On the Periodic Solutions in One Dimensional Cellular Nonlinear Networks Based on Josephson Junctions (JJ’s)

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Abstract—In this paper we consider an autonomous one dimensional Cellular Nonlinear Network (CNN) that consists of chain of N simple cells based on Josephson Junctions (JJ’s) coupled by linear inductors. In fact the subject is the well known Josephson Transmission Line (JTL) that is used in many applications in superconductor electronics and more specially in Rapid Single Flux Quantum (RSFQ) technique. When the last cell is connected with the first one a ring of Josephson Junctions is formed. Using the framework of the Cellular Nonlinear Network we consider the JTL under the umbrella of the system theory. Based on describing functions method we rigorously prove the existence and stability of periodic solutions in the cellular nonlinear network model considered. The results are confirmed by simulations.

Index Terms—Cellular nonlinear networks, Describing functions, Josephson junctions, Periodic solutions.

I. INTRODUCTION

THE relation between CNN’s and Josephson Junction arrays has been considered in many publications (see [1, 2, 3, 4] and references there). In [1] and [2] authors show that one-dimensional array of JJ’s come very close to the CNN paradigm; therefore, these arrays are suitable for modeling and simulating with CNN’s. They consider the dynamics of one-dimensional JJ’s array, stability of equilibrium states and show the dependence of the stability on the coupling parameter. Self-organizing behavior of array of non-ideal JJ’s is investigated in [3]. It is shown that the array exhibits spatio-temporal chaos. Two-dimensional array of JJ’s is considered in [4]. Authors report the results of a Floquet analysis of such arrays of resistively and capacitively shunted JJ’s in an external transverse magnetic field. The Floquet analysis indicates stable phase locking of the active junctions over a finite range of values of the bias current and junction capacitance, even in the absence of an external load.

In this contribution we investigate an autonomous one-dimensional CNN composed by inductively coupled JJ’s. The last JJ is connected to the first one, and thus the ring shaped oscillator is formed. We apply the describing functions method and rigorously prove the existence and stability of periodic solution in the model considered.

The paper is organized as follows. In the next section we present the mathematical model of the JJ. In section 3 we show the CNN model of one-dimensional array of JJ’s. Then in section 4 we utilize the describing functions method and investigate the dynamic behavior of the CNN model. Thus we prove the existence and the stability of the periodic solutions. In section 5 we show simulation results and finish up the paper with conclusion remarks given in section 6.

II. MATHEMATICAL MODEL OF THE JOSEPHSON JUNCTION

In its general form, the Josephson junction is a weak connection between two superconductors, designated as $S_1$ and $S_2$ in Fig. 1.

![Fig. 1. A tunnel Josephson junction -- structure (left), electrical symbol (middle) and equivalent schematic (right).](image)

Let $\theta_1$ and $\theta_2$ be the phases of the complex pair wave functions of the both superconductors and $\phi = \theta_1 - \theta_2$ be their difference. Let $I_c$ be the lossless supercurrent flowing through the Josephson junction, $I_c$ - its maximum value (also called critical current of the Josephson junction) and $U(t)$ - the voltage drop over the junction. Then:

$$I_s = I_c \sin \phi(t)$$

and

$$\frac{d\phi}{dt} = \frac{2eU(t)}{\Phi_0}$$

where $\Phi_0$ is the flux quantum.
where $\Phi_0=2.07mVps$ is a fundamental constant named the Single Flux Quantum (SFQ) and $I_c$ is the critical current for the junction. These relations are first predicted by B. D. Josephson [5] and are popular as the dc (1) and ac (2) Josephson effect, respectively.

For RSFQ applications, the tunnel Josephson junction should be properly shunted by an external resistor $R_{\text{ext}}$, and thus the equivalent resistance of the model of Fig. 1 will be $R=R_{\text{eq}}|R_{\text{ext}}$ [6]. If the resulting structure is biased by a dc current $I_{\text{bias}}$, and a non-dc excitation $I_{\text{ext}}(t)$ is applied on it, two additional components of the current through the junction appear: the capacitive current $I_{\text{cap}}(t)=C\frac{dU(t)}{dt}$ through the capacitance $C$, formed by the superconductor contact electrodes, and the dissipation current $I_d(t)=U(t)/R$. The balance of the currents over the junction becomes:

$$I_0 = I_{\text{bias}} + I_{\text{cap}}(t) = I_c \sin \phi(t) + \frac{U(t)}{R} + C\frac{dU(t)}{dt} \tag{3}$$

Taking into account (2), (3) is transformed into:

$$I_c \sin \phi(t) + \frac{1}{R} \frac{d\phi(t)}{dt} + C\frac{\Phi_0}{2\pi} \frac{d^2\phi(t)}{dt^2} = I_b \tag{4}$$

which is the main differential equation describing the time-domain behavior of the tunnel Josephson junction. Based on the above equation the JJ could be considered as biased by current source $I_b=I_{\text{bias}}+I_{\text{ext}}(t)$.

With a proper choice of its parameters, this equation has equilibrium solutions $\phi=2\pi n \cdot \arcsin(I_b/I_c)$ about the superconductive phase drop $\phi(t)$, with $n$ - nonnegative integer. The transition between two neighboring equilibrium states is named a switching of the Josephson junction, and according to (2), such a $2\pi$ phase change causes a voltage pulse with a picosecond duration. Its properties can be derived by integrating (2):

$$\int_0^t U(t)dt = \Phi_0 \tag{5}$$

Due to its quantized area, this pulse is named a Single Flux Quantum (SFQ) pulse and is used as a data carrier within the RSFQ technique. The typical shape of an SFQ pulse is shown in Fig. 2.

The speed of the Josephson junction’s switching (i.e. the speed of RSFQ digital circuits) is determined by the parameters of (4).

An important parameter determining the solution of (4) (i.e. the junction’s dynamic behaviour) is the Stewart-McCumber parameter $\beta$, defined as:

$$\beta = \frac{2\pi CJR^2}{\Phi_0}. \tag{6}$$

Depending on the value of this parameter the dynamics of JJ is different. For a single JJ the optimal value of $\beta$ is 1. In this case only a single SFQ pulse is generated. For $\beta > 1$ a continuous generation of SFQ pulses occurs. For most commercial fabrication technologies $\beta$ is about 100. Because of this for many RSFQ applications the tunnel Josephson junction is properly shunted by an external resistor, and thus the equivalent resistance of the JJ is $R=R_{\text{eq}}$.

III. CELLULAR NONLINEAR NETWORK (CNN) MODEL

Let us consider one-dimensional array of N identical JJ’s shown in Fig. 3.

Every JJ could be considered as a simple cell connected to its neighboring cells by superconductive strips represented by their inductances $L$. The currents through every single JJ are given by

$$I_{j+1} = \frac{1}{R} \frac{\Phi_0}{2\pi} \frac{d\phi_j(t)}{dt} + C\frac{\Phi_0}{2\pi} \frac{d^2\phi_j(t)}{dt^2} + I_c \sin \phi_j(t) \tag{7}$$

$$j=1,2,...,N$$

The external current sources $I_b$ ensure the correct operation of every single JJ. Generally the excitation currents $I_{\text{ext}}(t)$ included into the external current sources $I_b$ are zero. To
invoke periodic pulse propagation we will excite by external
SFQ pulse only one junction and this pulse will propagate
periodically through the array of JJ’s considered.

Using the KCL the current through the \(j\)-th JJ is

\[
I_{\text{jy}} = I_{\text{y}},
\]

where \(I_{\text{y}}\) and \(I_{\text{y}}\) are the currents through \((j-1)\)-th and \(j\)-th
inductances connected with \(j\)-th JJ’s. Using the flux
quantization [9] these currents could be expressed as

\[
I_{\text{y}} = \Phi_0 \frac{2\pi}{2\pi L} (\phi_{j-1} - \phi_j),
\]

\[
I_{\text{y}} = \Phi_0 \frac{2\pi}{2\pi L} (\phi_j - \phi_{j+1}).
\]

where \(\phi_{j-1}\) and \(\phi_j\) and \(\phi_{j+1}\) are phase differences across \((j-1)\)-th,
\(j\)-th and \((j+1)\)-th JJ’s.

Combining (7), (8), and (9) the equations that govern the
behaviour of the model considered are:

\[
\Phi_o \frac{d^2 \phi (t)}{dt^2} + \Phi_o \frac{d \phi (t)}{dt} + \sin \phi (t) = \frac{\Phi_o}{2\pi L_c} \left[ \phi_{j-1}(t) - 2\phi_j(t) + \phi_{j+1}(t) \right] + I_y
\]

\[
 j = 1, 2, ..., N.
\]

Changing the variable \(t\) with dimensionless time \(i\)

\[
i = 2\pi R L t
\]

we obtain

\[
\Phi_o \frac{d^2 \phi (i)}{dt^2} + \Phi_o \frac{d \phi (i)}{dt} + \sin \phi (i) = \frac{\Phi_o}{2\pi L_c} \left[ \phi_{j-1}(i) - 2\phi_j(i) + \phi_{j+1}(i) \right] + I_y
\]

\[
 j = 1, 2, ..., N.
\]

If we introduce constants

\[
\beta = \frac{2\pi L L_c}{\Phi_o}, \quad I_j = \frac{I_j}{I_c}
\]

the differential equations describing the behavior of the
one-dimensional array of JJ’s finally are:

\[
\frac{d^2 \phi (i)}{dt^2} + \frac{1}{\beta_c} \frac{d \phi (i)}{dt} + \frac{1}{\beta_c} \sin \phi (i) = \frac{1}{\beta \beta_c} \left[ \phi_{j-1} - 2\phi_j + \phi_{j+1} \right] + I_j
\]

\[
 j = 1, 2, ..., N.
\]

To simplify the notations in what follows we will use \(t\) to
denote the dimensionless time. If \(a = \frac{1}{\beta_c}, \quad b = \frac{1}{\beta \beta_c}, \quad c = I_j\)
and introducing state variables \(u_j = \phi_j\) and \(v_j, j=1,2,...,N\) for
every single JJ, the state space model of the CNN considered is described by

\[
\frac{du_j}{dt} = -au_j + b(v_{j-1} - 2v_j + v_{j+1}) - \sin v_j + c
\]

\[
\frac{dv_j}{dt} = u_j
\]

\[
 j = 1, 2, ..., N.
\]

IV. DYNAMIC BEHAVIOR OF THE CNN MODEL

In this section we apply the describing functions method in
order to study the dynamics of the CNN obtained model. This
method is well known in control theory and in the study of
electronic oscillators [10], [11]. The state variables \(u_j(t)\) and \(v_j(t)\) are functions depending on two arguments: discrete one —
space \(j\) and continuous one — time \(t\).

We shall study the dynamics and the stability properties of
(15) by using the describing function method. Applying the
double Fourier transform:

\[
F(s, z) = \sum_{k=0}^{\infty} z^k \int f_k(t) \exp(-st)dt
\]

to the CNN model (15) we obtain:

\[
sU(s, z) = -aU(s, z) + b(z^{-1}V(s, z) - 2V(s, z)) + zV(s, z) - \sin V(s, z) + c
\]

\[
sV(s, z) = U(s, z)
\]

where \(s = io_i, \quad z = \exp(D_\Omega), \quad \omega_s\) is a temporal frequency
and \(\Omega_s\) is a spatial frequency.

If we consider the nonlinearity of the form

\[
N(s, z) = c - \sin V(s, z)
\]
	hen from (16) it is easy to express the double Fourier transform \(V(s, z)\) of \(v_j(t)\) as a function of this nonlinearity:

\[
V(s, z) = \frac{s + a}{s^2 + sa - b(z^{-1} - 2 + z)} N(V(s, z))
\]
According to the describing function method [10], the transfer function \( H(s,z) \) in this case is:

\[
H(s,z) = \frac{s + a}{s^2 + sa - b(z^{-1} - 2 + z)} .
\]  

(17)

The above transfer function can be presented in terms of \( \omega_0 \) and \( \Omega_0 \), i.e. \( H(s,z) = H_{\Omega_0}(\omega_0) \):

\[
H_{\Omega_0}(\omega_0) = \frac{U_{\Omega_0}(\omega_0)}{V_{\Omega_0}(\omega_0)}. 
\]  

(18)

We are looking for possible periodic state solutions of system (16) of the form:

\[
egin{align*}
U_{\Omega_0}(\omega_0) &= U_{m_0} \sin(\omega_0 t + j\Omega_0) , \\
V_{\Omega_0}(\omega_0) &= V_{m_0} \sin(\omega_0 t + j\Omega_0) .
\end{align*}
\]  

(19)

Following the describing function method we take the first harmonic of (19), i.e. \( j = 0 \)

\[
egin{align*}
U_{\Omega_0}(\omega_0) &= U_{m_0} \sin \omega_0 t , \\
V_{\Omega_0}(\omega_0) &= V_{m_0} \sin \omega_0 t .
\end{align*}
\]  

(20)

(21)

Thus from (18) we get

\[
H_{\Omega_0}(\omega_0) = \frac{U_{\Omega_0}(\omega_0)}{V_{\Omega_0}(\omega_0)} = \frac{U_{m_0}}{V_{m_0}} .
\]  

(22)

On the other side if we substitute \( s = i\omega_0 \) and \( z = \exp(i\Omega_0) \) in (17) we obtain:

\[
H_{\Omega_0}(\omega_0) = \frac{i\omega_0 + a}{-\omega_0^2 + i\omega_0 - b(2\cos\Omega_0 - 2)} .
\]  

(23)

According to the describing function method the following constraints hold:

\[
\Re(H_{\Omega_0}(\omega_0)) = \frac{U_{m_0}}{V_{m_0}} ,
\]  

(24)

\[
\Im(H_{\Omega_0}(\omega_0)) = 0 .
\]  

(25)

Because our CNN model (15) is a finite circular array of \( N \) cells we have finite set of frequencies:

\[
\Omega_0 = \frac{2\pi k}{N}, \quad 0 \leq k \leq N-1.
\]  

(25)

Remark 1. As \( v(t) \) is assumed to be periodic, with minimal period \( T_0 = 2\pi/\omega_0 \), one has

\[
\xi(\Omega_0 j + \omega_0 t) = \xi(\Omega_0 j + \omega_0 t + k\omega_0 T_0) ,
\]  

for any \( k \in \mathbb{N} \). On the other hand, since we take periodic boundary conditions of our CNN model:

\[
\begin{align*}
v_y(t) &= v_y(t) , \\
v_{N-1}(t) &= v_0(t) .
\end{align*}
\]  

Making the array circular we impose that

\[
\xi(\omega_0 t) = \xi(\Omega_0 N + \omega_0 t) .
\]

Combining the above equalities with \( j = 0 \) we get:

\[
\Omega_0 = \frac{k}{N} \omega_0 T_0 = \frac{2\pi k}{N}, \quad 0 \leq k \leq N-1.
\]

(25)

Thus (23), (24) and (25) give us necessary set of equations for finding the unknowns \( U_{m_0} \), \( V_{m_0} \), \( \omega_0 \), \( \Omega_0 \). As we mentioned above we are looking for a periodic wave solution of (15), therefore \( U_{m_0} \) and \( V_{m_0} \) will determine approximate amplitudes of the waves, and \( T_0 = 2\pi/\omega_0 \) will determine the wave speed.

Based on the above considerations the following proposition holds:

**Proposition 1.** CNN model (15) of the circular array of \( N \) identical, inductively coupled JJ has periodic state solution \( u_j(t), v_j(t) \) with a finite set of spatial frequencies \( \Omega = 2\pi k/N, 0 \leq k \leq N-1 \).

It is possible to use the describing function method not only to get an indication about the existence of the periodic solution, but also about their stability. Using the graphical criterion and following [12], [13] we can give the analytical condition for the stability of periodic solution, which is: for a real valued describing function, periodic solution is predicted to be stable if:

\[
\frac{\partial}{\partial \omega} \text{Im}[H_{\Omega_0}(\omega)] - \frac{\partial}{\partial V_m} D(V_m) < 0.
\]  

(26)

For our CNN model we can calculate that

\[
\frac{\partial}{\partial \omega} \text{Im}[H_{\Omega_0}(\omega)] = 0 ,
\]

and the first harmonic of the describing function \( D(V_m) \) is positive

\[
\frac{\partial}{\partial V_m} D(V_m) = \frac{\pi}{2} > 0
\]

Therefore, condition (26) is satisfied and thus the following proposition holds:
Proposition 2. CNN model (15) of the circular array of N identical, inductively coupled JJ’s has stable periodic solutions for all N.

We shall continue our stability analysis with definition of the equilibrium points of our model considered. For the sake of simplicity let us rewrite system (15) as follows:

\[
\begin{align*}
\frac{du}{dt} &= -au + B * u - a \sin v + c = F_1(u, v) \\
\frac{dv}{dt} &= u = F_2(u, v),
\end{align*}
\]

(27)

where \( u = u_j, v = v_j, B \) is the cloning template \([b, -2b, b]\) and we define convolution operator \(*\) according to the following definition:

Definition 1. For any cloning template \( B \) which defines the dynamic rule of the cell circuit, we define the convolution operator \(*\) by

\[
B * x_j = \sum_{c=0}^{N} B(c-j)x_c
\]

(28)

where \( B(m) \) denotes the entry in the \( m^{th} \) row of the cloning template, \( m = -1, 0, 1 \).

Then we shall define the equilibrium points of our model:

Definition 2. The equilibrium points of (27) are points \((\bar{u}, \bar{v}) \in \mathbb{R}^N\) such that

\[
\begin{align*}
F_1(\bar{u}, \bar{v}) &= -a\bar{u} + B * \bar{v} - a \sin \bar{v} + c = 0 \\
F_2(\bar{u}, \bar{v}) &= \bar{u} = 0
\end{align*}
\]

(29)

The solutions of (29) are

\[
\bar{u} = 0, \quad \bar{v} = k\pi + \arcsin \left( \frac{c + B}{a} \right), \quad k = 0, \pm 1, \pm 2, \ldots
\]

The associated linearized system in the sufficient small neighborhood of the equilibrium points is given by

\[
\frac{dz}{dt} = DF(\bar{u}, \bar{v})z,
\]

(30)

where

\[
z = \text{col}(z_1, z_2), \quad z_1 = u - \bar{u}, \quad z_2 = v - \bar{v}, \quad F = \text{col}(F_1, F_2)
\]

and \( DF(\bar{u}, \bar{v}) = J \) is the Jacobian matrix evaluated at the equilibrium point.

According to the theory of dynamical systems if all of the eigenvalues of the Jacobian matrix \( J \) evaluated at the equilibrium point have negative real part, then the corresponding equilibrium point \((\bar{u}, \bar{v})\) are asymptotically stable. If the \( \text{tr} J \) of the Jacobian matrix is negative, this means that there are some eigenvalues with negative real part and some with positive real part. In this case we have partial stability.

We shall prove the following theorem:

Theorem 1. CNN model (15) of the circular array of N identical, inductively coupled JJ’s has periodic state solutions \( u_j(t), v_j(t) \) which are partially stable for all N. In other words the trajectories will converge to stable periodic solutions depending on the initial conditions.

Proof:

In our particular case the Jacobian matrix \( J \) evaluated at the equilibrium points is

\[
J = \left( \frac{\partial F}{\partial (u,v)} \right)_{u=0,v=0} = \left( \begin{array}{cc}
\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\
\frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}
\end{array} \right)_{u=0,v=0}
\]

(31)

where

\[
A = \begin{pmatrix}
-a & \ldots & 0 \\
0 & \ldots & -a
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
-2 & -a \cos \bar{v} & \ldots & 1 \\
1 & \ldots & -2 & -a \cos \bar{v} \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix},
\]

and \( I_N \) is \( N \times N \) identity matrix.

Based on presentation (31) the \( \text{tr} J \) of the Jacobian matrix is negative and this proves the theorem.

Remark 2. By applying the describing function method we have been able to obtain a characterization of the periodic steady state solutions of our CNN model. In order to validate the accuracy of the achieved result it would be useful to have a possible initial condition from which the network will reach, a steady state solution characterized by the desired value of \( \Omega_b \). One such possibility is to take an initial condition

\[
u_j(t) = \sin(\Omega_b j), \quad j = 1, 2, \ldots, N.
\]

It should be mentioned that the continuous counterpart of the considered model is a perturbed sine-Gordon equation. In comparison with the classical sine-Gordon equation the perturbed one has two extra terms that include first derivative with respect to time and a constant term. A CNN model of the original sine-Gordon equation is presented in [14]. The author proves the existence of periodic solutions that are unstable. In comparison with this, the considered cellular nonlinear network model, has stable periodic solutions.

V. Simulation Results

We have designed a chain of identical JJ’s with different number of cells. The obtained ring-shaped oscillator has been used in [15] for direct experimental determination of RSFQ circuit performance. The parameters of JJ’s are taken in
accordance with the 4 μmNb/Al2O3–Al/Nb technology of PTB Braunschweig [16] by using the cell library [17].

A single SFQ pulse is generated by the DC/SFQ converter. This pulse is used as initial condition for our chain of JJ’s. It goes through the merger [17] into the one-dimensional array of JJ’s (ring-shaped oscillator). The periodic appearance of this pulse, respectively the periodic change of phase differences over the JJ’s is shown in Fig. 4 and Fig. 5.

![Fig. 4. The periodic pulse circulation in one-dimensional circular array of JJ’s with N=100. Pulses connected with J1 and J30.](image1)

![Fig. 5. The periodic change of phase differences of the JJ’s in one-dimensional circular array with N=100 JJ’s. Phase differences connected with J1 and J30.](image2)

VI. CONCLUSION

In this contribution we investigate an autonomous one-dimensional CNN composed by identical, inductively coupled JJ’s. The last JJ is connected to the first one, and thus the ring shaped oscillator is formed. We apply the describing functions method and rigorously prove the existence and stability of periodic solution in the model considered.

The continuous counterpart of this model is a perturbed sine-Gordon equation. In comparison with the CNN model of original sine-Gordon equation, the model considered has stable periodic solutions.

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