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EXACT SOLUTION OF THE FSI FOUR-EQUATION MODEL

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ABSTRACT

The so-called "FSI four-equation model" describes the axial vibration of liquid-filled pipes. Two equations for the liquid are coupled to two equations for the pipe, through terms proportional to the Poisson contraction ratio, and through mutual boundary conditions. Skalak (1955/1956ab) defined this basic model, which disregards friction and damping effects.

The four equations can be solved with the method of characteristics (MOC). The standard approach is to cover the distance-time plane with equidistantly spaced grid-points and to time-march from a given initial state. This approach introduces error, because either numerical interpolations or wave speed adjustments are necessary.

This paper presents a method of exact calculation in terms of a simple recursion. The method is valid for transient events only, because the calculation time grows exponentially with the duration of the event. The calculation time is proportional to the temporal and spatial resolution. The exact solutions are used to investigate the error due to numerical interpolations and wave speed adjustments, with emphasis on the latter.

KEYWORDS
FSI, axial pipe vibration, waterhammer, MOC, interpolation

1 INTRODUCTION

1.1 FSI four-equation model

Classical waterhammer (fluid) and beam (structure) theories adequately describe the low-frequency vibration of liquid-filled pipe systems. The liquid is added mass in the lateral pipe vibration and it is neglected in the torsional pipe vibration. The role of the liquid in the axial pipe vibration has always been a point of discussion. Is it just (frequency-dependent) added mass? Has its full elastic behaviour to be considered? And what fluid-structure interaction (FSI) mechanisms have to be taken into account? To answer these questions, the so-called "FSI four-equation model" has to be solved. The model describes the coupled axial vibration of liquid and pipe, where the coupling is through terms in the equations and through boundary conditions.

The four equations, governing fluid pressure, $P$, fluid velocity, $V$, axial pipe stress, $\sigma_z$, and axial pipe velocity, $u_z$, are

$$\frac{\partial V}{\partial t} + \frac{1}{\rho_f} \frac{\partial P}{\partial z} = 0 \quad (1)$$

$$\frac{\partial V}{\partial z} + \left( \frac{1}{K} + \frac{2R}{E e} \right) \frac{\partial P}{\partial t} - \frac{2v}{E} \frac{\partial \sigma_z}{\partial t} = 0 \quad (2)$$
\[
\frac{\partial \ddot{u}_z}{\partial t} - \frac{1}{\rho_s} \frac{\partial \sigma_z}{\partial z} = 0 \tag{3}
\]
\[
\frac{\partial \ddot{u}_z}{\partial z} - \frac{1}{E} \frac{\partial \sigma_z}{\partial t} + \frac{v_R}{E} \frac{R \partial P}{\partial t} = 0 \tag{4}
\]

Notation: \( E \) = Young modulus, \( e \) = wall thickness, \( K \) = bulk modulus, \( R \) = inner pipe radius, \( t \) = time, \( z \) = distance along pipe, \( \nu \) = Poisson ratio, \( \rho \) = mass density; the subscripts \( f \) and \( s \) refer to fluid and structure, respectively. The model is valid for the low-frequency acoustic behaviour of straight, thin-walled, linearly elastic, liquid-filled, prismatic pipes of circular cross-section.

Skalak (1955/1956ab) derived the FSI four-equation model as an extension of Joukowski's (1898) method and as the low-frequency limit of two-dimensional fluid and shell representations. He showed that the model permits solutions which are waves of arbitrary shape travelling without dispersion at the phase velocity of either the liquid (\( \lambda_1 \)) or the pipe (\( \lambda_3 \)), but he made no attempt to solve the four equations in general. The validity of the FSI four-equation model has been demonstrated by many researchers, but most prominently by Vardy and Fan (1989). For more information on the subject the reader is referred to review papers by Tijsseling (1996) and Wiggert and Tijsseling (2001).

1.2 Conventional approach

The method of characteristics (MOC) is the preferred method to solve the FSI four-equation model, because the wave speeds are constant (no dispersion) and, unlike finite difference (Schwarz 1978) and finite element (Zhang et al 1994) methods, steep wave fronts can be properly dealt with. The distance-time plane is covered with a rectangular computational grid. To avoid interpolations and have Courant numbers equal to one, Schwarz (1978), Wiggert et al (1985, 1987), Bürmann et al (1987), Bürmann and Thienen (1988ab) and others assumed wave speed ratios (\( \lambda_3/\lambda_1 \)) that are whole numbers (integers). This strong assumption was relieved by Tijsseling (1993), Tijsseling et al (1996), Bergant and Tijsseling (2001) by allowing the wave speed ratios (\( \lambda_3/\lambda_1 \)) to be rational numbers at the expense of refined grids. Fan (1989), Elansary and Contractor (1990), Bouabdallah and Massouh (1997) and others used interpolations on fine grids.

The conventional MOC approaches introduce phase error if wave speeds are adjusted, and numerical dispersion and damping if interpolations are employed. Both types of error accumulate in time. Furthermore, for all approaches, interpolations are necessary when numerical data is required in between grid points, for example at the location of measuring devices.

1.3 New approach

The new approach presented in this paper has no interpolations, no adjustments (of wave speeds) and no approximations. It is valid for linear, non-dispersive, non-dissipative, hyperbolic systems with linear (or quadratic) time-dependent boundary conditions. It gives exact solutions without the errors of the conventional approaches.

The only previous exact solutions of the FSI four-equation model known to the author are due to Bürmann (1975), Williams (1977), and Wilkinson and Curtis (1980). Bürmann (1975) used the MOC to find all possible transmission and reflection coefficients in coaxial pipe systems. Williams (1977) applied jump conditions to calculate the initial effect of precursor waves (axial-stress-induced pressure waves) on waterhammer. Wilkinson and Curtis (1980) used jump conditions and reflection coefficients to calculate and explain the events in their laboratory experiment (Figure 1). The applicability of jump conditions and reflection/transmission coefficients is limited because all wave fronts have to be tracked to find exact solutions.

Citing Williams (1977, p. 242): "each wave, be it precursor or waterhammer, in general gives rise to two reflected waves, one of each type: the resulting exponential growth in the number of separate waves in the pipe causes obvious analytical difficulties". Wilkinson and Curtis (1980, p. 240) simulated a time period of \( 5L/(2\lambda_1) \) in which 50 wave fronts occurred. Tracking of these wave fronts and their strengths was done by hand (and not without error). Edwards and Please (1988) proposed a discrete analogue of the MOC, which was less laborious than the method of Wilkinson and Curtis, but they had to apply some averaging (interpolation) at the boundaries.

The present method automatically tracks wave fronts backward in time by means of a simple recursion. The exact solutions thus obtained can be used: to check numerical results and schemes, to confirm the accuracy of previous results, to serve
as reference solutions in benchmark problems, and to perform parameter variation studies without parameter changes generated by the numerical method itself (e.g. changed wave speeds). The method does not require a conventional computational grid. It is general and can be applied to analogous four-equation systems encountered in, for example, the theory of linearised waves in two-phase flows, two-layer density flows, and liquid-saturated porous media. The method works also for higher-order systems like those developed by Bürmann (1975).

2 THEORY

The analytical development follows Zhang et al (1999).

2.1 General equations

The general equations

\[ A \frac{\partial}{\partial t} \phi(z,t) + B \frac{\partial}{\partial z} \phi(z,t) + C \phi(z,t) = 0 \]  

describe linear wave propagation in one spatial dimension. The constant matrices \( A \) and \( B \) are invertible and \( A^{-1}B \) is diagonalisable. The constant matrix \( C \), which may be singular, causes frequency dispersion (if \( C \neq 0 \)). The \( N \) dependent and coupled variables \( \phi_i \), constituting the state vector \( \phi \), are functions of the independent variables \( z \) (space) and \( t \) (time). Herein \( N = 4 \) and \( C = 0 \).

2.2 Method of characteristics

The MOC introduces a new set of dependent variables through

\[ \eta_i(z,t) = S^{-1} \phi(z,t) \quad \text{or} \quad \phi(z,t) = S \eta(z,t) \]  

so that each \( \eta_i \) is a linear combination of the original variables \( \phi_i \). Substitution of (6) into (5), with \( C = 0 \), gives

\[ AS \frac{\partial}{\partial t} \eta(z,t) + BS \frac{\partial}{\partial z} \eta(z,t) = 0 \]  

Multiplication by \( S^{-1}A^{-1} \) yields

\[ \frac{\partial}{\partial t} \eta(z,t) + A \frac{\partial}{\partial z} \eta(z,t) = 0 \]  

in which

\[ \Lambda = S^{-1}A^{-1}B S \]  

(9)

A set of decoupled equations is obtained when \( \Lambda \) is diagonal:

\[ \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \]  

(10)

Substitution of (10) into (9) and solving for \( S \) reveals that a non-trivial solution exists only when the diagonal elements of \( \Lambda \) are eigenvalues satisfying the characteristic equation

\[ \det(B - \lambda A) = 0 \]  

(11)

in which case \( S \) consists of the eigenvectors \( \xi_i \) belonging to \( \lambda_i \):

\[ S = (\xi_1, \xi_2, \xi_3, \xi_4) \]  

(12)

The decoupled equations (8),

\[ \frac{\partial}{\partial t} \eta_i(z,t) + \lambda_i \frac{\partial}{\partial z} \eta_i(z,t) = 0, \quad i = 1, 2, 3, 4 \]  

(13)

transform to

\[ \frac{d}{dt} \eta_i(z,t) = 0, \quad i = 1, 2, 3, 4 \]  

(14)

when they are considered along characteristic lines in the \( z-t \) plane defined by

\[ \frac{dz}{dt} = \lambda_i, \quad i = 1, 2, 3, 4 \]  

(15)

The solution of the ordinary differential equations (14) and (15) is

\[ \eta_i(z,t) = \eta_i(z - \lambda_i \Delta t, t - \Delta t), \quad i = 1, 2, 3, 4 \]  

(16)
when a numerical time step $\Delta t$ is used, or, more general and with reference to Figure 2,

$$\eta_i(P) = \eta_i(A_i), \quad i = 1, 2, 3, 4 \quad \text{or}$$

$$\eta(P) = \begin{pmatrix} \eta_1(A_1) \\ \eta_2(A_2) \\ \eta_3(A_3) \\ \eta_4(A_4) \end{pmatrix}$$

(17)

The value of the unknown variable $\eta_i$ does not change along the line $A_i P$.

The original unknowns in $\phi$ are obtained from $\eta$ through (6). This gives

$$\phi(P) = \sum_{i=1}^{4} S R_i S^{-1} \phi(A_i)$$

(18)

where in the matrix $R_i$, the $i$th diagonal element is 1 and all other elements are 0.

### 2.3 Boundary conditions

At the boundaries, the relations (16) or (17) provide two equations (see Figure 3). To find the four unknowns $\eta_i(P)$ at the boundary $z = z_0$ ($z_0 = 0$ or $z_0 = L$), two additional equations are required. These are given by the linear boundary conditions

$$D_\eta(t) \phi(z_0, t) = q_\eta(t) \quad \text{or} \quad D_\eta(t) S \eta(z_0, t) = q_\eta(t)$$

(19)

where $D_\eta$ is a 2 by 4 matrix of coefficients and the 2-vector $q_\eta$ is the (boundary) excitation. For convenience, the 4 by 4 matrix $D$ combines the two matrices $D_0$ and $D_L$ by alternately stacking the rows of $D_0$ and $D_L$, and the 4-vector $q$ combines the two vectors $q_0$ and $q_L$ by alternately stacking the rows of $q_0$ and $q_L$. See Section 2.5 for an example.

### 2.4 Initial conditions

The initial condition at $t = 0$ can be the steady-state solution $\phi(z, 0) = \phi_0$, where the constant state $\phi_0$ is consistent with the boundary conditions, or it can be any non-equilibrium state $\phi(z, 0) = \phi_0(z)$ exciting the system (e.g. sudden release of pressure and/or stress). The once-only matrix inversion in $\eta(z, 0) = S^{-1} \phi_0(z)$ has been done numerically herein, although it could be done algebraically. Matrix inversion is not needed for zero initial conditions.

### 2.5 FSI four-equation model

In terms of the general equation (5), the axial vibration of a liquid-filled pipe (equations (1-4)) can be represented by the state vector

$$\phi = \begin{pmatrix} V \\ P \\ \bar{u}_z \\ \sigma_z \end{pmatrix}$$

(20)

and the matrices of coefficients

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (\rho_f c_f^2)^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \nu R (E e)^{-1} & 0 & -(\rho_s c_s^2)^{-1} \end{pmatrix}$$

(21)

$$B = \begin{pmatrix} 0 & \rho_f^{-1} & 0 & 0 \\ 1 & 0 & -2\nu & 0 \\ 0 & 0 & 0 & -\rho_s^{-1} \end{pmatrix}$$

(22)

The constants

$$c_f^2 = \left( \frac{\rho_f}{K} + (1 - \nu^2) \frac{2\rho_f R}{E e} \right)^{-1} \quad \text{and} \quad c_s^2 = \frac{E}{\rho_s}$$

(23)

are the squares of the classical pressure and axial-stress wave speeds, respectively.

The characteristic (dispersion) equation (11), corresponding to the matrices (21) and (22), is

$$\lambda^4 - \gamma^2 \lambda^2 + c_f^2 c_s^2 = 0$$

(24)

where

$$\gamma^2 = (1 + 2\nu^2 \frac{\rho_f R}{\rho_s e}) c_f^2 + c_s^2$$

(25)
This leads to (slightly) modified (because of FSI) squared wave speeds:
\[ \lambda_{1,2}^2 = \frac{1}{2} \left\{ \gamma^2 - \left( \gamma^2 - 4 c_1^2 c_2^2 \right)^{1/2} \right\} \] (26a)
\[ \lambda_{3,4}^2 = \frac{1}{2} \left\{ \gamma^2 + \left( \gamma^2 - 4 c_1^2 c_2^2 \right)^{1/2} \right\} \] (26b)
where \( \lambda_1 \) and \( \lambda_3 \) are positive, and \( \lambda_2 \) and \( \lambda_4 \) are negative.

The transformation matrix used (equation (12)) is \( S = (TA)^{-1} \) with \( T \) defined by
\[ \text{row}_i\{T\} = \begin{pmatrix} 1 & \frac{1}{2} \lambda_i & 2v^2 c_1^4 - 2v^2 c_2^4 c_1^2 - \lambda_i^2 \end{pmatrix} \quad i = 1, 2 \]
\[ \text{row}_i\{T\} = \begin{pmatrix} \rho_f v R \frac{c_1^4}{c_2^4} & \rho_f v R \frac{c_1^4 c_2^2}{c_2^4} & \frac{\lambda_i^2}{c_1^2} & \lambda_i \end{pmatrix} \quad i = 3, 4 \] (27)

It is noted that transformation matrices are not unique. For example, \( S = (TB)^{-1} \) is an equally valid transformation matrix.

The boundary conditions are given by coefficient matrices \( D \) and excitation vectors \( q \) (see equation (19)). For example, a reservoir at \( z = 0 \) and an unrestrained massless valve at \( z = L \), as in the Sections 4.1 and 4.2, give matrices
\[ D_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad q_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \] (28)
\[ D_L = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & A_f & 0 & -A_s \end{pmatrix} \quad \text{and} \quad q_L = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \] (29)

where \( A \) is a cross-sectional area. In the author's previous work, e.g. Zhang et al (1999), the above matrices and vectors have been simply stacked to form one matrix \( D \) and one vector \( q \). Herein, the boundary matrix \( D \) has rows with alternating the boundary conditions at \( z = 0 \) and \( z = L \). Hence (28) and (29) are combined to give:
\[ D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & A_f & 0 & -A_s \end{pmatrix} \] (30)

3 ALGORITHM

This section is the heart of the paper. It describes an algorithm for finding the exact value of the state vector \( \phi \) in any point \((z, t)\) in the distance-time plane. The algorithm is formulated in terms of the Riemann invariants \( \eta_i \) and it is based on a "coast-to-coast" approach, which is explained in words now. Figure 3 is essential. The vector \( \eta \) in point \( P \) on the left boundary at \( z = 0 \) at time \( t \) consists of the four components \( \eta_1, \eta_2, \eta_3 \), and \( \eta_4 \). The components \( \eta_2 \) and \( \eta_4 \) are assumed to be known because, according to equation (17), these are equal to \( \eta_2 \) and \( \eta_4 \) in the points \( A_2 \) and \( A_4 \), respectively, on the right boundary at \( z = L \). The components \( \eta_1 \) and \( \eta_3 \) follow from the boundary conditions (19); they depend on \( D_0, S, q, \eta_2 \) and \( \eta_4 \). Naturally, the same story holds for any point \( P \) on the right boundary. For the calculation of \( \eta \) in point \( P \) on the left boundary, one needs information from the "earlier" points \( A_2 \) and \( A_4 \) on the right boundary; and for the calculation of \( \eta \) in the points \( A_2 \) and \( A_4 \) on the right boundary, one needs information from "earlier" points (e.g. \( A_1 \) and \( A_3 \)) on the left boundary. This whole process can nicely be captured in a simple recursion that stops when characteristic lines intersect the \( t = 0 \) line, at which \( \eta \) has a given initial value. The recursion is presented below.

3.1 Input

The coefficient matrices \( A, B, C, D(t) \), the excitation vector \( q(t) \), and the initial condition \( \phi_0(z) \) describe the complete system in general terms. It is stressed once again that \( C = 0 \), so that the algorithm does not apply to Timoshenko beams and systems with friction, viscous damping, etc.

3.2 Output

The output is the state vector \( \phi = S\eta \) as a function of distance and time.

3.3 Recursion

3.3.1 Boundary points
See Figure 3. The subroutine BOUNDARY calculates \( \eta \) in the boundary points (finally) from constant initial values. In pseudocode it reads:

```
BOUNDARY (input: z, t; output: \( \eta \))
```

if \( (t \leq 0) \) then

\[ \eta := \eta_0 \]

else

if \( (z = 0) \) then

```
CALL BOUNDARY (L, t + L / \lambda_4; \eta)
\eta^4 := \eta_4
```

CALL BOUNDARY (L, t + L / \lambda_2; \eta)
\( \eta_2 := \eta_2 \)

\[ \eta_1 := \alpha_{12}(t) \eta_2 + \alpha_{14}(t) \eta_4 + \beta_{11}(t) q_1(t) + \beta_{13}(t) q_3(t) \]
\[ \eta_2 := \eta_2 \]

\[ \eta_3 := \alpha_{32}(t) \eta_2 + \alpha_{34}(t) \eta_4 + \beta_{31}(t) q_1(t) + \beta_{33}(t) q_3(t) \]
\[ \eta_4 := \eta_4 \]

if \( (z = L) \) then

CALL BOUNDARY (0, t - L / \lambda_3; \eta)
\( \eta^3 := \eta_3 \)

CALL BOUNDARY (0, t - L / \lambda_1; \eta)
\( \eta_1 := \eta_1 \)

\[ \eta_1 := \eta_1 \]

\[ \eta_2 := \alpha_{21}(t) \eta_1 + \alpha_{23}(t) \eta_3 + \beta_{22}(t) q_2(t) + \beta_{24}(t) q_4(t) \]
\[ \eta_3 := \eta_3 \]

Note that \( \lambda_2 \) and \( \lambda_4 \) are negative numbers. The coefficients \( \alpha \) and \( \beta \) are given in Table 1. They define the unknowns \( \eta_1, \eta_3 \) and \( \eta_2, \eta_4 \) at the left and right boundaries, respectively, as the exact solution of the 2 by 2 linear system of equations (19).

### 3.3.2 Interior points

See Figure 2. The subroutine INTERIOR calculates \( \eta \) in the interior points from the boundary values. In pseudocode it reads:

```
INTERIOR (input: z, t; output: \( \eta \))
```

if \( (t \leq 0) \) then

\[ \eta := \eta_0 \]

else

if \( (0 < z < L) \) then

```
CALL BOUNDARY (0, t - z / \lambda_4; \eta)
\eta^1 := \eta_1
```

CALL BOUNDARY (L, t - (z - L) / \lambda_2; \eta)
\( \eta^2 := \eta_2 \)

CALL BOUNDARY (0, t - z / \lambda_3; \eta)
\( \eta^3 := \eta_3 \)

CALL BOUNDARY (L, t - (z - L) / \lambda_4; \eta)
\( \eta^4 := \eta_4 \)
\[ \eta_1 := \eta^1 \]
\[ \eta_2 := \eta^2 \]
\[ \eta_3 := \eta^3 \]
\[ \eta_4 := \eta^4 \]

\text{else}

"z is not an interior point"

\text{end}

3.3.3 Implementation

The algorithm described above has been implemented in Mathcad 8; the interested reader may find the Mathcad 8 worksheets in the Appendices A1 and A2.

4 RESULTS

Some important previous results are recalculated, but now exactly. The exact results give details as fine as the chosen resolution in time and/or distance.

4.1 Wilkinson and Curtis

Wilkinson and Curtis (1980) performed laboratory experiments in a very thin-walled, vertical, steel pipe in which an upward moving water column collided with an unrestrained closed end (Figure 1). The recorded transient clearly exhibited the occurrence of a precursor wave. Wilkinson and Curtis showed that simplified theory without Poisson coupling was not able to satisfactorily describe the experiment. The more exact FSI four-equation model gave good agreement with the measurement, except for dispersion effects. Unfortunately, the exact solutions given by Wilkinson and Curtis contain errors. For example, and with reference to their Table 2 and their Figures 8 and 9, the pressure rise caused by the reflected precursor was predicted well, so that the error is most likely due to the fact that the calculation and the drawing were done by hand. Also, the calculated pressure at the lower transducer is wrong 12 ms after impact. The automated calculation introduced herein gives the results displayed in Figure 4. The shown pressures at the two transducer positions reveal the small mistakes by Wilkinson and Curtis and they confirm the later simulation by Tijsseling and Lavooij (1989). The data used in the present simulation were: pipe length \( L = 6.10 \text{ m} \), \( R = 12.486 \text{ mm} \), \( e = 0.276 \text{ mm} \), \( E = 175.4 \text{ GPa} \), \( \rho_s = 7900 \text{ kg/m}^3 \), \( \nu = 0.28 \), \( K = 2.141 \text{ GPa} \), \( \rho_f = 997.5 \text{ kg/m}^3 \), \( V_0 = 5 \text{ m/s} \); the corresponding wave speeds and their ratio are \( \lambda_1 = 1008.9 \text{ m/s} \), \( \lambda_3 = 4816.7 \text{ m/s} \) and \( \lambda_3/\lambda_1 = 4.774 \).

4.2 Delft Hydraulics Benchmark Problem A

The Delft Hydraulics Benchmark Problems A to F have been used to test numerical methods and FSI software (Tijsseling and Lavooij 1990; Lavooij and Tijsseling 1991). Problem A concerns a reservoir-pipe-valve system defined by: \( L = 20 \text{ m} \), \( R = 398.5 \text{ mm} \), \( e = 8 \text{ mm} \), \( E = 210 \text{ GPa} \), \( \rho_s = 7900 \text{ kg/m}^3 \), \( \nu = 0.30 \), \( K = 2.1 \text{ GPa} \), \( \rho_f = 1000 \text{ kg/m}^3 \), \( V_0 = 1 \text{ m/s} \); so that the wave speeds and their ratio are \( \lambda_1 = 1024.7 \text{ m/s} \), \( \lambda_3 = 5280.5 \text{ m/s} \) and \( \lambda_3/\lambda_1 = 5.153 \). The instantaneously closing valve may be structurally fixed, implying zero displacement (and velocity), or free and closed, in which case \( V = \ddot{u}_z \) and \( A_f P = A_f \sigma_z \) at the valve. It is noted that the benchmark problems are numerical test cases only; experimental data does not exist.

4.2.1 Effect of interpolations

Figure 5 shows the dissipative effect of interpolations. The wave speed ratio \( \lambda_3/\lambda_1 (5.153) \) is close to 5 in this case, so that the interpolation error is relatively small: a ratio of 5.5 will give a stronger smearing of wave fronts. Dr David Fan (1989) provided the result obtained with (time-line) interpolations.

4.2.2 Effect of wave speed adjustments

To avoid interpolations, the author has applied wave speed adjustment in all of his previous work through modified mass densities \( \rho_f \) and \( \rho_s \): Figure 6 shows typical results. If the wave
speed ratio is 5, 5 structural waves fit in 1 fluid wave. This is nicely exhibited in the broken line, which depicts the pressure at the valve. If the wave speed ratio is 67/13, 67 structural waves fit in 13 fluid waves. This is the more exact solid line. If integer wave speed ratios are employed, the solution cannot converge to the exact solution. If rational wave speed ratios are used, the solution converges if the rational numbers are taken closer to the real number representing the exact wave speed ratio. It is noted that Liou (1983) presented an original method to correct for the adjusted wave speeds. Unfortunately his method introduces some numerical damping.

4.2.3 Exact solutions

The algorithm (Mathcad worksheet) producing the exact solutions has been verified against existing Fortran code. For rational wave speed ratios $\lambda_3/\lambda_1$, the new algorithm and the old code give practically the same results. The small relative differences, which are of the order of $10^{-7}$, may be attributed to numerical matrix inversion in the Fortran code.

No coupling

The new algorithm gives exact waterhammer (without FSI, $v = 0$) solutions for the Eqs. (1) and (2) - with $\sigma_z$ either constant or proportional to $P$ - which is nothing special, except that the exact solutions are not necessarily in equidistantly spaced grid points. The classical waterhammer solutions are used as a reference for the FSI solutions.

Historical note

Classical waterhammer theory and the $\Delta P = \rho_f c_f \Delta V$ formula are attributed to Joukowsky (1898). However, the waterhammer community has overlooked the important contribution by Von Kries (1883), who also derived this famous formula and who validated it experimentally.

Poisson coupling

If the valve is structurally fixed, the only FSI mechanism is Poisson coupling. The exact pressure at the valve (solid line) in Figure 7 and the result obtained with a wave speed ratio adjusted to 67/13 are - on this plotting scale - visibly identical, thus confirming the 67/13 result of Tijsseling (1997). The growing pressure amplitudes are the result of a beat phenomenon, which also appears in FSI computations of systems with fixed junctions (elbows, branches), see Tijsseling and Heinsbroek (1999), and sometimes in waterhammer measurements, see Budny et al (1991, Fig. 4a) and Vennatør (1999, Fig. 16).

Poisson and junction coupling

If the valve is unrestrained, its axial vibration provides a strong mechanism for FSI (junction coupling). The exact pressure at the valve (solid line) in Figure 8 again confirms the result obtained with a rational wave speed ratio 67/13: one needs to zoom in, as in Figure 8a, to see that the differences are small in timing and negligible in magnitude. The Figures 9 and 9a display the corresponding pressures at the midpoint obtained with subroutine INTERIOR. Each of the graphs in the Figures 6 to 9 consists of 1374 points.

5 CONCLUSION

The FSI four-equation model (Eqs. (1-4)) has been solved exactly for time-dependent boundary and constant (steady state) initial conditions. The strength of the method is the simplicity of the algorithm (recursion) and the fast and accurate (exact) calculation of transient events ($t < 8L/c_f$). Its weakness is the exponential calculation time needed for longer events: the time used by Mathcad 8 to produce the results in Section 4.1 on a 400 MHz PC was in the order of minutes, but for the results in Section 4.2 this was in the order of days.

The solutions are exact for the selected points of calculation in the distance-time plane, but they do not give information on what exists in between these points (no guaranteed constant values as in the wave tracking method of Wilkinson and Curtis (1980)). The efficiency of the algorithm can be much improved because many repeat calculations occur, but this will be at the expense of simplicity and clarity. The method is ideal for parallelisation and adaptivity.
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\[ z = 0 \]

\[
\det_{13}(t) := DS(1)_{1,1}DS(1)_{3,3}DS(1)_{1,3}DS(1)_{3,1}DS(1)_{1,3}DS(1)_{3,2}
\]

\[
\alpha_{12}(t) := \left( \frac{DS(1)_{1,2}DS(1)_{3,3}DS(1)_{1,3}DS(1)_{3,2}}{\det_{13}(t)} \right)
\]

\[
\alpha_{14}(t) := \left( \frac{DS(1)_{1,4}DS(1)_{3,3}DS(1)_{1,3}DS(1)_{3,4}}{\det_{13}(t)} \right)
\]

\[
\alpha_{42}(t) := \left( \frac{DS(1)_{3,2}DS(1)_{1,1}DS(1)_{3,1}DS(1)_{1,2}}{\det_{13}(t)} \right)
\]

\[
\alpha_{43}(t) := \left( \frac{DS(1)_{3,4}DS(1)_{1,1}DS(1)_{3,1}DS(1)_{1,4}}{\det_{13}(t)} \right)
\]

\[
\beta_{11}(t) := \left( \frac{DS(1)_{3,3}}{\det_{13}(t)} \right)
\]

\[
\beta_{13}(t) := \left( \frac{-DS(1)_{1,3}}{\det_{13}(t)} \right)
\]

\[
\beta_{31}(t) := \left( \frac{-DS(1)_{3,1}}{\det_{13}(t)} \right)
\]

\[
\beta_{33}(t) := \left( \frac{DS(1)_{1,1}}{\det_{13}(t)} \right)
\]

\[
\beta_{22}(t) := \left( \frac{DS(1)_{4,4}}{\det_{24}(t)} \right)
\]

\[
\beta_{41}(t) := \left( \frac{-DS(1)_{4,1}}{\det_{24}(t)} \right)
\]

\[
\beta_{23}(t) := \left( \frac{-DS(1)_{4,3}}{\det_{24}(t)} \right)
\]

\[
\beta_{34}(t) := \left( \frac{DS(1)_{2,3}}{\det_{24}(t)} \right)
\]

\[
\beta_{42}(t) := \left( \frac{DS(1)_{2,4}}{\det_{24}(t)} \right)
\]

\[
\beta_{44}(t) := \left( \frac{DS(1)_{2,2}}{\det_{24}(t)} \right)
\]

**Table 1**  Definition of the coefficients \( \alpha \) and \( \beta \) in subroutine BOUNDARY.  

\( DS(t) \) is the matrix product of \( D(t) \) and \( S \).

Fig. 2  Interior point P and "feeding" characteristic lines in the distance-time plane.

Fig. 3  Boundary point P and "feeding" characteristic lines in the distance-time plane.
Fig. 4 Exact solutions for the impact test of Wilkinson and Curtis (1980). See Fig.1. Pressure at impact end (upper line) and pressure 4.90 m away from impact end (lower line).

Fig. 5 Effect of interpolations. Pressure at valve for Delft Hydraulics Benchmark Problem A. Solid line: without interpolations, $\Delta z = L$, $\Delta t = 0.29$ ms. Broken line: with interpolations, $\Delta z = L/8$, $\Delta t = 0.47$ ms.

Fig. 6 Effect of wave speed adjustments. Pressure at valve for Delft Hydraulics Benchmark Problem A with free valve. Solid line: $\lambda_3 / \lambda_1 = 67/13$. Broken line: $\lambda_3 / \lambda_1 = 5$.

Fig. 7 Poisson coupling. Pressure at valve for Delft Hydraulics Benchmark Problem A with fixed valve. Solid line: $\lambda_3 / \lambda_1 = \text{exact}$ and $\lambda_3 / \lambda_1 = 67/13$. Broken line: no FSI.

Fig. 8 Poisson and junction coupling. Pressure at valve for Delft Hydraulics Benchmark Problem A with free valve. Solid line: $\lambda_3 / \lambda_1 = \text{exact}$ and $\lambda_3 / \lambda_1 = 67/13$. Broken line: no FSI.

Fig. 9 Poisson and junction coupling. Pressure at midpoint for Delft Hydraulics Benchmark Problem A with free valve. Solid line: $\lambda_3 / \lambda_1 = \text{exact}$ and $\lambda_3 / \lambda_1 = 67/13$. Broken line: no FSI.
Fig. 8a Poisson and junction coupling. Pressure at valve for Delft Hydraulics Benchmark Problem A with free valve. Detail. Solid line: $\lambda_3 / \lambda_1 = \text{exact}$. Broken line: $\lambda_3 / \lambda_1 = 67/13$.

Fig. 9a Poisson and junction coupling. Pressure at midpoint for Delft Hydraulics Benchmark Problem A with free valve. Detail. Solid line: $\lambda_3 / \lambda_1 = \text{exact}$. Broken line: $\lambda_3 / \lambda_1 = 67/13$. 
### NOMENCLATURE

#### Scalars

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>cross-sectional area, m$^2$</td>
</tr>
<tr>
<td>$c$</td>
<td>classical wave speed, m/s</td>
</tr>
<tr>
<td>det</td>
<td>determinant</td>
</tr>
<tr>
<td>$E$</td>
<td>Young modulus of pipe wall material, Pa</td>
</tr>
<tr>
<td>$e$</td>
<td>pipe wall thickness, m</td>
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<td>FSI</td>
<td>fluid-structure interaction</td>
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<tr>
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<td>fluid bulk modulus, Pa</td>
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<td>$L$</td>
<td>pipe length, m</td>
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<td>method of characteristics</td>
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<tr>
<td>$P$</td>
<td>fluid pressure, Pa</td>
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<td>$R$</td>
<td>inner radius of pipe, m</td>
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<tr>
<td>$t$</td>
<td>time, s</td>
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<tr>
<td>$\dot{u}_z$</td>
<td>axial pipe velocity, m/s</td>
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<tr>
<td>$V$</td>
<td>fluid velocity, m/s</td>
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<td>$z$</td>
<td>axial coordinate, m</td>
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#### Matrices and Vectors

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<tr>
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<tr>
<td>$A$</td>
<td>coefficients - see Eqs. (5) and (21)</td>
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<td>$B$</td>
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<td>$C$</td>
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<td>$D$</td>
<td>boundary condition coefficients - see Eqs. (19) and (30)</td>
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<td>$q$</td>
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<td>$R$</td>
<td>matrix - see Eq. (18)</td>
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<td>diagonal matrix of eigenvalues - see Eq. (10)</td>
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<td>$\eta$</td>
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<td>eigenvector - see Eq. (12)</td>
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<td>$\phi$</td>
<td>dependent variables - see Eq. (5)</td>
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#### Subscripts

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<td>numerical step size, change in magnitude</td>
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<td>$\lambda$</td>
<td>eigenvalue, wave speed, m/s</td>
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<tr>
<td>$\nu$</td>
<td>Poisson ratio</td>
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<tr>
<td>$\rho$</td>
<td>mass density, kg/m$^3$</td>
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<td>$\sigma_z$</td>
<td>axial pipe stress, Pa</td>
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<tr>
<td>$f$</td>
<td>fluid, flow</td>
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<tr>
<td>$L$</td>
<td>position $z = L$</td>
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<tr>
<td>$s$</td>
<td>structure, solid</td>
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<td>$t$</td>
<td>tube, pipe</td>
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<tr>
<td>$z$</td>
<td>axial direction</td>
</tr>
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<td>$0$</td>
<td>position $z = 0$, initial state</td>
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APPENDICES

A1 and A2
Appendix A1

Axial vibration of liquid-filled straight pipe

Wilkinson and Curtis (1980)  (Section 4.1 and Figure 1)

Input data:

\[ L : = 6.10 \text{ m} \quad R : = 12.486 \text{ mm} \quad e e : = 0.276 \text{ mm} \quad e = 2.718 \]

\[ E : = 175.4 \times 10^9 \text{ Pa} \quad \rho_\text{t} : = 7900 \frac{\text{kg}}{\text{m}^3} \quad \nu : = 0.28 \]

\[ K : = 2.141 \times 10^9 \text{ Pa} \quad \rho_\text{f} : = 997.5 \frac{\text{kg}}{\text{m}^3} \quad \frac{\rho_\text{t}}{\rho_\text{f}} = 7.919799 \quad \frac{R}{e e} = 45.23913 \]

Cross-sectional areas

\[ A_\text{f} : = \pi R^2 \quad A_\text{t} : = \pi \left[ (R + e e)^2 - R^2 \right] \]
\[ A_\text{f} = 4.898 \text{ cm}^2 \quad A_\text{t} = 0.219 \text{ cm}^2 \]

Axial wave speeds

\[ c_2_\text{f} : = \left[ \frac{\rho_\text{f}}{E} \left[ \frac{1}{K} + \frac{2R}{E e e} \right] \right]^{-1} \quad c_\text{f} : = \sqrt{c_2_\text{f}} \quad c_\text{f} = 1031.359 \text{ m} \text{s}^{-1} \quad (23) \]
\[ c_2_\text{t} : = \frac{E}{\rho_\text{t}} \quad c_\text{t} : = \sqrt{c_2_\text{t}} \quad c_\text{t} = 4711.956 \text{ m} \text{s}^{-1} \quad (23) \]

\[ \gamma 2 : = c_2_\text{f} + c_2_\text{t} + 2 \nu \frac{\rho_\text{f} R}{\rho_\text{t} e e} c_2_\text{f} \quad \gamma 2 = 2.422 \times 10^7 \text{ m}^2 \text{s}^{-2} \quad (25) \]

\[ a a : = 1 \quad b b : = -\gamma 2 \quad c c : = c_2_\text{f} c_2_\text{t} \quad (24) \]

\[ a a = 1 \quad b b = -2.422 \times 10^7 \text{ m}^2 \text{s}^{-2} \quad c c = 2.362 \times 10^{13} \text{ m}^4 \text{s}^{-4} \]

\[ \lambda 2_1 : = \frac{-b b - \sqrt{b b^2 - 4 a a c c}}{2 a a} \quad \lambda 2 _2 : = \lambda 2 _1 \quad (26a) \]

\[ \lambda 2_3 : = \frac{-b b + \sqrt{b b^2 - 4 a a c c}}{2 a a} \quad \lambda 2 _4 : = \lambda 2 _3 \quad (26b) \]

\[ \lambda 1 : = \sqrt{\lambda 2_1} \quad \lambda 2 : = -\sqrt{\lambda 2_1} \quad \lambda 3 : = \sqrt{\lambda 2_3} \quad \lambda 4 : = -\sqrt{\lambda 2_3} \]
\[ \lambda 1 = 1008.92192989173 \text{ m} \text{s}^{-1} \quad \lambda 2 = -1008.92192989173 \text{ m} \text{s}^{-1} \quad \lambda 3 = 4816.74493094659 \text{ m} \text{s}^{-1} \quad \lambda 4 = -4816.74493094659 \text{ m} \text{s}^{-1} \]
Matrices of coefficients (for axial vibration)
from Zhang et al (1999)

Note:
The matrices are made dimensionless through multiplication with the appropriate unit.

\[ \text{ORIGIN = 1} \]

\[
\begin{align*}
A_{1,1} := 1 & \quad & A_{2,2} := \frac{1}{\rho \, f \, c^2} \cdot \text{Pa} & \quad & A_{3,3} := 1 & \quad & A_{4,4} := \frac{1}{\rho \, t \, c^2} \cdot \text{Pa} \\
A_{4,2} := \frac{\nu \, R}{E \, \epsilon c} \cdot \text{Pa} & \quad & (21)
\end{align*}
\]

\[
\begin{align*}
B_{2,1} := 1 & \quad & B_{1,2} := \frac{1}{\rho \, f \, m^3} \cdot \text{kg} & \quad & B_{4,3} := 1 & \quad & B_{3,4} := \frac{1}{\rho \, t \, m^3} \cdot \text{kg} \\
(22)
\end{align*}
\]

\[
A := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 9.425 \times 10^{10} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 7.222 \times 10^{-11} & 0 & -5.701 \times 10^{12}
\end{bmatrix}
\]

\[
B := \begin{bmatrix}
0 & 0 & 0 & 0 \\
0.001 & 0 & 0 & 0 \\
0 & 0 & 1.266 \times 10^{-4} & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

Transformation matrix \( T \)

\[
\begin{align*}
\alpha_1 := & \frac{c_2 \, t - \lambda_2}{\lambda_2} \\
\alpha_2 := & \frac{c_2 \, t - \lambda_2}{\lambda_2} \\
\alpha_2 := & \alpha_1 \\
\alpha_1 := & \frac{c_2 \, t - \lambda_2}{\lambda_2} \\
\alpha_2 := & \frac{c_2 \, t - \lambda_2}{\lambda_2} \\
\alpha_2 := & \alpha_1 \\
t_{11} := & 1 \\
t_{12} := \lambda_1 \, \text{sec} \, \text{m} \\
t_{13} := & \frac{2 \, \nu}{\alpha_1} \\
t_{14} := \frac{c_2 \, t}{\lambda_1} \, \text{sec} \, \text{m} \\
t_{21} := & 1 \\
t_{22} := \lambda_2 \, \text{sec} \, \text{m} \\
t_{23} := & \frac{2 \, \nu}{\alpha_2} \\
t_{24} := \frac{c_2 \, t}{\lambda_2} \, \text{sec} \, \text{m} \\
t_{21} := & t_{11} \\
t_{22} := -t_{12} \\
t_{23} := t_{13} \\
t_{24} := -t_{14}
\end{align*}
\]
\[\alpha_3 := \frac{c^2 f - \lambda_2^3}{\lambda_2^3}\]
\[\alpha_4 := \frac{c^2 f - \lambda_4^4}{\lambda_4^4}\]
\[\alpha_4 := \alpha_3\]

\[t_{34} := \alpha_3 \frac{\text{sec}}{m}\]
\[t_{44} := \alpha_4 \frac{\text{sec}}{m}\]
\[t_{44} := -t_{34}\]

\[t_{33} := \frac{\lambda_2^3}{c_2 f}\]
\[t_{43} := \frac{\lambda_4^4}{c_2 f}\]
\[t_{43} := t_{33}\]

\[t_{32} := \left(\frac{\rho}{E} f \frac{R}{ee} c_2 f \frac{\lambda_3^3}{\alpha_3}\right) \frac{\sec}{m}\]
\[t_{42} := \left(\frac{\rho}{E} f \frac{R}{ee} c_2 f \frac{\lambda_4^4}{\alpha_4}\right) \frac{\sec}{m}\]
\[t_{42} := -t_{32}\]

\[t_{31} := t_{32} \frac{m}{\lambda_3^3} \frac{\sec}{\text{sec}}\]
\[t_{41} := t_{42} \frac{m}{\lambda_4^4} \frac{\sec}{\text{sec}}\]
\[t_{41} := t_{31}\]

\[t_{31} - t_{31A} = 0\]
\[t_{41} - t_{41A} = 0\]

\[T := \begin{bmatrix}
 t_{11} & t_{12} & t_{13} & t_{14} \\
 t_{21} & t_{22} & t_{23} & t_{24} \\
 t_{31} & t_{32} & t_{33} & t_{34} \\
 t_{41} & t_{42} & t_{43} & t_{44}
\end{bmatrix}\]
\[T = \begin{bmatrix}
 1 & 1008.922 & 0.027 & 592.144 \\
 1 & -1008.922 & 0.027 & -592.144 \\
 -0.08 & -386.822 & 1.045 & 4816.745 \\
 -0.08 & 386.822 & 1.045 & -4816.745
\end{bmatrix}\]

\[\text{Transformation matrix } S \text{ (for axial vibration)}\]
\[S := (T A)^{-1}\]
\[S := \text{numerical matrix inversion}\]
Constant initial conditions

Note:
Vectors are made dimensionless through multiplication with the appropriate unit.

\[
\begin{bmatrix}
\phi_{IC} = \\
\eta_{IC} = S^{-1} \phi_{IC} \\
\end{bmatrix}
\]

Boundary-condition matrices

Note:
Matrices are made dimensionless through multiplication with the appropriate unit.

\[
D(t) = \\
\]

Excitation vector

Note:
Vectors are made dimensionless through multiplication with the appropriate unit.

\[
\begin{bmatrix}
q(t) = \\
\end{bmatrix}
\]

Matrix DS

\[
DS(t) := D(t) \cdot S
\]

\[
DS(1. \text{ sec}) =
\begin{bmatrix}
5.022 \cdot 10^5 & -5.022 \cdot 10^5 & -6.173 \cdot 10^4 & 6.173 \cdot 10^4 \\
0.461 & 0.461 & -0.49 & -0.49 \\
0.038 & 0.038 & 0.477 & 0.477 \\
252.637 & -252.637 & 367.537 & -367.537 \\
\end{bmatrix}
\]
Coefficients $\alpha$ and $\beta$

**$z = 0$**

$\det_{13}(t) := \boldsymbol{DS}(t)_{1,1}\boldsymbol{DS}(t)_{3,3} - \boldsymbol{DS}(t)_{1,3}\boldsymbol{DS}(t)_{3,1}$

$\alpha_{12}(t) := \frac{\boldsymbol{DS}(t)_{1,2}\boldsymbol{DS}(t)_{3,3} - \boldsymbol{DS}(t)_{1,3}\boldsymbol{DS}(t)_{3,2}}{\det_{13}(t)}$

$\alpha_{14}(t) := \frac{\boldsymbol{DS}(t)_{1,4}\boldsymbol{DS}(t)_{3,3} - \boldsymbol{DS}(t)_{1,3}\boldsymbol{DS}(t)_{3,4}}{\det_{13}(t)}$

$\alpha_{32}(t) := \frac{\boldsymbol{DS}(t)_{3,2}\boldsymbol{DS}(t)_{1,1} - \boldsymbol{DS}(t)_{3,1}\boldsymbol{DS}(t)_{1,2}}{\det_{13}(t)}$

$\alpha_{34}(t) := \frac{\boldsymbol{DS}(t)_{3,4}\boldsymbol{DS}(t)_{1,1} - \boldsymbol{DS}(t)_{3,1}\boldsymbol{DS}(t)_{1,4}}{\det_{13}(t)}$

$\beta_{11}(t) := \frac{\boldsymbol{DS}(t)_{3,3}}{\det_{13}(t)}$

$\beta_{11}(1\text{-sec}) = 1.972 \times 10^{-6}$

$\beta_{13}(t) := \frac{-\boldsymbol{DS}(t)_{1,3}}{\det_{13}(t)}$

$\beta_{13}(1\text{-sec}) = 0.255$

$\beta_{31}(t) := \frac{-\boldsymbol{DS}(t)_{3,1}}{\det_{13}(t)}$

$\beta_{31}(1\text{-sec}) = -1.584 \times 10^{-7}$

$\beta_{33}(t) := \frac{\boldsymbol{DS}(t)_{1,1}}{\det_{13}(t)}$

$\beta_{33}(1\text{-sec}) = 2.074$

**$z = L$**

$\det_{24}(t) := \boldsymbol{DS}(t)_{2,2}\boldsymbol{DS}(t)_{4,4} - \boldsymbol{DS}(t)_{2,4}\boldsymbol{DS}(t)_{4,2}$

$\det_{24}(1\text{-sec}) = -293.174$

$\alpha_{21}(t) := \frac{\boldsymbol{DS}(t)_{2,1}\boldsymbol{DS}(t)_{4,4} - \boldsymbol{DS}(t)_{2,4}\boldsymbol{DS}(t)_{4,1}}{\det_{24}(t)}$

$\alpha_{21}(1\text{-sec}) = -0.155$

$\alpha_{23}(t) := \frac{\boldsymbol{DS}(t)_{2,3}\boldsymbol{DS}(t)_{4,4} - \boldsymbol{DS}(t)_{2,4}\boldsymbol{DS}(t)_{4,3}}{\det_{24}(t)}$

$\alpha_{23}(1\text{-sec}) = 1.229$

$\alpha_{41}(t) := \frac{\boldsymbol{DS}(t)_{4,1}\boldsymbol{DS}(t)_{2,2} - \boldsymbol{DS}(t)_{4,2}\boldsymbol{DS}(t)_{2,1}}{\det_{24}(t)}$

$\alpha_{41}(1\text{-sec}) = 0.794$

$\alpha_{43}(t) := \frac{\boldsymbol{DS}(t)_{4,3}\boldsymbol{DS}(t)_{2,2} - \boldsymbol{DS}(t)_{4,2}\boldsymbol{DS}(t)_{2,3}}{\det_{24}(t)}$

$\alpha_{43}(1\text{-sec}) = 0.155$

$\beta_{22}(t) := \frac{\boldsymbol{DS}(t)_{4,4}}{\det_{24}(t)}$

$\beta_{22}(1\text{-sec}) = 1.254$

$\beta_{24}(t) := \frac{-\boldsymbol{DS}(t)_{2,4}}{\det_{24}(t)}$

$\beta_{24}(1\text{-sec}) = 0.002$

$\beta_{42}(t) := \frac{\boldsymbol{DS}(t)_{4,2}}{\det_{24}(t)}$

$\beta_{42}(1\text{-sec}) = -0.862$

$\beta_{44}(t) := \frac{\boldsymbol{DS}(t)_{2,2}}{\det_{24}(t)}$

$\beta_{44}(1\text{-sec}) = 0.002$
Wave travel times

\[ \Delta t_3 = \frac{L}{\lambda} \quad \Delta t_4 = \Delta t_3 \quad \Delta t_4 = 0.001266415408632 \text{ s} \]

\[ \Delta t_1 = \frac{L}{\lambda} \quad \Delta t_2 = \Delta t_1 \quad \Delta t_2 = 0.006046057498874 \text{ s} \]

Recursion "coast to coast"

\[ \eta \text{ BOUNDARY} (z, t) := \]

if \( t < \varepsilon \)

\[ \eta_{1} \leftarrow \eta_{1} \text{ IC}_{1} \]

\[ \eta_{2} \leftarrow \eta_{2} \text{ IC}_{2} \]

\[ \eta_{3} \leftarrow \eta_{3} \text{ IC}_{3} \]

\[ \eta_{4} \leftarrow \eta_{4} \text{ IC}_{4} \]

if (\( t \geq \varepsilon \))

if \( z < 0 \)

\[ \eta \leftarrow \eta \text{ BOUNDARY} (L, t - \Delta t_4) \]

\[ \eta_{4} \leftarrow \eta_{4} \]

\[ \eta \leftarrow \eta \text{ BOUNDARY} (L, t - \Delta t_2) \]

\[ \eta_{2} \leftarrow \eta_{2} \]

\[ \eta_{1} \leftarrow \alpha_{12} (t) \cdot \eta_{2} + \alpha_{14} (t) \cdot \eta_{4} + \beta_{11} (t) \cdot q(t)_{1} + \beta_{13} (t) \cdot q(t)_{3} \]

\[ \eta_{3} \leftarrow \eta_{3} \]

\[ \eta_{4} \leftarrow \alpha_{32} (t) \cdot \eta_{2} + \alpha_{34} (t) \cdot \eta_{4} + \beta_{31} (t) \cdot q(t)_{1} + \beta_{33} (t) \cdot q(t)_{3} \]

if \( z = L \)

\[ \eta \leftarrow \eta \text{ BOUNDARY} (0, t - \Delta t_3) \]

\[ \eta_{3} \leftarrow \eta_{3} \]

\[ \eta \leftarrow \eta \text{ BOUNDARY} (0, t - \Delta t_1) \]

\[ \eta_{1} \leftarrow \eta_{1} \]

\[ \eta_{1} \leftarrow \eta_{1} \]

\[ \eta_{3} \leftarrow \alpha_{21} (t) \cdot \eta_{1} + \alpha_{23} (t) \cdot \eta_{3} + \beta_{22} (t) \cdot q(t)_{2} + \beta_{24} (t) \cdot q(t)_{4} \]

\[ \eta_{4} \leftarrow \eta_{4} \]

\[ \eta_{4} \leftarrow \alpha_{41} (t) \cdot \eta_{1} + \alpha_{43} (t) \cdot \eta_{3} + \beta_{42} (t) \cdot q(t)_{2} + \beta_{44} (t) \cdot q(t)_{4} \]

return "z is not at a boundary" otherwise
Calculation intervals

\[ T := 0.015 \text{ -sec} \quad \Delta t := 0.0000207 \text{-sec} \quad \Delta t = 2.070000000000 \cdot 10^{-5} \text{ s} \]

\[ \Delta t \text{ as in SINGALT FORTRAN code} \]

\[ N := \text{ceil} \left( \frac{T}{\Delta t} \right) \quad N = 725 \quad i := 1..N + 1 \quad t_i := (i - 1) \cdot \Delta t \]

Results (in nested arrays)

\[ \eta_i^0 := \eta \text{ BOUNDARY} \left[ 0, t_i \right] \quad \eta_i^L := \eta \text{ BOUNDARY} \left[ L, t_i \right] \]

Pressures

\[ P_0 := \left( S \eta_i^0 \right)_i \quad P_L := \left( S \eta_i^L \right)_i \]

Fluid velocities

\[ V_0 := \left( S \eta_i^0 \right)_i \quad V_L := \left( S \eta_i^L \right)_i \]
**Stresses**

\[ \sigma_0^i := \langle S \cdot \eta^0 \rangle_{1/4} \]

\[ \sigma_L^i := \langle S \cdot \eta^L \rangle_{1/4} \]

**Structural velocities**

\[ \dot{u}_0^i := \langle S \cdot \dot{\eta}^0 \rangle_{1/3} \]

\[ \dot{u}_L^i := \langle S \cdot \dot{\eta}^L \rangle_{1/3} \]
Results (write to file)

PRNPRECISION = 16
PRNCOLWIDTH = 32

RESV0 := augment(t[sec], V0)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_V0.prn") := RESV0

RESP0 := augment(t[sec], P0)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_P0.prn") := RESP0

RESud0 := augment(t[sec], ud0)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_ud0.prn") := RESud0

RES σ0 := augment(t[sec], σ0)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_σ0.prn") := RES σ0

RESVL := augment(t[sec], VL)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_VL.prn") := RESVL

RESPL := augment(t[sec], PL)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_PL.prn") := RESPL

RESudL := augment(t[sec], udL)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_udL.prn") := RESudL

RES σL := augment(t[sec], σL)
WRITEPRN ("c:\winmcad\mcad80\results\RANA1_σL.prn") := RES σL

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<td>3.105·10^-4</td>
<td>1.817</td>
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Comparison with results of SINGALT FORTRAN code

\[ S10 := C:\..\Wicu01.z1p \quad S1L := C:\..\Wicu01.z3p \]
Solution in interior points

\[ \eta_{\text{INTERIOR}}(z, t) := \begin{cases} \eta_1 \leftarrow \eta_{IC_1} \\ \eta_2 \leftarrow \eta_{IC_2} \\ \eta_3 \leftarrow \eta_{IC_3} \\ \eta_4 \leftarrow \eta_{IC_4} \\ \eta_1 \leftarrow \eta_{1} \\ \eta_2 \leftarrow \eta_{2} \\ \eta_3 \leftarrow \eta_{3} \\ \eta_4 \leftarrow \eta_{4} \\ \eta \leftarrow \eta_{\text{BOUNDARY}} \left( 0, t - \frac{z}{\lambda_1} \right) \\ \eta \leftarrow \eta_{1} \\ \eta \leftarrow \eta_{\text{BOUNDARY}} \left( L, t - \frac{z - L}{\lambda_2} \right) \\ \eta \leftarrow \eta_{2} \\ \eta \leftarrow \eta_{\text{BOUNDARY}} \left( 0, t - \frac{z}{\lambda_3} \right) \\ \eta \leftarrow \eta_{3} \\ \eta \leftarrow \eta_{\text{BOUNDARY}} \left( L, t - \frac{z - L}{\lambda_4} \right) \\ \eta \leftarrow \eta_{4} \\ \eta_1 \leftarrow \eta_1 \\ \eta_2 \leftarrow \eta_2 \\ \eta_3 \leftarrow \eta_3 \\ \eta_4 \leftarrow \eta_4 \\ \text{return } "z \text{ is not an interior point}" \\ \text{otherwise} \end{cases} \]

\[ \eta_{\text{INTERIOR}}(0, 0.02 \cdot \text{sec}) = "z \text{ is not an interior point}" \]

\[ \eta_{\text{INTERIOR}}(L, 0.02 \cdot \text{sec}) = "z \text{ is not an interior point}" \]
**Calculation intervals (repeated)**

\[
z := 1.20 \text{ m} \\
T = 0.015 \text{ s} \\
\Delta t = 2.070000000000\times10^{-5} \text{ s} \\
N := \text{ceil}\left(\frac{T}{\Delta t}\right) \\
N = 725 \\
i := 1..N+1 \\
t_i := (i-1)\Delta t \\
t = 1.035\times10^{-4} \text{ s}
\]

**Results (in nested array)**

\[
\eta_i := \eta_{\text{INTERIOR}}(z, t_i)
\]

**Pressure** \( z = 1.2 \text{ m} \)

\[
P_{z_i} := \left( S \eta_{z_i} \right)_{i/2}
\]

**Fluid velocity** \( z = 1.2 \text{ m} \)

\[
V_{z_i} := \left( S \eta_{z_i} \right)_{i/1}
\]

**Stress** \( z = 1.2 \text{ m} \)

\[
\sigma_{z_i} := \left( S \eta_{z_i} \right)_{i/4}
\]

**Structural velocity** \( z = 1.2 \text{ m} \)

\[
u_{dz_i} := \left( S \eta_{z_i} \right)_{i/3}
\]
Results (write to file)

PRNPRECISION = 16 		 PRNCOLWIDTH = 32

\[ \text{PRNPRECISION} = 16 \quad \text{PRNCOLWIDTH} = 32 \]

\[ \text{RESVz} := \text{augment} \left( \frac{t}{\text{sec}}, V_z \right) \]

\[ \text{RESPz} := \text{augment} \left( \frac{t}{\text{sec}}, P_z \right) \]

\[ \text{RESudz} := \text{augment} \left( \frac{t}{\text{sec}}, udz \right) \]

\[ \text{RES} \sigma_z := \text{augment} \left( \frac{t}{\text{sec}}, \sigma_z \right) \]

\[ \text{WRITEPRN} \ ("c:\winmcad\mcad80\results\RANA1_Vz.prn") := \text{RESVz} \]

\[ \text{WRITEPRN} \ ("c:\winmcad\mcad80\results\RANA1_Pz.prn") := \text{RESPz} \]

\[ \text{WRITEPRN} \ ("c:\winmcad\mcad80\results\RANA1_udz.prn") := \text{RESudz} \]

\[ \text{WRITEPRN} \ ("c:\winmcad\mcad80\results\RANA1_sz.prn") := \text{RES} \sigma_z \]

Comparison with results of SINGALT FORTRAN code

\[ \text{SIz} := C:\..\..\Wicu01.z2 \]

\[ \text{SIZ} <\rangle = C:\..\..\Wicu01.z2 \]

\[ \text{Vz} <\rangle = C:\..\..\Wicu01.z2 \]

\[ \text{SIZ} <\rangle = C:\..\..\Wicu01.z2 \]
Appendix A2

Axial vibration of liquid-filled straight pipe

Delft Hydraulics Benchmark Problem A (Section 4.2) with free valve

Input data:

L := 20·m  
R := 398.5·mm  
ee := 8·mm  
e = 2.718

E := 210·10^9·Pa  
ρ ′ := 7900·kg/m³  
v := 0.30

K := 2.1·10^9·Pa  
ρ ″ := 1000·kg/m³  
ρ ′/ρ ″ = 7.9  
R/ee = 49.8125

Cross-sectional areas

A′ := π·R²  
A″ := π·[(R + ee)² - R²]

A′ := 4988.92·cm²  
A″ := 202.319·cm²

Axial wave speeds

\[ c_2' := \sqrt{\frac{\rho′}{K} \left(1 - \frac{v^2}{E}\right)} \]  
\[ c_2'' := \frac{E}{\rho′} \]  
\[ c_f := \sqrt{c_2'} \]  
\[ c_0 := \frac{c_2'}{c_f} = 0.994.497\,m/s \]  
\[ c_t := \sqrt{\frac{c_2'}{c_2''}} \]  
\[ c_0 = 5155.8\,m/s \]  
\[ \gamma := c_f + c_2'' + 2v^2\frac{\rho′}{\rho″} \frac{R}{ee} c_f \]  
\[ \gamma = 2.489\cdot10^7\,m²/s² \]  
\[ \alpha := 1 \]  
\[ \beta := \gamma \]  
\[ \alpha = 1 \]  
\[ \beta = -2.489\cdot10^7\,m²/s² \]  
\[ \gamma = 2.928\cdot10^{13}\,m^4/s^4 \]

\[ \lambda_{2,1} := \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \]  
\[ \lambda_{2,2} = \lambda_{2,1} \]  
\[ \lambda_{2,3} := \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \]  
\[ \lambda_{2,4} = \lambda_{2,3} \]  
\[ \lambda_{1} := \sqrt{\lambda_{2,1}} \]  
\[ \lambda_{2} := -\sqrt{\lambda_{2,1}} \]  
\[ \lambda_{3} := \sqrt{\lambda_{2,3}} \]  
\[ \lambda_{4} := -\sqrt{\lambda_{2,3}} \]  
\[ \lambda_{1} = 1024.71110374254\,m/s \]  
\[ \lambda_{2} = -1024.71110374254\,m/s \]  
\[ \lambda_{3} = 5280.51081175748\,m/s \]  
\[ \lambda_{4} = -5280.51081175748\,m/s \]
Matrices of coefficients (for axial vibration)
from Zhang et al (1999)

Note:
The matrices are made dimensionless through multiplication with the appropriate unit.

\[ A_{1,1} := 1 \quad A_{2,2} := \frac{1}{\rho \, c^2} \text{Pa} \quad A_{3,3} := 1 \quad A_{4,4} := \frac{1}{\rho \, t \, c^2} \text{Pa} \]
\[ B_{2,1} := 1 \quad B_{1,2} := \frac{1}{\rho \, f \, m^3} \quad B_{3,3} := 1 \quad B_{4,4} := \frac{1}{\rho \, t \, m^3} \]

\[
 A = \begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 9.079 \times 10^{10} & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 7.116 \times 10^{-11} & 0 & -4.762 \times 10^{12}
\end{bmatrix}
\]

\[
 B = \begin{bmatrix}
 0 & 0.001 & 0 & 0 \\
 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1.266 \times 10^{-4} \\
 0 & 0 & 1 & 0
\end{bmatrix}
\]

Transformation matrix \( T \)

\[
\alpha_1 := \frac{c^2 t - \lambda_2}{\lambda_2} \\
\alpha_2 := \frac{c^2 t - \lambda_2}{\lambda_2} \\
\alpha_2 := \alpha_1
\]

\[
t_{11} := 1 \\
t_{21} := 1 \\
t_{21} := t_{11}
\]

\[
t_{12} := \frac{\lambda_1 \, \text{sec}}{m} \\
t_{22} := \frac{\lambda_2 \, \text{sec}}{m} \\
t_{22} := -t_{12}
\]

\[
t_{13} := \frac{2 \, \nu}{\alpha_1} \\
t_{23} := \frac{2 \, \nu}{\alpha_2} \\
t_{23} := t_{13}
\]

\[
t_{14} := \frac{c^2 t}{\lambda_1 \, t \, \text{sec}} \\
t_{24} := \frac{c^2 t}{\lambda_2 \, t \, \text{sec}} \\
t_{24} := -t_{14}
\]
\[ \alpha_3 := \frac{c_2 f - \lambda_2 3}{\lambda_2 3} \quad \alpha_4 := \frac{c_2 f - \lambda_2 4}{\lambda_2 4} \quad \alpha_4 = \alpha_3 \]

\[ t_{34} := \frac{\lambda_2 3}{c_2 t} \sec \frac{m}{\lambda_3} \]

\[ t_{44} := \frac{\lambda_2 4}{c_2 t} \sec \frac{m}{\lambda_4} \]

\[ t_{44} := -t_{34} \]

\[ t_{33} := \frac{\lambda_2 3}{c_2 t} \]

\[ t_{43} := \frac{\lambda_2 4}{c_2 t} \]

\[ t_{43} := t_{33} \]

\[ t_{32} := \left( \rho \frac{v}{E e} c_2 f \frac{\lambda_3}{\alpha_3} \sec \frac{m}{\lambda_3} \right) \]

\[ t_{42} := \left( \rho \frac{v}{E e} c_2 f \frac{\lambda_4}{\alpha_4} \sec \frac{m}{\lambda_4} \right) \]

\[ t_{42} := -t_{32} \]

\[ t_{31} := \rho \frac{v}{E e} c_2 f \lambda_2 3 + \frac{\lambda_3}{c_2 f} t_{32} \sec \frac{m}{\lambda_3} \]

\[ t_{41} := \rho \frac{v}{E e} c_2 f \lambda_2 4 + \frac{\lambda_4}{c_2 f} t_{42} \sec \frac{m}{\lambda_4} \]

\[ t_{41} := t_{31} \]

\[ t_{31} = t_{31A} = 0 \quad \Rightarrow \quad t_{41} = t_{41A} = 0 \]

\[
\begin{bmatrix}
  t_{11} & t_{12} & t_{13} & t_{14} \\
  t_{21} & t_{22} & t_{23} & t_{24} \\
  t_{31} & t_{32} & t_{33} & t_{34} \\
  t_{41} & t_{42} & t_{43} & t_{44}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 1024.711 & 0.025 & 640.112 \\
  1 & -1024.711 & 0.025 & -640.112 \\
 -0.082 & -430.905 & 1.049 & 5280.511 \\
 -0.082 & 430.905 & 1.049 & -5280.511
\end{bmatrix}
\]

\[ T = (T_A)^{-1} \]

**Transformation matrix S (for axial vibration)**

\[
\begin{bmatrix}
  0.499 & 0.499 & -0.012 & -0.012 \\
  5.114 \times 10^5 & -5.114 \times 10^5 & -6.199 \times 10^4 & 6.199 \times 10^4 \\
  0.039 & 0.039 & 0.476 & 0.476 \\
 -3.143 \times 10^5 & 3.143 \times 10^5 & -1.985 \times 10^7 & 1.985 \times 10^7
\end{bmatrix}
\]


**Constant initial conditions**

Note:
Vectors are made dimensionless through multiplication with the appropriate unit.

\[
\phi_{IC} := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \eta_{IC} := S^{-1} \phi_{IC} \quad \eta_{IC} := \begin{bmatrix} 1 \\ 1 \\ -0.082 \\ -0.082 \end{bmatrix}
\]

**Boundary-condition matrices**

Note:
Matrices are made dimensionless through multiplication with the appropriate unit.

\[
D(t) := \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
0 & \frac{A_f}{m^2} & 0 & -\frac{A_t}{m^2}
\end{bmatrix}
\]

(30)

**Excitation vector**

Note:
Vectors are made dimensionless through multiplication with the appropriate unit.

\[
q(t) := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

(19)

**Matrix DS**

\[
DS(t) := D(t) \cdot S
\]

\[
DS(1, \text{sec}) = \begin{bmatrix}
5.114 \cdot 10^5 & -5.114 \cdot 10^5 & -6.199 \cdot 10^4 & 6.199 \cdot 10^4 \\
0.46 & 0.46 & -0.487 & -0.487 \\
0.039 & 0.039 & 0.476 & 0.476 \\
2.615 \cdot 10^5 & -2.615 \cdot 10^5 & 3.706 \cdot 10^5 & -3.706 \cdot 10^5
\end{bmatrix}
\]
Coefficients $\alpha$ and $\beta$

$z = 0$

$$
\text{det}_{13}(t) := \frac{\mathbf{DS}(t)_{1,1} \mathbf{DS}(t)_{3,3} - \mathbf{DS}(t)_{1,3} \mathbf{DS}(t)_{3,1}}{	ext{det}_{13}(t)}
$$

$$
\alpha_{12}(t) := \frac{\mathbf{DS}(t)_{1,2} \mathbf{DS}(t)_{3,3} - \mathbf{DS}(t)_{1,3} \mathbf{DS}(t)_{3,2}}{	ext{det}_{13}(t)}
$$

$$
\alpha_{14}(t) := \frac{\mathbf{DS}(t)_{1,4} \mathbf{DS}(t)_{3,3} - \mathbf{DS}(t)_{1,3} \mathbf{DS}(t)_{3,4}}{	ext{det}_{13}(t)}
$$

$$
\alpha_{32}(t) := \frac{\mathbf{DS}(t)_{3,2} \mathbf{DS}(t)_{1,1} - \mathbf{DS}(t)_{3,3} \mathbf{DS}(t)_{1,2}}{	ext{det}_{13}(t)}
$$

$$
\alpha_{34}(t) := \frac{\mathbf{DS}(t)_{3,4} \mathbf{DS}(t)_{1,1} - \mathbf{DS}(t)_{3,3} \mathbf{DS}(t)_{1,4}}{	ext{det}_{13}(t)}
$$

$$
\beta_{11}(t) := \frac{\mathbf{DS}(t)_{3,3}}{	ext{det}_{13}(t)}
$$

$$
= \frac{-\mathbf{DS}(t)_{1,3}}{\text{det}_{13}(t)}
$$

$$
\beta_{13}(t) := \frac{-\mathbf{DS}(t)_{3,3}}{\text{det}_{13}(t)}
$$

$$
= \frac{\mathbf{DS}(t)_{1,3}}{\text{det}_{13}(t)}
$$

$$
\beta_{31}(t) := \frac{-\mathbf{DS}(t)_{3,1}}{\text{det}_{13}(t)}
$$

$$
= \frac{\mathbf{DS}(t)_{3,1}}{\text{det}_{13}(t)}
$$

$z = L$

$$
\text{det}_{24}(t) := \frac{\mathbf{DS}(t)_{2,2} \mathbf{DS}(t)_{4,4} - \mathbf{DS}(t)_{2,4} \mathbf{DS}(t)_{4,2}}{	ext{det}_{24}(t)}
$$

$$
\alpha_{21}(t) := \frac{\mathbf{DS}(t)_{2,1} \mathbf{DS}(t)_{4,4} - \mathbf{DS}(t)_{2,4} \mathbf{DS}(t)_{4,1}}{\text{det}_{24}(t)}
$$

$$
\alpha_{23}(t) := \frac{\mathbf{DS}(t)_{2,3} \mathbf{DS}(t)_{4,4} - \mathbf{DS}(t)_{2,4} \mathbf{DS}(t)_{4,3}}{\text{det}_{24}(t)}
$$

$$
\alpha_{41}(t) := \frac{\mathbf{DS}(t)_{4,1} \mathbf{DS}(t)_{2,2} - \mathbf{DS}(t)_{4,2} \mathbf{DS}(t)_{2,1}}{\text{det}_{24}(t)}
$$

$$
\alpha_{43}(t) := \frac{\mathbf{DS}(t)_{4,3} \mathbf{DS}(t)_{2,2} - \mathbf{DS}(t)_{4,2} \mathbf{DS}(t)_{2,3}}{\text{det}_{24}(t)}
$$

$$
\beta_{22}(t) := \frac{\mathbf{DS}(t)_{4,4}}{\text{det}_{24}(t)}
$$

$$
= \frac{-\mathbf{DS}(t)_{2,4}}{\text{det}_{24}(t)}
$$

$$
\beta_{24}(t) := \frac{\mathbf{DS}(t)_{4,4}}{\text{det}_{24}(t)}
$$

$$
= \frac{-\mathbf{DS}(t)_{2,4}}{\text{det}_{24}(t)}
$$

$$
\beta_{42}(t) := \frac{\mathbf{DS}(t)_{4,2}}{\text{det}_{24}(t)}
$$

$$
= \frac{-\mathbf{DS}(t)_{2,2}}{\text{det}_{24}(t)}
$$

$$
\beta_{44}(t) := \frac{\mathbf{DS}(t)_{4,2}}{\text{det}_{24}(t)}
$$

$$
= \frac{-\mathbf{DS}(t)_{2,2}}{\text{det}_{24}(t)}
$$
Wave travel times

\[ \Delta t_2 := \frac{L}{\lambda} \]
\[ \Delta t_4 := \Delta t_3 \]
\[ \Delta t_1 := \frac{L}{\lambda} \]
\[ \Delta t_2 := \Delta t_1 \]
\[ \Delta t_2 = 0.019517696184763 \ s \]
\[ \Delta t_4 = 0.003787512366317 \ s \]

Recursion "coast to coast"

\[ \eta \text{ BOUNDARY}(x, t) := \]
\[ \text{if } t < \varepsilon \]
\[ \eta_1 := \eta \text{ IC}_1 \]
\[ \eta_2 := \eta \text{ IC}_2 \]
\[ \eta_3 := \eta \text{ IC}_3 \]
\[ \eta_4 := \eta \text{ IC}_4 \]
\[ \text{if } t \geq \varepsilon \]
\[ \text{if } z = 0 \]
\[ \eta := \eta \text{ BOUNDARY}(L, t - \Delta t_4) \]
\[ \eta_4 := \eta_4 \]
\[ \eta := \eta \text{ BOUNDARY}(L, t - \Delta t_2) \]
\[ \eta_2 := \eta_2 \]
\[ \eta_1 := \alpha_{12}(t) \cdot \eta_2 + \alpha_{14}(t) \cdot \eta_4 + \beta_{11}(t) \cdot q(t)_1 + \beta_{13}(t) \cdot q(t)_3 \]
\[ \eta_3 := \eta_3 \]
\[ \eta_4 := \eta_4 \]
\[ \text{if } z = L \]
\[ \eta := \eta \text{ BOUNDARY}(0, t - \Delta t_3) \]
\[ \eta_3 := \eta_3 \]
\[ \eta := \eta \text{ BOUNDARY}(0, t - \Delta t_1) \]
\[ \eta_1 := \eta_1 \]
\[ \eta_1 := \eta_1 \]
\[ \eta_2 := \alpha_{21}(t) \cdot \eta_1 + \alpha_{23}(t) \cdot \eta_3 + \beta_{22}(t) \cdot q(t)_2 + \beta_{24}(t) \cdot q(t)_4 \]
\[ \eta_3 := \eta_3 \]
\[ \eta_4 := \alpha_{41}(t) \cdot \eta_1 + \alpha_{43}(t) \cdot \eta_3 + \beta_{42}(t) \cdot q(t)_2 + \beta_{44}(t) \cdot q(t)_4 \]

return "z is not at a boundary" otherwise

\[ \eta \]
Calculation intervals

\[ T := 0.100 \text{ sec} \quad \Delta t := 0.000145673544858331 \text{ sec} \quad \Delta t = 1.45673544858331 \times 10^{-4} \text{ s} \]

\( \Delta t \) as in SINGALT FORTRAN code

\[ N := \text{ceil} \left( \frac{T}{\Delta t} \right) \quad N = 687 \]

\[ t_i := (i - 1) \cdot \Delta t \]

Results (in nested arrays)

\[ \eta_i^0 := \eta_{\text{BOUNDARY}}(0, t_i) \quad \eta_i^L := \eta_{\text{BOUNDARY}}(L, t_i) \]

Pressures

\[ P_0_i := \left( S \cdot \eta_i^0 \right)_i \quad P_L_i := \left( S \cdot \eta_i^L \right)_i \]

Fluid velocities

\[ V_0_i := \left( S \cdot \eta_i^0 \right)_i \quad V_L_i := \left( S \cdot \eta_i^L \right)_i \]
Stresses

\[ \sigma_{0i} := \langle S \eta \theta \rangle_{i/4} \]

\[ \sigma_{L_i} := \langle S \eta L \rangle_{i/4} \]

Structural velocities

\[ u_{d0i} := \langle S \eta \theta \rangle_{i/3} \]

\[ u_{dL_i} := \langle S \eta L \rangle_{i/3} \]
Results (write to file)

PRNPRECISION  = 16
PRNCOLWIDTH  = 32

\[\text{RESV0} := \text{augment} \left( \frac{t}{\text{sec}}, V0 \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_V0.prn") := RESV0

\[\text{RESP0} := \text{augment} \left( \frac{t}{\text{sec}}, P0 \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_P0.prn") := RESP0

\[\text{RESud0} := \text{augment} \left( \frac{t}{\text{sec}}, \text{ud0} \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_ud0.prn") := RESud0

\[\text{RES}\sigma0 := \text{augment} \left( \frac{t}{\text{sec}}, \sigma0 \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_s0.prn") := RES\sigma0

\[\text{RESVL} := \text{augment} \left( \frac{t}{\text{sec}}, \text{VL} \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_VL.prn") := RESVL

\[\text{RESPL} := \text{augment} \left( \frac{t}{\text{sec}}, \text{PL} \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_PL.prn") := RESPL

\[\text{RESudL} := \text{augment} \left( \frac{t}{\text{sec}}, \text{udL} \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_udL.prn") := RESudL

\[\text{RES}\sigmaL := \text{augment} \left( \frac{t}{\text{sec}}, \sigmaL \right)\]

WRITEPRN ("c:\winmcad\mcad80\results\RANA2_sL.prn") := RES\sigmaL
Comparison with results of SINGALT FORTRAN code

\( S10 := \text{C:\..\Simaje13.z1p} \quad \text{SIL} := \text{C:\..\Simaje13.z3p} \)
Solution in interior points

$\eta_{\text{INTERIOR}}(z, t) := \begin{cases} \eta_1 \leftarrow \eta_{\text{IC}_1} \\ \eta_2 \leftarrow \eta_{\text{IC}_2} \\ \eta_3 \leftarrow \eta_{\text{IC}_3} \\ \eta_4 \leftarrow \eta_{\text{IC}_4} \\ \text{if } t < \varepsilon \end{cases}$

$\begin{cases} \eta_1 \leftarrow \eta_1 \\ \eta_2 \leftarrow \eta_2 \\ \eta_3 \leftarrow \eta_3 \\ \eta_4 \leftarrow \eta_4 \\ \text{if } t \geq \varepsilon \end{cases}$

$\begin{cases} \eta \leftarrow \eta_{\text{BOUNDARY}}(0, t - \frac{z}{\lambda_1}) \\ \eta_1 \leftarrow \eta_1 \\ \eta_2 \leftarrow \eta_2 \\ \eta_3 \leftarrow \eta_3 \\ \eta_4 \leftarrow \eta_4 \\ \text{if } 0 < z < L \end{cases}$

$\begin{cases} \eta \leftarrow \eta_{\text{BOUNDARY}}(L, t - \frac{z - L}{\lambda_2}) \\ \eta_1 \leftarrow \eta_1 \\ \eta_2 \leftarrow \eta_2 \\ \eta_3 \leftarrow \eta_3 \\ \eta_4 \leftarrow \eta_4 \\ \text{if } 0 < z < L \end{cases}$

$\begin{cases} \eta \leftarrow \eta_{\text{BOUNDARY}}(0, t - \frac{z}{\lambda_3}) \\ \eta_1 \leftarrow \eta_1 \\ \eta_2 \leftarrow \eta_2 \\ \eta_3 \leftarrow \eta_3 \\ \eta_4 \leftarrow \eta_4 \\ \text{if } 0 < z < L \end{cases}$

$\begin{cases} \eta \leftarrow \eta_{\text{BOUNDARY}}(L, t - \frac{z - L}{\lambda_4}) \\ \eta_1 \leftarrow \eta_1 \\ \eta_2 \leftarrow \eta_2 \\ \eta_3 \leftarrow \eta_3 \\ \eta_4 \leftarrow \eta_4 \\ \text{if } 0 < z < L \end{cases}$

return "z is not an interior point" otherwise

$\eta_{\text{INTERIOR}}(0 \cdot \text{m}, 0.02 \cdot \text{sec}) = "z \text{ is not an interior point}"$

$\eta_{\text{INTERIOR}}(L, 0.02 \cdot \text{sec}) = "z \text{ is not an interior point}"$
**Calculation intervals (repeated)**

\[ \begin{align*}
  z &:= \frac{L}{2} \\
  T &= 0.1 \text{ s} \quad \Delta t = 1.45673544858331 \times 10^{-4} \text{ s} \\
  N &:= \text{ceil}\left( \frac{T}{\Delta t} \right) \quad N = 687
\end{align*} \]

\[ i := 1..N+1 \quad t_i := (i - 1) \cdot \Delta t \]

\[ \eta_z := \eta \text{ INTERIOR}(z, t_i) \]

**Results (in nested array)**

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \eta_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.001</td>
</tr>
<tr>
<td>2</td>
<td>0.001</td>
</tr>
<tr>
<td>3</td>
<td>0.001</td>
</tr>
<tr>
<td>4</td>
<td>0.001</td>
</tr>
<tr>
<td>5</td>
<td>0.002</td>
</tr>
<tr>
<td>6</td>
<td>0.003</td>
</tr>
<tr>
<td>7</td>
<td>0.004</td>
</tr>
<tr>
<td>8</td>
<td>0.005</td>
</tr>
<tr>
<td>9</td>
<td>0.006</td>
</tr>
<tr>
<td>10</td>
<td>0.007</td>
</tr>
<tr>
<td>11</td>
<td>0.008</td>
</tr>
<tr>
<td>12</td>
<td>0.009</td>
</tr>
</tbody>
</table>

**Pressure** \( z = 10 \text{ m} \)

\[ P_z := \{ S \eta_z \}_{i,2} \]

**Fluid velocity** \( z = 10 \text{ m} \)

\[ V_z := \{ S \eta_z \}_{i,1} \]

**Stress** \( z = 10 \text{ m} \)

\[ \sigma_z := \{ S \eta_z \}_{i,4} \]

**Structural velocity** \( z = 10 \text{ m} \)

\[ udz := \{ S \eta_z \}_{i,3} \]
**Results (write to file)**

PRNPRECISION  =  16  
PRNCOLWIDTH  =  32

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.457·10^-4</td>
</tr>
<tr>
<td>3</td>
<td>2.913·10^-4</td>
</tr>
<tr>
<td>4</td>
<td>4.37·10^-4</td>
</tr>
<tr>
<td>5</td>
<td>5.827·10^-4</td>
</tr>
<tr>
<td>6</td>
<td>7.284·10^-4</td>
</tr>
<tr>
<td>7</td>
<td>8.74·10^-4</td>
</tr>
<tr>
<td>8</td>
<td>0.001</td>
</tr>
<tr>
<td>9</td>
<td>0.001</td>
</tr>
<tr>
<td>10</td>
<td>0.001</td>
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<tr>
<td>11</td>
<td>0.001</td>
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<tr>
<td>12</td>
<td>0.002</td>
</tr>
<tr>
<td>13</td>
<td>0.002</td>
</tr>
<tr>
<td>14</td>
<td>0.002</td>
</tr>
<tr>
<td>15</td>
<td>0.002</td>
</tr>
<tr>
<td>16</td>
<td>0.002</td>
</tr>
</tbody>
</table>

RESVz := augment\( \left( \frac{t}{\text{sec}}, V_z \right) \)

```
WRITEPRN ("c:\\winmcad\\mcad80\\results\\RANA2_Vz.prn") := RESVz
```

RESPz := augment\( \left( \frac{t}{\text{sec}}, P_z \right) \)

```
WRITEPRN ("c:\\winmcad\\mcad80\\results\\RANA2_Pz.prn") := RESPz
```

RESudz := augment\( \left( \frac{t}{\text{sec}}, udz \right) \)

```
WRITEPRN ("c:\\winmcad\\mcad80\\results\\RANA2_udz.prn") := RESudz
```

RES\( \sigma \_z \) := augment\( \left( \frac{t}{\text{sec}}, \sigma_z \right) \)

```
WRITEPRN ("c:\\winmcad\\mcad80\\results\\RANA2_sz.prn") := RES\( \sigma \_z \)
```

**Comparison with results of SINGALT FORTRAN code**

SIz:=C:\.\Simajc13.z2p