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Spectral properties of decaying turbulence in electron magnetohydrodynamics

T. M. Abdalla^{a)}

FOM Institute for Plasma Physics Rijnhuizen, Association EURATOM-FOM, Trilateral Euregio Cluster, P.O. Box 1207, 3430 BE Nieuwegein, The Netherlands

V. P. Lakhin

RRC "Kurchatov Institute," Institute of Nuclear Fusion, Kurchatov Sqr. 1, 123182 Moscow, Russia

T. J. Schep

Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

E. Westerhof

FOM Institute for Plasma Physics Rijnhuizen, Association EURATOM-FOM, Trilateral Euregio Cluster, P.O. Box 1207, 3430 BE Nieuwegein, The Netherlands

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The spectral properties of decaying turbulence in $2\frac{1}{2}$ -dimensional electron magnetohydrodynamics are studied numerically. In the range $kd_e < 1$ the energy exhibits a direct cascade while mean square momentum exhibits an inverse cascade. Their spectra are characterized by $k^{-7/3}$ and $k^{-13/3}$, respectively. The self-similar decay state of the turbulence is reached after an initial phase of fast exchange between the axial and poloidal magnetic energies. The time behavior $t^{-2/3}$ of the total energy is found to be consistent with that obtained from selective decay. The maximum of the energy spectrum shifts towards low mode numbers and decays in time as t^{-1} , in agreement with the infrared scaling of the turbulence. In the large d_e limit, both energy and mean square generalized momentum exhibit direct cascades. No stationary turbulent state could be found as long as the axial kinetic energy is large as compared to the poloidal kinetic energy initially. The global physical quantities decay well before turbulent macroscopic quantities have established similar space-time behavior, and the turbulence is infected by the lack of stationarity. The system decouples into a Navier-Stokes equation and a passive scalar equation only if the poloidal kinetic energy is larger than or equal to the axial kinetic energy. In this limit the $k^{-5/3}$ and k^{-3} spectra of the poloidal kinetic energy are recovered. © 2003 American Institute of Physics. [DOI: 10.1063/1.1588308]

I. INTRODUCTION

In a magnetized plasma, the dynamics of processes with length scales below the ion inertial skin depth and time scales in between the ion and electron plasma and gyro periods are governed by the motion of the electrons and can be described by electron magnetohydrodynamics (EMHD).^{1,2} The EMHD model is used to analyze nonlinear vortices.^{3,4} It is also used to model collisionless magnetic reconnection⁵ and to study vortex interactions in the wake of a laser pulse.⁶ Since EMHD describes phenomena on time scales just above the electron-gyro period, it is used to study the possible generation of small scale structures during electron cyclotron resonant heating (ECRH).⁷ Special interest in EMHD has arisen recently to study plasma turbulence.^{8–12}

Turbulence in a magnetized plasma is in general strongly anisotropic. The scale lengths of the perturbation along the strong field are typically much longer than the scale lengths in the perpendicular plane. This anisotropy tends to restrict the dynamics to two dimensions while vector quantities may still have all three components. As a result phenomena taking

place in such systems are predominantly $2\frac{1}{2}$ dimensional ($2\frac{1}{2}$ D).

In case of uniform equilibrium density and in the absence of density perturbations and heating effects, the $2\frac{1}{2}$ D EMHD model contains a single length scale, the electron inertial skin depth d_e . Turbulent processes in EMHD exhibit different character for scales much smaller or much larger than d_e . The $2\frac{1}{2}$ D ideal model conserves the total energy and possesses two sets of Casimirs. The quadratic versions of the latter are the mean square of the generalized parallel momentum and the mean generalized helicity. These ideal invariants enforce the system to exhibit both direct and inverse spectral cascades.

In this paper we will study the properties of the turbulence in $2\frac{1}{2}$ D EMHD model by investigating the spectral decay laws, the cascade directions of the quadratic ideal invariants of the model and the time behavior of the energy. The emphasis is on the numerical investigations of $2\frac{1}{2}$ D decaying EMHD turbulence. Decaying turbulence may be affected by the lack of stationarity since there is no continuous source of energy input. However, if the spectra of the conserved quantities are established on time scales that are short as com-

^{a)}Electronic mail: abdalla@rijnh.nl

pared to the time scale on which the global physical quantities decay, then the turbulence is quasistationary.

Previous numerical studies of $2\frac{1}{2}$ D decaying EMHD turbulence^{8,9,12} have focused on investigating the spectral properties of the energy. These studies have shown that the energy spectrum follows a $k^{-7/3}$ decay law for $kd_e < 1$ and $k^{-5/3}$ for $kd_e > 1$. In Refs. 8 and 9 it is also shown that in the phase of fully developed turbulence, energy dissipation rates are independent of the diffusion coefficients. The issue of the validity of a Kolmogorov-type law in EMHD has been the subject of study in Ref. 10. Upon assuming that the energy flux is independent on the scales in the inertial range, the analogue of the Kolmogorov four-fifth law is derived for $2\frac{1}{2}$ D homogeneous, isotropic EMHD. Numerical simulations¹⁰ show that, in the inertial range, the energy flux is towards small scales and does not depend on wave number.

In the present paper we will report numerical simulations of $2\frac{1}{2}$ D decaying isotropic homogeneous EMHD turbulence. We will concentrate on investigating the spectral properties of energy and mean square generalized momentum, whereas net generalized helicity will be equal to zero. The time behavior of the turbulent energy is analyzed. In addition, we will investigate the role of the ratio of the energies related with the two fields that govern the turbulence. The equations are solved numerically on a quadratic box of size $2\pi \times 2\pi$ with double periodic boundary condition, using a pseudospectral method with 512×512 modes. The initial perturbations are centered around a mode number k_0 in the inertial range. The mode number k_0 is larger than the global mode number k_g associated with the global scale and smaller than the mode number k_μ associated with the dissipation scale.

The inertial range is too small to cover accurately both $kd_e < 1$ and $kd_e > 1$ ranges in one simulation. For this reason our numerical simulations will be performed such that over the entire inertial range either $kd_e < 1$ or $kd_e > 1$.

A Kolmogorov-type of scaling for the spectra of the ideal invariants is obtained by using the scaling symmetry of the equations and taking the spectral flux of energy to be the invariant quantity.¹³ Because the EMHD model contains the scale length d_e , such a symmetry only exists in the limits $kd_e \ll 1$ and $kd_e \gg 1$. Equilibrium statistics are used to obtain indications about the cascade directions of the quadratic ideal invariants in the long scale range $kd_e < 1$ as well as in the small scale range $kd_e > 1$.¹¹

The spectral decay laws obtained from the numerical simulations are analyzed and compared with those obtained from the scaling symmetries of the governing equations. Cascade directions of the ideal invariants are determined by following the evolution of the global quantities (energy and mean square generalized momentum) in time together with the time trace of the centroid wave number for each of the conserved quantities, see for example, Ref. 14. These cascade directions are found to agree with those obtained from equilibrium statistics.

The time behavior of the turbulent energy is discussed on the bases of two methods that were previously applied in Navier–Stokes as well as in MHD turbulence. The first of

these is selective decay,^{16–19} which is applicable if certain quantities decay at slower rate than others and can thus be considered constant. The second method is based on the assumption that the turbulence possesses an infrared asymptotic scaling.²³ The former method is used to determine the time behavior of the total energy. The latter method together with a constant linear momentum, is used to obtain the time behavior of the peak of the energy spectrum in k -space during the early evolution of the turbulence.

Power law spectra are established when the turbulence is fully developed. In the small d_e regime, the self-similarity of the turbulence shows up in the constancy in time of the ratio of the energies E_b , which consist of the axial magnetic energy and poloidal kinetic energy, and E_ψ , which is the sum of the poloidal magnetic energy and the axial kinetic energy. Further, the numerical investigations reveal that the time scale at which power law spectra are established depends on the initial value of the energy ratio E_b/E_ψ . The smaller the initial value of their ratio, the shorter is the characteristic time on which the spectra are established.

We do not take into account a background magnetic field. Therefore, our system does not describe linear whistler modes and the associated tendency towards equipartition of energy.¹² However, the large scale turbulent magnetic field may act as a mean field for the small scale dynamics and our system may contain nonlinear whistler modes. In this paper we will show that if E_b is small as compared to E_ψ initially in the small as well as in the large d_e regimes, the nonlinear exchange between the energies E_b and E_ψ leads to an approximate constant ratio $E_b/E_\psi < 1$. In the opposite limit when E_b is larger than or equal to E_ψ in the large d_e regime, the system decouples into a Navier–Stokes equation and a passive scalar equation. In this limit the two energies behave independently.

II. MODEL EQUATIONS

The $2\frac{1}{2}$ D EMHD model^{8,9} consists of two coupled equations for the generalized axial vorticity $\Omega = b - d_e^2 \nabla^2 b$ and the generalized parallel momentum $\Psi = \psi - d_e^2 \nabla^2 \psi$. The $2\frac{1}{2}$ D representation of the magnetic field is

$$\mathbf{B} = B_0[(1+b)\mathbf{e}_z + \nabla\psi \times \mathbf{e}_z]. \quad (1)$$

Here $b(x,y,t)$ is the perturbation of the axial magnetic field, $\mathbf{e}_z$ is the unit vector in the ignorable z -direction, B_0 is a large constant, and $\psi(x,y,t)$ is the poloidal flux function. In case of uniform density and for length scales larger than the magnetic Debye radius $(\omega_{Be}/\omega_{pe})d_e$ where density perturbations are negligible, the electron velocity to leading order can be expressed in terms of the magnetic field and is given by

$$\mathbf{v} = -\frac{c}{4\pi n_0 e} \nabla \times \mathbf{B} = -\omega_{Be} d_e^2 (\nabla b \times \mathbf{e}_z + \nabla^2 \psi \mathbf{e}_z). \quad (2)$$

In the absence of pressure perturbations,⁷ the $2\frac{1}{2}$ D model equations are given by^{3,8}

$$\frac{\partial \Omega}{\partial t} = -[b, \Omega] - [\psi, \nabla^2 \psi] - \mu_r (-\nabla^2)^\nu b, \quad (3)$$

$$\frac{\partial \Psi}{\partial t} = -[b, \Psi] - \mu_\nu (-\nabla^2)^\nu \psi. \quad (4)$$

Here the brackets denote the Jacobian $[f, g] \equiv \mathbf{e}_z \cdot \nabla f \times \nabla g$, time and space coordinates are normalized as $t \rightarrow \omega_{Be} d_e^2 t$ and $(x, y) \rightarrow (x/L, y/L)$, where $d_e = c/(L\omega_{pe})$ and L is the size of the calculation box. In Eq. (3), Ω is convected by the streaming potential b . The second term on the right-hand side describes the coupling between the fields b and ψ . Equation (4) describes the convection of Ψ by the streaming function b . Dissipation is included in the form of hyperviscosity. In numerical studies of turbulence one often uses higher order “diffusion” with $\nu > 2$ in order to concentrate dissipation at the smallest scales. Note that $\nu = 1$ corresponds to resistivity and $\nu = 2$ to viscosity. Recently¹⁵ it has been shown that for magnetohydrodynamic (MHD) turbulent flows with helicity, hyperdiffusivity causes the dynamo-generated magnetic field to saturate faster than in case of normal diffusivity. In our case hyperviscosity restricts both energy and mean square momentum to dissipate at small scales. Hyperviscosity may result in either dissipation or generation of generalized helicity at the viscosity scale k_μ^{-1} . Thus one may argue that helicity generated by hyperviscosity at small scales affects the large scales in case, $d_e^{-1} \geq k_\mu$, where helicity has an inverse cascade. On the other hand in case $d_e^{-1} < k_\mu$, generalized helicity generated by hyperviscosity at small scales will not affect the large scales because in the range $kd_e > 1$ helicity has a direct cascade.

Upon neglecting dissipation, the EMHD equations conserve the energy $E = E_b + E_\psi$, where

$$E_b = \frac{1}{2} \int d\mathbf{r} [b^2 + d_e^2 (|\nabla b|^2)],$$

$$E_\psi = \frac{1}{2} \int d\mathbf{r} [|\nabla \psi|^2 + d_e^2 (\nabla^2 \psi)^2]. \quad (5)$$

The first term in the integral of E_b is the axial magnetic energy and the second one the poloidal kinetic energy. In the integral of E_ψ , the first term represents the poloidal magnetic energy and the second term the axial kinetic energy. The second ideal invariant is a quantity containing only parallel momentum $F = \int d\mathbf{r} g(\Psi)$, where g is an arbitrary function. Finally, the third ideal invariant is $He = \int d\mathbf{r} \Omega f'(\Psi)$ where f is an arbitrary function. As quadratic invariants are the most important in turbulence theory, we will make the choice $f(x) = g(x) = x^2$. The quadratic Casimirs F and He are given by

$$F = \int d\mathbf{r} \Psi^2, \quad He = \int d\mathbf{r} \Omega \Psi. \quad (6)$$

The ideal version of Eqs. (3) and (4) can be cast in the Hamiltonian form,

$$\frac{\partial \xi_k}{\partial t} = J_{ki} \frac{\delta H}{\delta \xi_i}. \quad (7)$$

Here $\xi_1 = \Omega$, $\xi_2 = \Psi$, the Hamiltonian functional, $H = (1/2) \int d\mathbf{r} [b\Omega - \Psi \nabla^2 \psi]$, is the total energy and $\mathbf{J}$ is the cosymplectic form,

$$\mathbf{J} = \begin{pmatrix} [\Omega, \cdot] & [\Psi, \cdot] \\ [\Psi, \cdot] & 0 \end{pmatrix}. \quad (8)$$

The variations of the Hamiltonian are $\delta H / \delta \Omega = b$, $\delta H / \delta \Psi = -\nabla^2 \psi$. Upon using these variations and Eq. (8) into Eq. (7), the ideal version of Eqs. (3) and (4) can be recovered.

Note that in Eqs. (3) and (4), the term $[\psi, \nabla^2 \psi]$ couples a quasigeostrophic (Hasegawa–Mima) equation for the axial vorticity Ω to a passive scalar equation for the generalized flux (momentum) Ψ . If the poloidal field $\psi = 0$ at the beginning of the evolution, then it will remain so at any later time. In this case, our system of equations reduces to a quasigeostrophic (Hasegawa–Mima) equation for the axial vorticity Ω .

In the limit $d_e^2 \nabla^2 \gg 1$, the electric field is negligible, and the dynamical equations (3) and (4) reduce to

$$\frac{\partial d_e^2 \nabla^2 b}{\partial t} = -[b, d_e^2 \nabla^2 b] + [\psi, \nabla^2 \psi] + \mu_\nu (-\nabla^2)^\nu b, \quad (9)$$

and

$$\frac{\partial \nabla^2 \psi}{\partial t} = -[b, \nabla^2 \psi] + \mu_\nu (-\nabla^2)^\nu \frac{\psi}{d_e^2}. \quad (10)$$

Upon neglecting dissipation, this system possesses three quadratic invariants,

$$\hat{F} = \frac{1}{2} \int d\mathbf{r} (\nabla^2 \psi)^2, \quad \hat{H} = \int d\mathbf{r} d_e^2 (\nabla^2 b)(\nabla^2 \psi), \quad (11)$$

and $\hat{H} = (1/2) \int d\mathbf{r} [d_e^2 (\nabla b)^2 - (\nabla \psi)^2]$. Here apart from a multiplicative constant, $\hat{F}$ is the large d_e limit of the mean square generalized momentum integral Eq. (6). Similarly, $\hat{H}$ is obtained from the helicity integral Eq. (6). The invariant $\hat{H}$ is the Hamiltonian integral of the system of Eqs. (9) and (10), and follows from $E - F/2d_e^2$ in the large d_e limit.

If initially $|\nabla \psi| < (d_e/l)|b|$, the coupling term is small, which implies that the poloidal magnetic energy is smaller than poloidal kinetic energy. As a result, the system of Eqs. (9) and (10) decouples into a Navier–Stokes equation for the axial vorticity $\nabla^2 b$ and a passive scalar equation for the axial current $\nabla^2 \psi$. In this limit the resulting set of equations possesses some constants of the motion that cannot be obtained from Eqs. (5) and (6).

III. CASCADE DIRECTIONS NEAR STATISTICAL EQUILIBRIUM

The method of equilibrium statistics is used to obtain an indication about the cascade directions of the quadratic ideal invariants. The statistical equilibrium spectra are obtained for non-dissipative turbulence on the basis of the truncated Fourier representation of the equations and of the quadratic ideal invariants. For this purpose, we use the Fourier representation of the fields

$$(b, \psi) = \sum_k (b_k, \psi_k) \exp(i\mathbf{k} \cdot \mathbf{r}). \quad (12)$$

The series is truncated at some maximum wave vector, so that a finite dimensional system remains. This system pos-

sesses the same quadratic invariants as the original infinite dimensional system, and obeys a (detailed) Liouville theorem. The method of equilibrium statistical mechanics are applied (see, e.g., Ref. 16), which results into the canonical probability density ρ in phase space,

$$\rho = \frac{1}{Z} \exp(-\alpha E - \beta He - \gamma F), \quad (13)$$

where $E, F, He = \int dk (E(k), F(k), He(k))$, the partition function Z is a normalizing constant, and α, β, γ are the Lagrange multipliers (or inverse temperatures) associated with the energy, generalized helicity, and mean square generalized momentum, respectively. From the canonical distribution the following absolute equilibrium spectra are obtained as function of the wave vector $\mathbf{k}$:¹¹

$$E(\mathbf{k}) = \frac{4\alpha k^2 + 2\gamma(1 + d_e^2 k^2)}{4\alpha[\alpha k^2 + \gamma(1 + d_e^2 k^2)] - \beta^2(1 + d_e^2 k^2)^2}, \quad (14)$$

$$F(\mathbf{k}) = \frac{4\alpha(1 + d_e^2 k^2)}{4\alpha[\alpha k^2 + \gamma(1 + d_e^2 k^2)] - \beta^2(1 + d_e^2 k^2)^2}, \quad (15)$$

$$He(\mathbf{k}) = \frac{2\beta(1 + d_e^2 k^2)^2}{4\alpha[\alpha k^2 + \gamma(1 + d_e^2 k^2)] - \beta^2(1 + d_e^2 k^2)^2}. \quad (16)$$

For given values of the invariants E , F , and He , the Lagrange multipliers are determined by summation of the spectra over the wave vector range $[k_{\min}, k_{\max}]$. Note that the convergence of the integrals of the probability density requires that $\alpha E + \beta He + \gamma F$ be positive definite. Energy and mean square generalized momentum in Eqs. (14) and (15) should be positive for all k . This requires that α , γ , and the common denominator should all have the same sign.

Equation (14) shows that for $kd_e < 1$ as well as for $kd_e > 1$, the energy tends to be equally distributed over the modes. As there are many more modes at large k , this implies that energy is concentrated at small scales. This can be interpreted as due to flow of energy towards small scales, which means a direct cascade of energy. The mean square generalized momentum in the long scale range $kd_e < 1$ behaves as k^{-2} and is peaked towards the larger scales, which can be interpreted as due to an inverse cascade of mean square generalized momentum. For the small scale range $kd_e > 1$, the spectrum of mean square generalized momentum weakly depends on k for a large range of k values, which means equipartition of mean square generalized momentum between modes. This implies that mean square generalized momentum is accumulated at small scales, suggesting a direct cascade of mean square generalized momentum. The spectrum of generalized helicity is peaked both towards large scales ($kd_e < 1$) and small scales ($kd_e > 1$) with a minimum at $kd_e = 1$. This suggests an inverse cascade for scales larger than d_e and a direct cascade for smaller scales.

IV. SPECTRAL PROPERTIES AND INERTIAL RANGES

In this section we will discuss the spectral properties of EMHD turbulence by means of the scaling symmetries of the governing equations in the inertial range. The turbulence is assumed to be isotropic and homogeneous. The inertial range

consists of the scales between the global scale k_g^{-1} and the dissipation scale k_μ^{-1} . At $k = k_\mu$ the “Reynolds” number, ratio between the non-linear terms and the dissipation terms, is equal to unity. The dissipation in our system is due to hyperviscosity. The system of equations (3) and (4) is characterized by a set of three “Reynolds” numbers, namely $(d_e^2 b_k / \mu k^2)$, $(\psi_k^2 / \mu b_k k^2)$ and $(b_k / \mu k^4)$. If $kd_e < 1$, the system is characterized by the “Reynolds” numbers $(\psi_k^2 / \mu b_k k^2)$ and $(b_k / \mu k^4)$. In the large d_e limit, the system of Eqs. (3) and (4) reduces to Eqs. (9) and (10), and the corresponding “Reynolds” numbers are $(\psi_k^2 / \mu b_k k^2)$ and $(d_e^2 b_k / \mu k^2)$. From the above discussion we see that in the small as well as in the large d_e regimes, the system is characterized by two “Reynolds” numbers. In both regimes, the nonlinear dynamics will be characterized by the largest one.

Equations (3) and (4) are not scale invariant because they contain the scale length d_e . They are scale invariant in the limits $kd_e \ll 1$ as well as $kd_e \gg 1$. Although these limits do not correspond to the full equations, the results are used to analyze turbulence in the ranges $kd_e < 1$ and $kd_e > 1$.

In the limit $kd_e \ll 1$ and for turbulent length scales l sufficiently far below the global scale length where boundary conditions have to be applied, the equations are invariant under the group of transformations,¹¹

$$l' = \alpha l, \quad t' = \alpha^{1-\rho} t, \quad b' = \alpha^{1+\rho} b, \quad \psi' = \alpha^{2+\rho} \psi, \quad (17)$$

for arbitrary values of the parameters α and ρ .

The scaling transformations (17) together with the assumption that the flux of one of the ideal invariants does not depend on the scale l , determine the spectral decay exponents uniquely. A Kolmogorov-type scaling for the spectra of the ideal invariants in the inertial range is obtained by assuming that in the inertial range, the flux of energy is independent of the scales.¹³ The spectra of the ideal invariants obtained in numerical simulations of decaying EMHD turbulence turn out to be characterized by decay exponents that are consistent with such invariant energy flux.

In the range $kd_e < 1$, dimensional analysis reveals that the energy flux ϵ_E in Eqs. (3) and (4), scales as $b\psi^2/l^4$. The scaling (17) implies $\epsilon'_E = \alpha^{3\rho+1} \epsilon_E$. The invariance of ϵ_E requires $\rho = -1/3$. As a result we obtain

$$\begin{aligned} \langle b^2 \rangle(l) &\propto \epsilon_E^{2/3} l^{4/3}, & \langle \psi^2 \rangle(l) &\propto \epsilon_E^{2/3} l^{10/3}, \\ \langle b\psi \rangle(l) &\propto \epsilon_E^{2/3} l^{7/3}. \end{aligned} \quad (18)$$

Here $\langle b^2 \rangle(l)$ denotes $\langle (b(l+x) - b(x))(b(l+x) - b(x)) \rangle$. Equation (18) leads to the following spectra of energy $E(k)$, mean square generalized momentum $F(k)$, and generalized helicity $H(k)$ integrated over angles in k -space,¹¹

$$\begin{aligned} E(k) &\propto \epsilon_E^{2/3} k^{-7/3}, & F(k) &\propto \epsilon_E^{2/3} k^{-13/3}, \\ He(k) &\propto \epsilon_E^{2/3} k^{-10/3}. \end{aligned} \quad (19)$$

In the large d_e limit discussed in Sec. II, the equations of motion are given by Eqs. (9) and (10). These equations are invariant under the group of transformation,

$$l' = \alpha l, \quad t' = \alpha^{1-\rho} t, \quad b' = \alpha^{1+\rho} b, \quad \psi' = \alpha^{1+\rho} \psi. \quad (20)$$

Note that the transformations (20) infer that b and ψ should have the same scaling with l .

Similar to what has been done in the previous case for the small d_e limit, power law spectra of the ideal invariants can be determined by using the scaling (20) together with the assumption that one of the ideal invariants has an invariant flux through the scales. Any of these invariant fluxes leads to spectra that are characterized by a unique decay exponent. However, the spectra obtained from the numerical calculations are characterized by decay exponents that are inconsistent with any of these invariant fluxes. Moreover, in case of $E_b \ll E_\psi$ initially, the numerical simulations reveals a tendency towards equipartition of E_b and E_ψ , which contradicts the self-similar scaling of these fields, Eq. (20). Thus, a self-similar turbulent state is never established.

V. ENERGY DECAY LAWS

In this section we discuss two possible scenarios for the time behavior of the total energy and the peak of the energy spectrum in decaying EMHD turbulence. Predictions of the time behavior obtained from these scenarios will be used to analyze the time behavior of the total energy and the peak of the energy spectrum obtained from the numerical calculations.

The first scenario is based on the concept of selective decay,^{19,20} which is applicable if certain quantities decay at slower rate than others and can thus be considered constant.

As will be shown in Sec. VI, the energy decays faster than the mean square generalized momentum in the limit $kd_e \ll 1$, in agreement with direct cascade of energy and inverse cascade of mean square generalized momentum. Upon considering mean square generalized momentum to be constant, one can usefully apply the selective decay approach. We define a scale length L by

$$F = L^2 E_\psi \propto L^2 E. \quad (21)$$

In Eq. (21) we have assumed that E_b and E_ψ have similar space-time behavior, which will be the case if b and $\nabla\psi$ obey the same scaling. It is assumed that the scale L is at the top of the inertial range so that the associated energy scales like $E(L) \propto \epsilon_E^{2/3} L^{4/3}$. Assuming further that the energy $E(L)$ dominates the total energy, then the total energy E may be replaced by $E(L)$ in Eq. (21). Substituting $E(L)$ in Eq. (21) and using $dE(L)/dt = -\epsilon_E$, one obtains $dE(L)/dt \propto -F^{-1} E^{5/2}(L)$. Provided F is time-independent, one obtains the following similarity relation for the scale $L(t)$ and the energy $E(t)$:

$$L(t) \propto t^{-1/3}, \quad E(t) \propto t^{-2/3}. \quad (22)$$

Next we investigate a different approach, which is based on the assumption that if the turbulence initially possesses the property of infrared asymptotic self-similarity with a scaling exponent $h < 0$, then it preserves this property for all later times with the same h .²³ In the small d_e regime, we take b and $\nabla\psi$ to have the scaling $b \propto L^h$, $\nabla\psi \propto L^h$, with $h < 0$ on large scales. Here, we require that proportionality constants are time-independent. The scaling of b and $\nabla\psi$ implies that for small values of k the energy spectrum $E(k)$

is proportional to k^{-1-2h} . The scaling of b and $\nabla\psi$ is valid at large scales, $L \gg l_0$ where l_0 is assumed to be the length scale at the maximum of the energy distribution. The energy $E(L)$ and the energy flux $\epsilon(L)$ at the scale L can be estimated as $E(L) \propto L^{2h}$ and $\epsilon(L) \propto L^{3h-2}$. For sufficiently negative value of h , the dominant energy $E(L)$ is at scale $L = l_0$. Substituting $E(L)$ and $\epsilon(L)$ into $dE(L)/dt \propto -\epsilon(L)$, we obtain a first order differential equation for the scale L , which give the following similarity solutions for $L(t)$ and $E(L, t)$:

$$L(t) \propto t^{1/(2-h)}, \quad E(L, t) \propto t^{2h/(2-h)}. \quad (23)$$

From Eq. (23), it is clear that in order to determine the time behavior of the turbulence, one needs to know the value of h . An additional physical quantity that is constant in time, is needed to determine the value of h .

In the literature it is argued that the determination of the time behavior of decaying isotropic turbulence, in MHD (Refs. 20, 21) as well as in Navier-Stokes,²² is based on the invariance of the Loitsianskii integral. The invariance of the Loitsianskii integral is a consequence of the general law of conservation of angular momentum.

In what follows we will discuss the invariant that determines the time behavior of decaying turbulence in EMHD. Upon comparing the obtained time behavior of the turbulence with those of Eq. (23), the value of h can thus be determined.

The linear momentum $\mathbf{P}$ and angular momentum M in $2\frac{1}{2}$ D EMHD are given by

$$\mathbf{P} = \int d\mathbf{r} \mathbf{e}_z \times \mathbf{r} \Omega, \quad M = \frac{1}{2} \int d\mathbf{r} r^2 \Omega. \quad (24)$$

The linear momentum generates translations and the angular momentum generates rotations. They are constants of the motion if the Hamiltonian does not depend explicitly on space coordinates. We adopt a double periodic domain as will be used later in our numerical calculations. In such a domain the net linear momentum is a constant of the motion, while the net angular momentum is not. We assume that the mean value of the fluctuations of the axial vorticity $\bar{\Omega}$ vanishes over the volume, $\int d\mathbf{r} \bar{\Omega}(\mathbf{r}) = 0$. This condition is satisfied in our numerical calculations, where the initial fluctuations of the axial vorticity are chosen to have vanishing mean values.

The mean of the square of the linear momentum $\langle P^2 \rangle = \langle P_x^2 + P_y^2 \rangle$ can be written as

$$\langle P^2 \rangle = -\frac{(2\pi)^2}{2} \int R dR \int dL L^3 \langle \Omega \Omega' \rangle. \quad (25)$$

Here $\langle \cdots \rangle$ denotes ensemble average and $\langle \Omega \Omega' \rangle$ denotes the correlation function between vorticities at two points a distance L apart. For isotropic flows, the quantity $\langle \Omega \Omega' \rangle$ depends only on L . The convergence of the second integral on the right-hand side of Eq. (25), $\int dL L^3 \langle \Omega \Omega' \rangle$, requires that the integrand diminishes rapidly with increasing L . Since the turbulence is homogeneous, this integral takes the same value everywhere in the fluid. Thus we conclude that, for isotropic homogeneous turbulence, the constancy of the lin-

FIG. 1. (a) Evolution of the energy spectrum, and (b) time behavior of the peak of the energy spectrum in k -space. Diamonds represent data points and solid line is a least square fit in the range $0.3 \leq t \leq 1.5$. Plasma parameters: $d_e = 0.01$, $k_0 = 10$, $\mu_3 = 10^{-10}$, and $b_0 = \psi_0 = 3.4 \times 10^{-3}$.

ear momentum and thus of $\langle P^2 \rangle$ implies that the integral $\int dLL^3 \langle \Omega \Omega' \rangle$ is also constant in time. This would require that

$$\Omega^2 L^4 = \text{constant}. \quad (26)$$

Note that Eq. (26) implies that $\langle \Omega \Omega' \rangle$ decays as L^{-4} . This behavior of the correlator is sufficient for the integral $\int dLL^3 \langle \Omega \Omega' \rangle$ to converge. Condition (26) is the analog of the condition obtained from the invariance of the Loitsianskii integral for 3D Navier–Stokes turbulence.²²

Condition (26) also follows from the angular momentum. Upon using the condition $\int d\mathbf{r} \tilde{\Omega}(\mathbf{r}) = 0$, the mean of the square of the angular momentum is given by

$$\langle M^2 \rangle = -1/8 \int d\mathbf{R} \int d\mathbf{L} (\mathbf{R} \cdot \mathbf{L})^2 \langle \Omega \Omega' \rangle.$$

Although this angular momentum is globally not conserved, it is an approximate constant of the motion in regions whose size is smaller than the global size of the system. In that case, we can perform the integration over angles and $\langle M^2 \rangle$ can be written as

$$\langle M^2 \rangle \approx -\frac{(2\pi)^2}{16} \int R^3 dR \int dLL^3 \langle \Omega \Omega' \rangle. \quad (27)$$

Requiring that $\langle M^2 \rangle$ is time independent leads again to condition (27), with L being smaller than the global scale of the turbulence.

Using relation (26) we can determine the time behavior of the turbulence.

For $kd_e \ll 1$, $\Omega \approx b$ and Eq. (26) implies that $b^2 L^4 = \text{constant}$. Assuming that the fields b and $\nabla \psi$ have similar space–time behavior, the energy $E(L)$ and energy flux $\epsilon(L)$ at the scale L can be estimated as $E(L) \propto L^{-4}$ and $\epsilon(L) \propto L^{-8}$. We assume further that the scale L is at the maximum of the energy distribution so that the energy $E(L)$ can be interpreted as the peak of the energy distribution. Substituting $E(L)$ and $\epsilon(L)$ into $dE(L)/dt \propto \epsilon(L)$, we obtain a first order differential equation for the scale L . The solution of this differential equation gives the following similarity solutions for $L(t)$ and $E(L, t)$,

$$L(t) \propto t^{1/4}, \quad E(L, t) \propto t^{-1}. \quad (28)$$

Upon comparing this time behavior with that of Eq. (23), we see that $h = -2$. The corresponding energy spectrum at large scales is $E(k) \propto k^3$.

Note that the k^3 spectrum of the energy at large scale is also observed in MHD turbulence.²⁰

In the large d_e limit with, the time behavior of the invariant quantity $\hat{E}$ and the peak of its spectrum can be determined using similar treatments as in the small d_e limit. In this limit, the Hamiltonian $\hat{H}$ decays at a slower rate than the invariant $\hat{E}$ and can be considered constant on the time scale of the decay of the latter quantity. Thus the time behavior of $\hat{E}$ can be predicted by applying the selective decay approach between the Hamiltonian $\hat{H}$ and the invariant $\hat{E}$. The peak of the spectrum of $\hat{E}$ can be determined by assuming that the turbulence possesses an invariant infrared scaling together with the invariance of the total linear momentum.

The relevant system is invariant under the group of transformation (20). The scaling transformations (20) infer that the fields b and ψ should have similar scaling with length. This is used in the derivation of the time behavior of $\hat{E}$ and the peak of its spectrum.

However, the time behavior of $\hat{E}$ and that of the peak of its spectrum that are obtained from the numerical calculations do not fit those obtained from the selective decay and the invariant infrared scaling, respectively. This is because the condition that the fields b and ψ have a common scaling with length, under which these time behaviors are derived analytically, is violated.

VI. NUMERICAL RESULTS

We have performed numerical studies of 2D decaying EMHD turbulence. The equations are solved numerically on a quadratic box of size $2\pi \times 2\pi$ with double periodic boundary condition, using a pseudospectral method with N^2 modes and dealiasing according to the 2/3 rule. We start with the initial conditions,

$$b_k = b_0 \exp(-(k - k_0)^2 / 2\Delta^2 + i\alpha_k),$$

$$\psi_k = \psi_0 \exp(-(k - k_0)^2 / 2\Delta^2 + i\gamma_k). \quad (29)$$

FIG. 2. (a) Evolution of $\langle k \rangle_E$ in the inertial range $8 \leq k \leq 40$, and (b) the energy spectrum at time $t=2$. Solid line is a least square fit in the inertial range $8 \leq k \leq 40$. Same plasma parameters as in Fig. 1.

Here, b_0, ψ_0 are the initial amplitudes of the perturbations, α_k, γ_k are random phases, $k^2 = k_x^2 + k_y^2$, $k_{x,y} = \pm 1, \pm 2, \pm 3, \dots$, Δ is the width, and k_0 is the dominant initial wave number.

The energy $E(k)$ and mean square generalized momentum $F(k)$ are calculated in the following way. We divide the k -space into bands around integer values of k . Each band contains those k -values that are nearest to the defining integer of the band. The values of the energy and mean square generalized momentum are taken to be the sum over the corresponding modes in the band $(k-0.5, k+0.5]$ around each integer value of k . Finally, the values of energy and mean square generalized momentum at any time t are taken to be the average of their values calculated in the last 100 time steps.

We have performed two calculations for two different bin widths in k -space and same initial conditions. Moreover, we have taken the values of the energy $E(k)$ and mean square momentum $F(k)$ at any time t to be the average of their values calculated in the last 50 and 100 time steps. In all cases, the spectra are found to be qualitatively and quantitatively similar. All changes in the decay exponents are within the uncertainties of the fits.

In addition, we have performed calculations for different initial phases. In all cases the global shape of the spectra is unaltered, which means that the decay exponent of the spectra are independent of the initial phases. On the other hand, we have observed that the scatter of points in k -space changes with the initial phases. The scatter of data points in the spectra correspond to different statistical realizations.

Upon performing proper statistical averaging, the scatter will disappear while the spectra preserve their global shape.

The decay exponents of the spectra of the physical quantities and their time behavior are determined by a least square fitting method. In all plots, diamonds represent data points and the solid line on top is a least square fit.

A. $kd_e < 1$, excitation at low mode number

In order to investigate processes associated with the direct cascade of the ideal invariants, the initial perturbations are centered around $k_0 = 10$. The largest “Reynolds” number, $(\psi^2/\mu b k^2)$, associated with the initial perturbations at $k_0 = 10$ is 3.4×10^5 . The first case we consider is when $kd_e < 1$ in the inertial range. Figure 1(a) shows the evolution of the energy spectrum, where initially the spectrum is peaked around $k = 10$. In the course of time the tail of the energy spectrum extends towards larger values of k whereas the peak decreases and propagates towards smaller values of k . At each time during the evolution most of the energy flows towards small scales whereas a small fraction flows towards large scales. The peak of the energy spectrum, that is the maximum of the energy spectrum, in k -space is shown to have a $t^{-1.01 \pm 0.05}$ time behavior as illustrated in Fig. 1(b). This time behavior is in agreement with the one given by Eq. (28), which was derived under the assumption of invariant linear momentum and of an invariant infrared asymptotic scaling of the turbulence possesses with exponent $h = -2$. The peak of the energy spectrum propagates towards smaller values of k , and is followed in time until it reaches the global scales and becomes too small. This leaves insufficient space

FIG. 3. (a) Time behavior of the total energy, and (b) the energy calculated in the inertial range. Solid line is a least square fit in the range $1 \leq t \leq 5$. Same plasma parameters as in Fig. 1.

FIG. 4. (a) Evolution of mean square generalized momentum, and (b) the spectrum at time $t = 2$. Solid line is a least square fit in the range $8 \leq k \leq 40$. Same plasma parameters as in Fig. 1.

to verify the predicted k^3 spectrum of the energy at large scales. The slow decay of the peak of the energy spectrum at later times is caused by the slow decay of the energy in the global modes.

At the time when the spectra are calculated, the smallest mode number k_μ where the “Reynolds” number is equal to unity, is at $k = 50$. To avoid boundary and dissipation effects, we consider the range $8 \leq k \leq 40$ as being representative of the inertial range. The average value $\langle k \rangle_E(t) = \sum_{k=8}^{40} k E_k / \sum_{k=8}^{40} E_k$ increases initially, reaches a maximum and decreases afterwards as shown in Fig. 2(a). The initial increase of $\langle k \rangle_E(t)$ is a clear indication that initially the energy transfer is towards small scales. The average value $\langle k \rangle_E(t)$ reaches its maximum when fully developed turbulence is established. The decay of the total energy in time shown in Fig. 3(a) indicates that energy is cascaded to large mode numbers and subsequently dissipated there by viscosity.

At time $t = 2$ the energy spectrum is consistent with the $k^{-7/3}$ spectrum of Eq. (19) obtained from scaling symmetry and invariant energy flux [see Fig. 2(b)]. After $\langle k \rangle_E(t)$ has reached its maximum, the energy spectrum becomes a bit steeper and is characterized by $k^{-2.55 \pm 0.07}$ at time $t = 5$. This goes together with the final slow decrease of $\langle k \rangle_E(t)$. This behavior could point to a reduced energy flux at large k values.

The time behavior of the total energy $E(t)$ and the inertial range energy $E_i(t)$ are shown in Fig. 3. Initially, the energy in the inertial range is leaking towards large scales. This is consistent with the initial steep decay of $E_i(t)$. The

total energy is shown to have a $t^{-0.62}$ temporal behavior. The mean square generalized momentum remains constant during the characteristic time of energy decay [see Fig. 4(a)]. Thus the condition discussed in Sec. V for the applicability of selective decay is met and the time behavior of energy is close to that of Eq. (22). The energy calculated in the fixed inertial range $8 \leq k \leq 40$ is shown to have a $t^{-0.90}$ temporal behavior. This indicates that the energy of the global modes decays at a slower rate than the inertial range energy.

The numerical calculations show that the total mean square generalized momentum remains constant in the course of time as shown in Fig. 4(a). This indicates that the mean square momentum is not cascaded to small scales where it would be dissipated, but either remains in the excitation scales or flows towards larger scales. At time $t = 2$, the spectrum of mean square generalized momentum is characterized by $k^{-4.4 \pm 0.08}$ as shown in Fig. 4(b). This is consistent with the $k^{-13/3}$ spectrum of Eq. (19) determined by an invariant energy flux.

The time evolution of the ratio between the total energies $E_b(t) = \int dk E_b(k, t)$ and $E_\psi(t) = \int dk E_\psi(k, t)$ in the fields b and ψ , respectively, is shown in Fig. 5(a). Initially the energy E_ψ is dominant ($E_b/E_\psi|_{t=0} = 9.7 \times 10^{-3}$). In the course of time the ratio increases and reaches a value of 0.4 at time $t = 1.0$. The initial increase of the energy ratio means that energy in the field b , E_b , is being generated. Afterwards the ratio remains relatively constant, which means that both energies have similar time behavior as required by the self-similar scaling Eq. (21). This happens roughly at the time span when the spectra are established.

FIG. 5. (a) Evolution of the ratio between the total energies in the fields b and ψ , E_b/E_ψ , evolution of the energies E_b and E_ψ for (b) $k = 1$, and (c) $k = 2$. Same plasma parameters as in Fig. 1.

FIG. 6. (a) Evolution of the ratio E_b/E_ψ at large scales $3 \leq k \leq 6$, the spectral energy ratio in the ranges (b) $1 \leq k \leq 8$, and (c) $8 \leq k \leq 40$ at $t=2$. Same plasma parameters as in Fig. 1.

The contributions from large scale modes to the energies E_b and E_ψ are small fractions of the corresponding total energies. The time evolution of the energies E_b and E_ψ at the modes $k=1,2$ is shown in Fig. 5(b) and Fig. 5(c), respectively. As can be seen from the figures, the energy E_b increases fast initially; subsequently it decreases and becomes almost equal to E_ψ . This means that the ratio $E_b(k)/E_\psi(k)$ for $k=1,2$, increases fast initially, subsequently it decays to a constant value close to unity around time $t=0.5$. For modes $3 \leq k \leq 6$, the energy ratio relaxes to a constant value close to 0.4 after a fast initial increase as demonstrated in Fig. 6(a).

The ratio $E_b(k)/E_\psi(k)$ at time $t=2$ is shown in Figs. 6(b) and 6(c). The oscillations of $E_b(k)/E_\psi(k)$ are caused by the scatter of points in the spectra of both $E_b(k)$ and $E_\psi(k)$. As we mentioned earlier the scatter of these points represents some statistical realization. At the global scales $k=1,2$ the ratio has a value larger than unity and decays to 0.5 at $k=8$. This is shown in Fig. 6(b). Figure 6(c) shows the ratio $E_b(k)/E_\psi(k)$ in the range $8 \leq k \leq 40$. It is shown that the ratio is increasing towards a value close to unity at $k=40$.

Thus from the preceding discussion, we conclude that the evolution of the ratio between total energies in the fields b and ψ is determined by the corresponding energies in the inertial range.

B. $kd_e < 1$, excitation at high mode numbers

To investigate processes associated with the possible inverse cascade of the mean square generalized momentum, the initial perturbations are centered around $k_0=40$. In this case we consider the range $8 \leq k \leq 35$ as a representative of

the inertial range. The electron inertial skin depth is taken to be $d_e=0.01$, so that in the inertial range $kd_e < 1$. The early time evolution of the spectrum of the mean square generalized momentum is shown in Fig. 7(a) and the one of the energy in Fig. 7(b). It is seen that at early stages and within a very short interval of time, spectral peaks propagate towards large scales. These peaks represent a relatively low amount of energy, but a large amount of mean square generalized momentum. In case of the mean square generalized momentum Fig. 7(a), the area under the curves is practically constant in time. The mean square momentum remains at large scales due to an inverse cascade. The associated energy decays in time. Energy has a direct cascade just as in the case where the initial perturbations are centered around $k_0=10$ and $d_e=0.01$. In the case under consideration, the spectra of energy and mean square momentum are close to those of Fig. 2(b) and Fig. 4(b), respectively. Also, in this case the peak of the energy spectrum in k -space is shown to have a t^{-1} time behavior as illustrated in Fig. 8(a). Together with the previous result (see Fig. 1), this indicates that the time behavior of the peak of the energy spectrum is independent of the mode number k_0 . Figure 8(b) illustrates that initially $\langle k \rangle_F(t)$ decreases fast, which is consistent with an inverse cascade of mean square generalized momentum. The average mode number $\langle k \rangle_F(t)$ becomes constant at the time when the system has established a power law spectrum.

The time evolution of the ratio between the total energies in the fields b and ψ , E_b/E_ψ , is shown in Fig. 9(a). Initially the energy E_ψ is dominant ($E_b/E_\psi|_{t=0}=6.24 \times 10^{-4}$). The ratio increases and reaches a value of 0.4 at time $t=0.01$.

FIG. 7. (a) Evolution of the mean square generalized momentum spectrum, and (b) the energy spectrum. Plasma parameters are: $d_e=0.01$, $k_0=40$, $\mu_3=10^{-10}$, and $b_0=\psi_0=3.4 \times 10^{-3}$.

FIG. 8. (a) Time behavior of the peak of the energy spectrum in k -space, and (b) the evolution of $\langle k \rangle_F$ in the inertial range $8 \leq k \leq 35$. Solid line is a least square fit in the range $0.01 \leq t \leq 1$. Same plasma parameters as in Fig. 7.

This process is much faster than in the case with $d_e = 0.01$ and $k_0 = 10$ due to the larger coupling term. Afterwards the ratio remains relatively constant around the value 0.4. The ratio $E_b(k)/E_\psi(k)$ at time $t = 2$ is shown in Figs. 9(b) and 9(c). In the range $1 \leq k \leq 8$, the ratio at the global mode $k = 1$ has a value equal to 2.75, and drops to 0.32 at $k = 8$. This is shown in Fig. 9(b). Figure 9(c) shows the ratio $E_b(k)/E_\psi(k)$ in the range $8 \leq k \leq 35$. It is seen that the ratio increases until it reaches a value of about 0.8 at $k = 35$.

Similar to the previous case, the evolution of the ratio between total axial and poloidal energies is determined by the corresponding energies in the inertial range.

C. $kd_e > 1$, excitation at low mode number

In this part we discuss numerical simulations for the large d_e limit where the system equations may be approximated by Eqs. (9) and (10). We stress that all numerical calculations are performed for the full Eqs. (3) and (4). We have considered two cases corresponding to two different values of d_e , namely, $d_e = 0.2$ and $d_e = 1.0$. In both cases the mode number k_0 is taken to be equal to 10. The initial conditions are chosen such that for the case with $d_e = 0.2$, the ratio $|d_e^2(k^2 b(k)^2)/(k^2 \psi(k)^2)| = O(10^{-2})$ at each k . This means that the initial poloidal magnetic energy $\int d\mathbf{r}(\nabla \psi)^2$ is two orders of magnitude larger than the initial poloidal kinetic energy $\int d\mathbf{r} d_e^2(\nabla b)^2$. For the case with $d_e = 1.0$, the initial ratio $|d_e^2(k^2 b(k)^2)/(k^2 \psi(k)^2)| = 1$ at each k , which means the convection term equals the coupling term and that the poloidal magnetic and kinetic energies are equal at t

$= 0$. In both cases, the ratio of these energies, $(\int d\mathbf{r} d_e^2(\nabla b)^2)/(\int d\mathbf{r}(\nabla \psi)^2)$, turn out to increase in time by an order of magnitude.

In the simulation of the case with $d_e = 0.2$, the hyperviscosity coefficient μ_3 is taken to be 10^{-10} . During the simulation of the case with $d_e = 1.0$, we have encountered some problems due to numerical instabilities at small scales. In order to suppress such instabilities, one needs to start the simulation with a large viscosity coefficient, namely $\mu_3 = 5 \times 10^{-7}$. The problem is that in the course of time when the amplitudes start to decay, the viscosity coefficient becomes large. To solve this problem we reduced the viscosity coefficient during the simulation. This recipe has been used in Ref. 9 for decaying turbulence in 2½D EMHD where it has been shown that the energy dissipation rate does not depend on the value of the viscosity coefficient. In our case, initially we use $\mu_3 = 5 \times 10^{-7}$, at time $t = 1$ it is reduced to $\mu_3 = 10^{-7}$, $\mu_3 = 5 \times 10^{-8}$ at $t = 2$, $\mu_3 = 10^{-8}$ at $t = 3$, and finally $\mu_3 = 5 \times 10^{-9}$ at $t = 4$.

We have considered the range $10 \leq k \leq 40$ as being representative of the inertial range. This means that $kd_e > 1$ over the entire inertial range for both values of d_e . The evolution of the average values of $\langle k \rangle_E(t)$ and $\langle k \rangle_F(t)$ in the inertial range $10 \leq k \leq 40$ is represented in Fig. 10. The curves $\langle k \rangle_E(t)$ and $\langle k \rangle_F(t)$ indicate that at early times the transfer of both energy and mean square generalized momentum is towards small scales. At later times, the average mode numbers $\langle k \rangle_E(t)$ and $\langle k \rangle_F(t)$ become relatively constant.

Figure 11 shows that the total energy and the total mean

FIG. 9. (a) Evolution of the energy ratio, the spectral energy ratio, $(E_b(k)/E_\psi(k))$ at time $t = 2$ in the ranges (b) $1 \leq k \leq 8$, and (c) $8 \leq k \leq 35$. Same plasma parameters as in Fig. 7.

FIG. 10. Evolution of the average values $\langle k \rangle_E$ and $\langle k \rangle_F$ in the inertial range $10 \leq k \leq 40$ for $d_e = 0.2$ (a) and $d_e = 1.0$ (b). Plasma parameters are: $k_0 = 10$, and $b_0 = \psi_0 = 3.4 \times 10^{-3}$.

square generalized momentum decay in the course of time. This decay together with the initial increase of the average mode numbers $\langle k \rangle_E(t)$ and $\langle k \rangle_F(t)$, means that both quantities exhibit direct cascade. At later times, the total energy for $d_e = 0.2$ decays according to $t^{-0.94}$ as shown in Fig. 11(a).

The time evolution of the ratio between the total energies in the fields b and ψ , E_b/E_ψ , for $d_e = 0.2$, and $d_e = 1.0$ are shown in Fig. 12(a). Note that in the large d_e limit, E_b is dominated by the contribution $\int d\mathbf{r} d_e^2 (\nabla b)^2$, and E_ψ is dominated by $\int d\mathbf{r} d_e^2 (\nabla^2 \psi)^2$. The energy E_ψ is dominant initially, $(E_b/E_\psi)_{t=0} = 9.41 \times 10^{-3}$ and $(E_b/E_\psi)_{t=0} = 9.52 \times 10^{-3}$ for $d_e = 0.2$ and $d_e = 1.0$, respectively. In both cases, the ratio remains small for some time. In the course of time the ratio increases and becomes approximately constant. In the case with $d_e = 0.2$, the ratio levels off at a value equal to 0.4 at time $t = 2.0$, while for $d_e = 1.0$, the ratio levels off at a relatively higher value 0.6 at the later time $t = 5.0$.

As mentioned before, the increase of the ratio E_b/E_ψ is due to flow of energy from E_ψ to E_b . This conversion of energy is done by the coupling term in Eq. (3). The different time behavior of the ratio E_b/E_ψ in the two cases under consideration, is related to the difference in evolution of the convection and coupling terms in Eq. (3). The time evolution of the ratio

$$R(t) = \frac{\sum_{x=1}^{n_x} \sum_{y=1}^{n_y} [b(x,y), \Omega(x,y)] + [\psi(x,y), \nabla^2 \psi(x,y)]}{\sum_{x=1}^{n_x} \sum_{y=1}^{n_y} [\psi(x,y), \nabla^2 \psi(x,y)]}, \quad (30)$$

where $n_x = n_y = n$ is the number of grid points, is shown in Fig. 12(b). Initially, the convection term is $O(10^{-2})$ with respect to the coupling term for the case with $d_e = 0.2$, while for the case with $d_e = 1.0$, these two terms are equal in magnitude. For $d_e = 0.2$, the initial value of $R(t)$ is equal to unity, and it remains so for some time. This goes together with E_b/E_ψ remaining unchanged at early times. For the case with $d_e = 1.0$, the ratio $R(t)$ decreases from a value equal to 1.5–1.02. This decrease is due to a partial cancellation between the two nonlinear terms in the right-hand side of Eq. (3), which also goes together with E_b/E_ψ remaining unchanged for longer times as compared with the case of $d_e = 0.2$. Subsequently, for both cases $d_e = 0.2$ and $d_e = 1.0$, $R(t)$ increases, becomes large and approximately constant at the time when the ratio E_b/E_ψ becomes constant. In the case with $d_e = 1.0$, the compensation between the convection and coupling terms is more pronounced than in the case with $d_e = 0.2$. As a result $R(t)$ increases at a slower rate and consequently it takes longer for the ratio E_b/E_ψ to become constant.

At early times where the ratio E_b/E_ψ is still small, the dynamics is dominated by the ψ -field and the coupling term. Upon comparing the evolution of the total energy in Figs. 11(a) and 11(c) with the evolution of the ratio E_b/E_ψ in Fig. 12(a), one observes that at the time when the ratio E_b/E_ψ becomes constant 26% of the total energy is already dissipated for $d_e = 0.2$ and even more than 50% in the case with $d_e = 1.0$. The energy losses are mainly in the E_ψ energy,

FIG. 11. (a) Time behavior of the total energy, (b) evolution of the total mean square generalized momentum for $d_e = 0.2$, and (c) evolution of the total energy and mean square generalized momentum for $d_e = 1.0$. Solid line is a least square fit in the range $2 \leq t \leq 5$. Same plasma parameters as in Fig. 10.

FIG. 12. (a) Evolution of the energy ratio, (E_b/E_ψ) , (b) evolution of $R(t)$. Same plasma parameters as in Fig. 10.

which means that the energy E_ψ spreads out and decays on a time scale where the energy E_b is still negligibly small, and as a result the turbulence never reaches a stationary state. As will be shown later, on these time scales the spectrum of E_ψ extends towards small scales, and as a result the energy E_ψ is damped by viscosity. Although the energy E_ψ is still the largest contribution to the total energy in the phase where E_b/E_ψ is constant, the ratio $R(t)$ is large so that the dynamics is mainly due to the convection term.

Numerical analysis of the early time evolution of the peak of the energy spectrum and the time behavior of the energy in the inertial range $10 \leq k \leq 40$ are given in Figs. 13. At early times the energy related with the b field is negligibly small and the total energy E is dominated by E_ψ . In both cases, $d_e = 0.2$ and 1.0, the peak is shown to propagate towards small mode numbers and to decay in time. This decay is rather fast. For $d_e = 1.0$, the peak has decayed by more than 60% of its initial value at time $t = 1.0$. Subsequently, the decay slows down and behaves like $t^{-1.11 \pm 0.05}$ as shown in Fig. 13(a).

Initially the energy contained in the inertial range, $E_i(t)$, flows towards large and small scales. It decays fast at early times. For the case with $d_e = 0.2$, at time $t = 2.0$ around 77% of $E_i(t=0)$ is already lost from the inertial range. At later times, $E_i(t)$ has a $t^{-1.15}$ temporal behavior as shown in Fig. 13(b). Upon comparing Figs. 11(a) and 13(b), it is seen that both the total energy $E(t)$ and the energy contained in the inertial range decay at almost the same rate at later times.

This is because there is hardly any energy left in the large scale modes.

The local ratio $E_b(k)/E_\psi(k)$ in k -space at time $t = 2$ is shown in Figs. 14(a) and 14(b). In the range $1 \leq k \leq 10$, the ratio decays from 0.8 at the global scale $k = 1$ to 0.1 at $k = 2$, and finally increases to 0.7 at $k = 10$. This is shown in Fig. 14(a). However, there is little energy in the low mode numbers, so that the evolution of the ratio E_b/E_ψ is determined by the corresponding energies in the inertial range. Figure 14(b) shows the ratio $E_b(k)/E_\psi(k)$ in the range $10 \leq k \leq 40$, where it oscillates around a value that increases from 0.6 at $k = 10$ –0.8 at $k = 40$.

The plots of the spectra of energy and mean square generalized momentum at time $t = 2$ are shown in Fig. 15. For the case with $d_e = 0.2$, the spectrum of the total energy is characterized by $k^{-1.64 \pm 0.05}$ as shown in Fig. 15(a). The decay exponent of the energy spectrum remains constant at later times, in agreement with the final steady behavior of $\langle k \rangle_E(t)$ in Fig. 10(a). The spectrum of mean square generalized momentum is proportional to $k^{-1.92 \pm 0.06}$, see Fig. 15(b). At later times, this spectrum becomes a bit flatter which goes together with the slow increase of $\langle k \rangle_F(t)$ [see Fig. 10(a)].

In the case where $d_e = 1.0$, the energy and mean square generalized momentum are characterized by almost the same spectrum. At time $t = 2.0$, the spectra are $E(k) \propto k^{-1.69 \pm 0.06}$ and $F(k) \propto k^{-1.67 \pm 0.06}$ as shown in Figs. 15(c) and 15(d). In the course of time the spectra become a bit flatter and at time $t = 5$ the spectra of the energy and mean square generalized

FIG. 13. (a) Time behavior of the peak of the energy spectrum in k -space, solid line is a least square fit in the range $1.0 \leq t \leq 5.0$. (b) Time behavior of the energy calculated in the inertial range, solid line is a least square fit in the range $2.0 \leq t \leq 5.0$. Same plasma parameters as in Fig. 10.

FIG. 14. Spectral energy ratio, $(E_b(k)/E_\psi(k))$ at time $t=2$ in the ranges (a) $1 \leq k \leq 10$ and (b) $10 \leq k \leq 40$. Same plasma parameters as in Fig. 10 for $d_e = 0.2$.

momentum are $E(k) \propto k^{-1.52 \pm 0.06}$ and $F(k) \propto k^{-1.49 \pm 0.06}$ (figures not shown). At later times, the decay exponents of both spectra remain constant in agreement with the final steady behavior of the mode numbers $\langle k \rangle_E(t)$ and $\langle k \rangle_F(t)$.

In the two cases discussed above, the total energy and the total mean square generalized momentum have established power law spectra at early times. At these times both spectra are associated with the $\nabla^2 \psi$ -field alone, which almost behaves as an advected scalar. The spectra are rather broad and extend to the viscous region so that these quantities are already strongly dissipated while the energy in the b -field, E_b , is still negligibly small. For the case with $d_e = 1.0$, the total energy, which is mainly E_ψ , has established a spectrum that extends towards small scales at time $t = 2.0$. At that time 68% of the energy in the inertial range is already lost. Moreover we point out that at time $t=2$ the energies E_b and E_ψ , and thus the fields b and ψ do not yet

have similar time behavior as can be seen from the time evolution of E_b/E_ψ for $d_e = 1.0$ given in Fig. 12(a).

The spectra of the poloidal magnetic energy and the poloidal kinetic energy for the case with $d_e = 1.0$ are displayed in Fig. 16. The spectrum of the poloidal magnetic energy at time $t=2$ is characterized by $k^{-3.65 \pm 0.06}$ [see Fig. 16(a)]. In the course of time the spectrum becomes a bit flatter and at time $t=5$, the spectrum follows $k^{-3.43 \pm 0.06}$ as shown in Fig. 16(b). This spectrum is consistently k^{-2} steeper than those of the energy and mean square momentum. This confirms that both energy and mean square generalized momentum are dominated by the $\int d\mathbf{r} (\nabla^2 \psi)^2$ term.

At time $t=2$ the spectrum of the poloidal kinetic energy is characterized by $k^{-2.40 \pm 0.05}$ as shown in Fig. 17(a). In the course of time this spectrum becomes a bit steeper and at time $t=5$ it follows $k^{-2.55 \pm 0.07}$ [see Fig. 17(b)].

The difference between the spectra of the poloidal mag-

FIG. 15. Spectra of the energy (left) and mean square generalized momentum (right) at time $t=2$. (a,b) $d_e = 0.2$, and (c,d) $d_e = 1.0$. Solid line is a least square fit in the range $10 \leq k \leq 40$. Same plasma parameters as in Fig. 10.

FIG. 16. Spectra of $(k\psi_k)^2$ (a) at $t=2$, and (b) at $t=5$. Solid line is a least square fit in the range $10 \leq k \leq 40$. Same plasma parameters as in Fig. 10 for $d_e = 1.0$.

netic and kinetic energies is due to the fact that the fields b and ψ have different scaling with length in contrast to Eq. (20), where b and ψ are required to have the same scaling with length. This difference decreases with time since the spectrum of the poloidal magnetic energy flatten while that of the poloidal kinetic energy steepen in the course of time. However, at time $t=5$ the spectrum of the poloidal magnetic energy is $k^{-0.88}$ steeper than that of the poloidal kinetic energy, while 86% of the energy in the inertial range $10 \leq k \leq 40$ has already dissipated.

For the case with $d_e = 0.2$, the spectra of $E_b(k) \approx d_e^2 k^3 b_k^2$ and $E_\psi(k) \approx d_e^2 k^5 \psi_k^2$ at time $t=2.0$ are $k^{-1.45 \pm 0.06}$ and $k^{-1.80 \pm 0.05}$, respectively (figures not shown). Again, this means that the fields b and ψ do not have a common scaling with length as required by Eq. (20).

The fact that the fields b and ψ do not have a common scaling with length means that the turbulence is not described by the class of self-similar solutions of Eqs. (9) and (10). This might be related to the nonstationary character of the turbulence.

Note that in both cases discussed above, the energy spectrum is close to k^{-a} , with $a=5/3$. In Refs. 8 and 9, in the large d_e limit, the total energy is found to exhibit a direct cascade and its spectrum is characterized by $k^{-5/3}$. This spectrum has been associated with the Kolmogorov spectrum of the energy as in hydrodynamic turbulence on the basis of the $-5/3$ exponent only. However, the initial conditions used in Refs. 8 and 9 amount to $E_b/E_\psi \ll 1$ initially. As we discussed earlier with such initial conditions, the coupling term

remains finite, although it decreases in time. Thus the system of Eqs. (3) and (4) do not split into a Navier–Stokes and a passive scalar equation.

The EMHD system reduces to a Navier–Stokes equation for the axial vorticity $\nabla^2 b$ and a passive scalar equation for the parallel current $\nabla^2 \psi$ in the large d_e limit and when initially the total energy E_b in the field b is larger than or equal to the total energy E_ψ in the field ψ . Figure 18 shows the evolution of the energies E_b and E_ψ for two cases where initially $E_b/E_\psi = 10.09$ (a) and $E_b/E_\psi = 0.94$ (b). In both cases $k_0 = 10$ and $d_e = 0.2$, as before. In the first case of Fig. 18(a), the coupling term is very small and the system is Navier–Stokes like. In the second case of Fig. 18(b), the energy E_b remains constant while the energy E_ψ continues to decay after an initial phase where E_b decays and E_ψ increases slightly. Consequently, the coupling term becomes negligibly small as compared to the convection term and the system of Eqs. (3) and (4) decouples. In this case the spectrum of the poloidal kinetic energy $\int dr d_e^2 (\nabla b)^2$ has been verified numerically to be k^{-3} . Upon choosing k_0 large such that inertial range extends towards small mode numbers, the spectrum of the poloidal kinetic energy is found to be $k^{-5/3}$. Thus the system behaves like 2D Navier–Stokes.

VII. CONCLUSIONS AND DISCUSSIONS

We have investigated the spectral properties of $2\frac{1}{2}$ D EMHD using numerical simulations of freely decaying turbulence. In the small d_e limit, the processes associated with

FIG. 17. Spectra of $(kb_k)^2$ (a) at $t=2$, and (b) at $t=5$. Solid line is a least square fit in the range $10 \leq k \leq 40$. Same plasma parameters as in Fig. 10 for $d_e = 1.0$.

FIG. 18. Evolution of energies. Plasma parameters are: $d_e=0.2$, $k_0=10$, $\mu_3=10^{-10}$, $b_0=3.4\times 10^{-3}$, $\psi_0=1.0\times 10^{-4}$, and $(E_b/E_\psi)_{t=0}=10.9$ (a) and $d_e=0.2$, $k_0=10$, $\mu_3=10^{-10}$, $b_0=3.4\times 10^{-3}$, $\psi_0=3.4\times 10^{-4}$, and $(E_b/E_\psi)_{t=0}=0.94$ (b).

the direct cascade of the conserved quantities are investigated by choosing the mode number k_0 of the initial turbulence to be small such that the inertial range extends towards large k values and $kd_e < 1$ over the entire inertial range. On the other hand to investigate processes associated with the possible inverse cascades of the ideal invariant quantities, we choose the mode number k_0 to be large, but $k_0 d_e < 1$, such that the inertial range extends towards small k values. In both cases, the cascade directions of energy and mean square generalized momentum are found to agree with the indications obtained from equilibrium statistics, and the spectra are consistent with those obtained from the scaling symmetries of the equations together with an invariant energy flux. Numerical calculations indicate that the energy exhibits a direct cascade and dissipates at small scales. The energy is characterized by a $k^{-7/3}$ spectrum, in agreement with the results of Refs. 8 and 9. The mean square generalized momentum remains constant in the course of time and flows towards large scales which means an inverse cascade. Its spectrum is characterized by $k^{-13/3}$ which is also consistent with an invariant spectral energy flux through the scales.

When the turbulence is fully established, the time behavior of the total energy is $t^{-2/3}$. This time behavior is consistent with that obtained from selective decay where energy decays faster than mean square generalized momentum. Further, the time behavior of the peak of the energy spectrum is shown to propagate towards large scales and decays in time as t^{-1} , in agreement with invariant linear momentum and infrared scaling of the turbulence.

The time evolution of the ratio between the total energies E_b and E_ψ in the fields b and ψ , respectively, is found to depend on the initial values of the ratios of these energies. If initially the energy E_b is less than the energy E_ψ , their ratio undergoes a fast increase. Afterwards this ratio remains approximately constant meaning that the energies acquire similar time behavior. This process takes place on short time scales as compared to the time scale of the total energy decay, and is roughly the time scale on which the spectra are established.

In the opposite limit when initially the energy E_b is equal to or exceeds the energy E_ψ , the coupling term is small and the ratio remains practically constant in the course of time.

In the large d_e limit, the mode number k_0 is chosen small, but $k_0 d_e > 1$. The inertial range extends towards large

k values. In this regime the numerical calculations are presented for two values of d_e and three different initial energy ratios.

In all cases where initially the energy E_b is smaller than the energy E_ψ , the calculations show that both the total energy and the total mean square generalized momentum flow towards small scales, which means that both quantities have a direct cascade. The direct cascade of mean square generalized momentum in the large d_e regime was predicted first in Ref. 11. This is in contrast with previous papers,^{8,9} where it is suggested that mean square generalized momentum exhibit an inverse cascade without making any distinction between different values d_e . The time behavior of the ratio E_b/E_ψ is studied for $d_e=0.2$ and $d_e=1.0$. The initial conditions are chosen such that in the former case the convection term is $O(10^{-2})$ smaller than the coupling term, while in the latter case these two terms are equal in magnitude. In both cases the time evolution of the ratio indicates that it increases at a slower rate as compared to that in the small d_e regime. Moreover, in the case with $d_e=0.2$, the ratio becomes constant at a value equal to 0.4 at time $t=2.0$, while for $d_e=1.0$, the ratio levels off at a relatively higher value 0.6 and on longer time $t=5.0$. The different time behavior of the ratio E_b/E_ψ in the two cases is related to the compensation between the convection and coupling terms. This compensation is more pronounced in the case with $d_e=1.0$. As a result the ratio increases at a slower rate as compared to the case with $d_e=0.2$, and it takes longer for the ratio to level off.

The peak of the energy spectrum propagates towards small mode numbers and decays in time. The early decay of the peak is rather fast, subsequently, the decay slows down and behaves according to t^{-1} . The energy contained in the inertial range decays much faster than the total energy at early times. This is because part of it flows to large scales. At later times, the decay rate of the energy in the range slows down and becomes almost equal to that of the total energy. In the case with $d_e=0.2$, the spectra of the energy and mean square generalized momentum at time $t=2.0$, when the ratio E_b/E_ψ becomes constant, are characterized by $k^{-1.46\pm 0.05}$ and $k^{-1.92\pm 0.06}$, respectively. Afterwards, the decay exponent of the energy spectrum remains constant, while that of the mean square generalized momentum spectrum decreases a bit.

For the case with $d_e=1.0$, both energy and mean square generalized momentum are almost characterized by the same

spectrum. At time $t=2$, their spectra are $k^{-1.69 \pm 0.06}$ and $k^{-1.67 \pm 0.06}$, respectively. Later at time $t=5$ when E_b/E_ψ becomes stationary, their spectra become a bit flatter and are characterized by $k^{-1.52 \pm 0.06}$ and $k^{-1.49 \pm 0.06}$, respectively. Afterwards the decay exponents of both spectra become constant.

In both cases presented here, $E_\psi \gg E_b$ initially. The energy E_ψ establishes a power law spectrum, k^{-a} where (a) is close to 5/3 at early times. This spectrum extends towards small scales and the energy E_ψ is dissipated considerably when the energy E_b is still negligibly small. At this stage $\nabla^2 \psi$ behaves like a passive scalar convected by the streaming potential b . In both cases, the energies E_ψ and E_b acquire similar time behavior at late time when more than 60% of the total energy in the inertial range is already lost. The energy losses are mainly in the E_ψ energy. Thus it can be concluded that the system never reaches a state of stationary turbulence. At later times, the energy E_b becomes of the order of the energy E_ψ . Analysis of the spectra of these energies reveals that the fields b and ψ do not have a common scaling with length as required by Eq. (20). This means that the numerically observed turbulence does not correspond to self-similar solutions of Eqs. (9) and (10).

In the limit where initially the energy E_b is equal to or exceeds the energy E_ψ , the system behaves rather differently. The energy E_b remains constant while the energy E_ψ decays, and the coupling term becomes negligibly small. As a result, the system decouples into a Navier–Stokes equation for the axial vorticity and a passively convected scalar equation for the parallel current. Upon choosing the mode number k_0 small such that the inertial range extends towards large k values, the spectrum of the poloidal kinetic energy $\int d\mathbf{r} d_e^2 (\nabla b)^2$ is verified to be k^{-3} . On the other hand when k_0 is chosen large such that the inertial range extends towards small k values the spectrum of the poloidal kinetic energy is found to be $k^{-5/3}$.

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