Matrix generators of $C_0$ semigroups in $l^2$

Citation for published version (APA):

Document status and date:
Published: 01/01/1989

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
MATRIX GENERATORS OF $C_0$ SEMIGROUPS IN $l_2$

by

Liu Gui-Zhong
MATRIX GENERATORS OF $C_0$ SEMIGROUPS IN $l^2$

Liu Gui-Zhong*
Department of Mathematics and Computing Science
Eindhoven University of Technology
5600 MB Eindhoven
The Netherlands

Abstract

Two criteria are given such that an infinite matrix generates a $C_0$ semigroup in $l^2$.

§1. Introduction

Let $(a_{jk})_{j,k \in \mathbb{N}_0}$ be an infinite matrix of complex numbers with $j,k$ the row index and column index respectively. We suppose that

\[ \{a_{j \cdot}\} \in l^2, \quad \forall k \in \mathbb{N}_0 ; \quad \{a_{\cdot k}\} \in l^2, \quad \forall j \in \mathbb{N}_0. \]

Corresponding to the matrix $(a_{jk})$ we can define an operator $A_{\text{max}} \equiv \text{Op}(a_{jk})$ in $l^2$ as follows:

\[ A_{\text{max}} u = \left\{ \sum_{k=0}^{\infty} a_{jk} u_k \right\}_{j \in \mathbb{N}_0} \]

with

\[ D(A_{\text{max}}) = \{ u = (u_k) \in l^2 \mid \sum_{k=0}^{\infty} a_{jk} u_k \text{ converges for all } j \in \mathbb{N}_0 \text{ and } \{ \sum_{k=0}^{\infty} a_{jk} u_k \}_{j \in \mathbb{N}_0} \in l^2 \}. \]

If instead of the matrix $(a_{jk})$ we use its complex conjugate $(\overline{a}_{jk})$, then we obtain another operator on $l^2$, denoted by $A_{\text{max}}^\dagger$. Under the above conditions it is easy to see that $l^2_2 \subset D(A_{\text{max}})$ and $l^2_2 \subset D(A_{\text{max}}^\dagger)$, where $l^2_2 = \{ u = (u_k) \mid u_k = 0 \text{ if } k \geq K \text{ depending on } u \}$, the subspace of finite sequences. We fix the notations $A_{\text{max}} \upharpoonright l^2_2 = A_{\text{min}}$ and $A_{\text{max}}^\dagger \upharpoonright l^2_2 = A_{\text{min}}^\dagger$. Thus both the operators $A_{\text{max}}$ and $A_{\text{max}}^\dagger$ are densely defined. Actually they are also closed and obey the following relations:

* Permanent address: Department of Mathematics, Xi’an Jiaotong University, Xi’an, China.
The above facts are simple to prove and can be found in standard text books, e.g. in [6].

In the present note we give two criteria which determine classes of matrices \((a_{jk})\) whose corresponding maximal operators \(A_{\text{max}}\) or restrictions thereof are infinitesimal generators of \(C_0\) semigroups on \(l^2\). Nontrivial criteria of such kind seem missing in the literature. We shall obtain our criteria by applying the perturbation theorem of Rellich-Kato-Gustafson-Chernoff and the auxiliary operator trick of De Graaf in §2 and §3 respectively. These abstract results read:

**Theorem I.** (Rellich-Kato-Gustafson-Chernoff) Suppose that \(A : D(A) \subset X \to X\) be an infinitesimal generator of a \(C_0\) semigroup on a Banach space \((X, \| \cdot \|)\). Let \(P\) be another operator in \(X\) such that \(D(P) :\supseteq D(A)\) and

\[
\|P u\| \leq M \|u\| + \alpha \|A u\|, \quad \forall u \in D(A)
\]

where \(M\) and \(\alpha\) are nonnegative constants with \(0 < \alpha < 1\). Then the operator \(A + P\) as defined on \(D(A)\) generates a \(C_0\) semigroup on \(X\).

If instead of (9) above a weaker condition

\[
\|P u\| \leq M \|u\| + \|A u\|, \quad \forall u \in D(A)
\]

is satisfied, then the operator \(A + P\) as defined in \(D(A)\) is closable and its closure \(\overline{A + P}\) generates a \(C_0\) semigroup on \(X\).

**Theorem II.** (De Graaf) Let \(A\) be a closable densely defined operator in a Hilbert space \((H, \langle \cdot, \cdot \rangle)\). Assume that there exists a strictly positive self-adjoint operator \(Q\) such that \(D(Q) \subset D(A), \quad R(Q) = H\) and

\[
\begin{align*}
\Re\langle u, Au \rangle & \leq \omega(u, u) \\
\Re\langle Qu, Au \rangle & \leq \omega(Qu, u)
\end{align*}
\]

\(\forall u \in D(Q)\).

Then \(D(Q)\) is a core for the operator \(A\) and \(A\) generates a quasi-contractive \(C_0\) semigroup on \(X\) (i.e. a \(C_0\) semigroup \(\{T(t) \mid t \geq 0\}\)) such that \(\{e^{-\omega t} T(t) \mid t \geq 0\}\) is contractive for some \(\omega \geq 0\). []

Theorem II was given in [3]. Theorem I is quite standard and can be found in standard books on functional analysis or monographs on operator semigroups, cf. e.g. [1], [2], [4] and [5].

For the sake of simplicity in formulation from now on we always assume that all the matrices are
tridiagonal. Extensions to cases involving more general matrices are immediate.

§2. Diagonal Dominant Generators

Theorem 1. Suppose that \((a_{jk})\) is a tridiagonal matrix with the diagonal elements \(a_{kk} \geq d > 0\) for all \(k \in \mathbb{N}_0\) and \((A_{\text{max}} - Q)^{1/2}\) quasi-accretive. Here \(Q = \text{Op}(\text{diag}(a_{kk}))\). Assume that there exist constants \(K \in \mathbb{N}_0\) and \(c_l \geq 0\) with \(c_l + c_u \leq 1\) such that

\[
|a_{k+1,k}|^2 \leq c_l, \quad |a_{k-1,k}|^2 \leq c_u, \quad \forall k \geq K. \quad (31)
\]

Then the closure of \(A_{\text{max}}^{1/2}\) generates a \(c_0\) semigroup of operators on \(l^2\).

Proof. From the condition \(a_{kk} \geq d > 0\) it follows easily that the operator \(Q\) generates a \(c_0\) semigroup on \(l^2\) with \(l_c^2\) as a core of \(Q\).

For \(u \in D(Q)\) we have the estimate

\[
\left( \sum_{k=0}^{\infty} |a_{k-1,k} u_{k-1} + a_{k+1,k} u_{k+1}|^2 \right)^{1/2} \\
\leq \left( \sum_{k=0}^{\infty} |a_{k-1,k} u_{k-1}|^2 \right)^{1/2} + \left( \sum_{k=0}^{\infty} |a_{k+1,k} u_{k+1}|^2 \right)^{1/2} \\
\leq M_k \|u\| + (c_l + c_u) \|Q u\|.
\]

where

\[
M_k = 2 \max_{0 \leq k \leq K} \{ |a_{k-1,k}|, |a_{k+1,k}| \}.
\]

From this estimate we have \(D(Q) \subset D(A_{\text{max}})\) and for \(u \in D(Q)\)

\[
\| (A - Q) u \| \leq 2 M_k \|u\| + (c_l + c_u) \|Q u\| \quad (32)
\]

and

\[
\| A u \| \leq 2 M_k \|u\| + (1 + c_l + c_u) \|Q u\|. \quad (33)
\]

By virtue of (32) with \(c_l + c_u \leq 1\) Theorem 1 ensures that the closure of \(A_{\text{max}}^{1/2}D(Q)\) generates a \(c_0\) semigroup on \(l^2\). Moreover (33) implies that \(l_c^2\) is a core for \(A_{\text{max}}^{1/2}D(Q)\), and hence also for its closure.

Corollary 2. If in the above theorem instead of condition (31) we assume the stronger condition
that

$$\limsup |a_{k+1,k}| a_{kk}^2 + \limsup |a_{k,k-1}| a_{kk}^2 < 1$$  \hspace{1cm} (34)$$

then $A_{\text{max}} \dagger D(Q)$ generates a $c_0$ semigroup in $l^2$ with $l_2^2$ as a core for $A_{\text{max}} \dagger D(Q)$.

We remark that if $(a_{jk}) = \text{diag}(a_{kk}) + (a_{jk}^{(1)}) + (a_{jk}^{(2)})$ with $(a_{jk}^{(1)})$ a bounded matrix on $l^2$ and $(a_{jk}^{(2)})$ skew symmetric, then $(A_{\text{max}} - Q) \dagger l_2^2$ is quasi-accretive.

**Example 3.** With $(a_{jk})$ given by $(u,v < 1)$

$$\text{diag}(1,2,3,...) + \begin{bmatrix}
0 & -1^v \\
1^v & 0 & -2^v \\
2^v & 0 & 3^v \\
3^v & 0 & \ddots
\end{bmatrix}$$

$$+ i \begin{bmatrix}
0 & 1^u \\
1^u & 0 & 2^u \\
2^u & 0 & 3^u \\
3^u & 0 & \ddots
\end{bmatrix} + \begin{bmatrix}
0 & 1 - \frac{1}{2} \\
2 + \frac{1}{2} & 0 & 1 - \frac{1}{2} \\
0 & 2 + \frac{1}{2} & 0 & 1 - \frac{1}{3} \\
2 + \frac{1}{3} & \ddots & \ddots & \ddots
\end{bmatrix}$$

the operator $A_{\text{max}} \dagger D(Q)$ generates a $c_0$ semigroup in $l^2$ with $l_2^2$ as a core of it. Here $Q$ is the maximal operator corresponding to diag $(1,2,3,...)$.

\[ \Box \]

§3. Skew-adjoint Like Generators

**Theorem 4.** For an infinite matrix $(a_{jk})$ assume there exists a diagonal matrix diag $(q_0,q_1,...)$ with $0 < q_k \to \infty$ such that its corresponding maximal operator $Q$ and the operators $A_{\text{max}}$ and $A_{\text{max}}^+$ satisfy the following conditions:

- \( \text{Op}(a_{jk} q_k^{-1}) \) is a Hilbert-Schmidt operator on $l^2$ \hspace{1cm} (35)
- \( \text{Op}(a_{jk} + \tilde{a}_{ij}) \uparrow D(Q) \) is quasi-accretive \hspace{1cm} (36)
\[ \text{Op}(q_j^{12} a_j q_k^{12} + q_j^{12} a_{kj} q_k^{12}) |D(Q^{12}) \] is quasi-accretive. \hspace{1cm} (37)

Then \( A_{\text{max}} |D(Q) \) is closable and its closure generates a \( c_0 \) semigroup on \( l^2 \).

**Proof.** Condition (35) implies in particular that \( D(Q) \subset D(A_{\text{max}}) \). Theorem II will yield the wanted conclusion if we can check the conditions that

\[ \text{Re}(u, A_{\text{max}} u) \geq \beta(u, u), \ \forall u \in D(Q) \hspace{1cm} (38) \]

\[ \text{Re}(Q u, A_{\text{max}} u) \geq \beta(Q u, u), \ \forall u \in D(Q) \hspace{1cm} (39) \]

where \( \beta \) is a constant.

Let \( u \in D(Q) \). We have

\[ (u, A_{\text{max}} u) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j \bar{a}_{jk} \bar{u}_k \]

and

\[ (\bar{u}, A_{\text{max}} \bar{u}) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \bar{u}_j a_{jk} u_k \]

\[ = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (q_j u_j) (q_j^{-1} a_j q_k^{-1}) (q_k u_k) \]

\[ = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (q_j u_j) (q_j^{-1} a_j q_k^{-1}) (q_k u_k) \]

\[ = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} u_j a_{jk} u_k \]

\[ = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j a_{kj} \bar{u}_k. \]

Note that the double summations above are interchangable because of the fact that \( (q_j^{-1} a_j q_k^{-1}) \) is a Hilbert-Schmidt matrix in \( l^2 \). Consequently

\[ \text{Re}(u, A_{\text{max}} u) = \frac{1}{2} \left[ (u, A_{\text{max}} u) + (\bar{u}, A_{\text{max}} \bar{u}) \right] \]

\[ = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j (\bar{a}_{jk} + a_{kj}) \bar{u}_k \]
This together with the condition (36) implies (38).

Similarly, for \( v \in D(Q^{-1/2}) \) we have

\[
(Q^{1/2} v, A_{\text{max}} Q^{-1/2} v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_j^{1/2} v_j^* a_{jk} q_k^{-1/2} \overline{v}_k.
\]

And

\[
\overline{(Q^{1/2} v, A_{\text{max}} Q^{-1/2} v)} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (q_j^{1/2} \overline{v}_j) (a_{jk} q_k^{-1}) q_k^{1/2} v_k
\]

\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (q_j^{1/2} \overline{v}_j) (a_{jk} q_k^{-1}) q_k^{1/2} v_k
\]

\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_j^{1/2} v_j a_{jk} q_k^{1/2} \overline{v}_k.
\]

The interchange of summations above is allowable since \( \text{Op}(a_{jk} q_k^{-1}) \) is Hilbert-Schmidt. Therefore for \( u = Q^{-1/2} v \in D(Q) \)

\[
\text{Re}(Q u, A_{\text{max}} u) = \text{Re}(Q^{1/2} v, A_{\text{max}} Q^{-1/2} v)
\]

\[
= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_j (q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} a_{jk} q_k^{1/2}) \overline{v}_k
\]

\[
= \frac{1}{2} (v, \text{Op}(q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} a_{jk} q_k^{1/2}) v).
\]

Condition (39) then follows from (37) and the above relation.

We have a few remarks on the conditions in the previous theorem. Condition (35) is satisfied if we take

\[
q_k = \alpha_k^{-1} \max \{ |a_{k-1,k}|, |a_{kk}|, |a_{k+1,k}| \}
\]

with \( \{ \alpha_k \} \) in \( l^2 \). Condition (36) is verified if \( (a_{jk}) \) is a sum of a skew-symmetric matrix and a bounded matrix. In case \( (a_{jk}) \) is skew-symmetric, then the entry at \( (k, k-1) \) of the matrix \( (c_{jk}) = (q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} a_{jk} q_k^{1/2}) \) is
Thus if $q_k/q_{k-1} \to 1$ the operator in (37) might be bounded and hence quasi-accretive in spite of $\{ | a_{k,k-1} | \}$ being unbounded.

**Example 5.** If

$$a_{k,k-1} \left[ \frac{q_k}{q_{k-1}} \right]^\frac{1}{4} - \left[ \frac{q_k}{q_{k-1}} \right]^{-\frac{1}{4}},$$

and $Q = \text{Op} \left[ \text{diag} \left( 1^\mu, 2^\mu, 3^\mu, \ldots \right) \right]$ ($\mu > \nu + \frac{1}{2}$), then the conditions in the above theorem are satisfied. So $\text{Op}(a_{jk}) \upharpoonright \text{D}(Q)$ essentially generates a $c_0$ semigroup in $l^2$. 

Acknowledgement:
The author wish to express his gratitude to Prof. J. de Graaf for his inspiring advices.

References


1980 Mathematics Subject Classification (1985 Revision): 47B37; 47B44; 47D05.
Key Words: Matrix Generators.