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State reduction in a dependent demand inventory model given by a time series

by

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Introduction

In practice we are very often confronted with dynamic inventory problems of dependent demand. However, until now not much work has been done on this subject. Usually, stochastic models assume that demand distributions of inventory in different periods are independent. Recently more attention has been given to dependent demand inventory models e.g. Blinder (1982), and Miller (1986). Usually the exponential smoothing forecasting model is considered. In this paper we consider a more general model for a dependent demand which is given by a time series. The time series will be given by a Stochastic Trend Component and a Seasonal Component. A Polynomial Trend Component and a Trading Day Effect will be neglected in the considered time series model.

Consideration of dependent demand leads to addition of a second state variable pertaining to demand in the dynamic programming formulation. The first state is as usual the inventory level. Obviously, taking into account the dependent demand, greatly increases the computational complexity of dynamic programming algorithms. Therefore the main attempt is to reduce the two-state variable dynamic programming problem under assumed demand and cost structure. The method of Scarf (1960) and Karlin (1960) which reduces a two-state variable Bayesian inventory model to a one-state variable model when demand is from exponential and range families can be used to solve the problem. Auzory (1985) has extended this result to the cases where demand is uniform and Weibull. Miller (1986) uses the same concept for the demand described by the exponential smoothing formula. The same approach will be used in this paper for the single item periodic review inventory problem with possible returns of a part of demand in the next period.
1. The model

Consider the single item, periodic review, dependent demand inventory model. Demands in each period are given by a time series

\[ q(n) = \theta q(n-1) + Y(n) + \varepsilon(n), \quad |\theta| < 1, \]  

(1)

where the first and last term are due to a Stochastic Trend Component given by the simple AR model and the second term is due to a long term seasonality (e.g. yearly), updated by one of known methods e.g. Holt et al. (1960) after each control horizon \( N \) (e.g. at the beginning of a new year), and \( \theta \) - the unknown parameter.

\( Y(n) \) - the average demand of the appropriate month one year ago.
\( Y(n) \equiv Z([n]) \cdot [n] \) - the appropriate month number.
\( [n] \equiv \left\lfloor \frac{n-1}{L} \right\rfloor + 1 \), \( |w| \) - the greatest integer less than \( w \), \( L \) - the number of days in a month.
\( \varepsilon(n) \) - i.i.d. in \( N(0, \sigma^2) \), \( \sigma^2 \) - the known variance.

We assume a finite planning horizon of \( N \) periods, and linear ordering, holding and shortage costs, and zero set-up costs. Namely

\begin{align*}
\text{c}(u) &= \begin{cases} 
0 & \text{if } u \leq 0 \\
\text{c}u & \text{if } u > 0
\end{cases} \quad \text{ordering costs.} \\
\text{h}(x - q) & \quad x - q \geq 0 \quad \text{shortage costs.} \\
\pi(q - x) & \quad x - q < 0 \quad \text{holding costs.}
\end{align*}

where

\( x(n) \in R \) - the initial inventory in period \( n \).
\( q(n) \in R \) - the demand in period \( n \).
\( u(n) \in R \) - the order in period \( n \).

We will denote

\( y(n) \in R \) - the inventory level + order in period \( n \) but before demand.

Holding and shortage costs are based on the amount of inventory at the end of each period. Future costs are discounted by a one-period discount factor \( \alpha, 0 < \alpha < 1 \). Excess demand in each period is completely backlogged and a time lag between ordering and delivery is assumed zero.

In most practical applications we possess a satisfactory knowledge about past demands. The history \( H_{m,n} \) of demands is available at the beginning of the period \( n + 1 \)

\[ H_{m,n} = \{ q(-m+1), q(-m+2), \ldots, q(0), q(1), \ldots, q(n) \} \]  

(2)

where \( m \) is the number of past demands occurred till the beginning of the control process. From now on we will denote: \( t = m+n-1 \), \( H(t) \equiv H_{m,n} \).
2. Dynamic programming formulation

Let $V_n(x, \mu(n))$ denote the expected value of discounted costs from the period $n$ to the end of the horizon, where the initial inventory level is $x$, and an optimal ordering policy is followed, and $\mu(n)$ is the time varying expected value of demand in the period $n$

$$\mu(n) = E(q(n)) = \theta q(n-1) + Y(n)$$

(3)

where $E$ is the expected value operator.

Notice that from (1) and (3)

$$\mu(n+1) = \theta \mu(n) + Y(n+1) + \theta \epsilon(n).$$

(4)

where $Y(n+1), n = 1,...,N$ are given numbers.

By (4), $V_n(x, \mu(n))$ satisfies the following equation

$$V_n(x, \mu(n)) = \min_{y \geq x} \left\{ c(y-x) + L_n(y, \mu(n)) \right\}$$

$$+ \alpha \int_{-\infty}^{\infty} V_{n+1}(y-q, \mu(n+1)) df(q | \mu(n))$$

(5)

with

$$V_{N+1}(x, \mu(N+1)) \equiv V_{N+1}(x) \equiv \begin{cases} k_1x & \text{if } x \geq 0 \\ k_2x & \text{if } x < 0 \end{cases}, \quad k_2 \geq k_1 > 0$$

(6)

where $k_1$ and $k_2$ are given numbers and

$$L_n(y, \mu(n)) = \begin{cases} \int h(y-q) df(q | \mu(n)) + \int_{-\infty}^{\infty} \pi(q-y) df(q | \mu(n)) & , y \geq 0 \\ \int_{-\infty}^{\infty} \pi(q-y) df(q | \mu(n)) & , y < 0 \end{cases}$$

(7)

is the expected one-period holding and shortage cost.

We will prove now that equations (5) - (7) are valid with respect to the normal distribution of demand with a mean $\mu(n)$ and variance $\sigma^2$. To do this let us first estimate the unknown parameter $\theta$ using the maximum likelihood method. With accuracy to constants we have

$$\ln L(\theta) = \sum_{l=1-m}^{n} [q(l) - \theta q(l-1) - Y(l)]^2$$

(8)

where $L(\theta)$ is the maximum likelihood function for $\theta$, based on the history $H_{m,n}$.

The problem of maximizing $L(\theta)$ is equivalent to a least-squares problem of minimizing (8) with respect to
According to Ljung and Söderström (1983), Chapter 2.2.1, the estimate $\theta(n)$ of $\theta$ in period $n$ is

$$\theta(n) = \theta(n-1) + K(n) \left[ q(n) - Y(n) - \theta(n-1)q(n-1) \right]$$

(10)

where $K(n)$ is a gain given by a recursive equation.

In the following we use the form (8) of $\ln L(\theta)$ and properties of consistent solutions of the likelihood equation

$$\frac{d \ln L(\theta)}{d \theta} = 0.$$  

Let us recall the well known result, which holds for (8), that the estimate $\theta(n)$ of $\theta$ is asymptotically normally distributed with the mean equal to the true value of the parameter $\theta$ and the variance

$$\text{var}(\theta) = E \left[ -\frac{d^2 \ln L(\theta)}{d^2 \theta} \right].$$  

(11)

Based on this result we can write for large $t$

$$\mu(n) = \theta(n-1)q(n-1) + Y(n)$$

(12)

and by (9)

$$\text{var}(q(n)) = E [(q(n) - \mu(n))^2] = E [\epsilon(n)^2] = \sigma^2$$

(13)

so for large $t$

$$q(n) \sim N(\mu(n),\sigma^2).$$

(14)

Because (14) is the asymptotical propriety, (5) should be formally written as follows

$$\hat{V}_{n+1}(x, \hat{\mu}_{1T}(n)) = \min_{y \geq x} \left\{ c(y-x) + \hat{L}_{n+1}(y, \hat{\mu}_{1T}(n)) + \alpha \int_{-\infty}^{\infty} \hat{V}_{n+1}(y-q, \hat{\mu}_{1T}(n+1))df(q | \hat{\mu}_{1T}(n)) \right\}$$

where

$\hat{\mu}_{1T}(n)$ - the maximum likelihood estimate of $\mu(n)$ based on the past history of demands $H(t)$,

$\hat{V}_{n+1}(x, \hat{\mu}_{1T}(n))$ - the expected value of discounted costs from the period $n$ to the end of the horizon, where the initial inventory level is $x$, and an optimal ordering policy is followed, and the maximum likelihood estimate $\hat{\mu}_{1T}$ of the expected value of demand is used.
\( \hat{L}_{n \mid r}(y, \hat{\mu}_{r}(n)) \) is given by (7) when replacing \( \mu(n) \) by \( \hat{\mu}_{r}(n) \).

We will show the convergence
\[
\hat{V}_{n \mid r}(x, \hat{\mu}_{r}(n)) \xrightarrow{a.s.} V_{n}(x, \mu(n)) \quad \text{as} \quad t \to \infty.
\] (15)

By a property of the maximum likelihood estimator
\[
\hat{\mu}_{r}(n) \xrightarrow{a.s.} \mu(n) \quad \text{as} \quad t \to \infty
\]
so obviously
\[
\hat{f}_{n \mid r}(q) \equiv f(q \mid \hat{\mu}_{r}(n)) \xrightarrow{a.s.} f(q \mid \mu(n)) \equiv f_{n}(q) \quad \text{as} \quad t \to \infty.
\]

To show (15) let us precede inductively. For \( n = N \) consider only the case \( x(N+1) \geq 0, y(N) < 0 \). We have
\[
0 \leq \hat{V}_{N \mid r}(x, \hat{\mu}_{r}(N)) - V_{N}(x, \mu(N))
\]
\[
\leq \max_{y \geq x} \left\{ \int_{-\infty}^{\infty} \pi(q-y) (\hat{f}_{N \mid r}(q) - f_{N}(q)) dq + \alpha \int_{-\infty}^{\infty} k_{1}(y-q) (\hat{f}_{N \mid r}(q) - f_{N}(q)) dq \right\}.
\]

By Glick's (1974) modification of the Lebesgue bounded convergence theorem for random functions all above integrals converge almost surely to zero as \( t \to \infty \) at almost all \( x \in \mathbb{R} \).

Assume that (15) is valid for \( n+1 \). We have then for the period \( n \) (for simplicity we take the case \( y(n) < 0 \))
\[
0 \leq \hat{V}_{n \mid r}(x, \hat{\mu}_{r}(n)) - V_{n}(x, \mu(n))
\]
\[
\leq \max_{y \geq x} \left\{ \int_{-\infty}^{\infty} \pi(q-y) (\hat{f}_{n \mid r}(q) - f_{n}(q)) dq \right. \right.
\]
\[
+ \alpha \int_{-\infty}^{\infty} \left. \left[ \hat{V}_{n+1 \mid r}(y-q, \hat{\mu}_{r}(n+1)) \hat{f}_{n \mid r}(q) - V_{n+1}(y-q, \mu(n+1)) f_{n}(q) \right] dq \right\}.
\]

By the inductive assumption and using again the Glick's theorem all above integrals converge almost surely to zero as \( t \to \infty \) at almost all \( x \in \mathbb{R} \).

We have proved

**Theorem 1**

For large history \( H(t) \) of past demands, i.e., for \( t \to \infty \), (5) - (7) are satisfied with respect to the normal demand distribution (14).

### 3. Reduction of the state space dimension

Based on the Theorem 1 we state

**Lemma 1**

\( L_{n}(y, \mu(n)) = L(y - \theta(n-1)q(n-1) - Y(n)) = L(y - \mu(n)) \), where
\[ L(y) = \begin{cases} \int_{-\infty}^{y} h(y-q)f(q)\,dq + \int_{y}^{\infty} \pi(q-y)f(q)\,dq & , \ y \geq 0 \\ \int_{-\infty}^{y} \pi(q-y)f(q)\,dq & , \ y < 0 \end{cases} \]

and \( f(q) \) is the probability density of the normal distribution \( N(0, \sigma^2) \).

**Proof:** Consider only the case \( y \geq 0 \), since the other case is simpler. By (12) and (14) we can write

\[ L_n(y, \mu(n)) = L_n(y, \theta(n-1)q(n-1) + Y(n)) \]

\[ = \int_{-\infty}^{y} h(y-q)f(q-\theta(n-1)q(n-1)-Y(n))\,dq \]

\[ + \int_{y}^{\infty} \pi(q-y)f(q-\theta(n-1)q(n-1)-Y(n))\,dq \]

\[ y-\theta(n-1)q(n-1)-Y(n) \]

\[ = \int_{-\infty}^{y} h(y-\tilde{q}-\theta(n-1)q(n-1)-Y(n))f(\tilde{q})\,d\tilde{q} \]

\[ + \int_{y}^{\infty} \pi(\tilde{q}-y+\theta(n-1)q(n-1)+Y(n))f(\tilde{q})\,d\tilde{q} \]

\[ \equiv L(y-\theta(n-1)q(n-1)-Y(n)) \]

where we used the substitution

\[ \tilde{q} = q - \theta(n-1)q(n-1)-Y(n). \]

Define a sequence of functions \( W_n(x), \ n = 1, \ldots, N \)

\[ W_n(x) = \min_{y \geq x} \left[ c(y-x) + L(y) \right. \]

\[ + \alpha \int_{-\infty}^{\infty} W_{n+1}(y-q-\theta q - \theta Y(n) - Y(n))f(q)\,dq \left. \right] \]

with

\[ W_{N+1}(x) \equiv V_{N+1}(x). \]

**Lemma 2**

For each \( n, \ n = 1, \ldots, N \)

a) \( W_n(x) \) has a continuous derivative and is a convex function of \( x \).

b) the optimal policy \( u^*(x) \) for the problem described in eq. (16) is characterized by a single critical number \( S \), so that

\[ u^*(x) = \begin{cases} S-x & \text{if } S > x \\ 0 & \text{if } S \leq x \end{cases} \]
Proof: The proof of the Lemma 2 is analogous to Karlin (1960).

Let us make the following assumptions

(i) \( \theta^2(q) + \theta Y(n) << \theta(q) + Y(n+1) \)

(ii) \( Y(n+1) = Y(n) \)

(iii) \( \theta(n) = \theta \)

The assumption (i) is justified by the definition of \( Y(n) \) because the Seasonal Component has the largest influence on the current demand in the model (3). The assumption (ii) states formally that the Trading Day Effect is neglected and (iii) is justified when \( t \to \infty \).

Assuming (i) - (iii) yields

Theorem 2

For each \( n = 1, \ldots, N \), \( V_n(x, \mu(n)) = W_n(x - \mu(n)) \).

Proof: The proof is by induction. By (6), it is clear that the Theorem 2 holds for \( n = N \).

Assume the Theorem 2 is true for \( n+1, \ldots, N \). Let us calculate the term under integral in equation (4). In the following we make use of two substitutions

\[
\tilde{\mu}(n) = \tilde{\mu} = \mu - \theta \mu(n-1) - Y(n)
\]

\[
\tilde{Y}(n) = \tilde{Y} = Y - \theta \mu(n-1) - Y(n)
\]

By the induction hypothesis and by (16)

\[
A = V_{n+1}(\mu(n+1)) f (\mu(n+1)) d\mu
\]

\[
= W_{n+1}(\mu(n+1)) f (\mu(n+1)) d\mu
\]

\[
= W_{n+1}(\mu(n+1)) f (\mu(n+1)) d\mu
\]

\[
= W_{n+1}(\mu(n+1)) f (\mu(n+1)) d\mu
\]

Using the assumptions (i) and (ii) we obtain

\[
A = W_{n+1}(\tilde{\mu} - \tilde{\mu} - \theta \tilde{\mu} - \theta Y(n) - Y(n)) f (\tilde{\mu}) d\tilde{\mu}
\]

Using the Lemma 1 and the result above in the functional equation (5) yields

\[
V_n(x, \mu) = \min_{\tilde{\mu} \geq x} \left\{ c(\tilde{\mu} - x) + L(\tilde{\mu}) \right\}
\]

\[
+ \alpha \int_{-\infty}^{\infty} W_{n+1}(\tilde{\mu} - \tilde{\mu} - \theta \tilde{\mu} - \theta Y - Y) f (\tilde{\mu}) d\tilde{\mu}
\]

\[
= W_n(\tilde{x})
\]
where
\[ \bar{z} = x - \theta q(n-1) - Y(n). \]  
(19)

It follows from the Lemma 2 that the minimum of the right-hand side of (18) is attained by a single critical number given by (17b). This yields that the optimal ordering level \( S_n \) in the period \( n \) for the model (5) satisfies
\[ S_n = S + \mu(n) = S + \theta q(n-1) + Y(n). \]  
(20)

4. Conclusions

The dependent demand inventory model given by the time series with a stochastic trend and seasonality was considered. For the assumptions: (i)-(iii) the two-state variable dynamic programming problem was reduced to the one-state variable problem under linear cost structure. The results are valid for large enough history of past demands, when the demand distribution was shown to be normal with the nonstationary mean and the constant variance. The optimal ordering levels are characterized by single critical numbers. The obtained solution is very attractive because of its simplicity.

References


