The use of the (x,T)-strategy for production to order

Citation for published version (APA):

Document status and date:
Published: 01/01/1987

Publisher Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
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Download date: 18. Feb. 2019
Memorandum COSOR 87-33

The use of the \((x,T)\)-strategy
for production to order

by

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November 1987
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Introduction

In a company producing steel pipes, we did a study for the planning of a part of the production (Dellaert and Wessels (1986)). In this part of the production process, different types of steel pipes are manufactured on the same machine. A part of the machine has to be rebuilt before the production of another type can be started. Because of the large variety in product types and because of the voluminous orders, no safety stocks can be kept. Therefore, we have to produce according to customer specifications. Due to this production to order, we have to work with delivery-times, since no orders can be delivered from stock. In this company, the average delivery-time was about four weeks. However, a lot of clients are not very interested in the average delivery-time, but prefer to be sure about the delivery-time before they order their products. Therefore, delivery-contracts are concluded with different groups of clients or for different groups of orders. There may be a group of clients, or orders, that always obtain a promised delivery-time of one week and another group that always obtain a promised delivery-time of two weeks and so on. These promised delivery-times have to be met as good as possible.

The situation as sketched above, is believed to occur quite often in companies in process industry. The reasons for production to order may be different, for instance they may produce perishable goods, and the nature of the set-up may be different, but nevertheless a comparable production scheduling will be possible.

In Dellaert (1987) a simple scheduling problem is considered, with one type of product on one machine. The main aspects of this problem are production to order in combination with a demand by different groups of clients, set-up times and unconstrained capacity. Four strategies are offered for this kind of situation: the optimal strategy, a Silver-Meal like approach, a Wagner-Whitin like approach and the \((x,T)\)-strategy, in which the known demand for the first \(T\) periods is produced if the demand for the current period is at least \(x\). The main advantage of the \((x,T)\)-strategy is that the average costs per period can be determined very easily, while these costs generally are less than a few percent higher than the average costs per period following the optimal strategy. Therefore, we will study this strategy more closely.
After a detailed description of the \((x,T)\)-strategy, we will give two properties, which enable us to find the optimal choices for \(x\) and \(T\) very quickly. The difference between the optimal strategy and the \((x,T)\)-strategy is caused by the fact that in the \((x,T)\)-strategy we only use information about the known demand for the first period. Therefore it is not so difficult to find an improvement on this strategy by using more information. In the improved strategy, which we will call the salvage-strategy, we consider two costs: the direct costs of an action and the expected costs of the left-demand. The left-demand is that part of the known demand that is not yet produced after the action. The expected costs of the left-demand will be determined using the assumption that in later periods the \((x,T)\)-strategy will be used. This one-step improvement will already yield a nearly optimal strategy.

Often the number of possible left-demands will be too large to allow calculating the expected costs of all of them. In those cases we can use the so called rule-strategy, consisting of a simple set of rules, which indicates when we have to deviate from the \((x,T)\)-strategy. The rules cover most of the improvements of the salvage-strategy. Calculating the average costs of these strategies may be quite difficult, but estimating them is very simple and accurate. An interesting result is that we can also estimate the average costs of other strategies, such as the Silver-Meal like strategy, comparing the actions and the salvage costs.

1. Description of the model

For the demand we assume that we have \(N\) different groups of clients, where clients of group \(i, 1 \leq i \leq N\), always require delivery-time \(i\) for their products, so the due-dates are independent of the production-schedule, but depend only on the priority-rules for the different clients and on the arrival-dates. Once a client is assigned to a certain group, his norm delivery-time is always the same. The order-flow, and thereby the costs, can be controlled by assign clients to certain groups. We assume that this assignment is made, therefore the demand distribution is known and stationary.

At the beginning of each period we know the exact demand for that period and a part of the demand for later periods. Now we can decide to produce or decide to wait with production. If we produce, then we have set-up costs \(S\) and holding costs \(h\) per unit of product per period for orders that are manufactured too soon. If we wait with production, we have penalty costs \(p\) per unit of product per period for orders that are delivered too late. We assume that once a decision is taken, it is not changed during that period. Furthermore we assume that we can express the demand in units of products and that the products are made according to customer specifications, so that only the known demand can be manufactured.

We call the demand we observe, \(r=(r_0, r_1, \ldots, r_{N-1})\), the residual demand-vector. This residual demand-vector contains the original demand for the current period and the next \(N-1\) periods, but the products that already have been manufactured for these periods are
left out. Because backlogging is permitted, \( r_0 \) also contains the demand of earlier periods which is not yet produced. The costs do not depend on the arrival-times of the orders, so therefore this residual demand provides all the necessary information.

Every period clients of group \( i \) can order a demand of \( 0.1, \ldots, M_i - 1 \) or \( M_i \) units of product. We assume that the probability that they order \( j \) units during one period, denoted by \( d_{ij} \), is known. Every period we have to take an action. Action \( \alpha \) means that we produce the known demand of the first \( \alpha \) periods. \(( \alpha = 0.1, \ldots, N)\).

Let \( \mathcal{Q}_0(r) \) be the first \( T \) elements of the next-period state, assuming we have taken action \( \alpha \) in state \( r \) and we have no new demand during the current period. Then, for instance, \( \mathcal{Q}_0(r) = (r_0 + r_1, r_2, \ldots, r_T) \) and \( \mathcal{Q}_3(r) = (0, 0, r_3, \ldots, r_T) \).

The one-stage costs of taking action \( \alpha \) on observing state \( r \) have the following form:

\[
g^\alpha_r = s + \sum_{i=1}^{\alpha-1} i \cdot r_i \cdot h \quad \text{if } \alpha > 0 \tag{1.1}
\]

\[
g^\alpha_r = r_0 \cdot p \quad \text{if } \alpha = 0. \tag{1.2}
\]

2. Description of the strategy

Following the idea of \((s,S)\)-strategies we consider the following approach: produce the residual demand of the first \( T \) periods as soon as the demand for the current period is at least \( x \). This strategy is called \((x,T)\)-strategy. In the unconstrained capacity situation it is rather easy to determine the average costs for given values of \( x \) and \( T \). Of course we are interested in finding the optimal choices for \( x \) and \( T \). We do this by computing the average costs per period for several values of \( x \) and \( T \).

In order to compute the average cost per period, \( g(x,T) \), for some pair \((x,T)\), we consider a finite recurrent Markov-chain with states \((i,j) \in \{1,2,\ldots,T-1\} \times \{0,1,\ldots,x\}\) or \( \{1\} \times \{0,1,\ldots,x\} \) for \( T = 1 \), where

- \( i = \min \) \{number of periods passed since the last production period, \( T-1 \)\}
- \( j = \min \) \{demand for the current period, \( x \)\}.

![Figure 2.1](image)

**Figure 2.1** The states of the Markov chain for the \((x,T)\)-strategy. \((T \geq 2)\).
We use the following notations:

- \( q_{ij} \) is the expected number of visits to state \((i,j)\) between two arbitrary production periods;
- \( c_{ij} \) is the expected cost in state \((i,j)\);
- \( b_{ij} \) is the probability that clients belonging to the groups \(1, 2, \ldots, i\) order a total amount of \(l\) units of product for one specific delivery period:

\[
b_{ij} = \sum_{j \in J} \left( \prod_{k=1}^{i} d_{ik} \right) \text{ with } \sum_{i=1}^{I} j_i = l.
\]

Because of the special structure of the Markov-chain we can determine \( q_{ij} \) very easily:

\[
q_{1j} = b_{1j} \quad 0 \leq j < x
\]

\[
q_{1x} = (1 - \sum_{k=0}^{x-1} b_{1k})
\]

\[
q_{ij} = \sum_{k=0}^{i-1} q_{i-1k} b_{ij-k} \quad 0 \leq j < x, 2 \leq i \leq T-1
\]

\[
q_{ix} = \sum_{k=0}^{x-1} q_{i-1k} - q_{ik} \quad 2 \leq i \leq T-2
\]

and then modifying \( q_{T-1j} \) for later periods:

\[
q_{T-1j} = (q_{T-1j} + \sum_{k=0}^{T-2} q_{T-1k} b_{Nj-k}) (1 - b_{N0})^{-1} \quad 0 \leq j < x
\]

\[
q_{T-1x} = 1 - \sum_{i=1}^{T-2} q_{ix}
\]

Using \( e_i \) as the expected value of the \(i+1\)-th component of the residual demand, if no part of this demand has been produced:

\[
e_i = \sum_{l=i+1}^{N} \sum_{j=1}^{M_i} j\delta_{ij}, \quad \text{the expected costs in state } (i,j) \text{ are:}
\]

\[
c_{ij} = pj \quad 1 \leq i \leq T-1, 0 \leq j < x
\]

\[
c_{ix} = s + h \sum_{j=1}^{T-1} e_j - h \sum_{j=i+1}^{T-1} (j-i) e_j
\]

\[
= s + h \sum_{j=1}^{T-1} e_j \min(i,j) \quad 1 \leq i \leq T-1
\]

The expected time between two production periods is given by:

\[
T(x,T) = \sum_{i=1}^{T-1} \sum_{j=0}^{x} q_{ij}
\]

and the expected costs between two production periods are given by:
The average costs of the strategy for this pair \((x, T)\) are now given by:

\[
g(x, T) = \frac{C(x, T)}{T(x, T)}
\]  

(2.13)

3. Some properties of the \((x,T)\)-strategy

To determine the optimal pair \((x, T)\), it is not necessary to determine the value of \(g(x, T)\) for all possible pairs \((x, T)\). To limit the number of pairs we have to consider, we make use of the following two properties:

**Property 1**: for a given value of \(T\) the optimal value of \(x\) satisfies:

\[
x \leq \left[ \frac{g(x, T)}{p} \right] + 1.
\]  

(3.1)

**Property 2**: for a given value of \(x\) the optimal value of \(T\) is less than or equal to \(k\) if for \(k\) the following holds:

\[
\sum_{j=0}^{x-1} q_{ij} (g^* - \bar{c}_i) > (k-1) c_i - 1
\]  

(3.2)

Here \(g^*\) is the upper bound of the optimal \(g(x, T)\) and \(\bar{c}_i\) is the average cost in the \(k\)-th period after the last period in which we produced. Now we will give the proofs of the properties:

**Proof property 1**: We want to show that if \(y\) does not satisfy (3.1), it cannot be optimal. The costs \(C(y-1, T)\) are less than \(C(y, T)\), because the penalty costs are less and the holding costs might be less, because we might produce sooner. Expressed in the \(q_{ij}\)'s associated with the pair \((y, T)\) we have

\[
C(y-1, T) = C(y, T) - p \sum_{i=1}^{T-1} (y-1)q_{iy-1} - h \sum_{i=1}^{T-2} q_{iy-1} (\sum_{j=i+1}^{T-1} c_j)
\]  

(3.3)

and

\[
T(y-1, T) = T(y, T) - \sum_{i=1}^{T-1} q_{iy-1}
\]  

(3.4)

If \(p (y-1) > g(y, T)\), that is \(y\) does not satisfy (3.1), we have that
Therefore $y$ can only be optimal if it satisfies (3.1).

**Note:** For $T \leq 2$ we also have that if $(y-1)p < g(y,T)$ then

$$g(y-1,T) = \frac{C(y,T) - p \sum_{i=1}^{T-1} (y-1) q_{iy-1}}{T(y,T) - \sum_{i=1}^{T-1} q_{iy-1}}$$

$$< \frac{T(y,T) C(y,T) - C(y,T) \sum_{i=1}^{T-1} q_{iy-1}}{T(y,T) (T(y,T) - \sum_{i=1}^{T-1} q_{iy-1})} = \frac{C(y,T)}{T(y,T)} = g(y,T)$$

This implies that for $T \leq 2$ the optimal value of $x$ satisfies:

$$x = \left\lfloor \frac{g(x,T)}{p} \right\rfloor + 1. \tag{3.7}$$

**Proof property 2:** Let $q_{ij}(K)$ be the probabilities based on the $(x,K)$-strategy and let $p_{ij}(K)$ be the probability that $i$ periods after the last production period the demand for the first period equals $j$, also assuming we follow the $(x,K)$-strategy. For these $p_{ij}(K)$ we also allow $i \geq K$. Then

$$b := T(x,K+1) - T(x,K) = \sum_{i=K-1}^{K} \sum_{j=0}^{x} q_{ij}(K) - \sum_{j=0}^{x} q_{K-1j}(K-1) \tag{3.9}$$

$$< \sum_{j=0}^{x-1} p_{Kj}(K) \tag{3.10}$$

because $\sum_{j=0}^{x-1} p_{ij}(K-1) > \sum_{j=0}^{x-1} p_{i+1j}(K)$ for all $i \geq K$.

Let $\bar{\mathcal{C}}_K = (p \sum_{i=0}^{x-1} p_{Ei}(K))^{-1} \sum_{i=0}^{x-1} p_{Ei}(K)$ be the average penalty-cost in the $K$-th period after the last period in which we produced. Then after period $K-1$, if we still do not produce, the average costs per period will be higher than $\bar{\mathcal{C}}_K$. Therefore:
Then
\[ g(x,K+1) = \frac{C(x,K+1)}{T(x,K+1)} > \frac{C(x,K) + Ke_k + b\tilde{c}_k}{T(x,K) + b} \] (3.12)

Now if \[ \sum_{j=0}^{x-1} P_{K_j}(K) < \frac{Ke_k}{g - \tilde{c}_k} \] (3.13)
then
\[ b < \frac{Ke_k}{g - \tilde{c}_k} \] (3.14)
and thus
\[ b\tilde{c}_k + Ke_k > bg^* \] (3.15)

Therefore
\[ g(x,K+1) > \frac{C(x,K) + bg^*}{T(x,K) + b} \] (3.16)

Now there are two possibilities:
1) \( g(x,K) \geq g^* \), that results in:
\[ g(x,K+1) > \frac{g^*(T(x,K) + b)}{T(x,K) + b} = g^* \] (3.17)
2) \( g(x,K) < g^* \), that results in:
\[ g(x,K+1) > \frac{C(x,K)(T(x,K) + b)}{T(x,K) + b} = \frac{C(x,K)}{T(x,K)} = g(x,K) \] (3.18)

and clearly the pair \((x,K+1)\) cannot be optimal.

Now to find the optimal pair \((x,T)\), we can start with \( x \) given by \( \left| \frac{X}{P} \right| + 1 \) and increase \( T \) starting from 1 until (3.2) does not hold any more for \( T \). Then we calculate \( g(x,T) \) for this pair using (2.13). This value is used in (3.1) to choose a new \( x \) and in (3.2) to decrease \( T \), if necessary. Most of the \( q_{ij}^t \)'s are not effected by this new choice, so calculating \( g(x,T) \) for this new pair will be very simple. We repeat this procedure until a further decaement of \( x \) or \( T \) yield higher costs.

4. The salvage-strategy

Given a residual demand \((r_0,r_1,\ldots,r_{N-1})\) the \((x,T)\)-strategy bases the decision only upon \( r_0 \). In the improved version we base the decision upon \((r_0,r_1,\ldots,r_T)\). In stead of choosing between the actions 0 and \( T \), we now consider the actions 0,1,\ldots,\( T+1 \). Of course it would

\[ C(x,K+1) > C(x,K) + Ke_k + b\tilde{c}_k \] (3.11)
be better to consider the actions 0,1,...,N, but the (x,T)-strategy does not provide the necessary information to do this. The idea behind the improvement is the following: determine the marginal cost of every possible left-demand $Q_a(r)$. These marginal costs are based upon the expected costs during the first $T$ periods following the (x,T)-strategy. The difference between the expected future costs of demand $r$ after action $a$ and the expected future costs after action $a$ over all possible demands is defined as the marginal cost of demand $r$ after action $a: L(Q_a(r))$. Now the strategy takes the following form:

On residual demand $r$ we take action $a$ if

$$q_a + L(Q_a(r))$$

is minimal over $a = 0,1,...,T+1$.

The function $L(.)$ is called the salvage-function and the resulting strategy the salvage strategy. In the next subsection we will describe how to determine the value of $L(.)$ for a given left-demand.

### 4.1. Determining the salvage costs

A lot of variables that have to be calculated to determine the salvage value for a given left-demand can also be used to determine other salvage values. First we will describe these variables:

The rest-value $R(i)$ is defined as the marginal cost if the demand for the first period equals $i$ and no part of the demand of the following periods has been produced:

$$R(x) = \bar{K}$$

$$R(i) = p_i + \sum_{j=0}^{x-i-1} b_{nj} R(i+j) + (1- \sum_{j=0}^{x-i-1} b_{nj}) R(x) - g^* \quad i = 0,...,x-1$$

where $\bar{K} = s + h \sum_{i=1}^{T-1} e_i$, the average producing costs if no part of the residual demand has been produced yet.

The production-probability $PP(i,j)$ is defined as the probability that given that sum of the first $j$ components of the left-demand equals $i$ and following the (x,T)-strategy, we produce during one of the first $j$ periods after the current period:

$$PP(x,j) = 1 \quad j = 1,...,T$$

$$PP(i,j) = 1 - \sum_{k=0}^{x-i-1} q_{jk} \quad j = 1,...,T; \quad i = 0,...,x-1$$

The expected penalty costs during the $j$-th period after the current period, given that the sum of the first $j$ components of the left-demand equals $i$, is written as $B(i,j)$ and can be described by:
Now given a left-demand \((w_0, w_1, \ldots, w_{T-1})\) we use the following definitions:

\[
  v_i = \min(x, \sum_{j=0}^{i-1} w_j)
\]

and for the expected production and holding costs \(i\) periods after the current period:

\[
  K_i = \bar{K} + \sum_{j=1}^{T-1} j(w_j - e_j)
  \\
  K_T = \bar{K}
\]

and for reasons of convenience:

\[
  K_{T+1} = 0
\]

The values of \(R(i), PP(i,j)\) and \(B(i,j)\) have to be determined only once, whereafter they can be used to determine all salvage-values. The values of \(K_i\) and \(v_i\) have to be determined for every possible left-demand. Then we can write the salvage value as:

\[
  L(w_0, \ldots, w_{T-1}) = \sum_{i=1}^{T} PP(v_i,i)(K_i - K_{i+1}) + \sum_{i=1}^{T-1} B(v_i,i)
  \\
  + \sum_{i=1}^{T-1} q_{T-1-i} R(i) - g^\ast (T - \sum_{i=1}^{T-1} PP(v_i,i))
\]

In some small examples, that are showed later on, this strategy showed to be nearly optimal. Although determining the marginal cost is quite simple, determining the average costs of the strategy is usually as complex as determining the average costs of the Silver-Meal strategy. In both cases we have to determine the probability that we are in state \((r_0, r_1, \ldots, r_T)\) for all possible values of \(r_0, r_1, \ldots, r_T\). Nevertheless estimating the average costs per period is quite simple as we will see in Section 6. If the number of possible left-demands is very large, a good alternative is offered by the following simplification of this strategy.

5. The rule-strategy

If the residual demand does not deviate much from the expected demand, the result of the \((x, T)\)-strategy will be the optimal action. However, in cases of an unexpectedly high or low demand the \((x, T)\)-strategy may choose a non-optimal action. For these cases we developed a number of rules that also cover most of the improvements that can be reached by the salvage-strategy. In this strategy the decisions are based on the assumption that in later periods we will follow the optimal \((x, T)\)-strategy.
5.1. Producing sooner in case of low holding cost

If $r_0 < x$ it may be interesting to produce if the producing costs are low. To determine whether it is preferable to produce, we compare the costs of producing in the current period and producing in the next period. If $r_0 + r_1 \geq x$ then it is correct to compare producing the current period or the next period, because following the $(x,T)$-strategy for later periods means that indeed we will produce in one of these periods. If $r_0 + r_1 < x$ then comparing these options is not completely correct. However, if producing in the current period is optimal for $r_1 = x - r_0$, then it will also be optimal for smaller $r_1$.

Let $g^*$ denote the costs $g(x,T)$ for the optimal pair $(x,T)$. Then the expected costs of the options producing during the current period (1) and producing during the next period (2) are given by:

1. $K + \sum_{i=1}^{T-1} (r_i - e_i) + (x - r_0 - r_1)^+$.
2. $p r_0 + K + \sum_{i=1}^{T-1} (i-1)(r_i - e_i) - g^*$.

This leads to the following rule:

**Rule 1:** We produce if $r_0 < x$ and $\sum_{i=1}^{T-1} (r_i - e_i) < p r_0 - g^* - (x - r_0 - r_1)^+$.

5.2. Postponing production in case of high holding costs

Sometimes it may be better to wait with production because the holding costs will be very high. If $r_0 \geq x$ and we do not produce during this period, we will certainly produce during the next period. Therefore we have the same options as in Rule 1, with expected costs:

1. $K + \sum_{i=1}^{T-1} (r_i - e_i)$.
2. $p r_0 + K + \sum_{i=1}^{T-1} (i-1)(r_i - e_i) - g^*$.

This leads to the second rule:

**Rule 2:** We do not produce if $r_0 \geq x$ and $\sum_{i=1}^{T-1} (r_i - e_i) > p r_0 - g^*$.

5.3. Produce for less than $T$ periods in case of high holding costs

Sometimes it will be better to produce for less than $T$ periods, for instance if $r_{T-1}$ is very high. Therefore we have to compare the options: produce for $T$ periods or produce for $T-z$ periods, ($z = 1,2,...,T-1$). To avoid the use of the marginal cost function $L(\cdot)$, we assume that if we produce for $T-z$ periods, the next time we will produce will be $T-z$ periods later. In this way we overestimate the costs of the second option, but now we can use the penalty-function $p(e)$, as defined in Dellaert (1987):
This penalty-function gives the expected penalty-costs during the next \( a - 1 \) periods, if we produce for \( a \) periods. Then the expected costs of the options 'producing for \( T \) periods' (1) and 'producing for \( T - z \) periods' (2) are:

1. \( K + \sum_{i=1}^{T-1} (r_i - e_i) i \).
2. \( K + \sum_{i=1}^{T-z-1} (r_i - e_i) i + p(T-z) + \sum_{i=1}^{T-z} i (r_{T-z+i} - e_{T-z+i}) - (T-z) g' - \sum_{i=T-z}^{T-1} i e_i \).

This leads to the third rule:

**Rule 3:** We produce only for \( T - z \) periods if

\[ p(T-z) + K - \sum_{i=1}^{T-z} i e_{T-z+i} < (T-z) (g' + \sum_{i=T-z}^{T-1} (r_i - e_i)) \]

*Note:* It is of course possible that Rule 3 holds for several values of \( z \). In that case we choose that \( z \) for which the difference between the right-hand term and the left-hand term is maximal.

In general these first three rules will cover most of the improvements of the salvage-strategy. Now we will give two more rules that usually offer little improvements.

### 5.4. Postponing production as an alternative for producing less than \( T \) periods

If Rule 3 indicates to produce less than \( T \) periods, we cannot use Rule 2 to postpone production. Now we have to compare the costs of producing for \( T-z \) periods (5.3.2) and producing the next period (5.2.2), in which case we assume that during the next period we produce the demand for \( T \) periods. This leads to the following rule:

**Rule 4:** We produce for \( T-z \) periods if

\[ p r_0 + (T-z-1) g' + (T-z) \sum_{i=T-z}^{T-1} r_i > K + \sum_{i=1}^{T-z} (r_i - e_i) + p(T-z) - \sum_{i=T-z}^{T-1} i e_{T-z+i} \]
and

\[ (2) \quad K + p r_0 - g^* + \sum_{i=2}^{T-1} (i-1) (r_i - e_i). \]

The fifth rule becomes:

**Rule 5:** We do not produce if Rule 1 holds but

\[ \sum_{i=1}^{T-1} (r_i - e_i) > p r_0 - g^* + (T-1)(r_{T-1} - e_{T-1}) - L^*(r_T) \]

and we do produce if Rule 2 holds but

\[ \sum_{i=1}^{T-1} (r_i - e_i) < p r_0 - g^* + (T-1)(r_{T-1} - e_{T-1}) - L^*(r_T) \]

In general we will deviate from Rule 1 if \( r_T \) is very small and deviate from Rule 2 if \( r_T \) is very big. Of course, these rules can not be tested in a random sequence. Therefore we will now give the algorithm of this rule-strategy.

5.6. Algorithm of the rule-strategy

Before we can start using the algorithm, we have to determine the optimal pair \((x, T)\) and the corresponding average costs \(g^* = g(x, T)\). The algorithm consists of the following 6 steps:

**Step 0:** if \( r_0 \leq \frac{x - \sum_{i=1}^{T-1} e_i}{p + 1} \) we will never produce, else **Step 1.**

**Step 1:** Test Rule 3. If we produce \( T \) periods rather than \( T - z \) periods then **Step 2** else **Step 5.**

**Step 2:** Test Rule 1. If this rule holds then **Step 4** else **Step 3.**

**Step 3:** Test Rule 2. If this rule holds then **Step 4** else we produce for \( T \) periods (stop).

**Step 4:** Test Rule 5 and produce according to this rule (stop).

**Step 5:** Test Rule 4 and produce for \( T - z \) periods if this rule holds (stop).

Every period we start with **Step 0** and continue, if necessary, until a decision is reached.

6. Estimation of the improvement

In Dellaert (1987) we have seen that the average costs of the \((x, T)\)-strategy usually are less than a few percent higher than those of the optimal strategy. The costs of the rule-strategy will be somewhere inbetween the costs of both strategies. Therefore we can use the rule-strategy without knowing the exact average costs per period. If we really are interested in the average costs we have two options: we can determine the average costs exactly or we can use an estimation for the improvement upon the \((x, T)\)-strategy. Calculating the average costs may be as complex as calculating the average costs of the Silver-Meal strategy and thereby not very interesting. A good estimation of the improvement compared with the \((x, T)\)-strategy can be obtained by the following technique: determine the probability of all possible demands \((r_0, r_1,..., r_T)\) by using the steady-state probabilities.
\( g_{ij} \) as determined in (2.2) until (2.7). Now the gain, if we deviate from the \((x,T)\)-strategy, is the positive difference between the right-hand term and the left-hand term of the rule(s) that gives us the deviation from the \((x,T)\)-strategy. The gain and probability are multiplied and summed over all possible demands with probability greater than some \( \epsilon \).

6.1. Estimating the average cost of the Silver-Meal like strategy

This method of estimating the average costs can also be used for other strategies. The average costs of the salvage-strategy can be estimated by summing the improvements upon the \((x,T)\)-strategy, where the improvements are based the probability of a certain demand, determined by the \((x,T)\)-strategy, and the difference in costs using the \(L(.)\)-function. We can also estimate the costs of the Silver-Meal like strategy in this way.

In this Silver-Meal like strategy, we divide the expected costs of an action by the number of periods involved in this action (see Dellaert (1987)). The direct cost of action \( a \) are given by \( q^a \), as defined in (1.1) and (1.2). Under the assumption that the next production will take place \( a \) periods after the current period, the expected penalty-costs are given by \( p(a) \), the penalty-function defined in (5.3.1). Using this penalty function, \( q^a \) for the direct costs and \( \delta(a) \) as a function that equals 1 for \( a = 0 \) and 0 elsewhere, the strategy takes the following form:

if we observe a state \( r \in R \), we take that action \( a \) for which

\[
\frac{q^a + p(a)}{a + \delta(a)}
\]

is minimal over \( a \in \{0,1,...,N\} \).

As a first estimate for the average costs of this Silver-Meal like strategy, we have the average costs of the \((x,T)\) strategy, and for those demands where the Silver-Meal like strategy is different from the \((x,T)\) strategy, we use the difference in costs using the \(L(.)\) function. This estimation of costs is much more easier than calculating the average costs and the accuracy is surprisingly good. In the next section we will give a few examples, where we compare the real and the estimated costs of the rule-strategy, the salvage-strategy and the Silver-Meal like strategy.

7. Numerical results

To compare the real and the estimated costs of the different strategies we use the same example as in Dellaert (1987). We will consider the following costs:

-\( OPT \), the average costs of the optimal strategy;
-\( XT \), the average costs of the optimal \((x,T)\)-strategy;
-\( SS \), the average costs of the salvage-strategy;
-\( SSE \), the estimated average costs of the salvage-strategy;
-RU  , the average costs of the rule-strategy;
-RUE  , the estimated average costs of the rule-strategy;
-SM  , the average costs of the Silver-Meal like strategy;
-SME  , the estimated average costs of the Silver-Meal like strategy.

In the set of examples we consider, we have $N = 4, p = 3, h = 1, d_i = 1 - d_{i0} = c$ for $i = 1, \ldots, 4$. For some different values of $c$ and $s$ we have the following results:

<table>
<thead>
<tr>
<th>c-value</th>
<th>0.25</th>
<th>0.25</th>
<th>0.50</th>
<th>0.50</th>
<th>0.75</th>
<th>0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>s-value</td>
<td>3.25</td>
<td>8.00</td>
<td>6.50</td>
<td>16.00</td>
<td>9.75</td>
<td>24.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>strategy</th>
<th>real or estimated average costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPT</td>
<td>1.8895 3.7147 4.5357 8.1705 7.0425 12.6002</td>
</tr>
<tr>
<td>SS</td>
<td>1.8896 3.7148 4.5357 8.1705 7.0426 12.6002</td>
</tr>
<tr>
<td>SSE</td>
<td>1.8894 3.7153 4.5356 8.1693 7.0424 12.6002</td>
</tr>
<tr>
<td>RU</td>
<td>1.8913 3.7148 4.5357 8.1732 7.0426 12.6002</td>
</tr>
<tr>
<td>RUE</td>
<td>1.8908 3.7153 4.5356 8.1725 7.0424 12.6001</td>
</tr>
<tr>
<td>SM</td>
<td>1.9002 3.7173 4.5392 8.1793 7.0445 12.6054</td>
</tr>
<tr>
<td>SME</td>
<td>1.9002 3.7177 4.5392 8.1790 7.0445 12.6054</td>
</tr>
</tbody>
</table>

Table 7.1 The real and estimated costs of the strategies in some examples.

Literature


