixed strategies of player 1 (player 2). Elements of \(S^m\) will be denoted by \(p\), elements of \(S^n\) will be denoted by \(q\). \(e_i\) is the element of \(S^m\) (or \(S^n\), no confusion will result) with \(i\)’th coordinate one.

If \((p, q), (p’, q’) \in S^m \times S^n\) we define the distance between \((p, q)\) and \((p’, q’)\) by

\[
d((p, q); (p’, q’)) := \max \{|p_i - p'_i|, |q_j - q'_j|; i \in N_m, j \in N_n\}
\]

Furthermore, for \(\varepsilon > 0\) and \((p, q) \in S^m \times S^n\) we define

\[
B_\varepsilon(p, q) := \{(p’, q’) \in S^m \times S^n; d((p, q); (p’, q’)) < \varepsilon\}.
\]

If \((A, B) \in G_{m \times n}\) and \((p, q) \in S^m \times S^n\) we define

\[
C_1(p) := \{i \in N_m; p_i > 0\},
\]

\[
M_1(A, q) := \{i \in N_m; \sum_{k \in N_m} e_i^T A q_{k} = \max_{k \in N_m} e_i^T A q_{k}\},
\]

\[
C_2(q) := \{j \in N_n; q_j > 0\} \quad \text{and,}
\]

\[
M_2(B, p) := \{j \in N_n; \sum_{l \in N_n} p^T B e_j = \max_{l \in N_n} p^T B e_j\}.
\]

We call \(C_1(p)\) the carrier of \(p\). \(p\) is completely mixed if \(C_1(p) = N_m\).

\(M_1(A, q)\) is (in the bimatrix game \((A, B)\)) the set of all pure best replies of player 1 against \(q\). The set of all best replies against \(q\) is

\[
\{p \in S^m; p^T A q = \max_{\pi \in S^m} \pi^T A q\} = \{p \in S^m; C_1(p) \subseteq M_1(A, q)\}.
\]

By \(|C_1(p)|\) and \(|M_1(A, q)|\) we denote the number of elements of \(C_1(p)\) and \(M_1(A, q)\), respectively. Of course, similar definitions are given for \(C_2(q)\) and \(M_2(B, p)\). A strategy pair \((p, q) \in S^m \times S^n\) is a Nash equilibrium point of the bimatrix game \((A, B)\) if \(p\) is a best reply against \(q\) and \(q\) is a best reply against \(p\) (hence \(C_1(p) \subseteq M_1(A, q)\) and \(C_2(q) \subseteq M_2(B, p)\)).
Nash [12] has shown that any bimatrix game has at least one (Nash) equilibrium point. By \( E(A,B) \) we denote the set of all equilibrium points of the bimatrix game \((A,B)\).

In the following we will give the definitions of some refinements of the equilibrium point concept for the special case of a bimatrix game \((A,B)\). The first refinement is a concept called essential equilibrium point. This concept was introduced by Wu Wen-tsün and Jiang Jia-he [16].

**Definition 2.1.**

\((p,q) \in E(A,B)\) is an essential equilibrium point of the bimatrix game \((A,B)\) if for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \((A',B') \in B_\varepsilon (A,B)\) there exists \((p',q') \in E(A',B') \cap B_{\delta} (p,q)\).

By \( E^e(A,B) \) we denote the set of all essential equilibrium points of the bimatrix game \((A,B)\). \( G^e_{m \times n} \) is the set of all \((A,B) \in G_{m \times n} \) which have the property that \( E^e(A,B) = E(A,B) \).

**Remark 2.2.**

i) the set \( E^e(A,B) \) may be empty as we see by considering the bimatrix game \((A,B)\) with \( A = B = (1 1) \).

ii) in [16] it is proved that \( G^e_{m \times n} \) is dense in \( G_{m \times n} \) and that \( E^e(A,B) \neq \emptyset \) whenever \((A,B)\) is a bimatrix game which has a finite number of equilibrium points.

Next, we consider quasi-strong equilibrium points. The concept of quasi-strong equilibrium points has been introduced by Harsanyi [2].

**Definition 2.3.**

\((p,q) \in E(A,B)\) is a quasi-strong equilibrium point of the bimatrix game \((A,B)\) if \( C_1(p) = M_1(A,q) \) and \( C_2(q) = M_2(B,p) \).

By \( E^{qs}(A,B) \) we denote the set of all quasi-strong equilibrium points of the bimatrix game \((A,B)\). \( G^{qs}_{m \times n} \) is the set of all \( m \times n \) bimatrix games for which all equilibrium points are quasi-strong.
Remark 2.4

(i) it is still an open question, whether there exists a bimatrix game \((A,B)\) with \(E^{qs}(A,B) = \emptyset\).

(ii) Harsanyi [3] has proved that the complement of \(G_{m \times n}^{qs}\) is a closed set with (Lebesgue) measure zero.

Next, we consider the perfectness concept, introduced by Selten [15].

Definition 2.5.

\((p,q) \in E(A,B)\) is a perfect equilibrium point of the bimatrix game \((A,B)\) if there exists a sequence \(\{(p(k),q(k))\}_{k \in \mathbb{N}}\) of elements of \(S^m \times S^n\) having the following properties:

(i) \(p(k)\) and \(q(k)\) are completely mixed, for all \(k \in \mathbb{N}\),

(ii) \(p\) is a best reply against \(q(k)\) and \(q\) is a best reply against \(p(k)\), for all \(k \in \mathbb{N}\),

(iii) \(\lim_{k \to \infty} (p(k),q(k)) = (p,q)\).

\(E_{pe}(A,B)\) will be used to denote the set of all perfect equilibrium points of the bimatrix game \((A,B)\). Selten [15] has shown that \(E_{pe}(A,B) \neq \emptyset\) for any bimatrix game \((A,B)\). \(G_{m \times n}^{pe}\) is the set of all \(m\)-by-\(n\) bimatrix games for which all equilibrium points are perfect.

Finally, we consider the properness concept, introduced by Myerson [11].

Definition 2.6.

Let \(\varepsilon > 0\). \((p(\varepsilon),q(\varepsilon)) \in S^m \times S^n\) is an \(\varepsilon\)-proper equilibrium point of the bimatrix game \((A,B)\) if

(i) \(p(\varepsilon)\) and \(q(\varepsilon)\) are completely mixed,

(ii) for all \(i,k \in \mathbb{N}\): \(\varepsilon_i Aq(\varepsilon) < \varepsilon_k Aq(\varepsilon)\) implies \(p_1(\varepsilon) \leq p_k(\varepsilon)\),

(iii) for all \(j,\ell \in \mathbb{N}\): \(p(\varepsilon)^T b_j \leq p(\varepsilon)^T b_{k,\ell}\) implies \(q_j(\varepsilon) \leq q_{k,\ell}(\varepsilon)\),

\((p,q)\) is a proper equilibrium point of \((A,B)\), if there exists a sequence \(\{(\varepsilon_k, p(\varepsilon_k), q(\varepsilon_k))\}_{k \in \mathbb{N}}\) which satisfies:

(i) \(\varepsilon_k > 0\) for all \(k \in \mathbb{N}\), \(\lim_{k \to \infty} \varepsilon_k = 0\)

(ii) \(\lim_{k \to \infty} (p(\varepsilon_k),q(\varepsilon_k)) = (p,q)\)

(iii) \((p(\varepsilon_k),q(\varepsilon_k))\) is an \(\varepsilon_k\)-proper equilibrium point of \((A,B)\), for all \(k \in \mathbb{N}\).
We will use $E^\text{pr}(A,B)$ to denote the set of all proper equilibrium points of the bimatrix game $(A,B)$. Myerson [11] has shown that $\emptyset = E^\text{pr}(A,B) \subseteq E^\text{pe}(A,B)$ for any bimatrix game $(A,B)$. Moreover, it is possible that $E^\text{pr}(A,B) \neq E^\text{pe}(A,B)$ (see e.g. example 5.4). We will use $G^\text{pr}_m$ to denote the set of all $m \times n$ bimatrix games for which all equilibrium points are proper.

3. Nondegenerate equilibrium points

In the sequel relations between the concepts introduced in section 2 will be derived. Some of these relations can already be obtained by examining a simple example.

**Example 3.1.**

Let the bimatrix game $(A,B)$ be given by:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0,0)</td>
<td>(1,0)</td>
</tr>
<tr>
<td>2</td>
<td>(0,1)</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

We have

$E(A,B) = \{(p e_1 + (1-p) e_2, e_1); p \in [0,1]\} \cup \{(e_1, q e_1 + (1-q) e_2); q \in [0,1]\}.$

$E^\text{qs}(A,B) = \{(p e_1 + (1-p) e_2, e_1); p \in (0,1)\} \cup \{(e_1, q e_1 + (1-q) e_2); q \in (0,1)\}.$

$E^\text{pe}(A,B) = E^\text{pr}(A,B) = \{(e_1, e_1)\}.$

$E^\text{e}(A,B) = \emptyset.$

Hence, we see that

- a proper equilibrium point need not be quasi-strong,
- a proper equilibrium point need not be essential,
- a quasi-strong equilibrium point need not be perfect, and
- a quasi-strong equilibrium point need not be essential.

For a concept to be useful as a refinement of the equilibrium concept, it should be easy to check whether an equilibrium point satisfies the requirements of this concept. Of the concepts introduced in section 2 only the requirements of the quasi-strongness concept are trivial to check.
In example 3.1 we have however seen, that quasi-strong equilibrium points do not possess all nice properties we want equilibrium points to have. In the sequel we will see that this is caused by the fact that a quasi-strong equilibrium point need not be isolated. On the other hand, it will turn out, that equilibrium points, which are both quasi-strong and isolated, possess very nice properties (also see [6] and [7]). Therefore, we will introduce a new name for these equilibrium points; we will call them nondegenerate equilibrium points). We will show that it is easy to check whether an equilibrium point is nondegenerate.

Definition 3.2.
Let \((A,B) \in G_{m \times n}\) and \((p,q) \in E(A,B)\).

\((p,q)\) is an isolated equilibrium point of \((A,B)\) if there exists an \(\varepsilon > 0\) such that \(B_\varepsilon (p,q) \cap E(A,B) \setminus \{(p,q)\} = \emptyset\).

\(E^i(A,B)\) is the set of all isolated equilibrium points of \((A,B)\) and \(G^i_{m \times n}\) is the set of all \(m\)-by-\(n\) bimatrix games for which all equilibrium points are isolated. In example 3.1 we see that \(E^i(A,B) = \emptyset\) for some \((A,B)\).

Definition 3.3.
Let \((A,B) \in G_{m \times n}\) and \((p,q) \in E(A,B)\).

\((p,q)\) is a nondegenerate equilibrium point of \((A,B)\) if \((p,q)\) is both a quasi-strong and an isolated equilibrium point of \((A,B)\).

\(E^{nd}(A,B)\) is the set of all nondegenerate equilibrium points of \((A,B)\) and \(G^{nd}_{m \times n}\) is the set of all \(m\)-by-\(n\) bimatrix games for which all equilibrium points are nondegenerate.

1) We have chosen for this terminology, since a bimatrix game which is nondegenerate in the sense of Lemke and Howson [19] is a game for which all equilibrium points are nondegenerate in this sense.
In order to obtain a characterization of nondegenerate equilibrium points, we will first derive a characterization of isolated equilibrium points. This latter characterization is obtained via the so-called maximal Nash subsets ([4],[10]).

**Definition 3.4**

Let \((A,B) \in G_{m \times n}\). Assume \((p,q), (p',q') \in E(A,B), S \subseteq E(A,B)\). \((p,q) \text{ and } (p',q')\) are \(S\)-interchangeable if \((p',q) \text{ and } (p,q')\) are also elements of \(S\). \(S\) is a Nash subset of \((A,B)\) if any two elements of \(S\) are \(S\)-interchangeable. \(S\) is a maximal Nash subset of \((A,B)\) if \(S\) is a Nash subset of \((A,B)\) and if there does not exist a Nash subset of \((A,B)\), which properly contains \(S\).

In [5] the proof of the following theorem can be found.

**Theorem 3.5**

Let \((A,B) \in G_{m \times n}\).

(i) if \(S\) is a maximal Nash subset of \((A,B)\), then \(S\) is a closed and convex set,

(ii) there are only finitely many maximal Nash subsets of \((A,B)\),

(iii) \(E(A,B)\) is the union of all maximal Nash subsets of \((A,B)\).

**Theorem 3.6**

Let \((A,B) \in G_{m \times n}\) and \((p,q) \in E(A,B)\).

\((p,q)\) is an isolated equilibrium point of \((A,B)\) if and only if \(\{(p,q)\}\) is a maximal Nash subset of \((A,B)\).

**Proof**

Assume \(\{(p,q)\}\) is not a maximal Nash subset. Assume \((p',q')\) is such that \(q' \neq q\) and that \((p,q)\) and \((p',q')\) are \(E(A,B)\)-interchangeable.

Define \(q(\lambda) := \lambda q + (1-\lambda) q'\) for \(\lambda \in [0,1]\).

We have \((p,q(\lambda)) \in E(A,B) \setminus \{(p,q)\}\) for \(\lambda \in [0,1]\).

Moreover \(\lim_{\lambda \to 1} (p,q(\lambda)) = (p,q)\). Hence \((p,q)\) is not an isolated equilibrium point of \((A,B)\).
Assume \((p,q)\) is not an isolated equilibrium point of \((A,B)\).
For \(n \in \mathbb{N}\), let \((p(n),q(n)) \in B_{1/n}(p,q) \cap E(A,B)\setminus\{(p,q)\}.
By (ii) and (iii) of theorem 3.5 we may assume that there exists a maximal Nash subset \(S\) of \((A,B)\) such that \((p(n),q(n)) \in S\) for all \(n \in \mathbb{N}\). Since \(S\) is closed \((p,q) \in S\).
Hence, \(\{(p,q)\}\) is not a maximal Nash subset of \((A,B)\).

In theorem 3.9 we give a characterization of nondegenerate equilibrium points of bimatrix games \((A,B) \in G_{m \times n}\), which satisfy

\[(3.7) \ a_{ij} > 0 \text{ and } b_{ij} > 0 \text{ for all } i \in \mathbb{N}_m, j \in \mathbb{N}_n.\]

This is no restriction, since we have

**Lemma 3.8**

Let \((A,B) \in G_{m \times n}\).
Define \((A^+, B^+) \in G_{m \times n}\) by

\[
a_{ij}^+ = a_{ij} + 1 + \max\{|a_{ij}| ; i \in \mathbb{N}_m, j \in \mathbb{N}_n\}. \\
b_{ij}^+ = b_{ij} + 1 + \max\{|b_{ij}| ; i \in \mathbb{N}_m, j \in \mathbb{N}_n\}.
\]

Then

\[
E(A,B) = E(A^+, B^+) \quad \quad \quad \quad \quad \quad E^e(A,B) = E^e(A^+, B^+)
\]
\[
E^{qs}(A,B) = E^{qs}(A^+, B^+) \quad \quad \quad \quad \quad \quad E^{pe}(A,B) = E^{pe}(A^+, B^+)
\]
\[
E^{pr}(A,B) = E^{pr}(A^+, B^+) \quad \quad \quad \quad \quad \quad E^i(A,B) = E^i(A^+, B^+).
\]

**Proof**

Elementary. 

Theorem 3.9
Let $(A,B) \in \mathbb{C}^{m \times n}$ such that (3.7) is satisfied. Let $(p,q) \in E(A,B)$.
Define matrices $\tilde{A}$ and $\tilde{B}$ by

$$
\tilde{A} = [a_{ij}]_{i \in M_1(A,q), j \in C_2(q)} \quad \tilde{B} = [b_{ij}]_{i \in C_1(p), j \in M_2(B,p)}.
$$

The following two statements are equivalent
(i) $(p,q) \in E^{nd}(A,B)$,
(ii) $|M_1(A,q)| = |C_1(p)| = |C_2(q)| = |M_2(B,p)|$ and the matrices $\tilde{A}$ and $\tilde{B}$ are nonsingular.

Proof
Assume $(p,q) \in E^{nd}(A,B)$. Assume $y \neq 0$ is such that $\tilde{A}y = 0$. Define $\tilde{y} \in \mathbb{R}^n$ by

$$
\tilde{y}_j = y_j \quad \text{if } j \in C_2(q) \quad \text{and} \quad \tilde{y}_j = 0 \quad \text{else}.
$$

For $\epsilon \in \mathbb{R}$, define $q(\epsilon) \in \mathbb{R}^n$ by

$$
q(\epsilon) = (I + \epsilon \sum_{j=1}^{n} \tilde{y}_j)^{-1}(q + \epsilon \tilde{y}) = (I + \epsilon \sum_{j=1}^{n} \tilde{y}_j)^{-1}q + \epsilon.
$$

Since $\tilde{A}y = 0$, we have

$$
e_i^T Aq(\epsilon) = e_k^T Aq(\epsilon) \quad \text{if } i,k \in M_1(A,q),
$$

Hence, if $\epsilon$ is sufficiently close to zero (say $\epsilon \in (-\epsilon_0,\epsilon_0)$) we have

$$
q(\epsilon) \in S^n; \quad C_2(q(\epsilon)) = C_2(q); \quad M_1(A,q(\epsilon)) = M_1(A,q).
$$

Hence, $(p,q(\epsilon)) \in E(A,B)$, if $\epsilon \in (-\epsilon_0,\epsilon_0)$. This in in contradiction with $(p,q) \in E^1(A,B)$. Hence, if $y \neq 0$, we have $\tilde{A}y \neq 0$ and so the columns of $\tilde{A}$ constitute a set of independent vectors. Therefore we must have

$$
|M_1(A,q)| \geq |C_2(q)| \quad (3.10)
$$
Applying a similar reasoning as above, we can prove

if \( x \neq 0 \) then \( x^T \mathbf{B} \neq 0 \).

Hence

\[
|M_2(B,p)| \geq |C_1(p)|. \tag{3.11}
\]

From (3.10), (3.11) and \((p,q) \in E^{qs}(A,B)\) we may conclude

\[
|C_1(p)| = |M_1(A,q)| \geq |C_2(q)| = |M_2(B,p)| \geq |C_1(p)|.
\]

Hence, the matrices \( \tilde{A} \) and \( \tilde{B} \) are square. The above reasoning shows that both matrices are nonsingular.

Assume that \(|M_1(A,q)| = |C_1(p)| = |C_2(q)| = |M_2(B,p)|\) and that the matrices \( \tilde{A} \) and \( \tilde{B} \) are nonsingular. It suffices to prove that \((p,q) \in \mathcal{E}^1(A,B)\).

Assume \( q' \neq q \) is such that \((p,q') \in E(A,B)\). We have \( C_2(q') \subseteq C_2(q)\).

Let \( \tilde{q} = [q_j]_{j \in C_2(q)} \) and \( \tilde{q}' = [q'_j]_{j \in C_2(q)} \).

Since \( M_1(A,q') \supseteq M(A,q) \), there exists a \( \lambda \in \mathbb{R} \) such that \( \tilde{A}\tilde{q}' = \lambda \tilde{A}\tilde{q} \).

Since \( \tilde{q}' \neq \tilde{q} \) and \( \sum_j \tilde{q}'_j = \sum_j \tilde{q}_j = 1 \), we have \( \tilde{q}' \neq \lambda \tilde{q} \).

Hence, \( \tilde{q}' - \lambda \tilde{q} \neq 0 \) and \( A(\tilde{q}' - q) = 0 \). Which is a contradiction.

Similarly, we can prove that there does not exist a \( p' \neq p \) such that \((p',q) \in E(A,B)\). By theorem 3.6 we have \((p,q) \in E^1(A,B)\).

4. Properties of nondegenerate equilibrium points

In this section we will prove that nondegenerate equilibrium points have very nice properties. Namely: nondegenerate equilibrium points are essential (theorem 4.5) and nondegenerate equilibrium points are proper (theorem 4.9) and hence also perfect. Furthermore, we show that for almost all games all equilibrium points are nondegenerate (theorem 4.11). The proof that nondegenerate equilibrium points are essential is only slightly different from the proof in [7] (thm.7.5).

Our proof is splitted into two lemmas, each of them interesting in his own right.
Lemma 4.1
Let $(A,B) \in G_{m\times n}$ and $(p,q) \in E(A,B)$. Define matrices $\tilde{A}$ and $\tilde{B}$ and vectors $\tilde{p}$ and $\tilde{q}$ by

\[
\tilde{A} = \begin{bmatrix} a_{ij} \end{bmatrix}_{i \in M_1(A,q), j \in M_2(B,p)} \quad \tilde{B} = \begin{bmatrix} b_{ij} \end{bmatrix}_{i \in M_1(A,q), j \in M_2(B,p)}
\]
\[
\tilde{p} = \begin{bmatrix} p_i \end{bmatrix}_{i \in M_1(A,q)} \quad \tilde{q} = \begin{bmatrix} q_j \end{bmatrix}_{j \in M_2(B,p)}
\]

We have: $(p,q) \in E^e(A,B)$ if and only if $(\tilde{p}, \tilde{q}) \in E^e(\tilde{A}, \tilde{B})$.

Proof
This follows almost immediately from the fact that there exists an $\varepsilon > 0$ such that for all $(A',B') \in B(A,B)$ and all $(p',q') \in B(p,q)$:

$M_1(A',q') \subset M_1(A,q)$ and $M_2(B',p') \subset M_2(B,p)$.

Lemma 4.2 (cf [14])
Let $(A,B) \in G_{m\times n}$ be such that $m = n$ and that $A$ and $B$ are nonsingular. Define $F(B) \in \mathbb{R}^n$ and $G(A) \in \mathbb{R}^n$ by

\[
F_i(B) = \sum_{j=1}^{n} B_{ij} \quad i, j = 1, \ldots, n \quad (i \in \mathbb{N}_n)
\]
\[
G_j(A) = \sum_{i=1}^{n} A_{ij} \quad i, j = 1, \ldots, n \quad (j \in \mathbb{N}_n),
\]

where $A_{ij}$ represents the co-factor of $a_{ij}$ in the matrix $A$.

If $F_i(B) > 0$ for all $i \in \mathbb{N}_n$ and $G_j(A) > 0$ for all $j \in \mathbb{N}_n$, then $(F(B),G(A))$ is the unique completely mixed equilibrium point of $(A,B)$.

If $F_i(B) \leq 0$ for some $i \in \mathbb{N}_n$ or $G_j(A) \leq 0$ for some $j \in \mathbb{N}_n$, then there does not exist a completely mixed equilibrium point of $(A,B)$.

Proof
This immediately follows from the fact that if $(p,q)$ is a completely mixed equilibrium point of $(A,B)$, then there exist $\lambda, \mu \in \mathbb{R}$ such that
(p,q,\lambda,\mu) \text{ is the unique solution of the system:}
\begin{align*}
p^T B &= \lambda e; & p^T e &= 1; & Aq &= \mu e; & q^T e &= 1.
\end{align*}
and Cramer's rule (e is the vector with all coefficients equal to one).

Theorem 4.5 (cf [7], thm. 7.5)
$$E^\text{nd}(A,B) \subset E^e(A,B).$$

Proof
Let \((A,B) \in C_{m \times n}\) and \((p,q) \in E^\text{nd}(A,B)\). By theorem 3.9 and lemma 4.1 we may assume that the matrices A and B are square and nonsingular and that p and q are completely mixed. Let \(\delta > 0\) be such that C and D are nonsingular, whenever \((C,D) \in B_\delta(A,B)\). By (4.3) and (4.4) a continuous mapping \((F,G): B_\delta(A,B) \to \mathbb{R}^n \times \mathbb{R}^n\) is defined. We have \((F,G)((A,B)) = (p,q)\).

Let \(\varepsilon \in (0, \frac{1}{\min\{p, q\}_i; i,j \in \mathbb{N}_n})\)

Since \((F,G)\) is continuous there exists a \(\eta \in (0, \delta)\) such that \((F,G)((C,D)) \in B_\varepsilon(p,q)\) whenever \((C,D) \in B_\eta(A,B)\).

By lemma 4.2: \((p,q) \in E^e(A,B)\).

Next, we will prove that nondegenerate equilibrium points are proper. Again, the proof is splitted into several lemmas.

Lemma 4.6
Let \((A,B) \in C_{m \times n}\). Assume \((p,q) \in E^\text{nd}(A,B)\). There exists a \(\delta_0 > 0\) such that for all \(\delta \in (0, \delta_0)\) and for all \(i \in \mathbb{N}_m \setminus C_1(p)\) and all \(j \in \mathbb{N}_n \setminus C_2(q)\) there exist \(f^i(\delta) \in S_m\) and \(g^j(\delta) \in S_n\) such that
\begin{align*}
C_1(f^i(\delta)) &= C_1(p) \cup \{i\} & C_2(g^j(\delta)) &= C_2(q) \cup \{j\}
f^i_1(\delta) &= \delta & g^j_1(\delta) &= \delta
M_2(B,f^i(\delta)) &= C_2(q) & M_1(A,g^j(\delta)) &= C_1(p) .
\end{align*}
Proof

We will only indicate how one can construct \( f^i(\delta) \). In a similar fashion one can construct \( g^j(\delta) \). By theorem 3.9 we may assume \( C_1(p) = \mathbb{N}_k = C_2(q) \).

Define matrices \( \tilde{B} \) and \( \bar{B} \) by:

\[
\tilde{B} = [b_{ij}]_{i=1,j=1}^{k,k}, \quad \bar{B} = [b_{ij}]_{i=1,j=1}^{m,k}.
\]

Since \( \tilde{B} \) is nonsingular there exists for all \( i \in \mathbb{N}_m \setminus \mathbb{N}_k \) a vector \( x^i \) such that \( (x^i)^T \tilde{B} = e^i \).

For \( i \in \mathbb{N}_m \setminus \mathbb{N}_k \) and \( \delta \in \mathbb{R}_+ \), define \( x^i(\delta) \in \mathbb{R}^m \) and \( \gamma_\delta^i \in \mathbb{R} \) by

\[
x^i(\delta) = \delta \\
x_j^i(\delta) = p_j - \delta x_j^i \quad \text{if } j \in \mathbb{N}_k \\
x_j^i(\delta) = 0 \quad \text{else} \\
\gamma_\delta^i = 1 - \delta(1 - \sum_{j=1}^{k} x_j^i).
\]

For \( \delta \) sufficiently close to zero we have \( \gamma_\delta^i > 0 \). In this case we define

\[
f^i(\delta) = \gamma_\delta^i x^i(\delta/\gamma_\delta^i).
\]

If \( \delta > 0 \) is sufficiently small, we have

\[
f^i(\delta) \in S^m, \\
C_1(f^i(\delta)) = C_1(p) \cup \{i\}, \\
f^i_1(\delta) = \delta.
\]

Moreover, we have \( \lim_{\delta \to 0} f^i(\delta) = p \).

Hence, if \( \delta \) is sufficiently close to zero: \( M_2(B,f^i(\delta)) \subset M_2(B,p) \).
Furthermore: 
\[(\xi^i(\delta))^T B = \sum_{j=1}^{k} \xi^i(\delta) e_j^T B + \delta e_i^T B = \gamma_0^i \sum_{j=1}^{k} p_j e_j^T B - (\delta/\gamma_0^i) \sum_{j=1}^{k} x_j^i e_j^T B + \delta e_i^T B = \gamma_0^i p^T B - \delta(x^i)^T B + \delta e_i^T B = \gamma_0^i \tilde{T}_i B.\]

Hence, if \(\delta\) is sufficiently small: \(M_2(B,\xi^i(\delta)) = M_2(B,p).\)

The idea of the proof that a nondegenerate equilibrium point \((p,q)\) is proper is the following: instead of the game in which the pure strategy sets of the players are \(N_m\) and \(N_n\), we look at a game in which the pure strategy sets are

\[C_1(p) = \{\xi^i(\delta) ; i \in N_m \setminus C_1(p)\} \quad \text{and} \quad C_2(q) = \{\xi^j(\delta) ; j \in N_n \setminus C_2(q)\}.\]

We then define a generalization of the \(\varepsilon\)-proper equilibrium point concept (definition 4.7) and show that a certain sequence of strategy pairs in this new game, which satisfy the requirements of this new concept, induces in the original game a sequence of \(\varepsilon\)-proper equilibrium points, converging to \((p,q)\).

**Definition 4.7**

Let \((A,B) \in G_{m \times n}\). Let \(P^1 = \{P_1^1, \ldots, P_\lambda^1\}\) be a partition of \(N_m\) and let \(P^2 = \{P_1^2, \ldots, P_\mu^2\}\) be a partition of \(N_n\). Let \((p,q) \in S^m \times S^n\) and let \(\varepsilon > 0\). \((p,q)\) is an \(\varepsilon\)-proper equilibrium point of \((A,B)\) with respect to \((P^1,P^2)\) if

(i) \(p\) and \(q\) are completely mixed,

(ii) for all \(i,k \in N_m\) and all \(\alpha, \beta \in N_n\):

- if \(\alpha > \beta\) and \(i \in P_\alpha^1\), \(j \in P_\beta^2\), then \(p_i \leq \varepsilon p_k\),
- if \(i,k \in P_\alpha^1\), then \(e_i^T Aq < e_k^T Aq\) implies \(p_i \leq \varepsilon p_k\).
(iii) for all \( j, \ell \in \mathbb{N} \) and all \( \alpha, \beta \in \mathbb{N}^u \):
- if \( \alpha > \beta \) and \( j \in P^2_\alpha, \ell \in P^2_\beta \), then \( q_j \leq \epsilon q_\ell \)
- if \( j, \ell \in P^2_\alpha \), then \( p^T_j e_j < p^T_\ell e_\ell \) implies \( q_j \leq \epsilon q_\ell \).

\((p, q)\) is a proper equilibrium point of \((A, B)\) with respect to \((P^1, P^2)\)
if there exists a sequence \( \{(\epsilon_k, p(\epsilon_k), q(\epsilon_k))\}_{k \in \mathbb{N}} \)
such that

\[ \begin{align*}
&\text{a) } \epsilon_k > 0 \ (k \in \mathbb{N}); \lim_{k \to \infty} \epsilon_k = 0, \\
&\text{b) } \lim_{k \to \infty} (p(\epsilon_k), q(\epsilon_k)) = (p, q), \\
&\text{c) } (p(\epsilon_k), q(\epsilon_k)) \text{ is an } \epsilon_k \text{-proper equilibrium point of } (A, B) \text{ with respect to } (P^1, P^2), \text{ for all } k \in \mathbb{N}.
\end{align*} \]

**Lemma 4.8**

Let \((A, B) \in \mathcal{G}_{m \times n}\). Let \( P^1 = \{P^1_1, \ldots, P^1_\lambda\} \) be a partition of \( \mathbb{N}_m \) and let \( P^2 = \{P^2_1, \ldots, P^2_\mu\} \) be a partition of \( \mathbb{N}_n \). Then

(i) if \( \epsilon \in (0, 1) \),

then there exists an \( \epsilon \)-proper equilibrium point of \((A, B)\) with respect to \((P^1, P^2)\),

(ii) there exists a proper equilibrium point of \((A, B)\) with respect to \((P^1, P^2)\).

**Proof**

(i) Let \( \epsilon \in (0, 1) \). Define \( \delta := \min\{\frac{1}{m}, \frac{1}{n}\} \)

\[ S^k(\delta) := \{x \in S^k; x_i \geq \delta \text{ for all } i \in \mathbb{N}_k \} \quad (k \in \{m, n\}) \]

Define a point-to-set map \( F_m : S^m(\delta) \times S^n(\delta) \to S^m(\delta) \) by

\[ F_m (p, q) := \begin{cases} x \in S^m(\delta) & \text{for all } i, k \in \mathbb{N}_m \text{ and all } \alpha, \beta \in \mathbb{N}_\lambda : \\
\quad \quad \quad \text{if } \alpha > \beta; i \in P^1_\alpha, j \in P^1_\beta, \text{ then } x_i \leq \epsilon x_k, \\
\quad \quad \quad \text{if } i, j \in P^1_\alpha, \text{ then } e^T_i A q < e^T_k A q \text{ implies } x_i \leq \epsilon x_k. \end{cases} \]
and define a point-to-set map $F : S^m(\delta) \times S^n(\delta) \to S^n(\delta)$ in a similar way (cf. definition 4.7 iii). Let $F(.) = F^m(.) \times F^n(.)$. Then $F$ is a point-to-set map from $S^m(\delta) \times S^n(\delta)$ to $S^m(\delta) \times S^n(\delta)$. It is not difficult to see that $F$ satisfies the conditions of the Kakutani fixed point theorem [8]. Hence, there exists a fixed point $(p,q)$ of $F$.

In this case $(p,q)$ is an $\varepsilon$-proper equilibrium point of $(A,B)$ with respect to $(P^1,P^2)$.

(ii) Follows immediately from (i) and the fact that $S^m \times S^n$ is a compact set. □

Theorem 4.9

$E^{nd}(A,B) \subset E^{pr}(A,B)$.

Proof

Let $(A,B) \in C_{m \times n}$. Assume $(p,q) \in E^{nd}(A,B)$. For $i \in N \setminus C_1(p)$, $j \in N \setminus C_2(q)$ and a certain fixed $\delta$, with $\delta$ sufficiently small, let $f^i(\delta)$ and $g^j(\delta)$ be defined as in lemma 4.6. To simplify the notation, we will write $f^i$ and $g^j$ for $f^i(\delta)$ and $g^j(\delta)$, respectively.

We define a partition $P^1 = \{P^1_1, \ldots, P^1_{\lambda}\}$ of $N \setminus C_1(p)$ by:

$$P^1_1 = \{i \in N \setminus C_1(p) ; e_i^T A \alpha = \max_{k \in N \setminus C_1(p)} e_k^T A \alpha\}$$

and (as long as $\bigcup_{\alpha=1}^{\lambda} P^1_\alpha = N \setminus C_1(p)$)

$$P^1_{\nu+1} = \{i \in N \setminus (C_1(p) \cup \bigcup_{\alpha=1}^{\lambda} P^1_\alpha) ; e_i^T A \alpha = \max_{k \in N \setminus (C_1(p) \cup \bigcup_{\alpha=1}^{\lambda} P^1_\alpha)} e_k^T A \alpha\}.$$

We define a partition $P^2 = \{P^2_1, \ldots, P^2_{\mu}\}$ of $N \setminus C_1(q)$ by:

$$P^2_1 = \{j \in N \setminus C_2(q) ; p^T B \epsilon_j = \max_{\epsilon \in N \setminus C_2(q)} p^T B \epsilon\}$$

We define a partition $P^3 = \{P^3_1, \ldots, P^3_{\nu}\}$ of $N \setminus C_1(p)$ by:

$$P^3_1 = \{i \in N \setminus C_1(p) ; e_i^T A \alpha = \max_{k \in N \setminus C_1(p)} e_k^T A \alpha\}$$

and (as long as $\bigcup_{\alpha=1}^{\nu} P^3_\epsilon = N \setminus C_1(p)$)

$$P^3_{\mu+1} = \{i \in N \setminus (C_1(p) \cup \bigcup_{\alpha=1}^{\nu} P^3_\epsilon) ; e_i^T A \alpha = \max_{k \in N \setminus (C_1(p) \cup \bigcup_{\alpha=1}^{\nu} P^3_\epsilon)} e_k^T A \alpha\}.$$
and (as long as $\bigcup_{\alpha=1}^{\nu} P^2_\alpha = N \setminus C_2(q)$):

$$P_{u+1}^2 = \left\{ j \in N \setminus (C_2(q) \cup \bigcup_{\alpha=1}^{\nu} P^2_\alpha) ; \max_{\alpha=1}^{\nu} p^T B e_j \right\}$$

We define a bimatrix game $(\tilde{A}, \tilde{B})$ by:

$$\tilde{a}_{ij} = (f^i)^T A g^j \quad (i \in N \setminus C_1(p); j \in N \setminus C_2(q)) .$$

$$\tilde{b}_{ij} = (f^i)^T B g^j$$

For $\varepsilon \in (0,1)$, let $(\tilde{x}(\varepsilon), \tilde{y}(\varepsilon))$ be an $\varepsilon$-proper equilibrium point of $(\tilde{A}, \tilde{B})$ with respect to $(P^1, P^2)$. We define a probability distribution on $\{f^i; i \in N \setminus C_1(p)\}$ by $x(\varepsilon) = \sum_{i=1}^{n} \tilde{x}(\varepsilon)f^i$.

Since $f^i \in S^m$, for all $i$, we have $x(\varepsilon) \in S^n$. In a similar way $y(\varepsilon) \in S^n$ is defined. We define

$$p(\varepsilon) = (1 + \varepsilon)^{-1} (p + \varepsilon x(\varepsilon))$$

$$q(\varepsilon) = (1 + \varepsilon)^{-1} (q + \varepsilon y(\varepsilon)) .$$

We will prove that $(p(\varepsilon), q(\varepsilon))$ is a $\sqrt{\varepsilon}$-proper equilibrium point of $(A, B)$ if $\varepsilon$ is sufficiently small. The proof is divided into several steps.

**Step 1**

If $i \in C_1(p)$ and $j \in N \setminus C_2(q)$, then $e_i^T A g^j = p^T A g^j$ and $e_i^T A y(\varepsilon) = p^T A y(\varepsilon)$. This immediately follows from the way in which $g^j$ was constructed and the definition of $y(\varepsilon)$.
Step 2
If \( i \in \mathbb{N}_m \setminus C_1(p) \) and \( j \in \mathbb{N}_n \setminus C_2(q) \), then

\[
\tilde{a}_{ij} = (f^i)^T a_j = \sum_{k=1}^{m} f^i_k e_k^j a_j = \\
= \sum_{k \in C_1(p)} f^i_k e_k^j + \sum_{k} f^i_k e_k^j = \\
= \sum_{k \in C_1(p)} f^i_k p a_j + \delta e_i^j a_j = \\
= (1 - \delta) p a_j + \delta e_i^j a_j.
\]

Step 3
If \( i \in \mathbb{N}_m \setminus C_1(p) \), then

\[
e_i^j a_y(\varepsilon) = \sum_{j \in \mathbb{N}_n \setminus C_2(q)} e_i^j a_y(j) = (\text{step 2}) = \\
= \sum_{j \in \mathbb{N}_n \setminus C_2(q)} \left[ \delta^{-1} \tilde{a}_{ij} y_j(\varepsilon) - \delta^{-1} (1 - \delta) p a_j y_j(\varepsilon) \right] \\
= \delta^{-1} e_i^j a_y(\varepsilon) - \delta^{-1} (1 - \delta) p a_y(\varepsilon).
\]

Step 4
If \( i \in \mathbb{N}_m \setminus C_1(p) \), then

\[
p_i(\varepsilon) = (1 + \varepsilon)^{-1} \varepsilon x(\varepsilon)^T e_i = \\
= (1 + \varepsilon)^{-1} \varepsilon \sum_{k \in \mathbb{N}_m \setminus C_1(p)} \tilde{x}_k(\varepsilon)(f^k)^T e_i = \\
= (1 + \varepsilon)^{-1} \varepsilon \tilde{x}_i(\varepsilon)(f^i)^T e_i = (1 + \varepsilon)^{-1} \varepsilon \tilde{x}_i(\varepsilon) \delta.
\]
Step 5
Assume \( i,k \in \mathbb{N} \) are such that \( e_i^TAq(\varepsilon) < e_k^TAq(\varepsilon) \).
If \( \varepsilon \) is sufficiently small, it follows that:

\[
(e_i^TAq < e_k^TAq) \quad \text{or} \quad (e_i^TAq = e_k^TAq \text{ and } e_i^TAy(\varepsilon) < e_k^TAy(\varepsilon)).
\]

Case (i): \( e_i^TAq < e_k^TAq \).
Then \( i \not\in C_1(p) \). If \( k \in C_1(p) \), we have for \( \varepsilon \) sufficiently small:

\[
p_i^-(\varepsilon) = (1 + \varepsilon)^{-1} \tilde{x}_i(\varepsilon) \delta \leq (1 + \varepsilon)^{-1} \varepsilon \leq (1 + \varepsilon)^{-1} \sqrt{\varepsilon} p_k \leq \sqrt{\varepsilon} p_k(\varepsilon).
\]
If \( k \not\in C_1(p) \), then \( p_i^-(\varepsilon) = (1 + \varepsilon)^{-1} \tilde{x}_i(\varepsilon) \delta \text{ and } p_k(\varepsilon) = (1 + \varepsilon)^{-1} \tilde{x}_k(\varepsilon) \delta \).
Since \((\tilde{x}_i(\varepsilon), \tilde{y}(\varepsilon))\) is an \( \varepsilon \)-proper equilibrium point of \((\tilde{A}, \tilde{B})\) with respect to \((\tilde{P}^1, \tilde{P}^2)\), we have \( \tilde{x}_i(\varepsilon) \leq \varepsilon \tilde{x}_k(\varepsilon) \).
Hence, \( p_i^-(\varepsilon) \leq \varepsilon p_k(\varepsilon) \).

Case (ii): \( e_i^TAq = e_k^TAq \) and \( e_i^TAy(\varepsilon) < e_k^TAy(\varepsilon) \).
By step 1 and the fact that \((p, q)\) is a quasi-strong equilibrium point, we have \( i,k \not\in C_1(p) \). By step 3: \( e_i^TAy(\varepsilon) < e_k^TAy(\varepsilon) \).
Since \((\tilde{x}(\varepsilon), \tilde{y}(\varepsilon))\) is an \( \varepsilon \)-proper equilibrium point of \((A, B)\) with respect to \((P^1, P^2)\): \( \tilde{x}_i(\varepsilon) \leq \varepsilon \tilde{x}_k(\varepsilon) \). By step 4: \( p_i^-(\varepsilon) \leq \varepsilon p_k(\varepsilon) \).
In a similar way as above we can prove that, \( q_j(\varepsilon) \leq \sqrt{\varepsilon} q_k(\varepsilon) \), whenever \( p(\varepsilon)_{Be_j} < p(\varepsilon)_{Be_k} \) \((i,k \in \mathbb{N})\).
Hence, \((p(\varepsilon), q(\varepsilon))\) is a \( \sqrt{\varepsilon} \)-proper equilibrium point of \((A, B)\) if \( \varepsilon > 0 \) is sufficiently small. Since \( \lim_{\varepsilon \to 0} (p(\varepsilon), q(\varepsilon)) = (p, q) \) we have that \((p, q)\) is a proper equilibrium point of \((A, B)\).

Corollary 4.10
\[ E_{nd}^{nd}(A, B) \subseteq E_{pe}^{pe}(A, B). \]

Theorem 4.11
For almost all games \((A, B) \in \mathbb{G}_{m \times n} \) we have \( E_{nd}^{nd}(A, B) = E(A, B) \).
Proof
Jansen ([7], thm ) proved that if \((p, q) \in E^{qs}(A, B)\), then \((p, q)\) is an element of the relative interior of some maximal Nash subset. Consequently, if all equilibrium points are quasi-strong, then all equilibrium points are isolated. Hence, if \(E(A, B) = E^{qs}(A, B)\), then \(E(A, B) = E^{nd}(A, B)\). Since, for almost all games \((A, B) \in G_{m \times n}\) we have that \(E^{qs}(A, B) = E(A, B)\) ([3] thm.2) the proof is complete.

5. Some characterizations
In this section we will derive a characterization of perfect equilibrium points of bimatrix games. Furthermore, we will derive a property of proper equilibrium points of bimatrix games. Finally, we will prove two theorems, which are more or less converses to the theorems 4.5 and 4.9.

Definition 5.1
Let \((A, B) \in G_{m \times n}\) and let \((p, q), (\pi, \sigma) \in S^m \times S^n\).
\(p\) is dominated by \(\pi\) if:
\[ p^T A e_j \leq p^T A e_j \] for all \(j \in N\) and \(p^T A e_j < p^T A e_j\) for some \(j \in N\).
\(q\) is dominated by \(\sigma\) if:
\[ e_i^T A q \leq e_i^T A \sigma \] for all \(i \in N\) and \(e_i^T A q < e_i^T A \sigma\) for some \(i \in N\).
\(p\) \((q)\) is dominated if there exists a \(\pi \in S^m\) \((\sigma \in S^n)\) such that \(p\) \((q)\) is dominated by \(\pi\) \((\sigma)\).

In [1] one can find the proof of the following theorem:

Theorem 5.2
Let \((A, B) \in G_{m \times n}\) and let \((p, q) \in E(A, B)\).
\((p, q) \in E^{pe}(A, B)\) if and only if \(p\) and \(q\) are not dominated.

Theorem 5.3
Let \((A, B) \in G_{m \times n}\) and let \((p, q) \in E(A, B)\). Define \(\tilde{A}, \tilde{B}, \tilde{\pi}\) and \(\tilde{\sigma}\) as in lemma 4.1. If \((p, q) \in E^{pr}(A, B)\), then \((\tilde{p}, \tilde{q}) \in E^{pe}(\tilde{A}, \tilde{B})\).
Proof

For $k \in \mathbb{N}$, let $(p(\varepsilon_k), q(\varepsilon_k))$ be an $\varepsilon_k$-proper equilibrium point of $(A, B)$ such that $\lim_{k \to \infty} \varepsilon_k = 0$ and $\lim_{k \to \infty} (p(\varepsilon_k), q(\varepsilon_k)) = (p, q)$.

If $j \notin M_2(B, p)$, $k \in M_2(B, p)$, we have for $k$ sufficiently large:

$$p(\varepsilon_k)^T B e_j < p(\varepsilon_k)^T B e_k.$$ Hence $q_j(\varepsilon_k) \leq \varepsilon_k q_k(\varepsilon_k)$.

Assume $\tilde{p}$ is (in $\bar{A}$) dominated by $\pi$. Let $j_0 \in M_2(B, p)$ and $\delta > 0$ be such that

$$\tilde{p}^T A e_{j_0} + \delta \leq \pi^T A e_{j_0}.$$

Define $\pi \in S^n$ by $\pi_i = \tilde{\pi}_i$ if $i \in M_1(A, q)$ and $\pi_i = 0$ else.

Let $\|A\| = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|$.

For $k$ sufficiently large, we have

$$p^T A q(\varepsilon_k) = \sum_{j \in M_2(B, p)} p^T A e_j q_j(\varepsilon_k) + \sum_{j \notin M_2(B, p)} p^T A e_j q_j(\varepsilon_k) \leq \sum_{j \in M_2(B, p)} \pi^T A e_j q_j(\varepsilon_k) - \delta q_{j_0}(\varepsilon_k) + \varepsilon_k \|A\| q_{j_0}(\varepsilon_k) \leq \pi^T A q(\varepsilon_k) + 2\varepsilon_k \|A\| q_{j_0}(\varepsilon_k) - \delta q_{j_0}(\varepsilon_k) < \pi^T A q(\varepsilon_k).$$

Hence, there exist $i \in C_1(p)$ and $k \in C_1(\pi)$ such that $e_i^T A q(\varepsilon_k) < e_k^T A q(\varepsilon_k)$ (if $k$ is sufficiently large).

Therefore $p_i = \lim_{k \to \infty} p_i(\varepsilon_k) \leq \lim_{k \to \infty} \varepsilon_k p_i(\varepsilon_k) = 0$, which is a contradiction. Similarly we can prove that $q$ cannot be dominated in $\bar{B}$.

A similar result as in theorem 5.3 is not true for perfect equilibrium points of $(A, B)$ as we see in example 5.4. In example 5.5 we see that the converse of theorem 5.3 is false in general.
Example 5.4
Let the bimatrix game \((A,B)\) be defined by:

\[
\begin{array}{ccc}
(1,1) & (0,0) & (-2,-2) \\
(0,0) & (0,0) & (-1,-1) \\
(-2,-2) & (-1,-1) & (-1,-1)
\end{array}
\]

Then \((e_2,e_2) \in \text{E}^\text{pe}(A,B)\). However \((e_2,e_2) \notin \text{E}^\text{pe}(\tilde{A},\tilde{B})\).

Example 5.5
Let the bimatrix game \((A,B)\) be defined by:

\[
\begin{array}{ccc}
(1,1) & (1,1) & (0,1) \\
(1,1) & (0,0) & (0,0) \\
(0,0) & (0,0) & (1,1)
\end{array}
\]

Then \((e_1,e_1) \in \text{E}^\text{pe}(\tilde{A},\tilde{B})\). However \((e_1,e_1) \notin \text{E}^\text{pr}(A,B)\).

Theorem 5.6
\[\text{E}^\text{nd}(A,B) = \text{E}^\text{i}(A,B) \cap \text{E}^\text{pr}(A,B).\]

Proof
We have already shown that a nondegenerate equilibrium point is isolated and proper. Assume \((p,q)\) is an isolated and proper equilibrium point of \((A,B)\). It is sufficient to show that \((p,q)\) is quasi-strong. Now by theorem 5.3 there exists a \(\sigma \in S^n\) such that \(C_2(\sigma) = M_2(B,p)\) and \(C_1(p) \subset M_1(A,\sigma)\). Hence, \((p,\sigma) \in E(A,B)\). Since \((p,q) \in E^i(A,B)\), we have \(q = \sigma\). Hence, \(C_2(q) = M_2(B,p)\). Similarly, we have \(C_1(p) = M_1(A,q)\). Hence, \((p,q) \in E^\text{qs}(A,B)\). \(\square\)
Remark 5.7
In example 5.4 we see that a perfect equilibrium point, which is isolated need not be nondegenerate.

Theorem 5.8 (cf [7], theorem 6.2)
\[ E^{nd}(A,B) = E^i(A,B) \cap E^e(A,B). \]

Proof
We have already seen that nondegenerate equilibrium points are isolated and essential. Assume \((p,q) \in E^i(A,B) \cap E^e(A,B).\) We have to prove \((p,q) \in E^{qs}(A,B).\) Assume \(C_2(q) = M_2(B,p).\) Since \((p,q) \in E^i(A,B)\) and because of theorem 5.2 we have: there exists a \(\pi \in S^m\) such that \(p^T A_1 \leq \pi^T A_1\) for all \(j \in M_2(B,p)\) and \(p^T A_1 < \pi^T A_1\) for some \(j \in M_2(B,p).\)

Choose \(i_0 \in N^m\) such that \(p_{i_0} < \pi_{i_0}.\) For \(k \in N,\) define an \(m\)-by-\(m\) matrix \(A(k)\) by:

\[
A_{i,j}(k) = a_{i,j}, \quad \text{if } i \neq i_0 \\
A_{i_0,j}(k) = a_{i_0,j} + 1/k.
\]

For \(j \in M_2(B,p),\) we have \(p^T A(k)e_j < \pi^T A(k)e_j.\)

For \(k \in N,\) let \((p(k),q(k)) \in E(A(k),B)\) such that \(\lim_{k \to \infty} (p(k),q(k)) = (p,q).\)

Then, for \(k\) sufficiently large: \(C_2(q(k)) \subset M_2(B,p(k)) \subset M_2(B,p).\)

Hence, \(p^T A(k)^T q(k) < \pi^T A(k)^T q(k).\) Hence, there exists an \(i \in C_1(p),\) such that \(i \notin M_1(A(k),q(k))\) for infinitely many \(k.\) But this is a contradiction, since for \(k\) sufficiently large: \(C_1(p) \subset C_1(p(k)) \subset M_1(A(k),q(k)).\)

Similarly, we can prove \(C_1(p) = M_1(A,q).\)

Hence, \((p,q) \in E^{qs}(A,B).\)

To complete the picture, we mention the following theorem. For a proof, see [7], theorem 7.4.

Theorem 5.9
\[ E^{nd}(A,B) = E^e(A,B) \cap E^{qs}(A,B). \]
Remark 5.10
We have proved that an equilibrium point which is essential and isolated is nondegenerate. Moreover, we have that an equilibrium point which is essential and quasi-strong is nondegenerate. By looking at simple examples it seems that only nondegenerate equilibrium points can be essential. Although we cannot prove it, we conjecture that the assertion "an equilibrium point is nondegenerate if and only if it is essential" is true.
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