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Polyhedral characterization of the Economic Lot-Sizing problem with Start-up costs
C.P.M. van Hoesel
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Eindhoven, November 1992
The Netherlands
Polyhedral characterization of the Economic Lot-Sizing problem with Start-up costs

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September 28, 1992

Abstract

A class of strong valid inequalities is described for the single-item uncapacitated economic lot-sizing problem with start-up costs. It is shown that these inequalities yield a complete polyhedral characterization of the problem. The corresponding separation problem is formulated as a shortest path problem. Finally, a reformulation as a plant location problem is shown to imply the class of strong valid inequalities, which shows that this reformulation is tight also.

Key words: Economic Lot-Sizing, Polyhedral Description, Plant Location Formulation, Separation.

AMS Subject classification: 90B.

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1 Introduction

Good integer linear programming formulations for $NP$-hard problems are a valuable aid in solving these problems with linear programming based solution methods. One way of improving formulations with strong valid inequalities is by looking at relaxations or substructures which are polynomially solvable. For such a relaxation, contrary to the original $NP$-hard problem, one may be able to find a complete linear description. This holds, for instance, for several economic lot-sizing problems. Research on strong valid inequalities for the uncapacitated single-item economic lot-sizing problem (ELS), as defined by Wagner and Whitin [17] and Manne [11], started with Barany et al. [1, 2] who developed the so-called $(I, S)$-inequalities. The polyhedral structure of generalizations of the economic lot-sizing problem has been the subject of several papers. Leung et al. [9] and Pochet and Wolsey [13] give strong valid inequalities for the capacitated economic lot-sizing problem. Pochet and Wolsey [12] consider the extension with backlogging. They describe an implicit characterization of the problem.

We consider the extension of the economic lot-sizing problem with start-up costs included (ELSS), i.e., costs for switching on a machine or changing over between different items. Schrage [15] introduced these costs to distinguish between normal set-up costs, which are incurred in each period that a certain item is produced, and costs which appear only in the first of a consecutive set of periods in which an item is produced. Problems in which start-up costs appear have been studied by Van Wassenhove and Vanderhenst [16], Karmarkar and Schrage [7] and Fleischmann [4]. The standard dynamic programming formulation of the economic lot-sizing problem with start-up costs can be solved in $O(T \log T)$ time, here $T$ is the length of the planning horizon (see van Hoesel [5]).

The polyhedral structure of several mixed integer programming formulations for the economic lot-sizing problem with start-up costs (ELSS) was first investigated by Wolsey [18]. For the formulation in a natural set of variables he derived a class of strong valid inequalities by generalizing the $(I, S)$-inequalities for the corresponding formulation of the economic lot-sizing problem (Barany et al. [2]). In this manuscript we generalize these inequalities further to the so-called $(I, R, S)$-inequalities. The main result that we present is that these inequalities imply a complete linear description for this formulation. We use a proof technique due to Lovász [10], which appears to be especially suitable for problems where a greedy algorithm solves the dual linear program arising from a complete linear description of the problem (see van Hoesel et al. [6]). Conditions under which the $(I, R, S)$-inequalities are facet-defining are provided in van Hoesel [5]). In addition, we discuss separation for the $(I, R, S)$-inequalities by formulating this problem as a set of $T$ shortest path problems on acyclic networks, each with $O(T^2)$ nodes.

A related formulation for ELS is the plant location reformulation, in which the production variables are split. This formulation has been introduced by Krarup and Bilde [8]. The plant location reformulation for ELSS is shown to be at least as strong as the formulation in the original variables. This is done by viewing the inequalities as di-cut inequalities (Rardin and Wolsey [14]) in a fixed-charge min-cost flow problem. For other related formulations and similar results see Wolsey [18] and Eppen and Martin [3].

In section 2, the economic lot-sizing problem with start-up costs is formulated as a mixed integer programming problem. The $(I, R, S)$-inequalities are introduced and shown to be valid.
In section 3, it is shown that the \((I, R, S)\)-inequalities provide a complete linear description. The separation algorithm for the \((I, R, S)\)-inequalities can also be found in this section. In section 4, the plant location model is discussed. Finally, in section 5, some concluding remarks are made.

## 2 Formulation of ELSS: the \((I, R, S)\)-inequalities

Consider the economic lot-sizing problem with start-up costs with a planning horizon consisting of \(T\) periods. For each period \(t\) a demand \(d_t\) must be satisfied by production in one or more of the periods in \(\{1, \ldots, t\}\). The costs for production in period \(t\) are \(c_t\) per unit. If production takes place in period \(t\) a set-up must be performed at a cost of \(f_t\). This is the formulation of the economic lot-sizing problem as defined by Wagner and Whitin [17]. In ELSS there are additional fixed costs for start-ups: each set of consecutive periods in which a set-up is performed should begin with a period \(t\) in which a start-up is performed at a cost of \(g_t\). For reasons of simplicity, in the formulation we present, the inventory variables are deleted (see for instance Wolsey [18] or van Hoesel [5]). ELSS can be modelled as a mixed integer program with the following parameters and variables.

**Parameters:**
- \(d_t\) \((1 \leq t \leq T)\): the demand of the item in period \(t\);
- \(c_t\) \((1 \leq t \leq T)\): the unit production cost of the item in period \(t\);
- \(f_t\) \((1 \leq t \leq T)\): the set-up cost of the item in period \(t\);
- \(g_t\) \((1 \leq t \leq T)\): the start-up cost of the item in period \(t\).

**Variables:**
- \(x_t\) \((1 \leq t \leq T)\): the production of the item in period \(t\);
- \(y_t\) \((1 \leq t \leq T)\): \(\begin{cases} 1 & \text{if a set-up of the item is incurred in period } t \\ 0 & \text{otherwise} \end{cases}
- \(z_t\) \((1 \leq t \leq T)\): \(\begin{cases} 1 & \text{if a start-up of the item is incurred in period } t \\ 0 & \text{otherwise} \end{cases}\)

(ELSS)

\[
\begin{align*}
\text{min} & \quad \sum_{t=1}^{T} (g_t z_t + f_t y_t + c_t x_t) \\
\text{s.t.} & \quad \sum_{t=1}^{T} x_t = d_{1,T} \\
& \quad \sum_{\tau=1}^{t} x_{\tau} \geq d_{1,t} \\
& \quad y_t \leq y_{t-1} + z_t \quad (y_0 = 0) \\
& \quad x_t \leq d_{t,T} y_t \\
& \quad (1 \leq t \leq T)
\end{align*}
\]
By \( d_{s,t} (1 \leq s \leq t \leq T) \) we denote the cumulative demand of the periods \( \{s, \ldots, t\} \), i.e., \( d_{s,t} = \sum_{r=s}^{t} d_r \).

Constraint 2 restricts production to the total demand over the planning horizon. Constraints 3 ensure that ending inventory in each period is nonnegative. Constraints 4 model the start-ups. Constraints 5 force a set-up in a period with positive production. Note that 2 and 3 (for \( t - 1 \)) imply the upper bound \( d_{t,T} \) on the production in each period \( t \) as mentioned in 5.

The remainder of this section is devoted to the description of the \((I, R, S)\)-inequalities and a proof of their validity. Take an arbitrary period \( I \in \{1, \ldots, T\} \), and let \( N_I = \{1, \ldots, I\} \). Let \( S \) be an arbitrary subset of \( \{1, \ldots, N_I\} \) and \( R \) be a subset of \( S \), such that the first element in \( S \) is also in \( R \). The corresponding \((I, R, S)\)-inequality is defined as follows:

\[
\sum_{t \in N_I \setminus S} x_t + \sum_{t \in R} d_{t,I} y_t + \sum_{t \in S \setminus R} d_{t,I} (z_{p(t)+1} + \ldots + z_t) \geq d_{1,I}
\]

where \( p(t) \equiv \max\{j \in S | j < t\} \). If \( S \cap \{1, \ldots, t - 1\} = \emptyset \), then \( p(t) \equiv 0 \).

**Example:** \( I = 15; S = \{2, 4, 5, 7, 10, 11, 14\}; R = \{2, 10\} \)

The coefficients of the left-hand side of the \((I, R, S)\)-inequality are given in the following table:

<table>
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<tr>
<th>( t )</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
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<tbody>
<tr>
<td>( x_t )</td>
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<td>( z_t )</td>
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</tbody>
</table>

The inequalities derived in Wolsey [18] are a special case of the \((I, R, S)\)-inequalities in the sense that each maximal set of consecutive periods in \( S \) should begin with a period in \( R \) there. The above example is not included, because the set \( \{4, 5\} \) does not begin with an element in \( R \) (4 \( \in S \setminus R \)).

**Lemma 1** The \((I, R, S)\)-inequalities are valid.

**Proof** Take an arbitrary \((I, R, S)\)-inequality as defined above. Denote an arbitrary feasible solution by \( (x, y, z) = \{x_t, y_t, z_t | t = 1, \ldots, T\} \). We prove that this solution satisfies the \((I, R, S)\)-inequality. We distinguish two cases.

**Case 1:** \( S \) does not contain a period with positive production, i.e. \( x_t = 0 \) for all \( t \in S \). Then

\[
\sum_{t \in N_I \setminus S} x_t = \sum_{t=1}^{I} x_t \geq d_{1,I}
\]
Case 2: $S$ contains a period with production. Let $s$ be the first such period, i.e., $x_s > 0$. First,

$$\sum_{t \in N \setminus S} x_t \geq \sum_{t \in N_{s-1}} x_t = \sum_{t \in N_{s-1}} x_t \geq d_{1,s-1}$$

Now choose $\tau$ as small as possible such there are set-ups in all periods $\{\tau, \ldots, s\}$, i.e., $z_\tau = y_\tau = \ldots = y_s = 1$. Since at least one of these variables appears in the left-hand side of the $(l, R, S)$-inequality with a coefficient which is at least $d_{s,t}$, the inequality is satisfied by this solution.

\[\square\]

3 Linear description of ELSS and separation for the $(l, R, S)$-inequalities

The main result in this section is that addition of the $(l, R, S)$-inequalities to the model for ELSS gives a complete polyhedral description. More precisely

**Theorem 2** The constraints 2, 4, 6, the variable bounding constraints $0 \leq y_t, z_t \leq 1$ ($1 \leq t \leq T$), and the $(l, R, S)$-inequalities 8 describe the convex hull of ELSS.

Note that the inequalities 3 are $(t, \emptyset, \emptyset)$-inequalities. The inequalities 5 can be derived from 2 and 8, where $S = R = \{t\}$ and $l = T$.

The technique we will use to prove the theorem is somewhat different from the usual techniques. Basically, the idea is to show that for an arbitrary objective function, denoted by $\sum_{t=1}^T (\alpha_t x_t + \beta_t y_t + \gamma_t z_t)$, the set of optimal solutions, denoted by $M(\alpha, \beta, \gamma)$, satisfies one of the inequality constraints as equality. Clearly, then the inequality constraints must include all facets of the convex hull of solutions.

We consider an arbitrary cost function $\sum_{t=1}^T (\alpha_t x_t + \beta_t y_t + \gamma_t z_t)$, and we denote its set of optimal solutions by $M(\alpha, \beta, \gamma)$.

**Case 0:** $\min\{\alpha_t | t = 1, \ldots, T\} = \delta \neq 0$

As $\sum_{t=1}^T x_t = d_{1,T}$ we can subtract $\delta$ times the inequality 2 from the objective function without changing the set of optimal solutions.

Thus, in the following, we can assume that $\min\{\alpha_t | t = 1, \ldots, T\} = 0$.

**Case 1:** $\gamma_t < 0$ for some $t \in \{1, \ldots, T\}$.

Any solution with $x_t = 0$ can be improved setting $z_t = 1$. Thus $M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | x_t = 1\}$.

**Case 2:** $\gamma_t \geq 0$ for all $t$, $\beta_t < 0$ for some $t$.

(i) $\beta_t + \gamma_t < 0$ for some $t$. 

\[5\]
Any solution with \( y_t = 0 \) can be improved by setting \( y_t = z_t = 1 \). Thus, \( M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | y_t = 1\}. \)

(ii) \( \beta_t + \gamma_t \geq 0 \) for all \( t \).

Let \( s = \min\{t | \beta_t < 0\} \). We show that \( M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | y_{s-1} + z_s = y_s\}. \) As \( \gamma_s > 0 \), any solution with \( y_{s-1} = z_s = 1 \) can be improved by setting \( z_s = 0 \). In addition, any solution with \( y_{s-1} + z_s = 1 \) and \( y_s = 0 \) can be improved by setting \( y_s = 1 \). Thus, the claim follows.

We are left with objective functions satisfying \( \min\{\alpha_t | t = 1, \ldots, T\} = 0; \beta_t \geq 0 (t = 1, \ldots, T); \gamma_t \geq 0 (t = 1, \ldots, T). \) In the following, \( l, S \) and \( R \) are determined, step by step, in this order. First, the period \( l \) is fixed. This is done such that the demands of the periods \( \{l + 1, \ldots, T\} \) can be produced at no cost, with respect to the given objective function. For notational convenience a period \( T + 1 \) is defined with \( \alpha_{T+1} = 0, \beta_{T+1} = 0, \gamma_{T+1} = 0, \) and the demand of \( T + 1 \) is supposed to be positive.

Choice of \( l \): Take \( k \) and \( m (1 \leq k \leq m \leq T + 1) \), both minimal, such that \( \gamma_k = \beta_k = \ldots = \beta_m = \alpha_m = 0. \) The period \( l \) is chosen as the last period in \( \{1, \ldots, m - 1\} \) with positive demand, i.e., \( d_l > 0, \) and \( d_{l+1} = \ldots = d_{m-1} = 0. \) If \( d_{1,m-1} = 0, \) then \( l \equiv 0. \)

Case 3: Since \( d_{l+1,m-1} = 0, \) any production in the periods \( \{l + 1, \ldots, T\} \) can be moved to period \( m \) at no cost, by taking \( z_k = y_k = \ldots = y_m = 1. \)

It follows that

(i) If \( \alpha_t > 0 \) for some \( t \geq l + 1, \) then \( M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | x_t = 0\}. \)

(ii) If \( \beta_t > 0 \) for some \( t \geq \min\{k, l + 1\}, \) then \( M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | y_t = 0\}. \)

(iii) If \( \gamma_t > 0 \) for some \( t \geq \min\{k, l + 1\}, \) then \( M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | z_t = 0\}. \)

If the contrary holds in one of these cases, then the solution can be improved by taking the solution mentioned above.

Note that, if \( l = 0, \) then we are finished. In that case the only objective function left is the one with zero cost coefficients for all variables. Thus, in the following we may assume that \( l > 0. \) Moreover, we may suppose that for \( t \geq \min\{k, l + 1\} \) we have \( \beta_t = \gamma_t = 0, \) and for \( t \geq l + 1, \) we have \( \alpha_t = 0. \) We proceed by specifying the choices of \( S \) and \( R. \)

Choice of \( S: S := \{t \leq l | \alpha_t = 0\}. \)
Choice of \( R: R := \{t \in S | \beta_t > 0\}. \)

With regard to the following case note that, if \( l = T, \) then \( S \) is not empty, since \( \min\{\alpha_t | t = 1, \ldots, T\} = 0. \)

Case 4: If \( S = \emptyset, \) then \( M(\alpha, \beta, \gamma) \subseteq \{(x, y, z) | \sum_{t=1}^{l} x_t = d_1, l\}. \)

\( S = \emptyset \) implies that \( l < T \) and \( \alpha_t > 0 \) for \( t \in \{1, \ldots, l\}. \) Therefore, if \( I_t > 0 \) (the inventory at the end of period \( t \)), then the production costs can be reduced by transferring units of
production from the last production period in \{1, \ldots , l\} to period \(l + 1\). Note that \(\alpha_{l+1} = \beta_{l+1} = \gamma_{l+1} = 0\). This proves the claim, since \(I_t = 0\) implies \(\sum_{t=1}^{I_t} x_t = d_{1,t}\).

In the following, we may suppose that \(S \neq \emptyset\).

Case 5: Suppose that, for some \(t \in S \setminus R\), there is an \(s \in \{p(t) + 1, \ldots , t\}\) with \(\beta_s > 0\). Then for this \(t\) the following holds. \(M(\alpha, \beta, \gamma) \subseteq \{(x,y,z)|y_s + z_{s+1} + \ldots + z_l = y_t\}\).

Note that the equation \(y_s + z_{s+1} + \ldots + z_l = y_t\) implies that \(y_{t-1} + z_t = y_t\) (\(s < t\) since \(t \notin R\)). Before we proceed, some properties of the coefficients of the variables are stated. First, \(\beta_s > 0\) and \(\beta_{s+1}, \ldots , \beta_t = 0\) (since \(t \notin R\)). Second, \(\alpha_s, \ldots , \alpha_{t-1} > 0\), since \(s \in \{p(t) + 1, \ldots , t-1\}\), and, since \(t \in S\) we have \(\alpha_t = 0\). Finally, if \(\gamma_t = 0\) for some \(\tau \in \{s + 1, \ldots , t\}\), then \(\gamma_{\tau} = \beta_{\tau} = \ldots = \beta_t = \alpha_t = 0\), which implies \(l < t\), a contradiction. It follows that \(\gamma_{s+1}, \ldots , \gamma_t > 0\). Concluding, the variables on the left-hand side of the constraint \(y_s + z_{s+1} + \ldots + z_l = y_t\) have positive cost coefficients, and the variable on the right-hand side has cost coefficient zero.

(i) First, suppose that \(y_s + z_{s+1} + \ldots + z_l \geq 2\). Let \(u\) be the first period in \(\{s + 1, \ldots , t\}\) in which the variable \(z_u\) has value 1. If \(y_u = 1\), then a cost reduction can be achieved by setting \(y_{u+1}, \ldots , y_u = 1, \) and \(z_u = 0\). If \(y_u = 0\), then let \(v\) be the first period after \(u\) with \(z_v = 1\). In this case, a cost reduction can be obtained by setting \(y_u = \ldots = y_v = 1\) and \(z_v = 0\).

(ii) Second, suppose that \(y_s + z_{s+1} + \ldots + z_l = 1\) and \(y_t = 0\). If no production takes place in \(s, \ldots , t-1\), then setting \(y_s + z_{s+1} + \ldots + z_l = 0\) leads to a cost reduction without losing feasibility. Otherwise, production takes place in one or more of the periods \(\{s, \ldots , t-1\}\). Let \(u\) be the last production period in \(\{s, \ldots , t-1\}\). If \(I_{t-1} > 0\), then production from period \(u\) can be transferred to \(t\), at a cost reduction, since \(\beta_{u+1} = \ldots = \beta_t = \alpha_t = 0\), and \(\alpha_s, \ldots , \alpha_{t-1} > 0\). Thus, we may assume that \(I_{t-1} = 0\). Then \(d_t\) must be zero, since \(y_t = 0\), which implies \(l > t\). Thus, the positive demand \(d_{t+1}/t\) is produced at a positive cost in some of the periods in the set \(\{t+1, \ldots , l\}\), otherwise \(l\) would have been chosen smaller (see choice of \(l\)). But now it is cheaper to produce this demand in period \(t\) at zero cost. Of course, this implies \(y_{u+1} = \ldots = y_t = 1\), where \(u\) is the last production period in \(\{s, \ldots , t-1\}\).

In the following we may assume that for each \(t \in S \setminus R\) we have \(\beta_{p(t)+1} = \ldots = \beta_t = 0\). The following case treats the problem, where the first element in \(S\) is in not in \(R\).

Case 6: If \(\beta_1 = \ldots = \beta_t = 0\), where \(t\) is the first element in \(S\), then we have the following: \(M(\alpha, \beta, \gamma) \subseteq \{(x,y,z)|\sum_{t=1}^{l-1} x_t = d_{1,t-1}\}\).

By definition of \(t\) we have \(\{1, \ldots , t-1\} \cap S = \emptyset\), and thus \(\alpha_1, \ldots , \alpha_{t-1} > 0\). Moreover, since \(t \in S\), we have \(\alpha_{t} = 0\). If \(y_1 + \ldots + y_t = 0\), then \(\sum_{t=1}^{l} x_t = 0 = d_{1,t-1}\). If \(y_1 + \ldots + y_t \geq 1\), then we can produce in \(t\) at no additional production costs, and thus, since \(\alpha_1, \ldots , \alpha_{t-1} > 0\), we have \(I_{t-1} = 0\). Therefore, \(\sum_{t=1}^{l-1} x_t = d_{1,t-1}\).

It follows from case 6, that we may assume that the first period in \(S\) is in \(R\).

Case 7: We claim that the following \((l,R,S)\)-inequality

\[
\sum_{t \in N \setminus S} x_t + \sum_{t \in R} d_{t,R} y_t + \sum_{t \in S \setminus R} d_{t,l}(x_{p(t)+1} + \ldots + z_t) \geq d_{1,l}
\]

(9)
is satisfied as equality for all points in \( M(\alpha, \beta, \gamma) \).

The following properties concerning the cost coefficients are valid.

For \( t \in N \setminus S \), \( \alpha_t > 0 \), by definition of \( S \).

For \( t \in R \), \( \beta_t > 0 \), by definition of \( R \).

For \( t \in S \setminus R \), \( \gamma_{p(t)+1, \ldots, \gamma_t} > 0 \), since \( \beta_{p(t)+1} = \ldots = \beta_t = \alpha_t = 0 \). Otherwise, \( I < t \).

(i) Suppose \( y_t = 0 \) for all \( t \in R \) and \( z_{p(t)+1} + \ldots + z_t = 0 \) for all \( t \in S \setminus R \). Then \( y_t = 0 \) for all \( t \in S \), which implies that there is no production in the periods of \( S \). Moreover, the contribution of the variables in the left-hand side of the periods in \( S \) is zero. It is left to prove that \( \sum_{t \in N \setminus S} x_t = d_{1,l} \). If \( I_l > 0 \), then the costs can be decreased by transferring production from the last production period in \( \{1, \ldots, I\} \) to \( l + 1 \). Recall that \( \alpha_{l+1} = \beta_{l+1} = \gamma_{l+1} = 0 \). Thus \( I_l = 0 \). This proves the claim.

Now take the minimal \( t \in S \), such that \( y_t = 1 \) (if \( t \in R \)) or \( z_{p(t)+1} + \ldots + z_t \geq 1 \) (if \( t \in S \setminus R \)).

(ii) If \( t \in R \), then by the minimality of \( t \) this gives the following.

(a) For \( \tau < t, \tau \in R \), \( y_\tau = 0 \).

(b) For \( \tau < t, \tau \in S \setminus R \), \( z_{p(\tau)+1} + \ldots + z_\tau = 0 \).

Since \( \alpha_t = 0 \), and \( y_t = 1 \), the production of \( d_{t,l} \) can take place at no additional cost in \( t \). Therefore, the following hold.

(c) For \( \tau > t, \tau \in N \setminus S \), \( x_\tau = 0 \), since \( \alpha_\tau > 0 \).

(d) For \( \tau > t, \tau \in R \), \( y_\tau = 0 \), since \( \beta_\tau > 0 \).

(e) For \( \tau > t, \tau \in S \setminus R \), \( z_{p(\tau)+1} + \ldots + z_\tau = 0 \), since \( \gamma_{p(\tau)+1, \ldots, \gamma_\tau} > 0 \).

Thus, the value of the left-hand side of the \((I, R, S)\)-inequality is reduced to the quantity \( \sum_{\tau \in N_{t-1} \setminus S} x_\tau + d_{1,l} \). It remains to be proved that \( \sum_{\tau \in N_{t-1} \setminus S} x_\tau \) equals \( d_{1,l} \). Suppose \( I_{t-1} > 0 \). From the minimality of \( t \), it follows that there are no production periods in \( S \cap N_{t-1} \), and therefore we can reduce the production costs by transferring production from the last production period in \( N_{t-1} \) to \( t \).

(iii) If \( t \in S \setminus R \), then let \( s \in \{p(t)+1, \ldots, t\} \) be the first period with \( z_s = 1 \).

By the minimality of \( t \) we have the following.

(a) For \( \tau < t, \tau \in R \), \( y_\tau = 0 \).

(b) For \( \tau < t, \tau \in S \setminus R \), \( z_{p(\tau)+1} + \ldots + z_\tau = 0 \).

Moreover, since \( t \) is minimal \( \{s, \ldots, t-1\} \cap S = \emptyset \), and therefore \( p(t) < s \). Since \( \beta_s = \ldots = \beta_t = \alpha_t = 0 \), the production of \( d_{s,l} \) can take place at no additional cost in \( t \). Therefore

(c) For \( \tau > t, \tau \in N \setminus S, x_\tau = 0 \), since \( \alpha_\tau > 0 \).

(d) For \( \tau > t, \tau \in R, y_\tau = 0 \), since \( \beta_\tau > 0 \).

(e) For \( \tau > t, \tau \in S \setminus R, z_{p(\tau)+1} + \ldots + z_\tau = 0 \), since \( \gamma_{p(\tau)+1, \ldots, \gamma_\tau} > 0 \).
Note that \(z_{t+1}, \ldots, z_t = 0\), for otherwise the costs can be reduced by setting \(y_{t+1} = \ldots = y_t = 1\). By the minimality of \(s\), it follows that \(z_{p(t)+1} = \ldots = z_{s-1} = 0\).

Since the coefficient of \(z_s\) is \(d_{t,l}\), the value of the left-hand side of the \((l, R, S)\)-inequality is reduced to the quantity \(\sum_{\tau \in N_{t-1} \setminus S} x_{\tau} + d_{t,l}\). It remains to be proved that \(\sum_{\tau \in N_{t-1} \setminus S} x_{\tau}\) equals \(d_{1,t-1}\). Suppose \(I_{t-1} > 0\). From the minimality of \(t\), it follows that there are no production periods in \(S \cap N_{t-1}\), and therefore we can reduce the production costs by transferring production from the last production period in \(N_{t-1}\) to \(t\).

This ends the proof of the theorem. The following theorem shows which of the \((l, R, S)\)-inequalities define facets of the convex hull of ELSS (for a proof see van Hoesel [5]).

**Theorem 3** Suppose that the demands \(d_t\) \((1 \leq t \leq T)\) are positive. The \((l, R, S)\)-inequalities define facets, if and only if the following conditions hold.

1. Period \(1 \notin S \neq \emptyset\).
2. \(l < T\) or \(|R| = 1\)

**A separation algorithm for the \((l, R, S)\)-inequalities.**

The separation algorithm for the \((l, R, S)\)-inequalities can be formulated as a shortest path problem as will be shown. A network is defined for a fixed \(l\), in which each \((l, R, S)\)-inequality corresponds to a unique path. The length of such a path corresponds to the value of the left-hand side of the \((l, R, S)\)-inequality for a given feasible solution \((x, y, z)\).

There are three types of nodes:

Type 1: \(u_t\) for \(t \in \{0, \ldots, l\}\);
Type 2: \(v_t\) for \(t \in \{1, \ldots, l\}\);
Type 3: \(w_t\) for \(t \in \{2, \ldots, l\}\).

Moreover, an end-node \(n\) is defined. Note that the nodes \(v_0, w_0\) and \(w_1\) are not defined.

There are also three types of arcs:

Type 1: \(\text{arcs } (u_{t-1}, u_t), (v_{t-1}, u_t), (w_{t-1}, u_t)\) with cost \(x_t\);
Type 2: \(\text{arcs } (u_{t-1}, v_t), (v_{t-1}, v_t), (w_{t-1}, v_t)\) with cost \(d_{t,l}y_t\);
Type 3: \(\text{arcs } (v_s, w_t), (w_s, w_t)\) with cost \(\sum_{\tau=s+1}^{t-1} x_{\tau} + \sum_{\tau=s+1}^{t-1} d_{t,l}z_{\tau}\).

These arcs are, of course, only defined for existing nodes. They are depicted in figure 1.

There are three arcs with head \(n\): \((u_1, n), (v_1, n)\) and \((w_1, n)\), all with zero costs.

Consider an \((l, R, S)\)-inequality. The path in the network corresponding to this inequality consists of the following nodes.

1. \(t \notin S\) and either \(\{t+1, \ldots, l\} \cap S = \emptyset\) or the first element in \(\{t+1, \ldots, l\} \cap S\) is in \(R\).

Then node \(u_t\) is entered by an arc with cost \(x_t\).
Figure 1: Arcs in the network for the separation problem.

costs $z_t$ and $d_t, y_t$
(2) \( t \in R \). Then node \( v_t \) is entered by an arc with cost \( d_t, y_t \).

(3) \( t \in S \setminus R \). Then node \( w_t \) is entered by an arc starting in \( v_{p(t)} \) or \( w_{p(t)} \) with cost \( \sum_{r=p(t)+1}^{t-1} d_r, z_r \).

If \( s \notin S \) and the first element in \( \{ s + 1, \ldots, l \} \cap S \), say \( t \), is in \( S \setminus R \). Then none of the nodes corresponding to \( s \) is entered, since an arc is in the path from a node corresponding to \( p(t) < s \) to a node corresponding to \( t \). The costs of the variables of period \( s \) are part of the costs of this arc.

The total costs of the path from \( u_0 \) to \( n \) are equal to the value of the left-hand side of the \((l, R, S)\)-inequality. The right-hand side of this inequality is \( d_{1,l} \). Therefore, the search for the most violated inequality for a given \( l \) amounts to finding the shortest path in the network. If this shortest path has a cost less than \( d_{1,l} \), then a violated inequality has been detected.

EXAMPLE: Consider the following four-period instance of ELSS.

<table>
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<td>( d_t )</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>( c_t )</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f_t )</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( g_t )</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

An optimal solution of the linear programming relaxation of ELSS for this instance is

<table>
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<th>3</th>
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<tbody>
<tr>
<td>( x_t )</td>
<td>$\frac{3}{5}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td></td>
</tr>
<tr>
<td>( y_t )</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td></td>
</tr>
<tr>
<td>( z_t )</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td></td>
</tr>
</tbody>
</table>

The network for the separation problem corresponding to \( l = 4 \) is given in figure 2.

The path \((u_0, u_1, v_2, w_4, n)\) has a cost of \( 3\frac{2}{3} \), and thus, since the right-hand side is \( d_{1,4} = 4 \), the inequality corresponding to the path \((u_0, u_1, v_2, w_4, n)\) is violated. This inequality is

\[
\begin{align*}
2x_1 + y_2 + 3z_3 & \geq 4 \\
+2x_3 + z_3 & \geq 4
\end{align*}
\]

This example is the smallest for which all inequalities described in Wolsey [18] are satisfied.

The complexity of the separation problem is determined as follows. There are \( O(l^2) \) arcs in the network, and since it is acyclic the shortest path problem in the network can be solved in \( O(l^2) \) time. Repeated for each period \( l \in \{1, \ldots, T\} \) this leads to an \( O(T^3) \) algorithm to find the most violated \((l, R, S)\)-inequality. This is to be compared with the single max-flow calculation on a graph with \( O(T^3) \) nodes derived in Rardin and Wolsey [14].
In this section, we consider the uncapacitated facility location reformulation of ELSS (ELSS-UFL) in which the production variables are split. The variables $x_{t,r}$ ($1 \leq t \leq \tau \leq T$) are introduced as: the production of the item in period $t$ to satisfy demand in period $\tau$. Clearly, the connection with the original production variables is $x_t = \sum_{\tau=t}^{T} x_{t,r}$. ELSS-UFL is modeled as follows.

$$\text{(ELSS-UFL)} \quad \min \sum_{t=1}^{T} (g_t z_t + f_t y_t + c_t(\sum_{\tau=t}^{T} x_{t,r}))$$

s.t.  
\begin{align*}
\sum_{\tau=t}^{T} x_{t,r} &= d_\tau \quad (1 \leq \tau \leq T) \quad (11) \\
y_t &\leq y_{t-1} + z_t \quad (y_0 = 0) \quad (1 \leq t \leq T) \quad (12) \\
x_{t,r} &\leq d_\tau y_t \quad (1 \leq t \leq \tau \leq T) \quad (13) \\
x_{t,r} &\geq 0 \quad (1 \leq t \leq \tau \leq T) \quad (14) \\
y_t, z_t &\in \{0, 1\} \quad (1 \leq t \leq T) \quad (15)
\end{align*}

The LP-relaxation of ELSS-UFL is not tight, in the sense that it still allows fractional solutions. By adding the following constraints, the so-called $(\tau,s,\tau)$-inequalities, we get a reformulation of ELSS which is at least as strong as the formulation given in the previous section.
Let \( 1 \leq r \leq s \leq \tau \leq T \).

\[
\sum_{t \in N_r \setminus \{r, \ldots, s\}} x_{t,r} + d_r(y_r + z_{r+1} + \ldots + z_s) \geq d_r
\]

These inequalities can be found in Wolsey [18]. The \((r, s, \tau)\)-inequalities can be viewed as cuts in the following fixed charge multi-commodity network flow problem. The flow network consists of the following vertices and arcs. There is a source \( s \), and there are three layers of nodes: \( \{u_t | 1 \leq t \leq T\} \), \( \{v_t | 1 \leq t \leq T\} \), and \( \{w_t | 1 \leq t \leq T\} \).

The arcs \((s, u_t), (1 \leq t \leq T)\), model the fixed start-up charges, i.e., if such an arc contains a positive flow, then \( z_t = 1 \).

The arcs \((u_t, v_t), (1 \leq t \leq T)\), model the fixed set-up charges.

The arcs \((v_t, u_{t+1}), (1 \leq t \leq T - 1)\), are included to allow for multiple set-ups in consecutive periods without a start-up in these periods.

The arcs \((v_t, w_t), (1 \leq t \leq T)\), model the production in period \( t \).

The production is exactly equal to the flow through the arc. Finally, the flows through the arcs \((w_t, w_{t+1}), (1 \leq t \leq T - 1)\), denote the inventory at the end of period \( t \), or equivalently, at the beginning of period \( t + 1 \). There are \( T \) different commodities \( \tau \), \( (1 \leq \tau \leq T) \), in the network, consisting of the source \( s \) and sink \( w_\tau \), and with a demand of \( d_{\tau} \) units of flow. For a commodity \( \tau \), the flow through the arcs \((v_t, w_t), (1 \leq t \leq \tau)\), is the production in period \( t \) for period \( \tau \), and therefore, it is denoted by \( x_{t,\tau} \). The arcs \((w_t, w_{t+1}), (1 \leq t \leq \tau - 1)\), contain the inventory at the end of period \( t \) for demand in period \( \tau \). It is denoted by \( I_{t,\tau} \).

EXAMPLE:

Consider a feasible flow of \( d_r \) units of a given commodity \( \tau \). The flow "through" a cut \((V, W)\) separating \( s \) and \( w_\tau \) is at least \( d_r \) units. For each type of arc one can derive an upperbound on the flow through such arcs as follows. The flow through an arc \((s, u_t)\) is bounded by \( d_r z_t \), the flow through an arc \((u_t, v_t)\) is bounded by \( d_r y_t \). Finally, the flow through an arc \((v_t, w_t)\) equals \( x_{t,\tau} \). We will consider cuts that "contain" these types of arcs only. In that case, it follows trivially that the sum of these upper bounds for all arcs in the cut, constitutes an upperbound on the flow \( d_r \).

The cut \((V, W)\) that constitutes the validity of the \((r, s, \tau)\)-inequality is the following:

\[
V = \{s, u_1, \ldots, u_r, u_{s+1}, \ldots, u_{\tau}, v_1, \ldots, v_{\tau-1}, v_{s+1}, \ldots, v_T\};
\]

\[
W = \{u_{r+1}, \ldots, u_s, v_r, \ldots, v_s, w_1, \ldots, w_T\}.
\]

Clearly, we may take more closed intervals \( \{r, \ldots, s\} \) as long as they are mutually disjoint. In fact for a given \( l, R \) and \( S \) we take the maximal closed intervals that start with a period in \( R \) and end with a period in \( S \setminus R \) such that no intermediate periods are in \( R \). This leads to the following valid inequality for \( \tau \).

\[
\sum_{t \in N_r \setminus S_r} x_{t,\tau} + \sum_{t \in R_r} d_r y_t + \sum_{t \in S_r \setminus R_r} d_r (z_{p(t)+1} + \ldots + z_t) \geq d_r
\]
Figure 3: Fixed-charge flow network.
Here \( R_\tau = N_\tau \cap R \) and \( S_\tau = N_\tau \cap S \). Addition of these inequalities for \( \tau = 1, \ldots, l \) results in the \((l, R, S)\)-inequality. This result can be found in Rardin and Wolsey [14]. There it has been shown that the \((r, s, \tau)\)-inequalities are so-called di-cut inequalities in the uncapacitated fixed charge network flow model.

**NOTE.** If the demands in all periods are positive one can get a more compact model than ELSS-UFL. The latter contains \( O(T^3) \) constraints. This can be reduced to \( O(T^2) \) as follows. ELSS has always an optimal solution in which \( x_{t,\tau}/d_\tau \) are non-increasing in \( \tau \) (\( t \) fixed). A simple exchange argument then shows that the \((r, s, s)\)-inequalities suffice.

5 Concluding remarks

We characterized the convex hull of the set of feasible solutions of the economic lot-sizing problem with start-up costs by use of the \((l, R, S)\)-inequalities, and we provided a separation algorithm for these inequalities. Besides we showed that the polynomial size uncapacitated facility location reformulation is tight. These results are of a purely theoretical nature. Computational testing should reveal which of the two formulations is best used in practical problems in which ELSS appears as a relaxation.

References


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<td>Product forms as a solution base for queueing systems</td>
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<td>L.C.G.J.M. Habets</td>
<td>A Reachability Test for Systems over Polynomial Rings using Gröbner Bases</td>
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<td>92-39</td>
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<td>G.J. van Houtum</td>
<td>The compensation approach for three or more dimensional random walks</td>
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<td>Bounds for expected loss in Bayesian decision theory with imprecise prior probabilities</td>
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<td>92-45</td>
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<td>C.P.M. van Hoesel A.P.M. Wagelmans B. Moerman</td>
<td>Using geometric techniques to improve dynamic programming algorithms for the economic lot-sizing problems and extensions</td>
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<td>92-46</td>
<td>November</td>
<td>C.P.M. van Hoesel A.P.M. Wagelmans L.A. Wolsey</td>
<td>Polyhedral characterization of the Economic Lot-sizing problem with Start-up costs</td>
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