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Controlled invariance of nonlinear systems: nonexact forms speak louder than exact forms
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Controlled invariance of nonlinear systems: nonexact forms speak louder than exact forms*

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Abstract

A general setting is developed which describes controlled invariance for nonlinear control systems and which incorporates the previous approaches dealing with controlled invariant distributions. The main feature of the theory developed is that it is able to clarify the controlled invariance of distributions which are not necessarily integrable. The latter are of major importance for the geometric description of e.g. dynamic feedback problems.

1 Introduction

During the last two decades, nonlinear control theory was developed thanks to the increasing number of researchers involved in this area. A main stream of the research in the 80's was the generalization, at least partially, of the so called geometric approach which proved to be particularly efficient for linear time-invariant systems (see [17] for an overview). In this linear theory, controlled invariance is a fundamental notion.

The study of controlled invariance for nonlinear systems of the form

\[ \dot{x} = f(x) + g(x)u \]  

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), was initiated in [2]. In this paper invariants were sought under feedback transformations

\[ u = \alpha(x) + v \]  

Later on, controlled invariance was tackled by various authors ([11],[7],[13],[14]). The group of feedback transformations acting on (1) was enlarged to transformations of the form

\[ u = \alpha(x) + \beta(x)v \]  

where \( \beta(x) \) is square and locally invertible. These works yielded the definition of a controlled invariant distribution. The key was found for the solution of synthesis problems, such as the disturbance decoupling problem and the noninteracting control problem, via regular (or invertible) static state feedback (see the textbooks [9],[15] for an overview).

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Some limits of this by now well established theory appeared at the end of the 80’s in the characterization of left- or right-invertibility for nonlinear systems or for synthesis problems involving dynamic feedback. A nice understanding of these problems is provided by a differential algebraic theory ([6]).

The goal of this paper is to introduce a generalized notion of controlled invariance by allowing an enlarged class of feedback transformations acting on (1). The motivation is to clarify a number of pathological situations. Relations exist with both the differential geometric and the differential algebraic approach, but these will not be outlined in this paper.

In the sequel we consider a nonlinear control system (1), where the entries of \( f(x) \) and \( g(x) \) are meromorphic functions of \( x \). It is assumed that \( \text{rank} \, g(x) = m \) and that \( n \geq 1 \).

In Section 2 we define the generalized notion of invariance with respect to the dynamics (1). Section 3 is devoted to controlled invariance and related properties.

## 2 Invariant subspaces

We follow the notations and setting of [5]. Let \( \mathcal{K} \) denote the field of meromorphic functions of \( x, u, \dot{u}, \ldots, u^{(n-1)} \). \( \mathcal{E} \) is the formal vector space spanned by \( \{dx, du, \dot{u}, \ldots, u^{(n-1)}\} \) over \( \mathcal{K} \). The notation \( dx \) stands for \( \{dx_1, \ldots, dx_n\} \) and \( du^{(k)} \) for \( \{du_1^{(k)}, \ldots, du_m^{(k)}\} \). Let \( \mathcal{X} := \text{span}_\mathcal{K}\{dx\} \) and \( \mathcal{U} := \text{span}_\mathcal{K}\{du, \dot{u}, \ldots, u^{(n-1)}\} \). Consider a subspace \( \Omega \subset \mathcal{X} \). Define

\[
\dot{\Omega} = \text{span}_\mathcal{K}\{\dot{\omega} \mid \omega \in \Omega\} \tag{4}
\]

where \( \omega = \sum_{i=1}^{n} \omega_i(x, u, \dot{u}, \ldots, u^{(n-1)})dx_i \) and time-derivation is defined by \( \dot{\omega} = \sum_{i=1}^{n}(\omega_i \dot{x}_i + \dot{\omega}_i dx_i) \). Thus \( \dot{\omega} \in \text{span}_\mathcal{K}\{dx, du\} \).

**Definition 2.1** A subspace \( \Omega \subset \mathcal{X} \) is said to be invariant with respect to (1) if

\[
\dot{\Omega} \subset \Omega + \text{span}_\mathcal{K}\{du\} \tag{5}
\]

**Remark 2.2** Let \( \mathcal{K}_k \) be the field of meromorphic functions of \( x, u, \ldots, u^{(k)} \) and define

\[
\mathcal{K}' = \bigcup_{k \in \mathbb{N}} \mathcal{K}_k
\]

Then (5) is equivalent to the statement that \( (\Omega + \text{span}_{\mathcal{K}'\{du^{(k)} \mid k \geq 0\}}) \) is a differential vector space, with the derivation defined above.

**Example 2.3** Let \( \Delta \) be an involutive invariant distribution for (1) and let \( (x_1, x_2) \) be a local system of coordinates such that \( \Delta = \text{span}\{\frac{\partial}{\partial x_2}\} \). Then in the coordinates \( (x_1, x_2) \), (1) takes the form (cf. [9],[15])

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)u \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u
\end{align*} \tag{6}
\]

Let \( \Omega := \text{span}_\mathcal{K}\{dx_1\} \). Then

\[
\dot{\Omega} = \text{span}_\mathcal{K}\{d\dot{x}_1\} = \text{span}_\mathcal{K}\{d(f_1(x_1) + g_1(x_1)u)\} \subset \Omega + \text{span}_\mathcal{K}\{du\} \tag{7}
\]

Hence \( \Omega \) is invariant in the sense of Definition 2.1. By means of \( \Omega \), the invariance of \( \Delta \) is described in a dual form.

When a given subspace is not invariant, it is interesting whether or not there exists a feedback transformation that renders it invariant. This is the topic of the next section.
In this section we define and characterize the controlled invariance of subspaces \( \Omega \subset \mathcal{X} \) under quasi-static state feedback. In Subsection 3.1 we first define quasi-static state feedback, based on [3],[4]. In Subsection 3.2 we give a definition of controlled invariance under quasi-static state feedback. We then investigate when subspaces are controlled invariant under a regular static state feedback (3). These feedbacks are a special case of quasi-static state feedback. It will turn out that the conditions for controlled invariance under regular static state feedback allow us to formulate a condition for controlled invariance of (not necessarily involutive) distributions via regular static state feedback. Subsection 3.2 is concluded with a condition for controlled invariance of subspaces \( \Omega \subset \mathcal{X} \) under quasi-static state feedback. In Subsection 3.3 we make some remarks about the smallest controlled invariant subspace containing some given subspace.

Throughout this section we employ the following terminology. A vector \( \omega \in \mathcal{E} \) is called exact if there exists a \( \phi \in \mathcal{K} \) such that \( \omega = d\phi \). A subspace \( \Omega \subset \mathcal{E} \) of dimension \( r \) is called exact if there exist functions \( \phi_1, \ldots, \phi_r \in \mathcal{K} \) such that \( \Omega = \text{span}_\mathcal{K}\{ d\phi_1, \ldots, d\phi_r \} \). Given subspaces \( \Omega_1 \subset \Omega_2 \subset \mathcal{E} \), \( \frac{\Omega_2}{\Omega_1} \) is said to be exact if there exist functions \( \phi_1, \ldots, \phi_d \in \mathcal{E} \), with \( d = \dim(\Omega_2) - \dim(\Omega_1) \), such that \( \Omega_2 = \Omega_1 \oplus \text{span}_\mathcal{K}\{ d\phi_1, \ldots, d\phi_d \} \), or, in other words, \( \frac{\Omega_2}{\Omega_1} \) is isomorphic to an exact subspace of \( \mathcal{E} \). Consider a subspace \( \Omega \subset \mathcal{E} \). Then clearly \( \{0\} \subset \Omega \) is exact. Furthermore, if \( \Omega_1 \subset \Omega \), \( \Omega_2 \subset \Omega \) are exact, then also \( \Omega_1 + \Omega_2 \subset \Omega \) is exact. Hence there exists a unique maximal exact subspace in \( \Omega \).

### 3.1 Quasi-static state feedback

Consider the nonlinear system (1). A generalized static state feedback for (1) is a feedback of the form

\[
u = \phi(x, v, \ldots, v^{(r)})
\]

where \( v \in \mathbb{R}^m \) denotes the new controls. Let \( \mathcal{K}_v \) denote the field of meromorphic functions of \( \{x, v^{(k)} | k \geq 0\} \) and define the formal vector space \( \mathcal{E}_v := \text{span}_{\mathcal{K}_v}\{ d\xi | \xi \in \mathcal{K}_v \} \). As in [3],[4], we define the following filtrations ([1]) of \( \mathcal{E}_v \):

\[
\mathcal{V}_{-1} := \text{span}_{\mathcal{K}_v}\{dx\} \\
\mathcal{V}_k := \text{span}_{\mathcal{K}_v}\{dx, dv, \ldots, dv^{(k)}\} \quad (k \geq 0)
\]

\[
\mathcal{U}_{-1} := \text{span}_{\mathcal{K}_v}\{dx\} \\
\mathcal{U}_k := \text{span}_{\mathcal{K}_v}\{dx, d\phi, \ldots, d\phi^{(k)}\} \quad (k \geq 0)
\]

The filtrations \( \mathcal{U}_k \) and \( \mathcal{V}_k \) are said to have bounded difference ([1]) if there exists an \( s \in \mathbb{N} \) such that for all \( k \geq -1 \)

\[
\mathcal{U}_k \subset \mathcal{V}_{k+s} \\
\mathcal{V}_k \subset \mathcal{U}_{k+s}
\]

**Definition 3.4** ([3],[4]) A feedback given by (8) is said to be a quasi-static state feedback for (1) if the filtrations \( \mathcal{U}_k \) and \( \mathcal{V}_k \) have bounded difference.

**Remark 3.5** It is easily verified that a regular static state feedback (3) is a quasi-static state feedback.
The following result is also easily proved.

**Proposition 3.6** \( \psi \) given by (8) is a quasi-static state feedback if and only if locally there exist an \( s \in \mathbb{N} \) and a function \( \psi(x, u, \ldots, u^{(s)}) \) such that locally (8) is equivalent to
\[
v = \psi(x, u, \ldots, u^{(s)})
\] (12)

### 3.2 Controlled invariance

#### 3.2.1 Definition of controlled invariance

Consider the control system (1) together with a quasi-static state feedback (8) and define \( \mathcal{V} := \text{span}_{\mathbb{K}_0}\{d_v^{(k)} | k \geq 0\} \).

**Definition 3.7** A subspace \( \Omega \subset \mathcal{X} \) is said to be controlled invariant for (1) if there exists a quasi-static state feedback (8) such that for (1,8) one has
\[
\dot{\Omega} \subset \Omega + \mathcal{V} \quad (13)
\]

The definition of controlled invariance given in Definition 3.7 is in accordance with the well known definition of a controlled invariant distribution. Recall from e.g. [9],[15] that a distribution \( \Delta \) is controlled invariant if there exists a regular static state feedback (3) such that
\[
[f + g_\alpha, \Delta] \subset \Delta \quad (14)
\]
where \((g_\beta)_{*i}\) denotes the \( i \)-th column of \((g_\beta)\). Define
\[
\Omega = \Delta^\perp = \{\omega | \langle \omega, \tau \rangle = 0, \forall \tau \in \Delta\} \quad (15)
\]

Then for \( \Omega \), (14) translates into
\[
\mathcal{L}_{f + g_\alpha} \Omega \subset \Omega \quad (16)
\]
\[
\mathcal{L}_{(g_\beta)} \Omega \subset \Omega \quad (i = 1, \ldots, m) \quad (17)
\]

Let \( \omega \in \Omega \). Then for (1,3) we have
\[
\dot{\omega} = \mathcal{L}_{f + g_\alpha} \omega + \sum_{i=1}^{m} (\nu_i \mathcal{L}_{(g_\beta)} \omega + \langle \omega, (g_\beta)_{*i} \rangle d v_i)
\]

From this equality it immediately follows that \( \Delta \) is controlled invariant if and only if (13) is satisfied for \( \Omega = \Delta^\perp \).
3.2.2 Controlled invariance via regular static state feedback

In this subsection we investigate under what conditions a subspace $\Omega \subset \mathcal{X}$ is controlled invariant under regular static state feedback. Recall from Subsection 3.2.1 that a regular static state feedback is a special sort of quasi-static state feedback. A first result is the following.

**Proposition 3.8** Consider a $d$-dimensional subspace $\Omega \subset \mathcal{X}$. Assume that $\Omega$ is controlled invariant under a quasi-static state feedback of the form $u = \phi(x, v)$. Then $\Omega$ admits a basis $\omega_1, \ldots, \omega_d$ with

$$\omega_i = \sum_{j=1}^{n} \omega_{ij}(x)dx_j \quad (18)$$

**Proof** Assume that $\Omega = \text{span}_\mathcal{K}\{\omega_1, \ldots, \omega_d\}$, with

$$\hat{\omega}_i = \sum_{j=1}^{n} \hat{\omega}_{ij}(x, u)dx_j \quad (19)$$

Let $A(x, u)$ be the matrix with entries $\hat{\omega}_{ij} (i = 1, \ldots, d; j = 1, \ldots, n)$. Viewing $\Omega$ as a linear subspace (over $\mathcal{K}$) of $\mathcal{X} \oplus \text{span}_\mathcal{K}\{du\}$, it may be characterized by

$$\Omega = \text{rowspan}_\mathcal{K}(A(x, u) 0) \quad (20)$$

Similarly, $\hat{\Omega}$ is characterized by

$$\hat{\Omega} = \text{rowspan}_\mathcal{K}(B(x, u, \hat{u}) (Ag)(x, u)) \quad (21)$$

where

$$B(x, u, \hat{u}) = \sum_{i=1}^{n} \frac{\partial A}{\partial x_i}(x, u)\hat{x}_i(x, u) + \sum_{j=1}^{m} \frac{\partial A}{\partial u_j} \hat{u}_j + A(x, u) \left( f_x(x) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \right) \quad (22)$$

with $f_x$ the Jacobian of $f$. Since $\Omega$ is rendered invariant via $u = \phi(x, v)$ there exist matrices $P(x, v, \hat{v})$ and $Q(x, v)$ such that

$$B(x, \phi, \phi)dx + (Ag)(x, v)dv = P(x, v, \hat{v})A(x, \phi)dx + Q(x, v)dv \quad (23)$$

or

$$B(x, \phi, \phi) = P(x, v, \hat{v})A(x, \phi) - (Ag)(x, \phi)\phi_v(x, v) \quad (24)$$

Since $\phi_v(x, v)$ is invertible, this yields

$$B(x, \phi, \phi) = P(x, v, \hat{v})A(x, \phi) - Q(x, v)\phi_v(x, v)^{-1}\phi_v(x, v) \quad (25)$$

Furthermore, by the Inverse Function Theorem there exists a function $\psi(x, u)$ such that $u = \phi(x, v)$ is equivalent to $v = \psi(x, u)$ and $\psi_x(x, u) = -\phi_v(x, \psi(x, u))^{-1}\phi_v(x, \psi(x, u))$. Hence (25) yields

$$B(x, u, \hat{u}) = \tilde{P}(x, u, \hat{u})A(x, u) + \tilde{Q}(x, u)\psi(x, u) \quad (26)$$
where \( \bar{P}(x,u,\dot{u}) = P(x,\psi(x,u),\dot{\psi}(x,u,\dot{u})) \) and \( \bar{Q}(x,u) = Q(x,\psi(x,u)) \). Taking partial derivatives with respect to \( \dot{u}_i \), we obtain

\[
\frac{\partial A}{\partial u_i} = \frac{\partial \bar{P}}{\partial \dot{u}_i} A(x,u) \quad (i = 1, \ldots, m)
\]  

Obviously,

\[
\frac{\partial^2 \bar{P}}{\partial u_i \partial u_j} = 0 \quad (i,j = 1, \ldots, m)
\]

Hence there exist matrices \( R_i(x,u) \) \((i = 1, \ldots, m)\) such that

\[
\frac{\partial A}{\partial u_i} = R_i(x,u)A(x,u)
\]  

Using arguments from the theory of linear time-varying ordinary differential equations this yields that \( A(x,u) \) is of the form

\[
A(x,u) = \Phi(x,u)\Psi(x)
\]

with \( \Phi(x,u) \) square invertible. Hence

\[
\Omega = \text{rowspan}_K(A(x,u) \ 0) = \text{rowspan}_K(\Psi(x) \ 0)
\]

which establishes our claim. If \( \Omega = \text{rowspan}_K\left(A(x,u, \ldots, u^{(\ell)}) \ 0\right) \) with \( \ell > 1 \), the claim is established by using the same arguments together with an induction argument. \( \blacksquare \)

For (1), let \( G \) denote the distribution spanned by the input vector fields. Define the subspace \( G^\perp \subset X \) by

\[
G^\perp = \{ x \in X \mid (x,g) = 0, \forall g \in G\}
\]

**Theorem 3.9** Let \( \Omega \subset X \) be a subspace which admits a basis satisfying (18). Then \( \Omega \) is controlled invariant if and only if

\[
(i) \quad (\Omega \cap G^\perp) \subset \Omega
\]

\[
(ii) \quad \frac{\Omega + \hat{\Omega}}{\Omega} \text{ is exact.}
\]

Moreover, if \( \Omega \) satisfies (i) and (ii), then it can be rendered invariant via a regular static state feedback.

**Proof** (sufficiency) Assume that (i) and (ii) hold. Note that \( \Omega + \hat{\Omega} \subset \text{span}_K\{dx, du\} \). Let \( \hat{\Omega} \subset X \) be such that \( \Omega = (\Omega \cap G^\perp) \oplus \hat{\Omega} \). Assume that \( \hat{\Omega} \cap X \neq \{0\} \). This implies that there is an \( \hat{\omega} \in \hat{\Omega}, \hat{\omega} \neq 0 \), such that \( \hat{\omega} \in X \) and hence \( \hat{\omega} \in (\Omega \cap G^\perp) \), which gives a contradiction. Thus

\[
\hat{\Omega} \cap X = \{0\}
\]

By (ii), there exists a \( v_1(x,u) \) such that

\[
\Omega + \hat{\Omega} = \Omega \oplus \text{span}_K\{dv_1\}
\]
Since \((i)\) and \((31)\) hold, we must have that \((\partial v_1/\partial u)\) has full row rank. Then there exists a \(v_2(u)\) such that \((\partial v/\partial u)\) is square and invertible, where \(v = (v_1^T\ v_2^T)^T\). By \((32)\) we now have that

\[
\hat{\Omega} \subset \Omega + V
\]  

(33)

Moreover, since \((\partial v/\partial u)\) is invertible, there exists a \(\psi(x,v)\) such that \(u = \psi(x,v)\). Hence \(\psi\) defines a quasi-static state feedback and thus \(\Omega\) can be rendered invariant via quasi-static state feedback. Since we are dealing with a control system \((1)\) that is affine in \(u\), it is easily seen that \(v\) can be taken affine in \(u\) and thus \(\psi\) can be taken affine in \(v\). This implies that \(\hat{\Omega}\) can be rendered invariant via a static state feedback \((3)\).

\((\text{necessity})\) Assume that \(\Omega\) is controlled invariant and that it is rendered invariant via a regular quasi-static state feedback \(u = \phi(x,v)\). It is easily verified that

\[
(\Omega \cap G^\perp) \subset \mathcal{X}
\]  

(34)

Hence condition \((i)\) must hold. Consider a basis \(\{\omega_1,\ldots,\omega_r\}\) of \(\Omega\) where

\[
\omega_i = \sum_{j=1}^{n} \alpha_{ij}(x)dx_j
\]  

(35)

and

\[
(\Omega \cap G^\perp) = \text{span}_K\{\omega_1,\ldots,\omega_d\}
\]  

(36)

Let \(A_1(x)\) be the matrix with entries \(\alpha_{ij}(x)\) \((i = 1,\ldots,d; j = 1,\ldots,n)\) and \(A_2(x)\) the matrix with entries \(\alpha_{ij}(x)\) \((i = d+1,\ldots,r; j = 1,\ldots,n)\). Viewing \(\Omega\) as a linear subspace (over \(K\)) of \(X \oplus \text{span}_K\{du\}\), it may be characterized by

\[
\Omega = \text{rowspan}_K\left( \begin{array}{c} A_1(x) \\ A_2(x) \end{array} \right)
\]  

(37)

Similarly, \(\hat{\Omega}\) is characterized by

\[
\hat{\Omega} = \text{rowspan}_K\left( \begin{array}{c} B_1(x,u) \\ B_2(x,u) \end{array} \right)
\]  

(38)

where

\[
B_i(x,u) = \dot{A}_i(x,u) + A_i(x)(f_\phi(x) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}u) \quad (i = 1,2)
\]  

(39)

with \(f_\phi\) the Jacobian of \(f\). Since \(\Omega\) is rendered invariant via \(u = \phi(x,v)\), there exist matrices \(P_1(x,v), P_2(x,v), Q(x,v)\) such that

\[
B_2(x,\phi(x,v))dx + (A_2g)(x)d\phi = P_1(x,v)A_1(x) + P_2(x,v)A_2(x) + Q(x,v)dv
\]  

(40)

or

\[
(B_2(x,\phi(x,v)) + (A_2g)(x)\phi_\phi(x,v))dx + (A_2g)(x)\phi_\phi(x,v)dv = (P_1(x,v)A_1(x) + P_2(x,v)A_2(x))dx + Q(x,v)dv
\]  

(41)
Hence
\[ B_2(x, \phi(x,v)) = P_1(x,v)A_1(x) + P_2(x,v)A_2(x) - (A_2 g)(x) \phi_2(x,v) \]
\[ (A_2 g)(x) \phi_v(x,v) = Q(x,v) \]  \hspace{1cm} (42)

Since \( \phi_v \) is invertible, there exists a function \( \psi(x,u) \) with the property that \( u = \phi(x,v) \) is equivalent to \( v = \psi(x,u) \). Moreover, by the Inverse Function Theorem, we have
\[ \phi_x(x,v) = -\psi_u(x,\phi(x,v))^{-1} \psi_2(x,\phi(x,v)) \]
\[ \phi_v(x,v) = \psi_u(x,\phi(x,v))^{-1} \]  \hspace{1cm} (43)

Then equations (42) and (43) yield
\[ B_2(x,u) = \tilde{P}_1(x,u)A_1(x) + \tilde{P}_2(x,u)A_2(x) + \tilde{Q}(x,u) \psi_2(x,u) \]
\[ (A_2 g)(x) = \tilde{Q}(x,u) \psi_u(x,u) \]  \hspace{1cm} (44)

where \( \tilde{P}_i(x,u) = P_i(x,\psi(x,u)) \) (i = 1, 2) and \( \tilde{Q}(x,u) = Q(x,\psi(x,u)) \). Using a similar argument as in the sufficiency-part, it can be shown that \( (A_2 g)(x) \) has full row rank. Hence there exists a regular static state feedback \( u = \beta(x)\tilde{u} \) such that \( (A_2 g\beta)(x) = (R(x)0) \), with \( R(x) \) invertible. This means that without loss of generality we may assume that \( (A_2 g)(x) \) itself is invertible. By (44) we then have that \( \tilde{Q}(x,u) \) is also invertible, and hence by (44),

\[ \text{rowspan}_K(B_2(x,u) (A_2 g)(x)) = \]
\[ \text{rowspan}_K(\tilde{Q}(x,u)^{-1}(B_2(x,u) (A_2 g)(x))) = \]  \hspace{1cm} (45)

\[ \text{rowspan}_K(\tilde{Q}(x,u)^{-1}(\tilde{P}_1(x,u)A_1(x) + \tilde{P}_2(x,u)) + \psi_2(x,u) \psi_u(x,u)) \]

Noting that
\[ \text{rowspan}_K(B_1(x,u) 0) \subset \text{rowspan}_K \begin{pmatrix} A_1(x) & 0 \\ A_2(x) & 0 \end{pmatrix} \]
the equality (45) yields
\[ \Omega + \hat{\Omega} = \text{rowspan}_K \begin{pmatrix} A_1(x) & 0 \\ A_2(x) & 0 \\ B_2(x,u) & (A_2 g)(x) \end{pmatrix} = \]
\[ \text{rowspan}_K \begin{pmatrix} A_1(x) & 0 \\ A_2(x) & 0 \\ \psi_2(x,u) & \psi_u(x,u) \end{pmatrix} \]  \hspace{1cm} (46)

which implies that \( (\Omega + \hat{\Omega})/\Omega \) is exact.

Theorem 3.9 has an interesting consequence. Recall from e.g. [9],[15] that necessary and sufficient conditions for (local) controlled invariance of distributions are only known in case the distribution under consideration is involutive. Using Theorem 3.9, one can also give conditions for controlled invariance of noninvolutive distributions.
Corollary 3.10 Consider a (not necessarily involutive) distribution $\Delta$. Define the distribution $\Delta_0$ by

$$\Delta_0 = \{ \tau \mid [f, \tau] \in \Delta, [g, \tau] \subset \Delta \} \cap \Delta$$

Then $\Delta$ is controlled invariant if and only if

\begin{enumerate}[(i)]
  \item $[f, \Delta] \subset \Delta + G$
  \item $[G, \Delta] \subset \Delta + G$
\end{enumerate}

(ii) $\Delta_0 = \Delta \cap \Delta_0$

where $\Delta_0$ denotes the involutive closure of $\Delta_0$.

Proof Define $\Omega := \Delta^\perp$. By Theorem 3.9, $\Omega$ is controlled invariant if and only if

\begin{equation}
(\Omega \cap \overline{G}^\perp) \subset \Omega
\end{equation}

and there exists a function $\phi$ such that

\begin{equation}
\Omega + \dot{\Omega} = \Omega \oplus \text{span}_K \{d\phi\}
\end{equation}

Note that (47) is equivalent to

\begin{equation}
\Omega + (\Omega \cap \overline{G}^\perp) \subset \Omega
\end{equation}

Now

\begin{equation}
(\Omega + (\Omega \cap \overline{G}^\perp))^\perp = \Delta \cap \{ \tau \mid \langle \omega, \tau \rangle = 0, \forall \omega \in \Omega \cap \overline{G}^\perp \} = \\
\Delta \cap \{ \tau \mid \langle L_{f+g}\omega, \tau \rangle = 0, \forall \omega \in \Omega \cap \overline{G}^\perp \} = \\
\Delta \cap \{ \tau \mid \langle L_{f+g}(\omega, \tau) - L_{f+g}\omega, \tau \rangle = 0, \forall \omega \in \Omega \cap \overline{G}^\perp \} = \\
\Delta \cap \{ \tau \mid \langle \omega, L_{f+g}\tau \rangle = 0, \forall \omega \in \Omega \cap \overline{G}^\perp \} = \\
\Delta \cap \{ \tau \mid [f, \tau] \in \Delta + G, [g, \tau] \subset \Delta + G \}
\end{equation}

(49) and (50) yield

\begin{equation}
\Delta \subset \{ \tau \mid [f, \tau] \in \Delta + G, [g, \tau] \subset \Delta + G \}
\end{equation}
which is equivalent to (i). Furthermore, we have

\[(\Omega + \Omega) = \{\tau | \omega_1 + \omega_2, \tau \geq 0, \forall \omega_1, \omega_2 \in \Omega\} = \]

\[\{\tau | \omega, \tau \geq 0, \forall \omega \in \Omega\} \cap \{\tau | \omega, \tau \geq 0, \forall \omega \in \Omega\} = \]

\[\Delta \cap \{\tau | \omega, \tau > 0, \forall \omega \in \Omega\} = \]

\[\Delta \cap \{\tau | L_{f+g} \omega, \tau > 0, \forall \omega \in \Omega\} = \]

\[\Delta \cap \{\tau | L_{f+g} \omega, \tau > 0, \forall \omega \in \Omega\} = \]

\[\Delta \cap \{\tau | f + g \omega, \tau \in \Delta\} = \]

\[\Delta \cap \{\tau | f, \tau \in \Delta, [a, \tau] \subset \Delta\} = \Delta_0 \]

Equalities (48) and (52) imply that

\[\Delta_0 = \Delta \cap \text{Ker } d\phi \]

Now define on \(\mathbb{R}^{n+m}\) the distributions \(\Delta^e := \Delta \times \{0\}\), \(\Delta^0 := \Delta_0 \times \{0\}\). Then according to (53) there exists an involutive distribution \(\Delta^0\) on \(\mathbb{R}^{n+m}\) such that

\[\Delta^e = \Delta^e \cap \Delta^0 \]

Let \(\Delta^e_0\) denote the involutive closure of \(\Delta^e_0\). By (54) we have that \(\Delta^e_0 \subset \Delta^0\) and hence, by involutivity of \(\Delta\), \(\Delta^e_0 \subset \Delta\). This implies that

\[\Delta^e \cap \Delta^e_0 \subset \Delta^e \cap \Delta = \Delta^e_0 \]

Since we clearly have \(\Delta^e_0 \subset \Delta^e \cap \Delta^0\), this means that \(\Delta^e_0 = \Delta^e \cap \Delta^0\). Condition (ii) now follows by noting that \(\Delta^e_0 = \Delta_0 \times \{0\}\).

**Remark 3.11** If \(\Delta\) is involutive, then it can be shown by using the Jacobi-identity that \(\Delta_0\) is also involutive. Hence for involutive \(\Delta\) condition (ii) in Theorem 3.9 is automatically satisfied and thus the result of Theorem 3.9 is in accordance with existing results (see e.g. [9],[15]).

### 3.2.3 The general case

Let us again consider a general subspace \(\Omega \subset \mathcal{X}\). Define by induction:

\[\dot{\Omega}_0 := \{0\}\]

\[\Omega_0 := \Omega\]

\[\dot{\Omega}_{k+1} := \text{maximal exact subspace in } \frac{\Omega_k + \dot{\Omega}_k}{\Omega_k}\]

\[\Omega_{k+1} := \Omega_k + \dot{\Omega}_{k+1}\]

Furthermore, define

\[k^* := \max\{k \geq 1 | \dim(\dot{\Omega}_k) > \dim(\dot{\Omega}_{k-1})\}\]
Theorem 3.12 Let $\Omega \subset X$. If

(i) $(\Omega \cap G^\perp) \subset \Omega$

(ii) \( \frac{\Omega_{k^* - 1} + \dot{\Omega}_{k^* - 1}}{\Omega_{k^* - 1}} \) is exact.

then $\Omega$ is controlled invariant for (1).

Proof From the definition of $k^*$, there exist vector valued $dv_1, \ldots, dv_{k^*}$ in $E$, where each $dv_i$ is non-empty, such that

\[
\begin{align*}
\dot{\Omega}_1 & = \text{span}\{dv_1\} \subset \frac{\Omega_0 + \dot{\Omega}_0}{\Omega_0} \\
\dot{\Omega}_2 & = \text{span}\{dv_1, dv_2\} \subset \frac{\Omega_1 + \dot{\Omega}_1}{\Omega_1} \\
& \vdots \\
\dot{\Omega}_{k^*} & = \text{span}\{dv_1^{(k^* - 1)}, dv_2^{(k^* - 2)}, \ldots, dv_{k^*}\} \subset \frac{\Omega_{k^* - 1} + \dot{\Omega}_{k^* - 1}}{\Omega_{k^* - 1}}
\end{align*}
\]

Note that from (ii) the last inclusion in (56) is in fact an equality. We now have

\[
\begin{align*}
\dot{\Omega} & \subset \Omega_0 + \dot{\Omega}_0 + \dot{\Omega}_1 + \dot{\Omega}_1 = \Omega_1 + \dot{\Omega}_1 + \cdots \\
\Omega_{k^* - 1} + \dot{\Omega}_{k^* - 1} & = \Omega_{k^* - 1} + \text{span}\{dv_1^{(k^* - 1)}, \ldots, dv_{k^*}\} \subset \\
\Omega + \text{span}\{dv^{(k)} \mid k \geq 0\}
\end{align*}
\]

It remains to be shown that $v$ defines a quasi-static state feedback. From the above construction, one has

\[
\begin{align*}
v_1 & = \phi_1(x, u) \\
v_2 & = \phi_2(x, v_1, \dot{v}_1, u) \\
& \vdots \\
v_{k^*} & = \phi_{k^*}(x, \{v_i^j \mid 1 \leq i \leq k^*, 1 \leq j \leq k^*-i\}, u)
\end{align*}
\]

From (i), $(\partial(\phi_1, \ldots, \phi_{k^*})/\partial u)$ has full row rank. Thus there exists $v_{k^* + 1} = \phi_{k^* + 1}(u)$ such that $(\partial(\phi_1, \ldots, \phi_{k^* + 1})/\partial u)$ is square invertible. From the Inverse Function Theorem, there exists a function $\psi$ such that $u = \psi(x, v, \dot{v}, \ldots, v_{k^*})$.

Remark 3.13 The above theorem only gives sufficient conditions for the controlled invariance of a subspace $\Omega \subset X$. It is easily proved that condition (i) is also a necessary condition. It is the authors' conjecture that condition (ii) is also necessary. For a proof of this conjecture a better understanding of quasi-static state feedback seems to be needed. This remains a topic for further research.
3.2.4 The smallest controlled-invariant subspace containing a subspace

Given a subspace $\Pi \subseteq \mathcal{X}$, it is unclear whether (or under what conditions) there exists a smallest controlled invariant subspace containing $\Pi$. This is due to the fact that for two controlled invariant subspaces $\Omega_1, \Omega_2 \subseteq \mathcal{X}$, we do not necessarily have that $\Omega_1 \cap \Omega_2$ is controlled invariant, so that we cannot use the "standard" arguments (as in e.g. [17],[9],[15]). In this subsection we will give some comments on this question. A first result is the following:

**Proposition 3.14** Consider a subspace $\Pi \subseteq \mathcal{X}$ and define $\Pi^* := \mathcal{X} \cap (\Pi + \Pi^{(1)} + \cdots + \Pi^{(n)})$. Let $\Omega \subseteq \mathcal{X}$ be a controlled invariant subspace containing $\Pi$. Then $\Pi^* \subseteq \Omega$.

**Proof** By definition, there exists a quasi-static state feedback such that $\dot{\Omega} \subseteq \Omega + \mathcal{V}$. It can easily be shown by induction that this implies that $\Omega^{(k)} \subseteq \Omega + \mathcal{V}$ ($k \in \mathbb{N}$). Hence also $\Pi + \Pi^{(1)} + \cdots + \Pi^{(n)} \subseteq \Omega + \mathcal{V}$. Thus,

$$\Pi^* = \mathcal{X} \cap (\Pi + \cdots + \Pi^{(n)}) \subseteq \mathcal{X} \cap (\Omega + \mathcal{V}) = \Omega$$

The subspace $\Pi^*$ defined in Proposition 3.14 is a candidate for being the smallest controlled invariant subspace containing $\Pi$. If $\Pi$ is exact, it can be shown that indeed it is. This may be shown in the following way. Let $r = \dim \Pi$ and choose meromorphic functions $h_1(x), \ldots, h_r(x)$ such that $\Pi = \text{span}_\mathcal{X}\{dh_1, \ldots, dh_r\}$. Next consider the system

$$\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}$$

(59)

Then for this system, $\Pi^* = \mathcal{X} \cap \mathcal{Y}$; where $\mathcal{Y} = \text{span}_\mathcal{X}\{dy, \ldots, dy^{(n)}\}$. (The subspace $\mathcal{X} \cap \mathcal{Y}$ was introduced in [18] for the study of the minimal order input-output decoupling problem.) If the system (59) is right-invertible, one can construct a quasi-static state feedback which renders $\Pi^*$ invariant by using the construction in [16]. If (59) is not right-invertible, the same construction, together with Lemma 1 from [12] may be used to show that $\Pi^*$ is controlled invariant. Summarizing, we have the following result:

**Theorem 3.15** Consider a subspace $\Pi \subseteq \mathcal{X}$ which is exact. Then $\Pi^* := \mathcal{X} \cap (\Pi + \cdots + \Pi^{(n)})$ is the smallest controlled invariant subspace containing $\Pi$.

4 Conclusions

A generalized notion of controlled invariance under quasi-static state feedback for nonlinear systems was introduced. It was shown that this notion coincides with the "classical" notion of a controlled invariant distribution under regular static state feedback. Using the generalized notion of controlled invariance, a condition for the controlled invariance of not necessarily involutive distributions was derived. For a subspace $\Omega \subseteq \mathcal{X}$, sufficient conditions for controlled invariance under quasi-static state feedback were given.

This paper leaves some interesting open questions, which are a topic for further research. A first question is whether the sufficient conditions for controlled invariance under quasi-static state feedback that were derived are also necessary. A second question is whether (or under
what conditions) there exists a smallest controlled invariant subspace containing some given subspace. It seems that for the answer to both questions a better understanding of quasi-static state feedback is needed.

Finally, let us remark that throughout we have restricted ourselves to "Kalmanian" systems and to subspaces $\Omega \subset X$. However, the definition of controlled invariance and the characterizations of controlled invariance in this paper can, *mutatis mutandis*, be translated to non-Kalmanian systems and subspaces $\Omega \subset X \times U$.

**References**


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