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Published: 01/01/1989

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Memorandum COSOR 89-30

Column reduction of polynomial matrices
an iterative algorithm

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COLUMN REDUCTION OF POLYNOMIAL MATRICES
an iterative algorithm

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INTRODUCTION

Recently BEELEN developed an algorithm, called KERPOL, to determine a minimal basis for the kernel of a polynomial matrix (see BEELEN [1], BEELEN-VELTKAMP [3]). In BEELEN-VAN DEN HURK-PRAAGMAN [2] this algorithm is used to find a column reduced polynomial matrix, unimodularly equivalent to a given polynomial matrix. As reported already in NEVEN [7] this algorithm can be improved considerably, by exploiting the special structure of the polynomial map to which the algorithm KERPOL is applied. In the first place this speeds up the procedure at least by a factor 4, and makes it possible to achieve an iteration, instead starting from scratch at each new step. In this paper, we show that it, moreover, enables us to drop the assumption that the original matrix should have full column rank.

In section 1 we give the basic definitions and recall the results from [2], section 2 is devoted to the structure indices of polynomial matrices. In section 3 we prove some results on the structure indices of the associated polynomials, and give the main result of this paper.

Let us finish this introduction by pointing out some conceptual differences between this paper and its predecessors [1,2,3,4,7]. In the latter a polynomial matrix was interpreted as a mapping between spaces of rational functions. We think it is more natural to see a polynomial matrix as a mapping between two free modules over the ring of polynomials.

§1 PRELIMINARIES

Let us start with defining the notions mentioned in the introduction:

Definition 1. Let $P \in \mathbb{R}^{m \times n}[s]$. Then $d(P)$, the degree of $P$ is defined as the maximum of the degrees of its entries, and $d_j(P)$, the $j$-th column degree of $P$ as the maximum of the degrees in the $j$-th column. $\delta(P)$ is the array of integers obtained by arranging the column degrees of $P$ in non-decreasing order.

Definition 2. Let $P \in \mathbb{R}^{m \times m}[s]$. Then $P$ is unimodular if $\det(P) \in \mathbb{R} \setminus \{0\}$.

Let $\Lambda^P(s) = \text{diag}(s^{-d_1(P)}, \ldots, s^{-d_n(P)})$, then $P \Lambda^P$ is a proper rational matrix.
Definition 3. Let $P \in \mathbb{R}^{m \times n}[s]$. Then the leading column coefficient matrix of $P$, $\Gamma(P)$ is defined as: $\Gamma(P) := PA^P(\infty)$. If $P = (0|P')^T$, $T$ a permutation matrix, and $\Gamma(P')$ has full column rank, then $P$ is called column reduced.

With a little abuse of terminology we will call a matrix $Q$ a basis for the module $M$, if the columns of $Q$ form a basis of $M$:

Definition 4. Let $M$ be a submodule of $\mathbb{R}^n[s]$. Then $Q \in \mathbb{R}^{n \times n}[s]$ is called a basis of $M$ if $\text{rank } Q = r$, and $M = \text{Im } Q$. If, moreover, $Q$ is column reduced, then $Q$ is called a minimal basis of $M$.

Note that if $Q(s)$ has full column rank for all $s \in \mathbb{C}$, then $M$ is a direct summand of $\mathbb{R}^n[s]$, so in that case $Q$ is a minimal polynomial basis in the sense of FORNEY [4], or [1].

For each polynomial matrix $P$ having full column rank there exists a unimodular matrix $U$, such that $PU$ is column reduced (see WOLOVITCH [9], KAILATH [6] or [4]). The proof, given in these references is constructive and does imply:

Lemma 1. Let $P$ and $Q$ be bases for $M$, and let $Q$ be minimal. Then $\delta(P) \geq \delta(Q)$ totally.

Unfortunately, the proof mentioned above, has awkward numerical properties, as was pointed out by VAN DOOREN [8]. The numerically more satisfying method in [2] is based on the following theorem:

Theorem 1. Let $P \in \mathbb{R}^{m \times n}[s]$ have full column rank, and let $\begin{bmatrix} U \\ R \end{bmatrix}$ be a minimal basis for $\text{Ker}(s^bP - I)$. Then $U$ is unimodular and if $b$ exceeds $(n-1)d$ then $s^{-b}R = PU$ is column reduced.

Proof. See [2].

The effort to determine a minimal basis for the kernel of a polynomial matrix with BEELEN's methods is proportional to the third power of the product of its size and its degree. So taking $b = (n-1)d$ leads to long computation time. Therefore the authors of [2] suggested as a possible alternative to run the procedure for $b = 0, 1, 2, \ldots$, successively,
until a column reduced solution was found. It turns out that in a lot of examples a rather small \( b \) already yields the desired result. A serious drawback in their proposal is that for each \( b \) the determination of the kernel starts from scratch. This motivated us to study the procedure \textsc{kerpol} in more detail to see whether some kind of iteration was achievable. This proved to be the case. To that end we had to study the structure of the algorithm for matrices of the form \((P|I)\). We found that for matrices of this type the procedure could be improved considerably, that it is possible to give an, a priori unknown, lower bound for the first value of \( b \) for which the procedure will work, and, moreover, that the condition that \( P \) should have full column rank can be dropped.

\section{Structure Indices of Polynomial Matrices}

In this section we introduce the concepts of left and right minimal indices and of elementary exponents, which will play a role in the next section:

\textbf{Definition 5.} Let \( P \in \mathbb{R}^{m \times n}[s] \). Then its \textit{right minimal indices} \( \kappa := (\kappa_1, ..., \kappa_q) \) are defined by \( \kappa = \delta(Q) \) where \( Q \) is a minimal basis for \( \text{Ker}(P) \). Its \textit{left minimal indices} are the right minimal indices of \( P^T \).

Clearly \( q \) equals \( n-r \), with \textit{rank} \( P = r \). Next we define the notion of elementary exponent, closely related to elementary divisors. Therefore we introduce the homogeneous polynomial associated to \( P \): Let \( P \in \mathbb{R}^{m \times n}[s] \) equal \( P(s) = P_0 s^d + P_1 s^{d-1} + \ldots + P_{m-1} s + P_n \), then \( P^h = \mathbb{R}^{m \times n}[s,t] \) is defined by \( P^h(s,t) = P_0 s^d + P_1 s^{d-1} + \ldots + P_{m-1} s + P_n t \). Let \( \Delta_i \) be the greatest common divisor of \( i \times i \) minors of \( P^h \), and define \( \Delta_0 = I \). Then \( \Delta_i \) divides \( \Delta_1 \), and let \( \Delta_i \Delta_{i-1} = c \Pi (as-bt)^{\ell_i} \) where the product is taken over all pairs \((I,b)\) and \((0,t)\) and \( 1 \) is denoted by \( \infty \).

\textit{Remark.} It is well known that there exist unimodular matrices \( U \) and \( V \) such that \( UPV(s) = \text{diag}(\Delta_i(s,I)\Delta_{i-1}(s,I)) \); moreover there exist unimodular \( S,T \), polynomial matrices in \( t \), such that \( S(t)P^h(I,t)T(t) = \text{diag}(\Delta_i(t,I)\Delta_{i-1}(I,t)) \).
**Definition 7.** Let $P$, then the *structure indices* of $P$ are its left and right minimal indices and its elementary exponents.

For each matrix polynomial $P \in \mathbb{R}^{m \times n}[s]$ of degree $d > 0$ define its *linearization* $LP \in \mathbb{R}^{m \times (n+(m-1)d)}[s]$ by:

$$LP(s) = \begin{bmatrix} I & P_d s \\ -sI & I \\ -sI & P_0 + P_1 s \end{bmatrix}.$$ 

There is a close relationship between the structure indices of a polynomial matrix, and those of its linearization:

**Theorem 2.** Let $P \in \mathbb{R}^{m \times n}[s]$ have degree $d$, and let $LP$ be its linearization. Then

- The right minimal indices of $P$ and $LP$ are equal;
- The elementary exponents of $P$ and $LP$ are equal;
- The left minimal indices of $LP$ equal those of $P$ augmented by $d-1$.

**Proof.** Premultiplying $LP$ by $C(s) = \begin{bmatrix} I & 0 & 0 \\ sI & I \\ s^{d-1}I & sI \end{bmatrix}$ yields

$$C(s)LP(s) = \begin{bmatrix} I & 0 & P_d \\ 0 & I & P_{d-1} s \\ 0 & 0 & P(s) \end{bmatrix}.$$ 

From this the first statement follows immediately, and the second in so far it concerns elementary exponents $\#(c) \neq \infty$.

Let $V^T$ be a minimal polynomial basis of $P^T$, then clearly $(0 \ 0 \ V)^T$ is a minimal polynomial basis for $(CLP)^T$, and hence it follows that $C^T(0 \ 0 \ V)^T = (s^{d-1}V \ V)^T$ is a minimal polynomial basis for $LP^T$, which yields the third statement.

Premultiplying $LP(l,t)$ with

$$\begin{bmatrix} I & l & \theta \alpha \\ 0 & l & \theta^{-1} \end{bmatrix}$$

yields the missing part of the second statement. 


As an immediate consequence of this theorem we find:

**THEOREM 3.** Let $P \in \mathbb{R}^{m \times n}[s]$ be a polynomial matrix of rank $r$ and degree $d$. Then the sum of its structure indices equals $r \cdot d$.

**PROOF.** It can be deduced immediately from the well known KRONECKER normal form for matrix pencils (GANTMACHER [5]) that the theorem holds for polynomials of degree 1. The rank of $LP$ equals $m(d-1)+r$, hence its number of left minimal indices (and that of $P$) is $m-r$.

From theorem 2 we conclude that the sum the structure indices of $P$ equals the sum of the structure indices of $LP$ minus $(m-r)(d-1)$, hence equals

$$md - m + r - md + m + rd - r = rd.$$ 

**Corollary.** $\kappa_i \leq r \cdot d$, $\ell_j \leq r \cdot d$.

§3 THE ASSOCIATED POLYNOMIAL MATRICES

For each $b \geq 1$ we associate to $P \in \mathbb{R}^{m \times n}[s]$ a matrix polynomial defined as:

$$P_b(s) := (s^bP(s) - I).$$

Note that $P_b$ has no left minimal indices, and that all its elementary divisors have the form $s^a$. Denote its right minimal indices by $\epsilon(b) = (\epsilon_1(b), \ldots, \epsilon_n(b))$, and its elementary exponents by $\omega(b) = (\omega_1(b), \ldots, \omega_m(b))$.

**LEMMA 4.** Let $P \in \mathbb{R}^{m \times n}[s]$ have rank $r$. Let $\kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_{n-r}$ be its right minimal indices and let $\ell_1 = \ell(\infty)$. Then the structure indices of the associated matrices $P_b$ satisfy for all $b$ and $i$:

- if $\kappa_i \leq b$ then $\epsilon_i(b) = \kappa_i$,
- $b \leq \epsilon_i(b) \leq \epsilon_i(b+l) \leq \epsilon_i(b)+1 \leq b+d+1$, if $i > n-r$,
- $\omega_i(b) = \min(\ell_i b+d)$, for $i \leq r$,
- $\omega_i(b) = b+d$, for $i > r$. 

PROOF. Let \( V_b = (U_b^T, R_b^T)^T \) be a minimal basis for \( \text{Ker}(P_b) \). Clearly \( s^b \) divides \( R_b \), from which the first statement follows. \( (U_b^T, sR_b^T)^T \) is a basis for \( \text{Ker}(P_{b+1}) \). From this the second statement follows immediately. The last two statements follow easily from the observation that

\[
P_b^h(s,t) = (P_b^h(s,t) t^{b+d})I.
\]

We are now in the position to prove the following generalization of theorem 1:

THEOREM 4. Let \( P \in \mathbb{R}^{m \times n}[s] \) have rank \( r \) and degree \( d \), and let \( V_b = (U_b^T, R_b^T)^T \) be a minimal polynomial basis for \( \text{Ker} P_b \). Then \( U_b \) is unimodular. If \( b \geq \max (\kappa_{n-r+1}, \ell-r-d) \) then \( R_b \) is column reduced, and if \( R_b \) is column reduced then \( b \geq \ell-r-d \).

PROOF. The unimodularity of \( U_b \) follows as in [2]. Since \( \sum \epsilon(b) + \omega(\kappa) = (b+d)m \) it is clear that for \( b \geq \max (\kappa_{n-r+1}, \ell-r-d) \) the structure indices \( \epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_1, \omega_{n-1}, \ldots, \omega_1 \) have to increase with increasing \( b \). Since \( s^b \) divides \( R_b \) we have in the first place that \( R_b = (0 | *) \) and further by the observation made in the preceding sentence that \( \text{diag}(I, s^h) \cdot V_b \) is a minimal basis for \( \text{Ker} P_{b+h} \). If \( h \) is large enough this implies that \( s^b R_b \), and hence \( R_b \), is column reduced.

If, on the other hand \( R_b \) is column reduced, then \( R_b \) contains \( n-r \) zero columns, implying that all the \( \kappa_i \) occur in \( \epsilon(b) \), and hence \( \omega(\kappa) = \ell \leq b + d \), implying the final statement.

Remark. The bound given here depends on \( \kappa_{n-r} \) and \( \ell_n \) which are in general unknown. As a direct consequence of this theorem and the corollary of theorem 3, we find that if \( b \) exceeds \( r \cdot d(P) \), then \( R_b \) is column reduced. If \( P \) has full column rank we find this yields \( b > n \cdot d(P) \), a worse bound than was found in [2]. But it is not hard to see that \( \max(\kappa_i, \ell_i - d(P)) \) never exceeds the bound given there. If \( r < n \) our bound can be much better.

§4 CALCULATION OF A MINIMAL BASIS

Let \( Q \in \mathbb{R}^{m \times n}[s] \), and assume that we want to calculate a minimal basis for \( \text{Ker} Q \).
The procedure described in [3] reads as follows:

1. Find orthogonal matrices $U$ and $V$, such that $ULQV$ is in a generalized Schur form: upper triangular staircase form, with constant right invertible matrices along the block diagonal.

2. Find a minimal basis for this matrix.

3. Calculate a minimal basis for $Ker Q$, starting from the minimal basis found in the preceding step.

Since in our case the polynomial matrix $P_b$ has some special features this procedure works extremely well if we bring in some minor modifications in the algorithm KERPOL, described in [3].

In the first place we introduce in a slightly different linearization of $P_b$: Let $P$ be given by $P(s) = P_d s^d + P_{d-1} s^{d-1} \ldots P_0$. Define

$$H_b(s) = A_b s - E_b = \begin{bmatrix}
sp_d & -I & 0 & \cdots \\
-Is & -I & 0 & \cdots \\
0 & Is & -I & \cdots \\
& & & & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & Is & -I
\end{bmatrix} \in L((b+d)m,n),$$

where $L(m,n) := \{ H \in \mathbb{R}^{m \times (n+m)} | \deg H = 1, H(0) = (0-I) \}$.

As in the proof of theorem 2 we see that

$$C_b(s)H_b(s) = \begin{bmatrix}
* & I & 0 & 0 \\
* & 0 & I & 0 \\
sbP & 0 & I
\end{bmatrix},$$

and if $Im V = Ker(H_b)$, then $Ker(sbP|-I) = Im(\begin{bmatrix}
I & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \cdot V)$. And again as in theorem 2, if $V$ is minimal, then $\begin{bmatrix}
I & 0 & 0 \\
0 & 0 & I
\end{bmatrix} \cdot V$ is minimal, too. So the problem reduces to finding a minimal basis for $Ker H_b$. Such a basis can be found by constructing an orthogonal matrix $U$, such that $UH_bU^\sim$, $U^\sim = diag(I,U^T)$, is in an upper staircase form, in which the constant part equals $E := (0,-I)$ as in $H_b$. Crucial in that respect is the following theorem.
THEOREM 5. Let $H = sA - E \in \mathbb{R}^{m \times n}[s]$. Then there exists an orthogonal matrix $U$, such that:

$$UHU^\sim = \begin{bmatrix}
A_{11}s & A_{12}s - I & \cdots & * \\
0 & A_{22}s & A_{23}s - I & \cdots & * \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & A_{11}s & A_{12}s - I & * \\
& & & & 0 & \cdots & A_{11}s & 0 & Ns - I
\end{bmatrix}$$

with $A_{jj} \in \mathbb{R}^{m_j \times m_j - 1}$ right invertible, and $N \in \mathbb{R}^{m + 1 \times m + 1}$.

Moreover, $m_{j-1} - m_j = \# \{ i \in \mathbb{N} | e_i = j - 1 \}$.

PROOF. By induction on $2n - m$. If $2n - m = m$, or $m = n$, then there is nothing to prove, so let $q = n - m > 0$. Partition $A$ in the following way:

$$A = (A_1 \ A_2),$$

with $A_2$ square, and $A_1 \in \mathbb{R}^{m \times q}[s]$, then there exists an orthogonal $U_1 \in \mathbb{R}^{m \times m}$ such that $U_1A_1 = \begin{bmatrix} A_{11} \\ 0 \end{bmatrix}$, with $A_{11} \in \mathbb{R}^{m_1 \times q}$ right invertible. It is easy to see that

$$U_1AU_1^\sim = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

with $A_{22} \in \mathbb{R}^{m' \times n'}$. $2n' - m' = 2(n - q) - (m - m_1) = 2n - m - (2q - m_1) < 2n - m$, since $q \geq m_1$, and $q > 0$. Using the induction hypothesis on $A_{22}$ yields the first statement. Let $K_{11}$ be a left invertible real matrix such that $Im \ K_{11} = Ker \ A_{11}$, let $A_{11}^\sim$ be a right inverse of $A_{11}$, and let $V_2$ be a minimal basis for $sA_{22} - E_{22}$, where we have used the analogous partitioning of $E$, then

$$V := \begin{bmatrix} K_{11} & A_{11}^\sim & (E_{12} - sA_{12})V_2 \\ 0 & 0 & sV_2 \end{bmatrix}$$

is a minimal basis for $UHU^\sim$, so $U^\sim V$ is a minimal basis for $H$, yielding the second statement.

Corollary. Clearly the numbers $m_i$ do not depend on the particular choice of $U$. Denote these invariants by $\mu_0(H), \ldots, \mu_j(H)$. 

In the proof of this theorem we already showed how a basis for $\text{Ker } H$ is constructed: Let $K_{ii}$ be a basis for $\text{Ker}(A_{ii})$, and $A_{ii}^\dagger$ a right inverse for $A_{ii}$. Define

$$V_{i}(s) := s^{j} K_{jp},$$
$$V_{j}(s) := s^{j} A_{ii}^\dagger \left( \sum_{k=j+1}^{i} s A_{jk} V_{k}(s) + V_{(j+1), k}(s) \right),$$

then

$$V := \begin{bmatrix} V_{11} & V_{12} \\ \emptyset & V_{22} \end{bmatrix}$$

is a minimal basis for $HU^\dagger$.

In the special case that $H = H_b$,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} := \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & U^T \end{bmatrix} V,$$

is a minimal basis for $(s^b P - I)$ (see [3]). Note that

$$V_1 = (V_{11} V_{22} \cdots V_{1n}),$$

and that $V_2 = s^b P V_1$.

Remark. The goal of our procedure is to find a $V_2$ which is column reduced. If we construct $V_1$ as above, first calculating $V_{11}$ and then $V_{12}$ etc, then we can calculate $V_2$ in the same way: $V_{2i} = P V_{1i}$, and $V_2 = s^b(V_{21} \cdots V_{2n})$. This enables us to check at each step whether $(V_{21} \cdots V_{2n})$ is column reduced, deciding as early as possible whether we should try a next $b$.

At this moment we can describe the skeleton of an algorithm:

\begin{enumerate}
  \item \textbf{step 1.} \quad b = 0.
  \item \textbf{step 2.} \quad b := b + 1, \ i := 0
  \item \textbf{step 3.} \quad i := i + 1.
\end{enumerate}
Calculate $V_{1i}$, $V_1 := (V_1 V_{1i})$.

$V_{2i} := PV_{1i}$, $V_2 := (V_2 V_{2i})$.

If $V_2$ is not column reduced go to step 2.

If $V_1$ is not square go to step 3.

$V_2$ is the column reduced matrix we are looking for.

Compared to the procedure described in [2], we already have gained about a factor 4 in computation speed, using the special structure of $H_b$. In the next section we will show that some information can be carried along if we increase $b$.

§5 THE ITERATIVE STEP

Assume that we have terminated the algorithm at step $b$, and we proceed with $a = b + h$. In step $b$ we have determined a $U_b$, and a $H_b$ such that

$$\tilde{H}_b = U_b H_b U_b^{-1},$$

where $\tilde{H}_b$ has in its leading columns a generalized Schur form structure. Clearly

$$H_{a} = \begin{bmatrix} H_b & 0 \\ 0 & 0 \end{bmatrix},$$

with

$$N = \begin{bmatrix} 0 & 0 & I \\ \vdots & \vdots & \vdots \\ 0 & \ldots \end{bmatrix} \in \mathbb{R}^{hm \times (n+(b+d)m)}$$

and

$$J = \begin{bmatrix} -I & 0 \\ 0 & I \\ \vdots & \vdots & \vdots \\ 0 & \ldots \end{bmatrix} \in \mathbb{R}^{hm \times hm[s]},$$

so if we define $U_a = \text{diag} (U_b, I)$, then

$$\tilde{H}_a := U_a H_a U_a^{-1} = \begin{bmatrix} \tilde{H}_b & 0 \\ 0 & J \end{bmatrix}. $$
If $U_b = (U_{ij})$, $U_{ij} \in \mathbb{R}^{m \times m}$, then

$$NU_b^\sim = \begin{bmatrix} 0 & U_{b+d,1} & \cdots & U_{b+d,b+d} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

so at first sight it seems that only the first block column of $\tilde{H}_a$ preserves the desired structure. But fortunately it turns out that we can select $U_b$ in such a way that $U_{b+d,1} = U_{b+d,2} = \ldots = U_{b+d,b} = 0$.

**Theorem 6.** Let $H \in L(k+hm,n)$, $m \leq k$, have the following structure:

$$H = \begin{bmatrix} H' & 0 \\ sN & J \end{bmatrix},$$

with $H' \in L(k,n)$, $N \in \mathbb{R}^{hm \times (n+k)}$ and $J \in \mathbb{R}^{hm \times hm} [s]$ with the block structure as above. Then there exists an orthogonal matrix $U$, such that $U^\sim H U$ has a generalized Schur form, and which displays the following structure:

$$U = U_{ij}, \quad U_{ij} \in \mathbb{R}^{m \times n_i},$$

where $m_i = \mu_i[H]$, $i = 1, \ldots, l$; $m_{l+1} = hm + k - \sum \mu_i[H]$, and $n_1 = k$, $n_2 = \ldots = n_{h+1} = m$, and $U_{ij} = 0$ if $j > i + k - h$.

**Proof.** By induction on $hm + k$. For $h = k = 0$ there is nothing to prove, so assume that $h + k > 0$. Let $sH_1$ be the matrix containing the first $n$ columns of $H'$, and let $H_1 = QR'$ be a $QR$–decomposition of $H_1$: $Q$ orthogonal, and $R'$ upper staircase: $R' = (R^T 0 0 0)^T$, with $R \in \mathbb{R}^{m \times n}$. Define $U' = \text{diag}(Q, J)$, let $N = \begin{bmatrix} N_1 \\ 0 \end{bmatrix}$, and $J = \begin{bmatrix} J_1 & 0 \\ sN_2 & J_2 \end{bmatrix}$, then

$$U^\sim H U = \begin{bmatrix} R & * & 0 \\ 0 & H' & H'' \\ 0 & sN_1 Q & J_1 \\ 0 & 0 & sN_2 J_2 \end{bmatrix}.$$

Define $H^* = \begin{bmatrix} H^* & 0 \\ sN^* & J_2 \end{bmatrix}$, with $H_1^* = \begin{bmatrix} H' & H'' \\ sN_1 Q & J_2 \end{bmatrix} \in L(k+m-m_1, m_1)$, and $N^* = \begin{bmatrix} 0 & N_2 \end{bmatrix} \in \mathbb{R}^{(h-1)m \times (k+hm)}$. Since $(h-1)m + m+k-m_1 < hm + k$ the induction hypothesis yields that there exists an orthogonal $U^*$ with the following structure:
\[
U^* = (U^*_{ij}), \quad U^*_{ij} \in \mathbb{R}^{m_i \times n_j},
\]
with
\[
m_i^* = \mu_i(H^*) \text{ for } i = 1, \ldots, l-1, \\
m_1^* = hm + k - m_1 - \sum \mu_j(H^*), \\
n_1 = k + m - m_1, \quad n_2 = \ldots = n_h = m,
\]
and
\[
U^*_{ij} = 0 \text{ if } j > i.
\]

Let \( V \) be the orthogonal matrix \( \text{diag}(I, U^*) \), and \( U = VU' \). Divide \( V \) into blocks as follows:
\[
\left[
\begin{array}{cc}
V_{11} & V_{12} \\
V_{l+1,1} & V_{l+1,2}
\end{array}
\right] = \left[
\begin{array}{c}
I \\
0 \\
0
\end{array}
\right] \\
\left[
\begin{array}{c}
0 \\
\end{array}
\right],
\]
with \( V_{11} \in \mathbb{R}^{k \times (k+hm)} \), \( V_{12} \in \mathbb{R}^{(k+hm) \times (k+mn)} \), etc.

then \( V_{ij} \in \mathbb{R}^{m_i \times n_j} \), with \( m_i = \mu_{i+1}(H^*) \), \( i \geq 2 \), \( n_i = k, \quad n_j = m, \quad j \geq 2 \), and moreover \( V_{ij} = 0 \) if \( j > i \). Since \( U_{ij} = V_{ij} \) if \( j > 1 \), and \( U_{1j} = V_{1j}Q \), we see that \( U \) has the right structure if we can prove that \( m_i = \mu_i(H) \). Since \( UHU' \) has a generalized Schur form, this follows immediately from theorem 5.

**Remark.** Note that the proof gives a constructive way to find \( U \). In the sequel we will use this several times.

As a consequence we find that if we have a generalized Schur form for \( H_b \), and we search one for \( H_a \), then we can divide the orthogonal \( U \), that we constructed, into blocks as above, and then
\[
\left[
\begin{array}{cc}
U & 0 \\
0 & I
\end{array}
\right] \cdot \left[
\begin{array}{cc}
I & 0 \\
0 & U'
\end{array}
\right] = \left[
\begin{array}{cccc}
sA_{11} & \cdots & \cdots & 0 \\
\vdots & sA_{bb} & \vdots & \vdots \\
0 & 0 & \ddots & \ddots \\
0 & 0 & \cdots & sA_{b+1,b+1}
\end{array}
\right],
\]
and hence we only have to work on the matrix in the box in the lower right corner,
having size \(((a+d)m - m_1 . -m_b) \times ((a+d)m - m_1 . -m_{b-1})\) instead of on
\(H_b \in L((a+d)m,n)\).

Remark. Note that in the algorithm from [2] the information cannot be carried over to
the next step, since there another linearization is used, destroying the structure of \(U\). Due
to this feature the \(b\)-th step in [2] is of order \((b+d)^3 m^3\), while in the algorithm in
section 4 each step is of order \(d^3 m^3\). Since \(b\) can increase up to \(rd\), this means that in
the worst case the algorithm in [2] is of order \(r^4 d^4 m^3\), while the algorithm presented
here is of order \(rd^4 m^3\)! Note that the one step procedure, i.e. starting with \(b = rd\), given
in [2] is of order \(r^3 d^3 m^3\), certainly not better not better than the worst case order of our
algorithm. Of course the one step procedure has much lower order in our set up, since
we thoroughly exploit the zero structure of the initial matrix. A quick calculation yields
that the one step procedure is of order \(r^4 d^4 m^3\), too. This means that the iterative
algorithm is even in the worst case of the same order as the one step procedure, unlike
the situation in [2]. As said before, at each step we gain at least a factor 4, since in the
first place we can use Householder transformations instead of Givens transformations,
and further we do not have to find the postmultiplication seperately, but we can use \(U^-\).

\section{Description of the Algorithm}

In this section we will give a more detailed description of the algorithm.

\begin{verbatim}
COLRED

Input: \(P = \sum_{i=1}^{d} P_i \in \mathbb{R}^{m \times n}[s]\)

Output: \(U = \sum_{i=1}^{d} U_i \in \mathbb{R}^{n \times n}[s]\) (the unimodular matrix \(U\))

\(R = \sum_{i=1}^{d} R_i \in \mathbb{R}^{m \times n}[s]\) (the column reduced polynomial matrix \(R = PU\))

Initialization: \(b := 0; h := 0; l := 0; \sigma := 0;\)
\(h(0) := 0; l(0) := 0; l(1) := n; \sigma(0) := 0;\)
\end{verbatim}
\[
H := \begin{bmatrix}
sP_0 & -I & 0 \\
. & sI & -I \\
. & . & sI -I
\end{bmatrix};
U := I \in \mathbb{R}^{md \times md}[s];
\]
\[
R(0) \in \mathbb{R}^{mx0}[s];
T(0) \in \mathbb{R}^{nx0}[s];
rdcd := false;
\]

\begin{verbatim}
begin
b := b+1;
rdcd := RED(P); (RED(P) returns true if P is columnreduced)
while rdcd = false do
begin
rdcd := true;
h := h(b-I); l := l(b-I); \( \sigma := \sigma(b-I) \);
R := R(b-I); T := T(b-I);
U := \text{diag}(U(b-I),I) \in \mathbb{R}^{(d+b) \times (d+b)}[s];
while rdcd = true do
begin
if I = I(b) then
begin
\( h(b) := h; l(b+1) := h; \sigma(b) := \sigma; \)
R(b) := R; T(b) := T;
end;
\[
h \frac{1}{\sqrt{2}} \begin{bmatrix}
K & L \\
0 & M
\end{bmatrix} := H;
\]
w := L(.,I);v := M(.,I)
if \| v \| \neq 0 do
begin
\[
V := \text{diag}(I,HH(v));
\]
(HH(v) returns the Householder transformation mapping v into \( e_1 \))
\[
V^\top := \text{diag}(I,V^T);
\]
H := VH\( V^\top \);
U := UV;
h := h+1; l := l+1;
end
end
end
\end{verbatim}
else
begin
\( x := \text{MNSL}(K,w); \) \hfill ((x'_{1+1}) \text{ is a minimal solution of } Kx' + wx'_{1+1})
\( t := (I \ 0 \ldots \ 0)x; \)
\( T := (T \ t); \)
\( r := Pu; \)
if \( \| r \| \neq 0 \) then \( \sigma := \sigma + 1; \)
\( R := (R \ r); \)
if \( \text{LCR}(R) \neq \sigma \) then
begin
\( rcd := \text{false}; \)
\( N := (0 \ldots 0 I)V^-; \)
\( H := \begin{bmatrix} H & 0 \\ sN & -I \end{bmatrix} \)
end
else \( l := l + 1; \)
end;
end;
end.

Clearly at each step the procedure LCR can use information from a previous step. To accomplish this we check the rank using a QR–decomposition.

References


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