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Published: 01/01/1994

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Link to publication

Citation for published version (APA):
EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

Memorandum COSOR 94-36

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Eindhoven, November 1994
The Netherlands
FITTING DISCRETE DISTRIBUTIONS ON THE FIRST TWO MOMENTS
IVO ADAN, MICHEL VAN EENIGE AND JACQUES RESING·

Abstract. In this paper we present a simple method to fit a discrete distribution on the first two moments of a given distribution. With the fitted distribution we solve approximately Lindley's equation for the discrete-time D/G/1 queue using a moment-iteration method. Numerical results show excellent performance of the method.

Key words. two-moment fits, discrete distributions, Lindley's equation, moment-iteration method

AMS subject classifications. 60K25, 90B22

1. Introduction. A common way to fit a continuous distribution on the mean, \(EX\), and the coefficient of variation, \(c_X\), of a given distribution is the following (see e.g. Tijms [9]). In case \(0 < c_X < 1\) one fits a mixture of Erlangian distributions with the same scale parameter. More specifically, if \(1/k \leq c_X^* \leq 1/(k - 1)\), for certain \(k = 2, 3, \ldots\), one can fit a so-called \(E_{k-1,k}\) distribution. A random variable having this distribution is with probability \(q\) (respectively \(1 - q\)) distributed as the sum of \(k - 1\) (respectively \(k\)) independent exponentials with common mean. In case \(c_X \geq 1\) one fits a hyperexponential with balanced means. A random variable having this distribution is with probability \(q_1\) (respectively \(q_2\)) distributed as an exponential variable with parameter \(\mu_1\) (respectively \(\mu_2\)) where \(q_1 + q_2 = 1\) and \(q_1/\mu_1 = q_2/\mu_2\).

Many applications involve discrete random variables. In such cases it is, of course, more natural to fit discrete distributions instead of continuous ones. The aim of this paper is to construct a discrete analogue of the method described above. It turns out that this problem is not trivial. In fact, in order to fit all possible values of \(EX\) and \(c_X\), we need four classes of distributions instead of only two. Ord [8] treats a problem related to the present one. He fits discrete distributions not on the first two moments, but on two parameters depending on the first three moments of a given distribution.

After the construction of the discrete fits on the first two moments, we use these fits for approximating the waiting-time characteristics of the discrete-time D/G/1 queue. This queueing model naturally arises, for example, in the analysis of a periodic review \((R, S)\) inventory system and in the analysis of a fixed-cycle traffic light queue. We use a discrete version of the moment-iteration method of de Kok [5], originally introduced for the continuous-time case. Furthermore, we will check the quality of this method by comparing it, for several service-time distributions, with an exact numerical analysis, Fredericks' approximation (see Fredericks [4]), and its continuous-time version.

The rest of the paper is organized as follows. In section 2 we answer the question how to fit a discrete distribution on the first two moments of a given distribution. Next, in section 3 we describe the discrete version of de Kok's iteration method for approximating the waiting-time characteristics of the discrete-time D/G/1 queue. Finally, section 4 is devoted to numerical results.
2. Fitting discrete distributions. Consider an arbitrary discrete distribution on the non-negative integers with first moment $EX$ and coefficient of variation $c_X$. We use four classes of distributions in order to choose a distribution which has the same mean and coefficient of variation as the given one. These classes of distributions are Poisson, mixtures of binomial, mixtures of negative-binomial and mixtures of geometric distributions with balanced means.

Let us first introduce some notation. A $GEO(p)$ random variable has probability distribution $p_i = (1 - p)^i p^i$, $i = 0, 1, 2, \ldots$, and an $NB(k, p)$ variable is the sum of $k$ independent $GEO(p)$ variables. A $POIS(\lambda)$ random variable is Poisson distributed with mean $\lambda$, and a $BIN(k, p)$ variable is binomially distributed, where $k$ is the number of trials and $p$ the success probability.

If we define $a := c_X^2 - 1/EX$, then we have that $a \geq -1$, because $EX \leq EX^2$ for random variables on the non-negative integers. Lemma 2.1 shows how to fit a distribution on the first two moments depending on the value of $a$.

**Lemma 2.1.** Let $X$ be a random variable on the non-negative integers with mean $EX$ and coefficient of variation $c_X$ and let $a = c_X^2 - 1/EX$. Then the random variable $Y$ matches the first two moments of $X$ if $Y$ is chosen as follows:

1. If $-1/k \leq a \leq -1/(k + 1)$ for certain $k = 1, 2, 3, \ldots$, then
   \[
   Y = \begin{cases} 
   BIN(k, p) & \text{w.p. } q, \\
   BIN(k + 1, p) & \text{w.p. } 1 - q,
   \end{cases}
   \]
   where
   \[
   q = \frac{1 + a(1 + k) + \sqrt{-ak(1 + k) - k}}{1 + a}, \quad p = \frac{EX}{k + 1 - q}.
   \]

2. If $a = 0$, then $Y = POIS(\lambda)$ with $\lambda = EX$.

3. If $1/(k + 1) \leq a \leq 1/k$ for certain $k = 1, 2, 3, \ldots$, then
   \[
   Y = \begin{cases} 
   NB(k, p) & \text{w.p. } q, \\
   NB(k + 1, p) & \text{w.p. } 1 - q,
   \end{cases}
   \]
   where
   \[
   q = \frac{(1 + k)a - \sqrt{(1 + k)(1 - ak)}}{1 + a}, \quad p = \frac{EX}{k + 1 - q + EX}.
   \]

4. If $a \geq 1$, then
   \[
   Y = \begin{cases} 
   GEO(p_1) & \text{w.p. } q_1, \\
   GEO(p_2) & \text{w.p. } q_2,
   \end{cases}
   \]
   where
   \[
   p_1 = \frac{EX[1 + a + \sqrt{a^2 - 1}]}{2 + EX[1 + a + \sqrt{a^2 - 1}]}, \quad q_1 = \frac{1}{1 + a + \sqrt{a^2 - 1}},
   \]
   \[
   p_2 = \frac{EX[1 + a - \sqrt{a^2 - 1}]}{2 + EX[1 + a - \sqrt{a^2 - 1}]}, \quad q_2 = \frac{1}{1 + a - \sqrt{a^2 - 1}}.
   \]

It is straightforward to check that the given distributions indeed have the same mean and coefficient of variation as $X$. Therefore, the proof of the lemma is omitted.
The result of Lemma 2.1 is illustrated in Figure 1.

**I:** Binomial

**II:** Poisson

**III:** Negative-binomial

**IV:** Geometric

\[ EX = \frac{1}{c_x + 1} \quad \text{(Bernoulli)} \]

\[ EX = \frac{1}{c_x} \quad \text{(Poisson)} \]

\[ EX = \frac{1}{c_x - 1} \quad \text{(Geometric)} \]

**Fig. 1.** The four classes of distributions used to match the first two moments of \( X \).

3. **Moment-iteration algorithm.** In this section, we describe a refined, discrete version of de Kok's moment-iteration method for approximating waiting-time characteristics of the discrete-time \( D/G/1 \) queue. Our interest in the discrete-time \( D/G/1 \) queue arose out of work on the fixed-cycle traffic light (see e.g. Darroch [2] and Newell [7]), and this queue also arises, for example, in the analysis of a periodic review \( (R, S) \) inventory system with finite production capacity (see e.g. de Kok [5]). The waiting-time characteristics considered here are the delay probability \( \Pi_W \), and the first two moments of the waiting time. The moment-iteration method is used in the next section to illustrate the method of fitting discrete distributions as presented in the previous section.

Consider a discrete-time \( D/G/1 \) queue. Customers arrive at the server with fixed interarrival time \( A \). The service-time probability distribution is given by \( \{b_i\}_{i=0}^{\infty} \) with mean \( b \). The service times of customers are assumed to be mutually independent and customers are served in order of arrival. Furthermore, we assume that the queue is stable, i.e., \( \rho := b/A < 1 \).

Let, for \( n = 0, 1, 2, \ldots \), the random variables \( B_n \) and \( W_n \) denote the service time, respectively, the waiting time of the \( n \)-th customer. Further, suppose that the 0-th customer arrives at an empty system, so that \( W_0 = 0 \). Then, it is easily seen that the following relation holds

\[ W_n = \max\{0, W_{n-1} + B_{n-1} - A\}, \quad n = 1, 2, 3, \ldots \]

This relation is the starting point for the approximation of the delay probability and the first two moments of the waiting-time distribution. Let the generic random variable \( B \) have probability distribution \( \{b_i\}_{i=0}^{\infty} \), then the moment-iteration method (for the discrete-time \( D/G/1 \) queue) can be described as follows:
Moment-iteration algorithm for the discrete-time $D/G/1$ queue.

1. Initialization.
Set $W_0 := 0$, and compute exactly
\[
P(W_1 > 0) = P(B - A > 0),
\]
\[
EW_1 = E(B - A)_+,\]
\[
EW^2_1 = E(B - A)^2_+,
\]
and set $n := 1$.

2. Iteration.
Set $V_n := (W_n|W_n > 0) - 1$, and compute
\[
EV_n = \frac{EW_n}{P(W_n > 0)} - 1,
\]
\[
EV^2_n = \frac{EW^2_n}{P(W_n > 0)} - \frac{2EW_n}{P(W_n > 0)} + 1.
\]
Fit a tractable probability distribution $\{v_i\}_{i=0}^\infty$ to the probability distribution of $V_n$ by matching the first two moments as described in the previous section. Compute
\[
EW_{n+1} = E(B - A)_+ P(W_n = 0) + E(1 + V_n + B - A)_+ P(W_n > 0)
\]
(1)
\[
= P(W_n = 0) \sum_{i=A+1}^{\infty} b_i(i - A) + P(W_n > 0) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_i b_j (1 + i + j - A)_+,
\]
and also
\[
P(W_{n+1} > 0) = P(W_n = 0) P(B - A > 0) + P(W_n > 0) P(1 + V_n + B - A > 0),
\]
\[
EW^2_{n+1} = E(B - A)^2_+ P(W_n = 0) + E(1 + V_n + B - A)^2_+ P(W_n > 0),
\]
where the right-hand sides can be worked out similarly as in (1).

3. Convergence.
If it holds that
\[
|EW_{n+1} - EW_n| < \epsilon_1 \quad \text{and} \quad |EW^2_{n+1} - EW^2_n| < \epsilon_2,
\]
then stop, otherwise set $n := n + 1$ and repeat step 2.

4. Stop.
Approximate $\Pi_W$ by $P(W_{n+1} > 0)$, and the first two moments of the waiting-time distribution by $EW_{n+1}$ and $EW^2_{n+1}$, respectively.

In the next section, this algorithm will be applied to the $D/G/1$ queue for several discrete service-time distributions.

4. Numerical results. In this section we present numerical results to illustrate the quality of the approximations obtained with the moment-iteration method. In Table 1 we list the moment approximations (mit) and the exact values (exa) of the delay probability $\Pi_W$ and the mean waiting time $EW$ for $D/G/1$ queues with binomial, negative-binomial and uniform service times, respectively. The performance characteristics $\Pi_W$ and $EW$ are evaluated for a range of values of the systems parameters.
We also compared the moment approximation with the well-known approximation of Fredericks (Fred) for the $D/G/1$ queue (see [4]). The results in Table 1 are based on Fredericks' approximation for discrete service-time distributions, which can easily be derived from the continuous version in [4]. The exact results for the $D/BIN(k,p)/1$ queue have been obtained with use of the matrix-geometric approach developed by Neuts [6] (see also [3]). The $D/NB(k,p)/1$ queue has been analyzed exactly by using an embedded Markov chain approach similar as in [1] and the exact results for the $D/U(0,k)/1$ have been obtained by numerically solving the embedded Markov chain using the approach of Tijms and van de Coevering [10], where $U(0,k)$ denotes the uniform distribution on $0,1,\ldots,k$.

**Table 1**

*The delay probability $P(W_n > i)$ and the mean waiting time $EW).*

<table>
<thead>
<tr>
<th>$A$</th>
<th>$k$</th>
<th>$p$</th>
<th>$\Pi_w$</th>
<th>$\text{exa}$</th>
<th>$\text{mit}$</th>
<th>Fred</th>
<th>$\text{exa}$</th>
<th>$\text{mit}$</th>
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</table>

The results in Table 1 show good performance of Fredericks' approximation and excellent performance of the moment approximation. Fredericks' approximation is exact for the $D/GEO(p)/1$ queue and the $D/BIN(k,p)/1$ and $D/U(0,k)/1$ queue with $k = A + 1$, since the waiting-time distribution for these queues is geometric.

Since we fit a tractable distribution to the distribution of $W_n | W_n > 0$, we are also able to compute an approximation for $P(W_{n+1} > i)$ for all $i = 1,2,\ldots$ in the same way as for $P(W_{n+1} > 0)$. From this, we can compute approximations for the $\alpha$-percentiles of the waiting-time distribution. For each $\alpha$ with $0 \leq \alpha \leq 1$ the $\alpha$-percentile is defined as the smallest $i$ satisfying $P(W_{n+1} \leq i) \geq \alpha$. In Table 2 we list the moment approximations and the exact values of the $\alpha$-percentiles for the $D/G/1$ queues in Table 1 with traffic load 0.9 and 0.95, where $\alpha$ is varied from 0.5, 0.9, 0.95 to 0.99. We see in Table 2 that the quality of the moment approximation is good.
Instead of using discrete distributions one may also fit a continuous distribution $W(t)$ on the first two moments of $W_n | W_n > 0$ as described in the introduction. Then equation (1) for $EW_{n+1}$ becomes

$$EW_{n+1} = E(\mathcal{B} - \mathcal{A}) + P(W_n = 0) + P(W_n > 0) \int_{t=0}^{\infty} \sum_{j=0}^{\infty} b_j (t + j - \mathcal{A}) dW(t)$$

and the equations for $P(W_{n+1} > 0)$ and $EW_{n+1}^2$ have to be adapted accordingly. In Table 3 we compare for several examples the quality of discrete (disc) and continuous (cont) fits. The results show that both fits perform excellent for the mean waiting time, but the discrete fits yield a significantly better approximation for the delay probability.

**Table 3**

*Comparison of the quality of discrete and continuous fits.*

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Acknowledgment. The authors thank Ton de Kok for stimulating discussions.

REFERENCES