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CONTROLLED INVARIANCE IN SYSTEMS
OVER RINGS

by

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Abstract. The definition of controlled invariant (i.e. (A,B)-invariant) subspaces of a linear system is extended to systems over rings. It is observed that in this more general setting, the equivalence of the geometric and the feedback characterization is no longer true. Particular attention is paid to the weakly unobservable space $V^*$, and conditions are given for this space to satisfy the feedback characterization. These conditions have the form of the existence of a factorization of the transfer function. An application to the disturbance rejection problem is given.

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1. Introduction

The concept of controlled invariant subspace (abbreviated C.I.S.) (see [2]) (or (A,B)-invariant subspace, see [16]) has played a significant role in the development of linear system theory.

In view of the great potentiality of the theory of systems over rings (see, e.g. [15]), it is tempting to generalize the concept of controlled invariance to systems over rings. However, efforts in this direction are met by a serious obstacle. There are various equivalent characterizations for a C.I.S., the most well-known being the geometric characterization \(AV \subseteq V + \text{im } B\) and the feedback characterization: "there exists F such that \((A + BF)V \subseteq V'\)" (see [2, 16]). These properties are no longer equivalent in the ring case! It is easily seen that the feedback characterization implies the geometric condition, but the converse is not true. The main reason of this difficulty is that for spaces over rings (i.e. modules), subspaces are not necessarily direct summands, so that the map F can be defined on V (supposing that V is free) but it cannot be extended to a map defined on the whole state space X. As a consequence of this state of affairs, we introduce in addition to a C.I.S (i.e. a space V satisfying \(AV \subseteq V + \text{im } B\)) another type of subspace, viz. a C.I.S of the feedback type, abbreviated C.I.S.F., i.e. a space for which there exists F such that \((A + BF)V \subseteq V\). A C.I.S. is more manageable than a C.I.S.F. and it behaves like in the field case. For example, the sum of two C.I.S.'s is again a C.I.S. and if \(K\) is an arbitrary subspace, there exists a largest C.I.S. contained in \(K\). Neither of these statements is true for C.I.S.F.'s! This is very inconvenient, because a C.I.S.F. is the type of space we need in applications.

We will spend most of our attention to a particular C.I.S., the space \(V^*\) of weakly unobservable states (compare [14]), which in the case of a strictly causal system reduces to the largest C.I.S. contained in ker C (for details on notation see section 2), and we will investigate the question of when \(V^*\) has the feedback property. A necessary and sufficient condition for this to be the case will be given in the form of a factorization condition on the transfer function, assuming that the system is reachable and injective. Under these assumptions, it will follow that for a single input system, \(V^*\) has always the feedback property. Also, it follows from the factorization condition that is does no depend on the realization whether \(V^*\) has the feedback property or not, as long as the realization is reachable. For a similar situation we refer to [5].

In section 5 a result by G. Conte and A. Perdon is given, which states that in the case when \(R\) is a principal ideal domain, \(V^*\) has the feedback property if and only if it is a direct summand.

Finally, in section 6 an application is given to the disturbance rejection problem.
2. Controlled invariance and the feedback property

In this section, $\mathbb{R}$ denotes an integral domain with unit element and $A, B, C, D$ are matrices over $\mathbb{R}$ of dimensions $n \times n$, $n \times m$, $r \times n$, $r \times m$, respectively. The matrix quadruple $(A, B, C, D)$ will be called a (free) system and denoted by $\Sigma$. We have in mind particularly the discrete time interpretation of $\Sigma$:

$$(2.1) \quad x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t.$$  

The quantities $u_t, x_t$ and $y_t$ are called input, state and output, respectively, and they are elements of $U := \mathbb{R}^m$, $X := \mathbb{R}^n$ and $Y := \mathbb{R}^r$, respectively. For a given input sequence $u = (u_t)_{t=0}^\infty$ and $x_0 \in X$ we denote by $x_t(x_0, u)$ the state at time $t$ resulting via (2.1) from initial value $x_0$ and input $u$. The corresponding output $C x_t(x_0, u) + Du_t$ is denoted as $y_t(x_0, u)$.

$\Sigma$ is called reachable if for every $\tilde{x} \in X$ a number $T > 0$ exists and an input $u$ such that $x_T(0, u) = \tilde{x}$. Necessary and sufficient for $\Sigma$ to be reachable is that the $m \times nm$ matrix $[B, AB, \ldots, A^{n-1}B]$ be right invertible.

A subspace $V \subseteq X$ is called a controlled invariant subspace (=: C.I.S.) if for each $x_0 \in V$ there exists an input sequence $u$ such that $x_t(x_0, u) \in V$ for $t = 0, 1, \ldots$. The following criterion is immediate:

$$(2.2) \quad \text{PROPOSITION.} \quad V \text{ is a C.I.S. iff } AV \subseteq V + \text{im } B.$$  

A subspace $V \subseteq X$ is called a controlled invariant subspace of the feedback type (=: C.I.S.F.) if there exists $F \in \mathbb{R}^{mn}$ such that $(A + BF)V \subseteq V$. A C.I.S.F. is easily seen to be a C.I.S. but the converse is not true.

$$(2.3) \quad \text{EXAMPLE.} \quad \text{Let } R := \mathbb{R}[\sigma], X := \mathbb{R}^2,$$

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \text{im } \begin{bmatrix} 0 \\ \sigma \end{bmatrix}.$$  

We have

$$A \begin{bmatrix} 0 \\ \sigma \end{bmatrix} = \begin{bmatrix} \sigma \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma \end{bmatrix} + \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  

Hence $AV \subseteq V + \text{im } B$. Now suppose that for

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$  

we have $(A + BF)V \subseteq V$, i.e.

$$\begin{bmatrix} 1 + \sigma f_{11} & 1 + \sigma f_{12} \\ f_{21} & 1 + \sigma f_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \sigma \end{bmatrix} = p(\sigma) \begin{bmatrix} 0 \\ \sigma \end{bmatrix}.$$  

for some polynomial \( p(\sigma) \). The first equation reads \( \sigma + \sigma^2 f_{12} = 0 \), i.e. \( f_{12} = -1/\sigma \), so that \( f_{12} \notin \mathbb{R} \). Notice that the system in this example is reachable.

A simple way of finding counterexamples to various conjectures about C.I.S.'s and C.I.S.F.'s is given by G. Conte and A.M. Perdon; see section 5.

We will concentrate on a special C.I.S.

(2.4) DEFINITION. Given \( \Sigma \), a state \( x_0 \) is called weakly unobservable if there exists an input \( u \) such that \( y_t(x_0,u) = 0 \) for \( t = 0,1, \ldots \). The set of weakly unobservable states is called the weakly unobservable space and is denoted by \( V^*(\Sigma) \), or simply by \( V^* \), if the underlying system is clear.

The following result is easily shown:

(2.5) PROPOSITION. \( V^* \) is a C.I.S. If \( D = 0 \), then \( V^* \) is the largest C.I.S. contained in \( \ker C \).

PROOF. That \( V^* \) is a subspace (i.e. a submodule of \( X \)) is immediately obvious. To show that \( V^* \) is a C.I.S. assume that \( x_0 \in V \) and \( u = (u_0,u_1, \ldots) \) are such that \( y_t(x_0,u) = 0 \) for \( t = 0,1,2, \ldots \). Then \( x_1 = Ax_0 + Bu_0 \) is also in \( V^* \) since \( y_t(x_1,\tilde{u}) = 0 \) for \( t = 0,1, \ldots \), where \( \tilde{u} := (u_1,u_2, \ldots) \). Hence \( Av^* \subseteq V^* + \text{im} B \). The second statement follows from the definition.

We denote by \( N(\Sigma) \) (or by \( N \)) the subspace of unobservable states, i.e. the set of initial states \( x_0 \) for which \( y_t(x_0,0) = 0 \) for \( t = 0,1, \ldots \), where \( 0 \) stands for the input sequence \( (0,0, \ldots) \). Obviously,

\[
N \subseteq V^*.
\]

Furthermore, \( N \) is well known and easily seen to be \( A \)-invariant (i.e. \( AN \subseteq N \)).

A feedback transformation has the form

\[
(2.7) \quad u_t = Fx_t + v_t,
\]

where \( F \in \mathbb{R}^{m \times n} \) and where \( v_t \) is considered a new input variable. Such a transformation yields a new system

\[
(2.8) \quad x_{t+1} = (A + BF)x_t + Bv_t,
\]

\[
y_t = (C + DF)x_t + Dv_t.
\]

It is easily seen that the set of state trajectories (i.e. state sequences resulting
from some input) for a given initial state is invariant under a feedback transformation. In particular, for a given \( x_0 \), if there exists an input \( u \) for \( \Sigma \) such that \( y_t(x_0, u) = 0 \) for all \( t \), then there exists an input \( v \) for system \( \Sigma_F \) such that the output is identically zero. Consequently,

(2.9) **PROPOSITION.** \( V^* \) is feedback invariant, i.e. \( V^*(\Sigma_F) = V^*(\Sigma) \) for all \( F \).

Combining this result with (2.6) we find that

\[
N(\Sigma_F) \subseteq V^*
\]

for all \( F \).

By definition, for every \( x_0 \in V^* \) there exists \( u \) such that \( y_t(x_0, u) = 0 \) for all \( t \). We say that \( V^* \) has the feedback property if there exists a feedback \( F \in \mathbb{R}^{m \times n} \) such that for each \( x_0 \in V^* \), the feedback input \( u \) defined by \( u_t = Fx_t \) for \( t = 0, 1, \ldots \) yields zero output. For systems over a field, \( V^* \) has always the feedback property (see [14]), but for rings this is not the case (for an example see Example 5.6). We have the following characterizations:

(2.10). **PROPOSITION.** The following statements are equivalent

i) \( V^* \) has the feedback property with feedback \( F \),
ii) \( V^* = N(\Sigma_F) \),
iii) \((A + BF)V^* \subseteq V^* \) and \( V^* \subseteq \ker(C + DF) \).

**PROOF.** i) \( \Rightarrow \) ii): \( V^* \) has the feedback property with feedback \( F \) iff the transformed system (2.8) yields zero output for zero input \( v_t \), for every \( x_0 \in V^* \). This will be the case iff \( V^* \subseteq N(\Sigma_F) \). The converse inclusion is always satisfied.

ii) \( \Rightarrow \) iii): Since \( N(\Sigma_F) \) is the largest \((A + BF)\)-invariant subspace contained in \( \ker(C + DF) \), iii) is equivalent to ii).

It follows in particular that, if \( V^* \) has the feedback property, it is a C.I.S.F. If \( D = 0 \), the converse is true, since in this case \( V^* \subseteq \ker(C) \).

Finally we mention

(2.11) **COROLLARY.** If \( V \) is a subspace for which there exists a feedback \( F \) such that

\[
(A + BF)V \subseteq V, \quad V \subseteq \ker(C + DF),
\]

then \( V \subseteq V^* \).

**PROOF.** Condition (2.12) implies \( V \subseteq N(\Sigma_F) \).

3. **Input-output conditions for the feedback property**

In this section we want to formulate conditions for \( V^* \) to have the feedback property in terms of the input-output behavior of \( \Sigma \), specifically, in terms of the transfer function
of $\Sigma$. For this it is convenient to identify input or output sequences with formal power series. We want to take a slightly more general point of view than in the previous section in the sense that we allow input sequences which start at an arbitrary, possibly negative, time. That is, an input sequence will be a doubly infinite sequence $(u_t)_{t \in \mathbb{Z}}$ with the property that a number $t_0$ exists such that $u_t = 0$ for $t < t_0$. Such a sequence will be identified with the formal Laurent series $\sum u_t z^{-t}$. Similarly we proceed for output sequences.

A rational function $\phi = n/d (\in \mathbb{R}(z))$ is called expandable if there exists a formal Laurent series $\psi$ such that $d\psi = n$. In this case we identify $\phi$ with $\psi$. Using long division one can show that $\phi$ is expandable if $d$ is monic, i.e., has leading coefficient equal to 1.

(3.1) **Lemma.** If $R$ is Noetherian then any expandable rational function has a representation $n/d$ with monic $d$.

For a proof see [15], or Corollary A.4.

A rational function $\phi$ is causal, if it is expandable and if its formal Laurent series is causal, i.e., has $u_t = 0$ for $t < 0$. It is easily seen that for an expandable $n/d$ to be causal it is necessary and sufficient that $\deg n \leq \deg d$. Finally, $\phi$ is called bicausal if $\phi$ and $1/\phi$ are causal. Similar terminology is used for rational matrices. In particular, a matrix $L$ is bicausal if $L$ and $L^{-1}$ are causal.

Let us return to the system $\Sigma$ given by (2.1). To $\Sigma$ is associated its transfer function

$$T(z) := \frac{C(zI - A)^{-1}B + D}{\det(zI - A)}.$$  

If $x_t = 0$ for sufficiently negative $t$, we have the relation

$$y(z) = T(z)u(z).$$

The matrix $T(z)$ has the representation $T(z) = N(z)/d(z)$ where $N(z)$ is a polynomial matrix and $d(z) := \det(zI - A)$ is monic. It follows that $T(z)$ is causal (see (3.2)).

(3.3) **Definition.** Let $T(z)$ be a rational matrix. Then $T(z)$ is called injective if $T(z)u(z) = 0$ implies $u(z) = 0$ for every formal Laurent series $u(z)$. Further, $T(z)$ is called strongly injective if every formal Laurent series $u(z)$ for which $T(z)u(z)$ is polynomial, is itself a polynomial.

If $T(z)$ is strongly injective, it is also injective, for, if $T(z)u(z) = 0$ then $T(z)(z^{-k}u(z)) = 0$ for all $k$. Hence $z^{-k}u(z)$ is polynomial for all $k$, which is only possible if $u(z) = 0$. In the definition of injectivity we could have used polynomial, or rational, or causal, or expandable rational functions $u(z)$ instead of formal Laurent series. This would have resulted in an equivalent concept of injectivity. The concept of strong injectivity is more sensitive, however. Instead of formal Laurent series we could have used expandable rational functions, as follows easily from
Lemma A1, but if we would have used polynomial or rational \( u(z) \)'s a different concept of strong injectivity would have resulted. The definition uses formal Laurent series in order that for transfer functions \( T(z) \) the system theoretic meaning be immediately obvious. In fact, we introduce:

(3.4) DEFINITION. System \( \Sigma \) of (2.1) is called injective if for any pair of inputs \( u \) and \( v \) and any \( x_0 \in X \) we have that \( y_t(x_0,u) = y_t(x_0,v) \) for all \( t \) implies that \( u = v \). System \( \Sigma \) is called strongly injective if for any pair of inputs \( u \) and \( v \), any \( x_0 \in X \) and any \( t_1 \geq 0 \) we have that \( y_t(x_0,u) = y_t(x_0,v) \) for \( t \geq t_1 \) implies that \( u \) = \( v \) for \( t \geq t_1 \).

It is straightforward that \( \Sigma \) is (strongly) injective iff \( T_\Sigma \) is (strongly) injective.

(3.5) REMARK. The concept of strong injectivity for systems over a field has appeared in literature under various names: A strongly invertible system is called strictly observable in [7], irreducible in [12] and feedback irreducible in [11]. For injective systems strong injectivity is equivalent to the absence of zeros and it is closely related to the concept of strong observability as discussed in [14].

The following result connects strong injectivity with the concepts of the previous section:

(3.6) THEOREM. Suppose that \( \Sigma \) is reachable. Then \( \Sigma \) is strongly injective iff \( \Sigma \) is injective and \( N(\Sigma) = \mathcal{V}(\Sigma) \).

PROOF. "if": Let \( y_t(0,u) = 0 \) for \( t \geq t_1 \). Then \( \tilde{x}_0 := x_{t_1} \in \mathcal{V} \). Consequently, \( x_{t} \in N(\Sigma) \) and hence \( y_t(\tilde{x}_0,0) = 0 \). On the other hand, \( y_t(\tilde{x}_0,\tilde{u}) = y_{t+t_1}(0,u) = 0 \) for \( t \geq t_1 \), where \( \tilde{u} := (u_{t_1},u_{t_1+1},...) \). Injectivity implies \( \tilde{u} = 0 \).

"only if": Let \( x_1 \in \mathcal{V} \). Since \( \Sigma \) is reachable there exists \( \tilde{u} \) and \( t_1 \geq 0 \) such that \( x_1 = x_{t_1}(0,\tilde{u}) \). In addition, there exists \( \bar{u} \) such that \( y_t(x_1,\bar{u}) = 0 \) (\( t \geq 0 \)). Concatenation of \( \tilde{u} \) and \( \bar{u} \) at \( t_1 \) yields the input sequence \( \underline{u} := (u_0,\tilde{u}_1,...,\tilde{u}_{t_1},\bar{u}_{t_1},\bar{u}_{t_1+1},...) \), which has the property that \( y_t(0,u) = 0 \) for \( t \geq t_1 \). By the strong injectivity of \( \Sigma \) this implies that \( u_t = 0 \) for \( t \geq t_1 \), i.e. \( \bar{u} = 0 \). Hence \( y(x_1,0) = 0 \). We see that \( x_1 \in N(\Sigma) \).

Now we are in the position to formulate a criterion in terms of the transfer function for \( \mathcal{V}^* \) to have the feedback property.

(3.7) THEOREM. Let \( \Sigma \) be injective and reachable and let \( T := T_\Sigma \). Then \( \mathcal{V}^* \) has the feedback property iff there exists a bicausal \( L \) such that \( TL \) is strongly injective.
PROOF. "only if": If \((A + BF)V^* \subseteq V^*, V^* \subseteq \ker (C + DF)\), then Proposition 2.10 and Theorem 3.6 imply that \(\Sigma_F\) is strongly injective. Since \(\Sigma_F = \Sigma_L\), where

\[
L := L_F := (I - FT^{-1})^{-1}
\]

and \(T_S(z) := (zI - A)^{-1}B\), and \(L\) is bicausal, the condition of the theorem follows.

"if": Let \(L\) be bicausal and \(TL = S\) be strongly injective. By the extension to systems over rings of [9, Thm. 5.7] (see also [5]) we know that \(L\) can be realized by feedback (i.e. there exists \(F\) such that \(L = L_F\)) iff for any polynomial \(u\) we have: If \(Ts_u\) and \(u\) are polynomial then \(L^{-1}u\) is polynomial. If \(Ts_u\) and \(u\) are polynomial then \(Tu = CT_su + Du\) is polynomial and hence \(S^{-1}u\) is polynomial. Since \(S\) is strongly injective, if follows that \(L^{-1}u\) is polynomial. Hence there exists \(F\) such that \(L = L_F\) and \(TL = T_F\). Because \(\Sigma_F\) is strongly injective it follows that \(N(\Sigma_F) = V^*(\Sigma_F) = V^*\). Hence \(V^*\) has the feedback property.

As a consequence of this theorem, it does not depend on the realization whether or not \(V(\Sigma)\) has the feedback property, as long as the realization is reachable.

A further conclusion can be drawn from Theorem 3.7. By definition, \(V^*\) has the feedback property if there exists a feedback control \(u_t = Fx_t\) such that the output will be identically zero for every \(x_0 \in V^*\). Now suppose we want to relax this condition by allowing dynamic state feedback, i.e. a system \(\Phi\) with input \(x\) and output \(y\) given by the relation \(u = F(z)x + v\) where \(F(z) = T_{\Phi}(z)\). This yields a combined system with transfer function \(S := TL\) where \(L(z) := (I - F(z)T_s(z))^{-1}\). We claim that the resulting system \(\Sigma_F\) is strongly injective. In fact, the compensator is chosen in such a way that the input \(v = 0\) yields \(\gamma = 0\) for every \(x_0 \in V^*\), so that \(V^* = N(\Sigma_F)\). Since \(TL = S\) is strongly injective and \(L\) bicausal, it follows from Theorem 3.7 that \(V^*\) has the feedback property, so that invariance could have been obtained by static state feedback. Nothing was gained by allowing dynamic feedback (compare [5]).

The following is a modified version of Theorem 3.7. The condition of Theorem 3.7 can be interpreted as the possibility to factorize the transfer function into \(T = SL^{-1}\), where \(S\) is strongly injective and \(L^{-1}\) is bicausal. Now we give a characterization in which less stringent condition imposed on the factorization.

(3.8) THEOREM. Let \(\Sigma\) be injective and reachable. Then \(V^*\) has the feedback property iff \(T := T_F\) can be factorized as \(T = PR\) where \(P\) is (not necessarily causal) strongly injective and \(R\) is causal and left invertible with an expandable (but not necessarily causal) left inverse \(S\).

Necessity is obvious since the factorization \(T = SL^{-1}\), mentioned before, satisfies the conditions. For sufficiency we decompose \(S\) as \(S = S_+ + S_-,\) where \(S_+\) is the polynomial part and \(S_-\) is strictly causal. Then \(S_+R = I - S_-R\) is rational and bicausal. Let \(L := (I - S_-R)^{-1}\). We have \(S_+RL = I\). It follows that \(RL\) is strongly injective. The result follows from Theorem 3.7.

\[\square\]
In this section we assume that $R$ is a N.U.F.D. (i.e. Noetherian unique factorization domain. (see [1, Ch. 4], [13])). Then $R[z]$ is also N.U.F.D. For this type of ring it is possible to give conditions for a rational matrix to be strongly injective.

A prime element $p$ of $R[z]$ is either essentially monic, i.e. of the form $p = ap$ where $a$ is a unit and $p$ is monic, or $p$ has a noninvertible leading coefficient. When multiplication by units is allowed we will always assume that prime factors have been chosen monic whenever possible. Any element $r \in R[z]$ can be factored as $r = r^- r^+$, where $r^-$ is the product of the monic prime factors of $r$ and hence monic, and $r^+$ is the product of the nonmonic prime factors of $r$. We call $r^-$ the monic part of $r$ and $r^+$ the nonmonic part (see [5] for somewhat more general concepts). We say that $p$ is completely nonmonic if $r^- = 1$. It is easily seen that $p \mid q$ ($p$ divides $q$) implies $p^- \mid q^-$. 

(4.2) THEOREM. Let $P = N/d$ be a rational $r \times m$ matrix with monic denominator $d$ and injective numerator matrix $N$. Then, $P$ is strongly injective if the monic part $\chi^-$ of the G.C.D. $\chi$ of the $m \times m$ minors of $N$ divides $d$.

PROOF. Let $u$ be a formal Laurent series and $Pu = v$ be a polynomial. Since $N$ is injective, it contains a nonzero $m \times m$ minor $\chi_1$. The equality $Nu = dv$ implies that $\chi_1 u$ is a polynomial, hence that $u$ is rational. But, since $u$ is formal Laurent series, it must be expandable. Hence (see Lemma (3.1)), $u$ has a representation of the form $u = w/\psi$ where $\psi$ is monic. The equality $Nw = \psi dv$ implies that for every $m \times m$ submatrix $N_i$ of $N$ we have $N_i w = \psi dv_i$ for some polynomial vector $v_i$. Multiplying by the adjoint matrix $\text{adj} N_i$, we find that $\psi d | \chi_1 w$, where $\chi_i$ denotes $\det N_i$. Since this is true for all $m \times m$ submatrices it follows that $\psi d | \chi w$. Taking monic parts we obtain $\psi d^- | \chi^- w^-$. Since, by assumption, $\chi^- d$ we have $\psi | w^-$ and a fortiori $\psi | w$. Hence $u = w/\psi$ is a polynomial.

The converse of this theorem is not true, not even when $R$ is a field. However, if $d = 1$, i.e. if $P = N$ is a polynomial matrix it can be shown that the condition $\chi^- = 1$ is necessary. In fact, let $\alpha$ be a (monic) prime factor of $\chi$ and let $\bar{P}$ denote the matrix with entries which are the residues modulo $\alpha$ in $R/(\alpha)$, which is an integral domain. Since all $m \times m$ determinants of $\bar{P}$ are zero there exists a nonzero $m$-vector $\bar{u}$ over $R/(\alpha)$ such that $\bar{P}\bar{u} = 0$. If $u$ is a representative of $\bar{u}$, then $\bar{u} \neq 0$ implies $\alpha | u$. We have $Pu = \alpha v$ for some polynomial $v$. Hence $P(u/\alpha)$ is a polynomial, $u/\alpha$ is expandable but not a polynomial.

We give some applications of the above result:

(4.2) COROLLARY. If $E$ is injective and reachable, and $m = 1$ then $V^*$ has the feedback property.
Proof. Let the $i$th entry of $T$ be $n_i/d_i$, where $d$ is monic. We have the factorization

$$T = \text{diag}(n_i) \text{col}(n_i/d_i),$$

where $\text{diag}(a_i)$ and $\text{col}(a_i)$ denote the diagonal matrix and column, respectively, with entries $a_i$. The polynomial matrix $P := \text{diag}(n_i)$ is strongly injective because of Theorem 4.1 since $\text{det} P$ is completely nonmonic. Also, the matrix $R := \text{col}(n_i/d_i)$ has an expandable left inverse. In fact, choose any $n_k \neq 0$ and $S := [0, \ldots, 0, d_i/n_k, 0, \ldots, 0]$ will do.

More generally, we have

(4.3) COROLLARY. If $E$ is reachable and injective, $T_E = N/d$ and $x^-$ (as defined in Theorem 4.1) satisfies

$$\deg x^- \leq \deg d,$$

then $\nu^*$ has the feedback property.

PROOF. We can factorize as follows:

$$T = (N/x^-)(x^-/d).$$

(4.4) EXAMPLE. Let

$$T(z) := z^{-6} \begin{bmatrix} z^5 - 1 & \sigma z^4 - 2z - \sigma \\ z^3 - 2z & \sigma z^2 - 1 \end{bmatrix}$$

be the transfer matrix of a reachable system $E$ over $R := \mathbb{R}[\sigma]$. Then the determinant of the numerator equals

$$\chi(z) = \sigma z^7 - z^5 + z^4 - 2z^3 - \sigma z^2 + 1.$$  

This polynomial is nonmonic, so that it contains a nonmonic part of degree at least one. Hence, $\deg x^- \leq 6$ so that Corollary 4.3 implies that $\nu^*$ has the feedback property. Actually, it can easily be seen that $\chi$ does not have a nonmonic factor of degree 1, so that $\deg x^- \leq 5$. Consequently, even if the denominator is $z^5$, $\nu^*$ has the feedback property.

One might be tempted to conjecture that $\nu^*$ always has the feedback property. This is not the case, as can been seen from Example 5.6.

Contrary to the theorems of the previous section, the results of this section are completely constructive, provided we have a constructive way of computing prime factors of polynomials over $R$. Not only conditions for $\nu^*$ to have the feedback property, but also explicit constructions of $\nu^*$ and the desired feedback can be derived from the results of this and the previous sections. In [5] and [10] it is
indicated how a feedback \( F \) can explicitly be constructed for a given bicausal \( L \) in Theorem 3.7. Furthermore, the space \( V^* \) is computed as the unobservable space of \( \Sigma_F \).

5. Systems over Principal Ideal Domains

The results of this section are mainly due to G. Conte and A.M. Perdon ([4]). We recall the following definition (see [3, Def. 1.9]).

\[(5.1) \text{DEFINITION. Given a subspace (i.e. an } R\text{-submodule) } V \text{ of } R^n, \text{ the closure } \overline{V} \text{ of } V \text{ is defined as the set of all } x \in R^n \text{ such that } ax \in V \text{ for some } a \in R. V \text{ is said to be closed if } \overline{V} = V.\]

We assume throughout this section that \( R \) is a principal ideal domain. Then we have:

\[(5.2) \text{PROPOSITION. A subspace } V \subseteq R^n \text{ is closed iff it is a direct summand (see [3, Prop. 1.10, iv]).}\]

The following simple observation is crucial:

\[(5.3) \text{PROPOSITION. If } V \text{ is a C.I.S.F. of } \Sigma \text{ (defined in (2.1)) then so is } \overline{V}.\]

\[\text{PROOF. If } (A + BF)V \subseteq V, \text{ then it is easily seen that } (A + BF)\overline{V} \subseteq \overline{V}. \]

A similar result for C.I.S.'s is not true. This gives us the possibility of verifying that a given C.I.S. is not a C.I.S.F. Let us reconsider Example 2.3. The space \( V \), which is shown to be a C.I.S. has a closure \( \overline{V} = \text{im}[0,1]' \) which is not a C.I.S., since \( AV = \text{im}[1,1]' \neq \overline{V} + \text{im}B = \text{im}B. \) It follows again that \( V \) is not a C.I.S.F. In a similar way counterexamples may be given to various conjectures one might have. For instance, it is possible to find two C.I.S.F.'s the sum of which is not a C.I.S.F. Also, one might think that \( V^* \) is closed if \( \text{im}B \) is, but an example can be given showing that this is not the case.

The main result of this section is

\[(5.4) \text{THEOREM. } V^* \text{ has the feedback property iff it is closed.}\]

\[\text{PROOF. If } V^* \text{ has the feedback property then so does } \overline{V^*}, \text{ since } \ker(C + DF) \text{ is closed. Consequently, } \overline{V^*} \subseteq V^* \text{ (see Corollary 2.11) and hence } \overline{V^*} = \overline{V^*}. \]

Conversely, if \( V^* \) is closed it is a direct summand. Definition 2.4 implies that for each \( x_0 \in V^* \) there exists \( u \in U \) such that

\[(5.5) \quad Ax_0 + Bu_0 = V^* \text{, } Cx_0 + Du_0 = 0.\]

Since \( V^* \) is free (being a submodule of free module over a P.I.D., see [6, Thm. 7.8]), it has a basis, say \( x^1, \ldots, x_k \). Define \( F_1 : V^* + U \) by \( Fx_i = u_i \), where \( u_i \) is chosen
according to (5.5). Since $V^*$ is a direct summand, $F_1$ can be extended to a map $F : X \to U$. Because of (5.5) we have $(A + BF)V^* \subseteq V^*$ and $(C + DF)V^* = 0$.

We conclude this section with an example of a reachable injective $\Sigma$ for which $V^*$ does not have the feedback property.

(5.6) Let $R = \mathbb{R}[\sigma]$

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B := \begin{bmatrix} \sigma & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D := 0.$$

It is easily seen that $\Sigma$ is injective and $V^* = \text{im}[0, \sigma, 0]'$. But $\bar{V}^* = \text{im}[0, 1, 0]' \neq V^*$ hence $V^*$ is not a C.I.S.F.

6. Disturbance rejection

In the system $\Sigma_1$:

$$x_{t+1} = Ax_t + Bu_t + Eq_t, y_t = Cx_t + Du_t,$$

where $q_t$ is a disturbance input, we try to find a feedback control $u_t = Fx_t$ such that in the resulting system, $y$ becomes independent of $q$. If we have found such an $F$ we say that we have solved the disturbance rejection problem and that we have obtained disturbance rejection by state feedback.

The following is a straightforward generalization of a well-known result for systems over fields.

(6.2) PROPOSITION. Disturbance rejection by a state feedback $F$ is achieved iff there exists a $(A + BF)$-invariant subspace $V$ such that

$$\text{im}E \subseteq V \subseteq \ker(C + DF).$$

The proof is straightforward and omitted. One can make this criterion for the solvability of the disturbance rejection problem more constructive if there exists a largest subspace $V$ for which there exists $F$ such that $(A + BF)V \subseteq V \subseteq \ker(C + DF)$. In general such a subspace does not exist (contrary to the field case). However we have the following result:

(6.3) THEOREM. Let $\Sigma := (A,B,C,D)$ be such that $V^*(\Sigma)$ has the feedback property. Then disturbance rejection by state feedback is possible iff

$$\text{im}E \subseteq V^*.$$
In fact, if $V^*$ has the feedback property, it is the largest space satisfying (2.12), see Corollary (2.11).

Next we give a frequency domain characterization for $V^*$ analogous to a characterization given in [8]. Introducing the formal power series

$$\omega(z) := \sum_{t=0}^{\infty} u_t z^{-t-1}, \quad \xi(z) := \sum_{t=0}^{\infty} x_t z^{-t-1},$$

$$\eta(z) := \sum_{t=0}^{\infty} y_t z^{-t},$$

the equations

$$x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t, \quad t \geq 0$$

with initial state $x_0$ can be written as

$$x_0 = (zI - A)\xi(z) - B\omega(z), \quad \eta(z) = C\xi(z) + D\omega(z).$$

Hence we can write: $x_0 \in V^*$ iff there exist strictly causal formal power series $\xi(z)$ and $\omega(z)$ such that

$$(zI - A)\xi(z) - B\omega(z) = x_0,$$

$$C\xi(z) + D\omega(z) = 0.$$  \hspace{1cm} (6.4)

Because of Lemma A.1, we see that $\xi$ and $\omega$ satisfying (6.4) can be chosen rational causal. Hence

(6.5) THEOREM. $x_0 \in V^*$ iff there exist strictly causal rational functions $\xi$ and $\omega$ satisfying (6.4). Equivalently, $x_0 \in V^*$ iff there exists a strictly causal rational $\omega$ such that

$$T(z)\omega(z) = -C(zI - A)^{-1}x_0$$

where $T := T_E$. The second statement of this theorem can be obtained by eliminating $\xi$ from (6.4).

If, in addition to $T$ we introduce $T_1(z) := C(zI - A)^{-1}E$, the disturbance to output transfer function, we can rewrite the condition $\text{im} E \subseteq V^*$ as: There exists a strictly causal rational $Q(z)$ such that

$$T_1(z) = T(z)Q(z).$$  \hspace{1cm} (6.6)

This can be seen applying (6.5) to each column of $E$. Combining this with Theorem 6.3 we have

(6.7) THEOREM. Let $E$ be such that $V^*(E)$ has the feedback property. Then, disturbance rejection is possible iff (6.6) has a strictly causal solution.
This result has a system theoretic interpretation. Suppose that instead of the state of $\Sigma_1$, the disturbance $q$ is available for measurement. Then one may attempt to achieve disturbance rejection by a strictly causal feedforward compensator $\Pi$.

For the problem of disturbance rejection it is no loss of generality to assume that the initial state of $\Sigma_1$ is zero. Then (6.1) yields

$$y(z) = T(z)u + T_1(z)q.$$  

Suppose that the transfer function of the compensator $\Pi$ is $R(z)$. Then, assuming (without loss of generality) that $\Pi$ also has initial state equal to zero, we have $u = R(z)q$. Substitution of this into (6.8) yields

$$y(z) = (T_1(z) + T(z)R(z))q(z).$$

Disturbance rejection will be achieved iff $T_1 + TR = 0$. Hence, the disturbance rejection problem by a feedforward compensator is solvable iff (6.6) has a strictly causal solution. Thus we obtain:

**COROLLARY.** Let $\Sigma$ be such that $V^*(\Sigma)$ has the feedback property. Then disturbance rejection by state feedback is possible iff disturbance rejection by a strictly causal feedforward compensator is possible.

**Appendix**

A result is given about the solvability of a linear equation over $R(z)$.

(A.1) **LEMMA.** Let $R$ be a Noetherian domain and let $A(z) \in R^{m \times n}(z)$, $b(z) \in R^m(z)$. Consider the linear equation

$$A(z)x(z) = b(z).$$

Then we have

i) If (A.2) has a formal Laurent series solution then it has a rational solution with monic denominator.

ii) If (A.2) has a causal formal series solution then it has a causal rational solution with monic denominator.
PROOF. i) is an easy consequence of ii). So, we restrict ourselves to the proof of ii). We denote by \( M \) the ring of causal rational functions with monic denominator. Without loss of generality we may assume that \( A \in M^{m \times n} \) and \( b \in M^m \), since we may multiply \((A.2)\) with any rational function. Let \( x(z) = \sum_{t=0}^{\infty} x_t z^{-t} \) be a causal formal solution of \((A.2)\) and define \( \xi_k(z) := \sum_{t=0}^{\infty} x_t z^{-t} \). Then

\[
(A.3) \quad b - A\xi_k = A(x - \xi_k) \in z^{-k}M^m.
\]

If \( N := M^m/A^m \)

and \( \bar{b} \) is the residue class of \( b \) in \( N \), we have to show that \( \bar{b} = 0 \), because this is equivalent to \( b \in A^m \). Relation \((A.3)\) implies that \( \bar{b} \in z^{-k}N \), since \( \xi_k \in M^m \). This holds for every \( k \). Hence

\[
\bar{b} \in \bigcap_{k=1}^{\infty} z^{-k}N.
\]

Krull's intersection theorem (see [1, Thm 6.2]) implies \( \bar{b} = 0 \).

(A.4) COROLLARY. Let in Lemma A.1 the matrix \( A(z) \) be nonsingular (i.e. \( A(z) \) is invertible over the quotient field of \( R(z) \)). If \( A^{-1}(z)b(z) \) is expandable, then it has the representation \( p(z)/q(z) \), where \( p(z) \) is a polynomial vector and \( q(z) \) is a monic (scalar) polynomial.

PROOF. If \( A^{-1}b \) is expandable, there exists a formal power series \( x(z) \) such that \((A.2)\) holds. By Lemma A.1 i) equation \((A.2)\) has a solution which is expressible as \( p(z)/q(z) \). But since this solution is unique, it follows that \( A^{-1}b = p/q \).

Specializing this result to the scalar case one obtains Lemma 3.1. More generally, the well-known result that a system over a Noetherian domain \( R \) is realizable over \( R \) if it is realizable over the quotient field of \( R \) (see [15, §3B]) is an immediate consequence of the foregoing. Finally, the fact that the existence of formal causal power series \( \xi \) and \( \omega \) satisfying \((6.4)\) implies the existence of a rational causal solution is a consequence of Lemma A.1.

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References


