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CHARACTERIZATIONS OF SHIFT-INVARIANT DISTRIBUTIONS BASED ON SUMMATION MODULO ONE.

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The Netherlands
Summary

For \( n \in \mathbb{N} \), let \( X, Y_1, \ldots, Y_n \) be independent random variables, and suppose that \( X \) is distributed in \([0,1)\), but not uniformly. We characterize the distributions of \( X \) and \( Y_s \) \((s=1, \ldots, n)\) satisfying the equation \( \{X+Y_1+\ldots+Y_n\} \overset{d}{=} X \), where \( \{Z\} \) denotes the fractional part of a random variable \( Z \).

In the case of full generality, \( Y_s \) is lattice, and \( X \) is shift-invariant with respect to a discrete uniform distribution on \([0,1)\). We also give a characterization of such shift-invariant distributions.

In addition, we consider some special cases of this equation: If \( X \overset{d}{=} Y_1 \), then \( X \) has a shifted discrete uniform distribution on \([0,1)\); further the case that \( Y_1, \ldots, Y_n \) are identically distributed, and a generalization of the equation with \( X, Y_1, \ldots, Y_n \) identically distributed is considered. Our results generalize results of Goldman (1968) and of Arnold and Meeden (1976).

Key words and phrases: Fourier-Stieltjes coefficients; distribution modulo 1; fractional parts.
1. Introduction.

Let $X$ be a random variable (rv). Let $\{X\} := X - \lfloor X \rfloor$ denote the fractional part of $X$, and $\lfloor X \rfloor$ the integer part of $X$. Throughout this paper $\mathbb{Z} \cdot \xi$ denotes the set $\{j\xi: j \in \mathbb{Z}\} (\xi \in \mathbb{R})$. Furthermore, $U$ denotes a rv with (continuous) uniform distribution on $[0,1)$; for $m \in \mathbb{N}$, $U_m$ denotes a rv with discrete uniform distribution on $[0,1)$, i.e.,

$$
P(U_m = \frac{j}{m}) = \frac{1}{m} \quad (j=0,1,\ldots,m-1).$$

If $X \overset{d}{=} U_m + \beta$ for some $0 \leq \beta < \frac{1}{m}$, then $X$ is called a shifted discrete uniform distribution on $[0,1)$; if $X$ is independent of $U_m$, and $X \overset{d}{=} \{X + U_m\}$, then $X$ is called $U_m$-shift-invariant. Here ‘$d$’ denotes equality in distribution.

In Goldman (1968), and Arnold and Meeden (1976) equations of the form

$$(1) \quad \{X + Y\} \overset{d}{=} X,$$

are studied, where $X$ and $Y$ are independent. It is of some interest to seek solutions of (1), where ‘solution’ means the determination of all distributions of the pair $(X,Y)$ for which (1) is true.

Arnold and Meeden prove the following result: If in (1) $Y$ does not have its distribution concentrated on $\mathbb{Z} \cdot \frac{1}{m}$ for some $m \in \mathbb{N}$, then $X \overset{d}{=} U$. Goldman considers the case that $X$ and $Y$ are identically distributed. Using topological groups he shows that in this case $U$ and $U_m$ are the only solutions of (1). However, he explicitly asks whether it is possible to prove this result by elementary means. In this paper we prove Goldman’s statement by elementary means. We also study the following generalization of (1): Let $n \in \mathbb{N}$. Let
Let $X, Y_1, \ldots, Y_n$ be independent rv's, and suppose that

\begin{equation}
\{X + Y_1 + \ldots + Y_n\} \overset{d}{=} X.
\end{equation}

In Section 2 we give properties of Fourier-Stieltjes Sequences (FSS's). These FSS's are a useful 'elementary' tool for studying distributions on $[0,1)$.

In Section 3, after an auxiliary result in Lemma 4, we characterize in Theorem 1 $U_m$-shift-invariant distributions. In Theorem 2 we consider equation (2) in full generality, and characterize the distributions of $X$ and $Y_s$ ($s=1, \ldots, n$).

Furthermore, in four corollaries we study special cases of equation (2). In Corollary 1 we consider the case $U \not\overset{d}{=} X \overset{d}{=} Y_1$; then $X$ has a shifted discrete uniform distribution on $[0,1)$. In Corollary 2 we consider the case that $Y_1, \ldots, Y_n$ are independent and identically distributed (iid); Corollary 3 with $n=1$, i.e. equation (1) studied by Arnold and Meeden, identifies the distributions of $X$ and $Y$. In Corollary 4 we study the equation

\begin{equation}
\{X_0 + \ldots + X_n + \alpha\} \overset{d}{=} X_0,
\end{equation}

where $X_0, \ldots, X_n$ are iid rv's, and $\alpha \in \mathbb{R}$; the case $n=1$ and $\alpha=0$ shows that $U$ and $U_m$ are the only solutions of equation (1) studied by Goldman, thereby answering his question.
2. Properties of Fourier-Stieltjes Sequences.

We recall the definition of the FSS of a rv.

**Definition.** Let $X$ be a rv. The FSS $c_X := \left( c_X(k) \right)_{k=-\infty}^{\infty}$ of $X$ is defined by

$$c_X(k) = \mathbb{E} e^{2\pi i k X} = \int_{-\infty}^{\infty} e^{2\pi i k x} dF_X(x) \quad (k \in \mathbb{Z}),$$

where $F_X$ denotes the distribution function of $X$.

Clearly $c_X(0) = 1$, $|c_X(k)| \leq 1$, and $c_X(-k) = \overline{c_X(k)}$ $(k \in \mathbb{Z})$.

Since $e^{2\pi i k x} = e^{2\pi i k}$ $(x \in \mathbb{R})$, we have for any rv $X$ the trivial but useful identity

$$\varphi_X(2\pi k) = \mathbb{E} e^{2\pi i k X} = c_X(k) \quad (k \in \mathbb{Z}),$$

where $\varphi_X$ denotes the characteristic function of $X$.

Next, we state the uniqueness and continuity theorems for FSS's. For the proofs we refer to Zygmund (1968).

**Proposition 1.** Let $X, Y$ be rv's. Then

$$c_X = c_Y \text{ iff } \{X\} \overset{d}{=} \{Y\}.$$

**Proposition 2.** Let $X, X_1$, and $X_2$ be rv's with $X_1$ and $X_2$ independent. Further let $c, c_1,$ and $c_2$ be the corresponding FSS's. Then

$$\{X\} \overset{d}{=} \{X_1 + X_2\} \text{ iff } c(k) = c_1(k) \cdot c_2(k) \quad (k \in \mathbb{Z}).$$

We shall need the following simple lemma.
Lemma 1. (i) $c_u(k)=0$ for all $k \neq 0$.

(ii) $c_{u(m)}(k)=\begin{cases} 1 & \text{if } k=0 \mod m \\ 0 & \text{otherwise.} \end{cases}$

Feller (1971) gives the following characterization of a lattice distribution.

Lemma 2. Let $X$ be a rv, and let $\lambda \neq 0$. Then the following statements are equivalent:

(i) $\varphi_X(\lambda)=1$.

(ii) $\varphi_X$ has period $\lambda$, i.e. $\varphi_X(\xi+\lambda)=\varphi_X(\xi)$ ($\xi \in \mathbb{R}$).

(iii) $X$ has its distribution concentrated on $\mathbb{Z} \cdot \frac{2\pi}{\lambda}$.

We now give a characterization of a lattice distribution in terms of FSS’s.

Lemma 3. Let $X$ be a rv, $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Then the following statements are equivalent:

(i) $c_X(m)=\exp(2\pi i \beta)$.

(ii) $c_X(k+m)=\exp(2\pi i \beta)c_X(k)$ ($k \in \mathbb{Z}$).

(iii) $X$ has its distribution concentrated on $\beta/m + \mathbb{Z} \cdot \frac{1}{m}$.

Proof: Using (3), (4) and lemma 2 we find the following equivalent statements:

\[ c_X(m)=\exp(2\pi i \beta); \]
\[ \mathbb{E}e^{2\pi im(X-\beta/m)}=1; \]
\[ \varphi_{X-\beta/m}(2\pi m)=1; \]
\[ X-\beta/m \text{ has its distribution concentrated on } \mathbb{Z} \cdot \frac{1}{m}; \]
\[ X \text{ has its distribution concentrated on } \beta/m + \mathbb{Z} \cdot \frac{1}{m}. \]
Thus (i)\iff(iii).

Furthermore, from $\varphi_{X+\beta_{2m}}(2\pi m) = 1$ it follows by lemma 2 that

$$\varphi_{X+\beta_{2m}}(2\pi m + 2\pi k) = \varphi_{X+\beta_{2m}}(2\pi k) \quad (k \in \mathbb{Z}),$$

whence

$$c_{X}(k+m) = \exp(2\pi i\beta) c_{X}(k) \quad (k \in \mathbb{Z}).$$

Thus (i)\Rightarrow(ii). Taking $k=0$ in (ii) we get (i), hence (ii)\Rightarrow(i).

\[\square\]

3. The main results.

In this section we characterize the distributions of $X$ and $Y_s$ ($s=1, \ldots, n$) satisfying equation (2). We start by giving an auxiliary result.

Lemma 4. (i) Let $Y$ be a rv independent of $U$. Then

$$\{U+Y\} \overset{d}{=} U.$$

(ii) Let $m \in \mathbb{N}$. Let $X$ and $Y$ be rv's such that the pair $(X, Y)$ is independent of $U_m$ and let $Y$ have its distribution concentrated on $\mathbb{Z}/m$. Then

$$\{X+Y+U_m\} \overset{d}{=} \{X+U_m\}.$$  

Proof: (i) By proposition 2 and Lemma 1(i) we have

$$c_{Y+U_{m}}(k) = c_{U}(k)c_{Y}(k) = c_{U}(k) \quad (k \in \mathbb{Z}),$$

which is equivalent with $\{U+Y\} \overset{d}{=} U$.

(ii) For all $k \in \mathbb{Z}$ with $k \neq 0$ (mod $m$) we have by Lemma 1(ii)

$$c_{X+Y+U_{m}}(k) = c_{X+Y}(k)c_{U_{m}}(k) = 0,$$
and for \( k = 0 \pmod{m} \) we have \( kY \in \mathbb{Z} \), whence

\[
    c_{X+Y+U_m}(k) = c_{X+Y}(k)c_{U_m}(k) = c_{X+Y}(k) = E e^{2\pi i kX} = c_X(k).
\]

So \( c_{X+Y+U_m} \) does not depend on \( Y \), and is therefore equal to \( c_{X+U_m} \).

By proposition 1 this implies \( \{X+Y+U_m\} \overset{d}{=} \{X+U_m\} \).

**Remark.** In part (ii) of this lemma \( X \) and \( Y \) need not be independent. This leads to the following result, which is of some interest by itself:

Let \( Z \) be a rv independent of \( U_m \). Then \( \frac{1}{m}[Z] \) has its distribution concentrated on \( \mathbb{Z} \cdot \frac{1}{m} \), and from part (ii) we find

\[
    \{U_m + \frac{1}{m}Z\} = \{U_m + \frac{1}{m}(Z) + \frac{1}{m}[Z]\} \overset{d}{=} \{U_m + \frac{1}{m}(Z)\} = \{U_m + \frac{1}{m}(Z)\}.
\]

In the following theorem we characterize \( U_m \)-shift-invariant distributions.

**Theorem 1.** Let \( m \in \mathbb{N} \), and let \( X \) have its distribution concentrated on \([0,1)\) and let \( X \) be independent of \( U_m \). Then the following statements are equivalent:

(i) \( c_X(k) = 0 \) if \( k \neq 0 \pmod{m} \).

(ii) \( \{X+U_m\} \overset{d}{=} X \).

(iii) \( X \overset{d}{=} U_m + \frac{1}{m}[Z] \) for some \( Z \) independent of \( U_m \).

**Proof:** Proof of \((i) \Rightarrow (ii)\): From Lemma 1(ii) we have

\[
    c_{X+U_m}(k) = c_X(k)c_{U_m}(k) = \begin{cases} c_X(k) & \text{if } k = 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases}.
\]
Hence \( c_{X+U} (k) = c_X (k) \) (\( k \in \mathbb{Z} \)), and by proposition 1 \( \{X+U_m\} \overset{d}{=} \{X\} \overset{d}{=} X \).

Thus (ii) holds.

**Proof of (ii) \( \Rightarrow \) (iii):** Since \( \frac{1}{m}[mX] \) has its distribution concentrated on \( \mathbb{Z} \) we have from Lemma 4(ii)

\[
X \overset{d}{=} \{X+U_m\} \overset{d}{=} \left\{ \frac{1}{m}[mX] + \frac{1}{m}[mX] + U_m \right\} \overset{d}{=} \left\{ \frac{1}{m}[mX] + U_m \right\} \overset{d}{=} U_m + \frac{1}{m}(mX).
\]

Since \( X \) is independent of \( U_m \), this is also true for \( (mX) \). Choosing \( Z:=mX \) yields \( X \overset{d}{=} U_m + \frac{1}{m}(Z) \). Thus (iii) holds.

**Proof of (iii) \( \Rightarrow \) (i):** From Lemma 2(ii) we find for all \( k \neq 0 \) (mod \( m \))

\[
c_{X}(k) = c_{U_m}(k) c_{\frac{1}{m}Z}(k)=0.
\]

Thus (i) holds. \( \square \)

Next, we prove the main result of this paper.

**Theorem 2.** Let \( n \in \mathbb{N} \), and let \( X, Y_1, \ldots, Y_n \) be independent rv's with \( X \overset{d}{=} U \). Then the following statements are equivalent:

(i) \( \{X+Y_1+\ldots+Y_n\} \overset{d}{=} X \).

(ii) There exist \( m \in \mathbb{N}, \beta_1, \ldots, \beta_n \in \mathbb{R}, \) and a rv \( Z \) independent of \( U_m \) such that \( Y_s \) has its distribution concentrated on \( \beta_s/m + Z - \frac{1}{m} \) (\( s=1, \ldots, n \)), \( \sum_{s=1}^{n} \beta_s \in \mathbb{Z} \), and \( X \overset{d}{=} U_m + \frac{1}{m}(Z) \).

**Proof:** Let (i) hold. Then \( c_X (k) = c_{X+Y_1+\ldots+Y_n} (k) = c_X (k) \prod_{s=1}^{n} c_{Y_s} (k), \) so

\[
c_X (k) \left( 1 - \prod_{s=1}^{n} c_{Y_s} (k) \right) = 0 \quad (k \in \mathbb{Z}).
\]

Since \( X \overset{d}{=} U \), we have \( c_X (k) \neq 0 \) for some \( k \in \mathbb{N} \), and therefore
Now let $m \in \mathbb{N}$ be such that

$$
\prod_{k=1}^{c_{Y_s}(k)} = 1.
$$

Then, since $|c_{Y_s}(m)| \leq 1$, we even have $|c_{Y_s}(m)| = 1$ for all $s$, and it follows that there exist $\beta_1, \ldots, \beta_n \in \mathbb{R}$ such that $c_{Y_s}(m) = \exp(2\pi i \beta_s)$ ($s=1, \ldots, n$) and $\sum_{s=1}^{n} \beta_s \in \mathbb{Z}$. Lemma 3 then yields that $Y_s$ has its distribution concentrated on $\beta_s/m + \mathbb{Z} \cdot \frac{1}{m}$ and that $c_{Y_s}(k+m) = \exp(2\pi i \beta_s) c_{Y_s}(k)$ ($k \in \mathbb{Z}, s=1, \ldots, n$). Hence

$$
\prod_{s=1}^{m} c_{Y_s}(k+m) = \prod_{s=1}^{m} c_{Y_s}(k) \quad (k \in \mathbb{Z}).
$$

From (6) we obtain $\prod_{s=1}^{m} c_{Y_s}(k) = 1$ if $k = 0 \pmod{m}$, and $\prod_{s=1}^{m} c_{Y_s}(k) \neq 1$ if $k \neq 0 \pmod{m}$; from this, using (5), we find $c_X(k) = 0$ for all $k \neq 0 \pmod{m}$. Theorem 1 now implies that there is a rv $Z$ independent of $U_m$ such that $X \overset{d}{=} U_m + \frac{1}{m}[Z]$. So (i)$\implies$(ii) has been proved.

Now let (ii) hold, and write $S = Y_1 + \ldots + Y_n$. Then $S$ has its distribution concentrated on $\mathbb{Z} \cdot \frac{1}{m}$. Applying Lemma 4(ii), and Theorem 1 twice we have

$$
\{X+S\} \overset{d}{=} \{X+U_m' + S\} = \{X+U_m'+S\} \overset{d}{=} \{X+U_m'\} \overset{d}{=} X,
$$

where $U_m'$ is a rv with discrete uniform distribution on $[0,1)$ independent of the pair $(X,S)$. So (ii)$\implies$(i) has been proved.

Clearly, from Lemma 4(i) we have that, for any $Y_1, \ldots, Y_n$, $X \overset{d}{=} U$ is a solution of equation (2). We now state some immediate consequences of Theorem 2.

**Corollary 1.** Let $n \in \mathbb{N}$, and let $X, Y_1, \ldots, Y_n$ be independent rv's such that $U \overset{d}{=} X \overset{d}{=} Y_1$, and $\{X+Y_1+\ldots+Y_n\} \overset{d}{=} X$. Then there exist $m \in \mathbb{N}$ and
\[ \beta \in (0,1) \text{ such that } X \overset{d}{=} U_m + \beta/m. \]

**Corollary 2.** Let \( n \in \mathbb{N} \), and let \( X \) be a rv with \( X \overset{d}{=} U \), and \( Y_1, \ldots, Y_n \) iid rv's independent of \( X \). Then the following statements are equivalent:

(i) \( \{X+Y_1+\ldots+Y_n\} \overset{d}{=} X. \)

(ii) There exist \( m \in \mathbb{N} \), \( p \in \mathbb{Z} \), and a rv \( Z \) independent of \( U_m \) such that \( Y_1 \) has its distribution concentrated on \( p/(nm) + Z \cdot \frac{1}{m} \) and \( X \overset{d}{=} U_m + \frac{1}{m}(Z) \).

The following corollary is the case \( n=1 \) of Corollary 2, and solves equation (1) studied by Arnold and Meeden.

**Corollary 3.** Let \( X \) and \( Y \) be independent rv's and \( X \overset{d}{=} U \). Then the following statements are equivalent:

(i) \( \{X+Y\} \overset{d}{=} X. \)

(ii) There exist \( m \in \mathbb{N} \) and a rv \( Z \) independent of \( U_m \) such that \( Y \) has its distribution concentrated on \( Z \cdot \frac{1}{m} \) and \( X \overset{d}{=} U_m + \frac{1}{m}(Z) \).

**Corollary 4.** Let \( n \in \mathbb{N} \), \( \alpha \in \mathbb{R} \), and let \( X_0, \ldots, X_n \) be iid rv's with \( X_0 \overset{d}{=} U \). Then the following statements are equivalent:

(i) \( \{X_0+X_1+\ldots+X_n+\alpha\} \overset{d}{=} X_0. \)

(ii) There exist \( m \in \mathbb{N} \) and \( s \in \{0, \ldots, n-1\} \) such that \( X_0 \overset{d}{=} U_m + \frac{1}{m}(s \cdot \frac{\alpha}{n}) \).

The special case \( n=1 \) of this corollary is the result proved by...
We now give some simple examples to illustrate the scope of these results.

Examples.

(a) Let \( n \in \mathbb{N} \). Let \( X \overset{d}{=} U_2 + \frac{1}{3} \), and let \( Y_1 \) be a rv with its distribution concentrated on the points \( \frac{1}{2n} \) and \( \frac{1}{2n} + \frac{1}{2} \). Let further \( Y_1, \ldots, Y_n \) be iid rv's independent of \( X \). Then \( \{Y_1 + \ldots + Y_n\} \) has its distribution concentrated on 0 and \( \frac{1}{2} \), and \( \{X + Y_1 + \ldots + Y_n\} \overset{d}{=} X \).

This also follows from Corollary 2 by taking a rv \( Z \) with \( \Pr(\{Z\} = \frac{2}{3}) = 1 \), and independent of \( U_2 \).

(b) Let \( k \in \mathbb{N} \), \( k \geq 3 \). Let \( (X_j)_{j=0}^{\infty} \) be a sequence of iid rv's with \( X_0 \overset{d}{=} U_2 + \frac{1}{k} \), and let \( S_n = \{X_0 + \ldots + X_n\} \) (\( n \in \mathbb{N} \)). We distinguish two cases: 1. \( k \) is even; 2. \( k \) is odd.

Then we find

1. Corollary 4 implies \( S_n \overset{d}{=} X_0 \) if \( n = 0 \pmod{\frac{1}{2}k} \) (i.e. \( \frac{1}{2}ks = 2n \)).

Furthermore, we have

\[
S_n \overset{d}{=} U_2 + \frac{j+1}{k} \quad \text{if} \quad n = j \pmod{\frac{1}{2}k} \quad \text{for some} \quad j \in \{0, 1, \ldots, \frac{1}{2}k-1\}.
\]

If we define \( \alpha_n = \frac{j}{k} \) if \( n = j \pmod{\frac{1}{2}k} \) for some \( j \in \{0, 1, \ldots, \frac{1}{2}k-1\} \), then we have

\[
X_0 \overset{d}{=} \{S_n - \alpha_n\} \quad (n \in \mathbb{N}).
\]

2. Similarly, we obtain \( S_n \overset{d}{=} U_2 + \frac{j+2}{2k} \) if \( n = s \pmod{k} \) for some \( s \in \{0, \ldots, k-1\} \), and \( j = 2s \) if \( s \leq \frac{1}{2}k-1 \), \( j = 2s+k \) if \( s > \frac{1}{2}k-1 \).
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6. References.

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