A two-priority fluid flow model: joint steady state distribution of the buffer content processes
Tzenova, E.I.; Adan, I.J.B.F.; Kulkarni, V.G.

Published: 01/01/2004

Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 02. Jan. 2019
A Two-Priority Fluid Flow Model:
Joint Steady State Distribution of the Buffer Content Processes

Elena I. Tzenova, Ivo J.B.F. Adan, Vidyadhar G. Kulkarni

February 5, 2004

Abstract

We consider a single-server two-class fluid model with a static priority service discipline under which type 1 fluid receives full service priority over type 2 fluid. The two types of fluid are stored in two separate infinite capacity buffers that are emptied at the constant rate of the server. The inputs to the buffers are governed by an external environment process which is taken to be an irreducible finite state continuous time Markov chain. We derive the Laplace-Stieltjes transform of the steady-state joint distribution of the two buffer content processes. The analytic results are illustrated by two simple examples and we display computational results for three larger scale examples.

1 Introduction

The importance of the study of fluid-flow models arises from their various applications. They arise as models of real-world systems that deal with the processing of continuous entities such as the ones used in the petroleum and chemical industries. They are also used as models of the asymptotic behavior of queues in heavy traffic. Fluid-flow models also provide an important tool for the performance analysis of high-speed data networks, or large-scale production systems where a large number of relatively small jobs are processed. Most of the classical research on stochastic fluid systems in the area of telecommunications is based on the work of Anick, Mitra and Sondhi [3] which is an extension of the pioneering work of Kosten [9]. They study a model with several identical and independent Markov on-off sources that transmit fluid to an infinite-capacity buffer. The fluid is then processed at a fixed rate. The limiting distribution of the buffer content process is computed as a solution to a set of ordinary differential equations. The identical input sources facilitate the analysis of the differential equations and the main result provides the system’s eigenvalues in explicit form. In [14, 15] Mitra generalizes this model by introducing multiple on-off switching. Most of the work in fluid queues deals with the steady-state distribution of the buffer content. See the survey paper by Kulkarni [11] for an extensive overview of the research in this area.
Depending on the service requirements of the traffic different service disciplines may be considered (for example, video/voice traffic is delay sensitive while data traffic is more loss sensitive). In particular, priorities may need to be employed to adequately model some versions of Internet protocol (IP) routers and asynchronous transfer mode (ATM) switches that support multiple priority classes.

In the present paper we study a single-node fluid-flow model with two classes of fluid that are served according to a static priority service discipline under which the higher priority traffic always takes precedence over the lower priority traffic. The service rate is assumed to be constant and the input flows are generated by an irreducible finite state Continuous Time Markov Chain. Under the static priority rule the service of the higher class is independent of the presence of the lower class of fluid in the system and therefore can be analyzed directly as in Kulkarni [11]. The goal of our work is to provide exact analytic results for the steady-state joint buffer content of the two priority classes. The main result is given in Theorem 7.1. It is worth mentioning that this two priority model provides the necessary tools for obtaining the steady-state distribution of the lowest class of fluid in a system with \( K > 2 \) priorities by reducing it to a two-priority system with the higher priority class defined as the aggregate of the first \( K - 1 \) priority classes of fluid. Another important observation is that the two-priority fluid model is equivalent to the tandem fluid model; see Elwalid and Mitra [7] and van den Berg and Mandjes [6]. Thus our results can also be used to analyse tandem fluid queues.

Fluid models with priorities have been studied before. In [10] Kulkarni and Narayanan study a model with two priority classes where each class of fluid originates from an on-off source with exponential on- and off-times and the server gives complete priority to type 1 fluid over type 2. They derive the steady-state marginal distributions of the buffer-content processes by noting that the down time of the server for priority 2 fluid is the same as the busy period of priority 1 fluid. First, they derive explicit expressions for the Laplace-Stieljes transform of the distribution of the first passage time to an empty buffer in a standard fluid flow model, which then is specialized to the case of a fluid model with an on-off source. A similar problem has been studied by Elwalid and Mitra [7] where they consider a two-priority model with the busy period of priority 1 being approximated by an exponential distribution. Zhang [21] has also studied a similar model in which the two types of fluid come from correlated on-off sources. He derives partial differential equations for the joint distribution of the buffer contents of both types. In general these differential equations are complicated and intractable. In [10] Kulkarni and Narayanan avoid these difficulties by considering the marginal distributions of the two buffer content processes.

A relatively recent work in the area of fluid models with priorities is the paper of Berger and Whitt [5]. The model, motivated by communication network applications, has constant processing rate and general input processes with stationary increments; the service policy is preemptive-resume. Upper (empty-buffer bound) and lower (reduced-service-rate bound) bounds are established on the steady-state workload and waiting time distributions of a given class. A relation between the lower priority waiting time and the total workload in the system is given under the additional assumption of independent increments of the higher
priority input process. It is exploited in the derivation of the bounds. An application of these bounds is to extend the notion of effective bandwidths for admission control (originally developed for FCFS discipline) to the setting of multiple priority classes, see Berger and Whitt [4].

The rest of the paper is organized as follows. In the next section we describe the model in detail. Then, in Section 3 we present some embedded analysis results that we use later in the paper. In Section 4 we derive preliminary results concerning the total increase of type two fluid in the buffer during off-periods (time periods when there is no service capacity available to the low priority fluid). We use these results in Section 5 to study the limiting distribution of the low priority fluid content during on-periods (time periods when there is positive leftover service capacity available to the low priority fluid). We then compute the limiting joint distribution of the two fluid content processes during off-periods in Section 6. Finally we combine these conditional distributions to compute the Laplace-Stieltjes transform of the complete steady state joint distribution of the buffer content processes in Section 7. These results are illustrated by analytical and numerical examples in section 8. The development involves several theorems whose proofs are rather technical. They are collected in the Appendix.

2 Model Description

We consider a general single-node fluid model with a server that is always on and works at its full capacity \( \mu \). There are two different priority classes of fluid entering two infinite capacity buffers that are emptied by the server. Their input rates are determined by the state of an external environment process \( \{I(t), \ t \geq 0\} \) that is taken to be an irreducible Continuous Time Markov Chain (CTMC) with finite state space \( S = \{1, \ldots, N\} \) and infinitesimal generator \( Q \). While \( I(t) = i, \ i \in S \), the input rate of type \( k \) fluid is \( p_k(i) \geq 0, \ k = 1, 2 \) and the overall net input rate to the two buffers is \( r(i) = p_1(i) + p_2(i) - \mu \). We assume that type 1 fluid has a higher priority than type 2 fluid, i.e. the server allocates as much of its capacity \( \mu \) as needed to serve type 1 fluid and then allocates the rest of the capacity to serve type 2 fluid. While there is type 1 fluid in the first buffer, type 2 can not be served. It is clear that the service of type 1 fluid does not depend on that of type 2. Let \( r_1(i) = p_1(i) - \mu \) denote the net input rate of type 1 fluid while the environment is in state \( i \in S \). The following notation will be used:

\[
S'_- := \{i \in S : r_1(i) < 0\}, \ N'_- := |S'_-|,
S'_+ := \{i \in S : r_1(i) \geq 0\}, \ N'_+ := |S'_+|,
S_- := \{i \in S : r(i) < 0\}, \ N_- := |S_-|,
S_0 := \{i \in S : r(i) = 0\}, \ N_0 := |S_0|,
S_+ := \{i \in S : r(i) > 0\}, \ N_+ := |S_+|.
\]

Denote by \( X_k(t) \) the amount of type \( k \), \( k = 1, 2 \), fluid in the buffer at time \( t \) and consider the three-dimensional Markov process \( \{(X_1(t), X_2(t), I(t)), \ t \geq 0\} \). The dynamics of the
system is described by the following:

\[
\frac{dX_1(t)}{dt} = \begin{cases} 
(p_1(i) - \mu)^+ & \text{if } X_1(t) = 0, I(t) = i, \\
(p_1(i) - \mu) & \text{if } X_1(t) > 0, I(t) = i,
\end{cases}
\]

and

\[
\frac{dX_2(t)}{dt} = \begin{cases} 
(p_2(i) - (\mu - p_1(i))^+) & \text{if } X_1(t) = 0, X_2(t) = 0, \text{ and } I(t) = i, \\
p_2(i) - (\mu - p_1(i))^+ & \text{if } X_1(t) = 0, X_2(t) > 0, \text{ and } I(t) = i, \\
p_2(i) & \text{if } X_1(t) > 0 \text{ and } I(t) = i,
\end{cases}
\]

where we use the notation \( x^+ := \max\{0, x\} \).

Let \( \pi_I \) denote the limiting distribution of the CTMC \( \{I(t), \ t \geq 0\} \). It is given by the unique solution to

\[
\pi_I Q = 0, \quad \sum_{j \in S} \pi_j^I = 1.
\]

Let \( R := \text{diag}[r(1), \ldots, r(N)] \) denote the net input rate matrix. Then it is clear that the system is stable if and only if

\[
\pi_I R e < 0.
\]

We assume that this condition is satisfied throughout the rest of the paper.
As mentioned earlier, under the described priority service discipline the service of type 1 fluid is not affected by the presence of type 2 fluid in the system. Therefore the results from the classical single buffer fluid model, as given in [11] and [13], can be directly applied to the analysis of $X_1(t)$. In the rest of the paper we study the steady-state joint distribution of $X_1(t)$ and $X_2(t)$.

**Remark:** There is an important relationship between the priority model and a tandem fluid queue. This will be explained by following the same reasoning as in Elwalid and Mitra [7]. Let

$$R_1(t) = \begin{cases} p_1(I(t)) & \text{if } X_1(t) = 0, \\ \mu & \text{if } X_1(t) > 0 \end{cases}$$

represent the rate of the output process of type 1 at time $t$. Analogously define the rate of the output process of type 2 at time $t$ as $R_2(t)$. Clearly, $R_1(t) + R_2(t) \leq \mu$ for $t \geq 0$, and in particular

$$R_2(t) = \begin{cases} p_2(I(t)) & \text{if } X_1(t) = 0, X_2(t) = 0, \\ \mu - p_1(I(t)) & \text{if } X_1(t) = 0, X_2(t) > 0, \\ 0 & \text{if } X_1(t) > 0. \end{cases}$$

The graphical representation of the priority model is given in Figure 1. The service rate available to type 2 fluid is $\mu - R_1(t)$ which is equivalent to saying that $R_1(t)$ supplies buffer 2 (in addition to $p_2(I(t))$, the input rate of type 2 at time $t$) and its service rate is $\mu$. Hence the priority model is equivalent to the tandem fluid queue shown in Figure 2.

### 3 Embedded Analysis

We proceed with some preliminary results that will be used later in the paper.

The second (lower priority) class of fluid can be served only when $I(t) = i \in S'_- \cup S'_+$ and $X_1(t) = 0$. Therefore, we call these periods of time on-periods (from the point of view of the second class of fluid). Every on-period is followed by an off-period which starts when the CTMC \( \{I(t), \ t \geq 0\} \) jumps to a state $I(t) = i \in S'_+$ so that there is no server capacity available to type 2. We assume that $S'_+ \neq \emptyset$, i.e. $p_1(i) \geq \mu$ for at least one state $i \in S$, so that there are off-periods. If $S'_+ = \emptyset$ then there is always some positive leftover service capacity and the marginal distribution of type 2 fluid in steady state can be obtained directly from the classical results for single-flow fluid models as given in Kulkarni [11].

Denote the beginning of an off-period by $T'$, i.e.,

$$T' := \min\{t \geq 0 : I(t) \in S'_+\}, \quad (3.1)$$

and then consider

$$\nu_{ij} := P(I(T') = j | I(0) = i), \quad i \in S, \ j \in S'_+, \quad (3.2)$$

i.e. $\nu_{ij}$ is the probability that the environment process is in state $j$ at the beginning of the next off-period given that its current state is $i$. It is clear that if $i \in S'_+$ then $T' = 0$ and hence $\nu_{ij} = \delta_{ij}$.
Theorem 3.1 The absorption probabilities $\nu_{kj}$, $k \in S'_-$, $j \in S'_+$ are given by the solution to
\[ \sum_{k \in S'_-} q_{ik}\nu_{kj} + q_{ij} = 0, \quad i \in S'_-, \quad j \in S'_+. \tag{3.3} \]

Proof: Follows by standard first-step analysis.

Next, define
\[ T := \min\{ t \geq 0 : I(t) \in S'_-, X_1(t) = 0 \}, \tag{3.4} \]
and
\[ \alpha_{jk}(x) := P(I(T) = k | I(0) = j, X_1(0) = x), \quad j \in S, \ k \in S'_-, \tag{3.5} \]
i.e. $T$ is the first time an on-period starts and $\alpha_{jk}(x)$ is the probability that the environment process is in state $k$ at the beginning of the next on-period given that its current state is $j$ and there is an amount $x$ of type one fluid in the buffer.

We use the following notation:
\[ \alpha_k(x) := [\alpha_{ik}(x), \ i \in S'_-], \quad \frac{d\alpha_k}{dx}(x) := \left[ \frac{d\alpha_{ik}}{dx}(x), \ i \in S'_- \right]^t, \ k \in S'_-. \]
\[ R_1 := \text{diag}[r_1(1), \ldots, r_1(N)], \]

Then the following result holds.

Theorem 3.2 For a fixed $k \in S'_-$ the column vector $\alpha_k(x)$ satisfies the following homogeneous system of differential equations
\[ R_1 \frac{d\alpha_k}{dx}(x) + Q\alpha_k(x) = 0, \quad x \geq 0, \ k \in S'_-, \tag{3.6} \]
with boundary conditions
\[ \alpha_{ik}(0) = \delta_{ik}, \ i \in S'_-. \tag{3.7} \]
Let $(\lambda_j, \phi_j)$ denote the eigenvalues and eigenvectors satisfying
\[ (\lambda_j R_1 + Q)\phi_j = 0. \]

There are exactly $|S'_-|$ eigenvalues with non-positive real part. Assume that these eigenvalues are distinct. Then the solution to (3.6) is given by
\[ \alpha_k(x) = \sum_{j: \Re(\lambda_j) \leq 0} a^k_j e^{\lambda_j x} \phi_j, \tag{3.8} \]
where the coefficients $a^k_j$ are found from the boundary conditions (3.7) as the solution to the linear system
\[ \sum_{j: \Re(\lambda_j) \leq 0} a^k_j \phi_{ij} = \delta_{ik}, \ i \in S'_-. \]
Figure 3: On- and off-periods for the two-priority fluid model; embedded processes

**Proof:** See the Appendix.

Suppose that an on-period starts at time 0, i.e. $X_1(0) = 0, I(0) \in S'_-$ and define recursively

\[
T_0 := 0,
\]

\[
T'_n := \min\{t > T_n : I(t) \in S'_+\}, \quad n \geq 0,
\]

and

\[
T_{n+1} := \min\{t > T'_n : I(t) \in S'_+, X_1(t) = 0\}, \quad n \geq 0,
\]

i.e. $T'_n$ and $T_n$ denote the beginning of the $n$-th off-period and the beginning of the the $n$-th on-period, respectively. See Figure 3.

Now, consider the \{\{I(t), t \geq 0\}\} process, embedded at the beginning of the $n$-th off-period and at the beginning of the $n$-th on-period and denote

\[
I'_n := I(T'_n +), \quad I_n := I(T_n +), \quad n \geq 0.
\]

**Theorem 3.3** \{\{I'_n, n \geq 0\}\} and \{\{I_n, n \geq 0\}\}, are irreducible time-homogeneous Markov chains with state spaces $S'_+$ and $S'_-$, respectively. Their respective transition probability matrices $P' = [p'_{kl}, k, l \in S'_+]$ and $P = [p_{ij}, i, j \in S'_-]$ are given by

\[
p'_{kl} = \sum_{i \in S'_-} \alpha_{ki}(0) \nu_{il}, \quad k, l \in S'_+,
\]

(3.9)

\[
p_{ij} = \sum_{k \in S'_+} \nu_{ik} \alpha_{kj}(0), \quad i, j \in S'_-.
\]

(3.10)

**Proof:** See the Appendix.

Let $\pi'_k = \lim_{n \to \infty} P(I'_n = k)$ and $\pi_i = \lim_{n \to \infty} P(I_n = i)$. The irreducibility of \{\{I'_n, n \geq 0\}\} and \{\{I_n, n \geq 0\}\} implies that the limiting probability vectors $\pi' = [\pi'_k, k \in S'_+]$ and $\pi = [\pi_i, i \in S'_-]$ are given by the unique solutions to

\[
\pi' = \pi' P', \quad \sum_{k \in S'_+} \pi'_k = 1,
\]

(3.11)

7
\[ \pi = \pi P, \quad \sum_{i \in S_-' } \pi_i = 1, \quad (3.12) \]

respectively.

The time between the beginning of two consecutive on-periods is called a cycle. Consider the mean cycle length conditional on the state of the environment \( I_0 \) at the beginning of the cycle
\[ \tau_i := E(T_1 | I_0 = i), \quad i \in S_-' . \]
In addition, denote by \( u_i \) the mean length of an on-period that starts in state \( i \) and by \( d_k \) the mean length of an off-period that starts in state \( k \), i.e.
\[ u_i := E(T'_0 | I_0 = i), \quad i \in S_-' , \]
\[ d_k := E(T_1 - T'_0 | I'_0 = k), \quad k \in S'_+ . \]

Clearly, by first-step analysis
\[ \tau_i = u_i + \sum_{k \in S'_+ } \nu_{ik} d_k, \quad i \in S_-' . \quad (3.13) \]

Next, we show how to compute \( u_i \) for \( i \in S_-' \) and \( d_k \) for \( k \in S'_+ \). Note that \( u_i \) is the mean first passage time into the set \( S'_+ \) of the CTMC \( \{I(t), \ t \geq 0\} \), and is given by the unique solution to \( \sum_{j \in S_-' } q_{ij} u_j = -1, \ i \in S_-' \), see Kulkarni [12]. Also, \( d_k \) can be found as the mean first passage time in a standard fluid model with only the first type of fluid. To be more precise, let
\[ T_A := \inf \{ t > 0 : X_1(t) = 0, I(t) \in A \}, \quad \text{where} \ A = S'_-, \quad (3.14) \]
and
\[ f^A_j(x) := E(T_A | X_1(0) = x, I(0) = j), \quad \text{for} \ j = 1, \ldots, N. \quad (3.15) \]

Then
\[ d_k = f^A_k(0), \quad k \in S'_+ . \quad (3.16) \]

where \( f^A_j(x) \) are as given in Theorem 4.2, Kulkarni and Tzenova [13]. Once the \( u_i \)'s and \( d_k \)'s are computed the mean cycle lengths \( \tau_i \) are obtained from Eq. (3.13).

Let
\[ Z(t) := \begin{cases} 1, & \text{if} \ X_1(t) = 0, \ I(t) \in S_-' , \\ 0, & \text{otherwise} , \end{cases} \]
i.e. \( Z(t) \) is equal to 1 if there is positive leftover service capacity at time \( t \) to serve type 2 (the system is in an on-period) and it is 0 if type 2 can not be served at time \( t \) (the system is in an off-period). The \( Z(t) \) process is an alternating renewal process with limiting distribution
\[ \gamma := \lim_{t \to \infty} P(Z(t) = 1) = \frac{\sum_{i \in S_-' } \pi_i u_i}{\sum_{i \in S_-' } \pi_i \tau_i}, \quad (3.17) \]
1 − γ = \lim_{t \to \infty} P(Z(t) = 0) = 1 - \lim_{t \to \infty} P(Z(t) = 1).

In this paper we study the limiting behavior of the joint distribution of \((X_1(t), X_2(t), I(t))\) by conditioning on \(Z(t)\) as follows:

\[
\lim_{t \to \infty} E(e^{-s_1 X_1(t)} e^{-s_2 X_2(t)}; I(t) = i) = \lim_{t \to \infty} E(e^{-s_2 X_2(t)}; I(t) = i|Z(t) = 1)P(Z(t) = 1) \\
+ \lim_{t \to \infty} E(e^{-s_1 X_1(t)} e^{-s_2 X_2(t)}; I(t) = i|Z(t) = 0)P(Z(t) = 0), \ i \in S,
\]

where \(E(X; A)\) means \(E(X \cdot 1_A)\) for any event \(A\).

In Section 5 we study \(\lim_{t \to \infty} E(e^{-s X_2(t)}; I(t) = i|Z(t) = 1)\) and in Section 6 we study \(\lim_{t \to \infty} E(e^{-s X_1(t)} e^{-s X_2(t)}; I(t) = i|Z(t) = 0)\). We put these two results together in Section 7 and present two examples in Section 8. We first need some additional preliminary results on the accumulation of the lower priority fluid during an off-period.

### 4 Total Accumulation of the Lower Priority Fluid in the Buffer during an Off-Period

During an off-period type 2 fluid is not served and it only accumulates in the buffer. We analyze the random increase of type 2 fluid during an off-period. Let \(A_2\) denote the amount of type 2 fluid that comes in the buffer during the interval \([0, T]\) where \(T\) is as defined in (3.4). We first compute the Laplace-Stieltjes Transform (LST) of \(A_2\) and then we obtain its mean.

Define

\[
\psi_{ji}(x, y) := P(A_2 \leq y, I(T) = i|I(0) = j, X_1(0) = x), \ x, y \geq 0, \ j, i \in S, \tag{4.18}
\]

and

\[
\tilde{\psi}_{ji}(x, s) := E(e^{-s A_2}; I(T) = i|I(0) = j, X_1(0) = x), \ Re(s) \geq 0; \ j, i \in S. \tag{4.19}
\]

Clearly, \(\psi_{ji}(x, y) = \tilde{\psi}_{ji}(x, s) = 0\) if \(i \in S'_+\). For \(i \in S'_-\) define the column vectors

\[
\tilde{\psi}_i(x, s) := [\tilde{\psi}_{1i}(x, s), \ldots, \tilde{\psi}_{Ni}(x, s)]^T, \ i \in S'_-.
\]

Then the following result holds.

**Theorem 4.1** For a fixed \(i \in S'_-\) the LST column vector \(\tilde{\psi}_i(x, s)\) satisfies the following system of differential equations

\[
R_1 \frac{\partial \tilde{\psi}_i}{\partial x}(x, s) + (Q - s P_2)\tilde{\psi}_i(x, s) = 0. \tag{4.20}
\]

The boundary conditions are

\[
\tilde{\psi}_{ji}(0, s) = \delta_{ji}, \ for \ j \in S'_-. \tag{4.21}
\]
Proof: See the appendix.

Next, we discuss the solution to (4.20) with boundary conditions (4.21). Suppose that for a fixed $s$, $\lambda_k(s)$ are the zeros of
\[
\det(\lambda R_1 + Q - sP_2) = 0,
\]
and $\phi_k(s)$ are the corresponding eigenvectors
\[
(\lambda_k(s)R_1 + Q - sP_2)\phi_k(s) = 0.
\]
The following result holds if all $\lambda_k(s)$ are distinct.

**Theorem 4.2** For a fixed $i \in S'_{-}$ the solution to the homogeneous system of differential equations in Theorem 4.1 is given by
\[
\tilde{\psi}_i(x, s) = \sum_{k : \text{Re}(\lambda_k(s)) \leq 0} a_i^k \phi_k(s) e^{\lambda_k(s)x},
\]
where the coefficients $a_i^k$ are determined from the boundary conditions (4.21) as the solution to
\[
\tilde{\psi}_{ji}(0, s) = \sum_{k : \text{Re}(\lambda_k(s)) \leq 0} a_i^k \phi_{jk}(s) = \delta_{ji}, \text{ for } j \in S'_{-}.
\]

**Remark:** For any fixed value of $s$ with $\text{Re}(s) \geq 0$, following a similar argument as the one in Theorem 4 of [8], it can be shown that there are exactly as many boundary conditions as needed
\[
|k : \text{Re}(\lambda_k(s)) \leq 0| = |S'_{-}| = N'.
\]
The important observation is that since $sP_2 = \text{diag}[sp_2(i), \ i \in S]$, with $\text{Re}(sp_2(i)) \geq 0$, the matrix $Q - sP_2$ has diagonal elements that in modulus are at least as large as those of $Q$ and therefore the proof of [8] can be applied.

Now, define
\[
y_j(x) := E[A_2|I(0) = j, X_1(0) = x], \ j \in S, \ x \geq 0, \quad (4.22)
\]
\[
y(x) := [y_1(x), \ldots, y_N(x)]^t,
\]
\[
P_2 := \text{diag}[p_2(1), \ldots, p_2(N)].
\]

**Theorem 4.3** $y(x)$ satisfies the following non-homogeneous system of differential equations,
\[
R_1 y'(x) + Q y(x) + P_2 \epsilon = 0. \quad (4.23)
\]
The boundary conditions are given by
\[
y_j(0) = 0, \text{ for } j \in S'_{-}. \quad (4.24)
\]
Proof: Note that

\[ y_j(x) = - \sum_{i \in S'_-} \frac{\partial}{\partial s} \tilde{\psi}_{ji}(x, s)|_{s=0} \]

and therefore Eq.(4.23) follows directly from Eq. (4.20) after summing over \( i \in S'_- \) and taking the derivative with respect to \( s \). The boundary conditions also follow directly from the boundary conditions (4.21) above, namely

\[ y_j(0) = - \sum_{i \in S'_-} \frac{\partial}{\partial s} \tilde{\psi}_{ji}(0, s)|_{s=0} = 0, \text{ for } j \in S'_- . \]

\( \diamond \)

The next result gives the solution \( y(x) \) to (4.23) with boundary conditions (4.24). We follow the proof of Theorem 3.3 in Kulkarni and Tzenova [13] and adopt an analogous notation. Let

\[ c := - \frac{\pi^t P_2 e}{\pi^t R_1 e}, \]

where \( \pi^t \) is the limiting distribution of the governing CTMC \( \{I(t), t \geq 0\} \). With this scalar \( c \), let \( b \) denote any solution to the linear system

\[ Qb = - P_2 e - c R_1 e . \]

Also, let \( \lambda_i \) be the solutions to

\[ \text{det}(\lambda R_1 + Q) = 0, \quad (4.25) \]

and \( \phi_i \) be the corresponding eigenvectors

\[ (\lambda_i R_1 + Q) \phi_i = 0 . \]

**Theorem 4.4** The solution \( y(x) \) to (4.23) with boundary conditions (4.24) is given by

\[ y(x) = \sum_{i: \text{Re}(\lambda_i) \leq 0} a_i e^{\lambda x} \phi_i + c x e + b, \]

assuming that all eigenvalues \( \lambda_i \) are distinct. The coefficients \( a_i \) are determined from the boundary conditions (4.24) as the solution to the linear system

\[ \sum_{i: \text{Re}(\lambda_i) \leq 0} a_i \phi_{ji} + b_j = 0 \text{ for } j \in S'_-. \]

Proof: Same as in Theorem 3.3 of Kulkarni and Tzenova [13]. The number of the boundary conditions is exactly equal to the number of the coefficients \( a_i \) that need to be determined as given by the Lemma in [13].

\( \diamond \)
5 Steady-State Distribution of the Lower Priority Fluid during an On-Period

In this section we compute \( \lim_{t \to \infty} E(e^{-s_2 X_2(t)}; I(t) = i | Z(t) = 1) \). Consider the distribution of the low priority fluid content during on-periods, i.e. while \( X_1(t) = 0 \) and \( I(t) \in S'_- \). Denote by \( \tau(t) \) the time spent in on-periods over \([0,t]\) and define the restricted processes \( \{X_2^{on}(t), t \geq 0\} := \{X_2(\tau(t)+), t \geq 0\} \) and \( \{I^{on}(t), t \geq 0\} := \{I(\tau(t)+), t \geq 0\} \). Clearly, \( I^{on}(t) \in S'_- \) for all \( t \geq 0 \). These stochastic processes describe the evolution of \( X_2 \) and \( I \) processes over the on-periods (skipping the off-periods). One can see that \( X_2^{on} \) is a fluid process with jumps as treated in [20]. A jump occurs at the end of an on-period and the size of the jump is equal to \( A_2 \), the total amount of type 2 fluid accumulated during the corresponding off-period. Let \( J_{ji}(z) \) denote the distribution of the jump size \( A_2 \) during an off-period that starts in state \( j \in S'_+ \) and ends in state \( i \in S'_- \):

\[
J_{ji}(z) := P(A_2 \leq z, I(T) = i | I(0) = j), \quad z \geq 0, \quad j \in S'_+, \quad i \in S'_-.
\]

Clearly, \( J_{ji}(z) = \psi_{ji}(0, z) \) where \( \psi_{ji}(0, z) \) is as in equation (4.18).

Let

\[
F_i(t, x) := P(X_2^{on}(t) \leq x, \quad I^{on}(t) = i), \quad t \geq 0, \quad x \geq 0, \quad i \in S'_-;
\]

\[
F_i(x) := \lim_{t \to \infty} F_i(t, x), \quad x \geq 0, \quad i \in S'_-;
\]

\[
F(x) := \{F_i(x), \quad i \in S'_-\}.
\]

For a given matrix \( M \) and subsets of indices \( A, B \), denote the sub-matrix

\[
M_{A,B} := [M_{ij}, \quad i \in A, \quad j \in B].
\]

To further simplify the notation we use

\[
R_{--} := R_{S'_+, S'_-}, \quad Q_{--} := Q_{S'_+, S'_-}, \quad Q_{++} := Q_{S'_+, S'_-}, \quad J_{--}(z) := J_{S'_+, S'_-}(z).
\]

The next theorem gives the differential equations satisfied by the row vector \( F(x) \) and uses matrix convolution defined as follows. Suppose that \( A(x) = [A_{ij}(x)] \) and \( B(x) = [B_{ij}(x)] \) are two matrices of functions where the number of columns of \( A(x) \) is equal to the number of rows of \( B(x) \). Then their convolution \( A \ast B(x) \) is the matrix \( C(x) = [C_{ij}(x)] \) with elements

\[
C_{ij}(x) = \sum_k \int_0^x A_{ik}(x-t)dB_{kj}(t).
\]

**Theorem 5.1** The distribution vector \( F(x) \) satisfies the system of differential equations

\[
\frac{dF}{dx}(x)R_{--} = F(x)Q_{--} + F \ast (Q_{--}J_{--})(x), \quad (5.27)
\]

12
with boundary conditions
\begin{align*}
F_j(0) &= 0, \text{ if } j \in S'_- \cap S_+ , \quad (5.28) \\
F(0)[Q_-]_{.,j} + F(0)[Q_+J_+(0)]_{.,j} &= 0, \text{ if } j \in S'_- \cap S_0 . \quad (5.29)
\end{align*}

Proof: We omit the proof since it is similar to that of Theorem 2.1 in Tzenova, Adan and Kulkarni [20].

Define the Laplace-Stieltjes transform
\[
\tilde{F}_i(s) := \int_0^\infty e^{-sx}dF_i(x)
\]
and consider the row vector \(\tilde{F}(s) = [\tilde{F}_i(s), i \in S'_-]\). After we take the Laplace-Stieltjes transform of both sides of equation (5.27) we have
\[
s(\tilde{F}(s) - F(0))R_- = \tilde{F}(s)Q_- + \tilde{F}(s)Q_+ J_+(s),
\]
which is equivalent to
\[
\tilde{F}(s)[sR_- - Q_- - Q_+ J_+(s)] = sF(0)R_- . \quad (5.30)
\]
Here the elements of the matrix \(\tilde{J}_+(s)\) are given by \(\tilde{J}_{ji}(s) = \tilde{\psi}_{ji}(0,s), j \in S'_+, i \in S'_-\), from equation (4.19). To find the remaining \(F_j(0), j \in S'_- \cap S_-\) first note that \(S'_- \cap S_- = S_-\) and therefore we need \(|S_-| = N_-\) equations. The following result is required to derive these equations.

**Theorem 5.2** The generalized eigenvalue problem
\[
(sR_- - Q_- - Q_+ \tilde{J}_+(s))\phi = 0 \quad (5.31)
\]
has exactly \(N_-\) solutions \((s_1, \phi_1), \ldots, (s_{N_-}, \phi_{N_-})\), with \(s_1 = 0, \text{Re}(s_i) > 0, i = 2, \ldots, N_-\) and \(\phi_i \neq 0\).

Proof: See the Appendix.

Define \(\Gamma_- := [\Gamma_{ij}, i, j \in S'_-]\) with elements \(\Gamma_{ij} := \sum_{k \in S'_+} q_{ik} m_{kj}\), where \(m_{kj}\) is the mean size of a jump that starts in \(k\) and ends in \(j\).

**Theorem 5.3** The LST row vector \(\tilde{F}(s)\) is given by the solution to
\[
\tilde{F}(s)(sR_- - Q_- - Q_+ \tilde{J}_+(s)) = sF(0)R_- ,
\]
where the unknowns \(F_i(0), i \in S'_-\) are determined from the following set of equations
\[
F_i(0) = 0, i \in S'_- \cap S_+ ,
\]
13
To be more precise, let
\[
\chi \in \mathcal{R}_{\omega}, i + F(0)[Q_{-i}] = 0, \; i \in S_{1} \cap S_0,
\]
\[
F(0)R_{-i} = 0, \; \text{for } i = 2, \ldots, N_{-},
\]
\[
F(0)R_{-i} = \pi(R_{-i} + \Gamma_{-i}).
\]

**Proof:** Same as the proof of Theorem 2.3 in Tzenova, Adan and Kulkarni [20].

To summarize, \(\lim_{t \to \infty} E(e^{-s_{2}X_{2}(t)}; I(t) = i|Z(t) = 1)\) is given by \(\tilde{F}_{i}(s_{2})\) in the above theorem and will be used in Section 7.

### 6 Steady-State Joint Distribution of the Two Buffer Content Processes during an Off-Period

While \(X_{1}(t) > 0\) or while \(X_{1}(t) \geq 0\) and \(I(t) = i \in S_{1}^{\prime}, i.e. r_{1}(i) \geq 0,\) the lower priority fluid is not being served. These periods of time are called off-periods (from the point of view of the lower priority fluid). In this section, we study the LST of the steady-state distribution of the \((X_{1}(t), X_{2}(t), I(t)), t \geq 0\) process during off-periods, i.e. we compute
\[
\lim_{t \to \infty} E(e^{-s_{1}X_{1}(t)}e^{-s_{2}X_{2}(t)}; I(t) = i|Z(t) = 0).
\]

To be more precise, let \(\chi(t)\) be time spent in off-periods over the interval \([0, t]\), and define the restricted processes
\[
X_{1}^{\omega}(t) := X_{1}(\chi(t) +), \quad X_{2}^{\omega}(t) := X_{2}(\chi(t) +), \quad I^{\omega}(t) := I(\chi(t) +).
\]

In addition, we introduce the process \(Y(t) := I(T_{n}^{\prime}), T_{n}^{\prime} \leq t < T_{n+1}^{\prime}\). Clearly, \(Y(t)\) stays the same within a given off-period and denotes the state in which the current off-period has started. Also define,
\[
Y^{\omega}(t) := Y(\chi(t) +).
\]

For a fixed \(k \in S_{1}^{\prime}\), we refer to the off-periods that start in \(k\) as type \(k\) off-periods. First, we consider an off-period of type \(k\) and study
\[
G_{k,j}(t, x_{1}, a_{2}) = P(X_{1}^{\omega}(t) \leq x_{1}, A_{2}^{\omega}(t) \leq a_{2}, I^{\omega}(t) = j, Y^{\omega}(t) = k),
\]

and
\[
G_{k,j}(x_{1}, a_{2}) = \lim_{t \to \infty} G_{k,j}(t, x_{1}, a_{2}), \quad (6.34)
\]

for \(x_{1} \geq 0, a_{2} \geq 0, j \in S\), where \(A_{2}^{\omega}(t)\) denotes the total amount of type two fluid accumulated in the buffer by time \(t\) during the current off-period (of type \(k\)). In order to simplify notation, in the rest of this section \(X_{1}, X_{2}, Y\) and \(A_{2}\) will represent \(X_{1}^{\omega}, X_{2}^{\omega}, Y^{\omega}\) and \(A_{2}^{\omega}\), unless indicated otherwise.

Define the Laplace Transform (LT) of \(G_{k,j}(x_{1}, a_{2})\) as
\[
G_{k,j}^{*}(s_{1}, s_{2}) := \int_{x_{1}=0}^{\infty} \int_{a_{2}=0}^{\infty} G_{k,j}(x_{1}, a_{2})e^{-s_{1}x_{1}}e^{-s_{2}a_{2}}dx_{1}da_{2}, \quad k \in S_{1}^{\prime}, j \in S,
\]

14
and consider the LST of the limiting distribution of $X_1$ and $A_2$ defined as

$$
\tilde{G}_{k,j}(s_1, s_2) := \lim_{t \to \infty} E(e^{-s_1 X_1(t)}e^{-s_2 A_2(t)}; I(t) = j, Y(t) = k), \quad k \in S'_+, \quad j \in S.
$$

Then

$$
\tilde{G}_{k,j}(s_1, s_2) = s_1 s_2 G^*_k(s_1, s_2), \quad k \in S'_+, \quad j \in S.
$$

In the next theorem we find $\tilde{G}_{k,j}(s_1, s_2)$ as a solution to a linear system of equations. First, we obtain

- $\mu_{k,j}$ - mean time between two consecutive type $k (\in S'_+) \text{ off-periods that end in } j (\in S'_-)$,
- $\mu_j$ - mean time between two consecutive off-periods that end in $j \in S'_-$.

**Lemma 6.1**

$$
\mu_{k,j} = \frac{\sum_{l \in S'_+} \pi'_l d_l}{\pi'_k \alpha_{kj}(0)}, \quad j \in S'_-, \quad k \in S'_+, \quad (6.35)
$$

$$
\mu_j = \frac{\sum_{l \in S'_-} \pi'_l d_l}{\pi_j}, \quad j \in S'_-, \quad (6.36)
$$

where $\pi'_k$, $\pi_j$, $\alpha_{kj}(0)$, and $d_l$ are as given in (3.11), (3.12), (3.8), and (3.16), respectively.

**Proof:** See the Appendix.

Note that

$$
\sum_{k \in S'_+} \pi'_k \alpha_{kj}(0) = \pi_j, \quad \sum_{j \in S'_-} \pi_j \nu_{jk} = \pi'_k,
$$

and

$$
\frac{1}{\mu_j} = \frac{1}{\sum_{k \in S'_-} \mu_{k,j}}
$$

The next theorem requires some additional notation. Let $\tilde{B}_{k,j}(s_2)$ be the LST of type two fluid that is added to the buffer within an arbitrary type $k \text{ off-period that finishes in state } j$, i.e.

$$
\tilde{B}_{k,j}(s_2) := E(e^{-s_2 A_2}; I(T) = j, Y(0) = k), \quad k \in S'_+, \quad j \in S,
$$

where $T$ is as defined in Eq. (3.4). Clearly, $\tilde{B}_{k,j}(s_2) = 0$ if $j \in S'_+$. We use the following matrices

$$
\tilde{B}_k(s_2) := [\tilde{B}_{k,j}(s_2), j \in S], \quad \text{for a fixed } k \in S'_+, \quad (6.37)
$$

$$
R_2 := \text{diag}[p_2(1), \ldots, p_2(N)], \quad (6.38)
$$

$$
L := [L_{ij}, \ i, j \in S] \quad \text{with elements} \quad L_{ij} := \begin{cases} 
\nu_{ij}, & \text{if } i \in S'_-, \ j \in S'_+ \\
0, & \text{otherwise}
\end{cases}, \quad (6.39)
$$
where \( \nu_{ij} \) are as in Theorem 3.1.

The following row vectors are also used in the next theorem,

\[
\tilde{G}_k(s_1, s_2) := [\tilde{G}_{k,j}(s_1, s_2), \ j \in S], \text{ for a fixed } k \in S^\prime_+,
\]

and

\[
l := [l_j, \ j \in S] \text{ with elements } l_j := \begin{cases} 
1 & \text{if } j \in S^\prime_- \\
\mu_j & \text{if } j \in S^\prime_+ \end{cases}.
\]

To further facilitate the notation in the sequel we use \( \alpha_{kj} := \alpha_{kj}(0) \).

**Theorem 6.2** The LST row vector \( \tilde{G}_k(s_1, s_2) \) is given by the solution to the following linear system of equations:

\[
\tilde{G}_k(s_1, s_2)(s_1R_1 + s_2R_2 - Q) = lLE_k - m\tilde{B}_k(s_2),
\]

where \( R_2 = P_2 = \text{diag}[p_2(i), \ i \in S] \), \( E_k \) is an \( N \times N \) matrix of zeroes with a single non-zero element \( (E_k)_{kk} = 1 \), \( m = 1/\sum_{i \in S^\prime_-} \pi'_i d_i \) and \( \tilde{B}_k(s_2) \) is as defined above by Eq. (6.37).

**Proof:** See the Appendix.

We proceed with the study of \( X_2 \) and \( I \) at the beginning of an arbitrary off-period. Consider a given state \( k \in S^\prime_+ \) and denote by \( H_k(x) \) the steady-state probability that \( X_2 < x \) and \( I = k \) at the beginning of an arbitrary off-period. Then \( H_k(x) \) is given by the following result.

**Theorem 6.3** For a fixed state \( k \in S^\prime_+ \) and buffer content of type 2, \( x \geq 0 \) at the beginning of an off-period

\[
H_k(x) = \frac{\sum_{i \in S^\prime_-} F_i(x)q_{ik}}{\sum_{i \in S^\prime_-} F_i(\infty)q_{ik}} \pi'_k,
\]

where \( F_i(x) \) is as in Eq. (5.26) and \( \pi'_k \) as in Eq. (3.11).

**Proof:** Consider an arbitrary off-period of type \( k \). Denote by \( M \) the total number of initiations per time unit of type \( k \) off-periods that start with \( X_2 < x \), i.e. from states \( (X_1 = 0, X_2 < x, I = i) \), for some \( i \in S'_- \). The long-run fraction of time the system is in such states is \( \gamma \sum_{i \in S'_-} F_i(x) \) (where \( \gamma \) is the long-run fraction of time the system is in an on-period) and the rate to \((X_1 = 0, X_2 < x, I = k)\), i.e. to the beginning of the type \( k \) off-period is \( q_{ik} \). Hence

\[
M = \gamma \sum_{i \in S'_-} F_i(x)q_{ik}.
\]

Now, let \( N \) denote the number of times per time unit an off-period starts in state \( k \). Clearly,

\[
N = \gamma \sum_{i \in S'_-} F_i(\infty)q_{ik}.
\]
Thus, the conditional probability that an off-period starts with $X_2 < x$ given that it starts in state $k$ is equal to $M/N$. We obtain the joint steady-state probability $H_k(x)$ as given in Eq. (6.41) by recalling that $\pi_k'$ denotes the long-run fraction of number of off-periods that start in $k$.  

After we take LST of Eq. (6.41) we get the following Corollary.

**Corollary 6.4** The LST of type 2 fluid in the buffer at the beginning of an arbitrary off-period that starts in state $k$ in steady-state is given by

$$
\tilde{H}_k(s) = \frac{\sum_{i \in S_i'} \tilde{F}_i(s) q_{ik}}{\sum_{i \in S_i'} F_i(\infty) q_{ik}} \pi_k'.
$$

Now, we are ready to formulate the main result of this section which gives the LST of the limiting joint distribution of the two buffer content processes during off-periods.

**Theorem 6.5**

$$
\lim_{t \to \infty} E(e^{-s_1 X_1(t)} e^{-s_2 X_2(t)}; I(t) = j | Z(t) = 0) = \sum_{k \in S'_+} \tilde{H}_k(s_2) \frac{\tilde{G}_{k,j}(s_1, s_2)}{\pi_k}, \ j \in S,
$$

where $\tilde{H}_k(s_2)$ are as in Corollary 6.4 and $\tilde{G}_{k,j}(s_1, s_2)$ are as in Theorem 6.2.

**Proof:** We introduce some temporary notation in order to facilitate the following argument. Let

- $Y$ - the type of an arbitrary off-period;
- $I$ - the state of the environment at an arbitrary point in an arbitrary off-period;
- $X_1$ - the amount of type 1 fluid at an arbitrary point in an arbitrary off-period;
- $X_2$ - the amount of type 2 fluid at an arbitrary point in an arbitrary off-period;
- $X_2^o$ - the amount of type 2 fluid at the beginning of an arbitrary off-period;
- $A_2^o$ - the amount of type 2 fluid that has accumulated in the buffer up to an arbitrary point in an arbitrary off-period.

Then $X_2 = X_2^o + A_2^o$. Furthermore, given $Y$, it is clear that $X_2^o$ and $(X_1, A_2^o, I)$ are independent. For a fixed $k \in S'_+$ we have computed

$$
\tilde{H}_k(s_2) = E(e^{-s_2 X_2^o}; Y = k), \ \text{in Lemma 6.4}, \ (6.42)
$$
and
\[ \tilde{G}_{k,j}(s_1, s_2) = E(e^{-s_1X_1}e^{-s_2A^2_2}; I = j; Y = k) \] in Theorem 6.2. \hfill (6.43)

Then
\[ E(e^{-s_1X_1}e^{-s_2X_2}; I = j) = \sum_{k \in S'_i} E(e^{-s_1X_1}e^{-s_2X_2}; I = j; Y = k) \]
\[ = \sum_{k \in S'_i} E(e^{-s_1X_1}e^{-s_2(X_2^2 + A^2_2)}; I = j; Y = k). \]

Also, note
\[ E(e^{-s_1X_1}e^{-s_2(X_2^2 + A^2_2)}; I = j; Y = k) = E(e^{-s_1X_1}e^{-s_2(X_2^2 + A^2_2)}; I = j|Y = k)P(Y = k) \]
\[ = E(e^{-s_2X_2^2}Y = k)P(Y = k)E(e^{-s_1X_1}e^{-s_2A^2_2}; I = j|Y = k) \]
\[ = E(e^{-s_2X_2^2}; Y = k)E(e^{-s_1X_1}e^{-s_2A^2_2}; I = j|Y = k) \]
\[ = \tilde{H}_k(s_2)\frac{\tilde{G}_{k,j}(s_1, s_2)}{\pi'_k} \]

Thus,
\[ \lim_{t \to \infty} E(e^{-s_1X_1(t)}e^{-s_2X_2(t)}; I(t) = jZ(t) = 0) = E(e^{-s_1X_1}e^{-s_2X_2}; I = j) \]
\[ = \sum_{k \in S'_i} \tilde{H}_k(s_2)\frac{\tilde{G}_{k,j}(s_1, s_2)}{\pi'_k} \]

\[ \diamond \]

7 The Limiting Distribution of \((X_1(t), X_2(t), I(t))\)

At this point we go back to our original goal as stated earlier, which is the study of the limiting behavior of \((X_1(t), X_2(t), I(t))\), i.e. we want to determine
\[ \lim_{t \to \infty} E(e^{-s_1X_1(t)}e^{-s_2X_2(t)}; I(t) = i) \]
\[ = \lim_{t \to \infty} E(e^{-s_2X_2(t)}; I(t) = i|Z(t) = 1)P(Z(t) = 1) \]
\[ + \lim_{t \to \infty} E(e^{-s_1X_1(t)}e^{-s_2X_2(t)}; I(t) = i|Z(t) = 0)P(Z(t) = 0), \ i \in S. \]

Note that \(\lim_{t \to \infty} E(e^{-s_2X_2(t)}; I(t) = i|Z(t) = 1)\) is given by Theorem 5.3. Furthermore, \(\lim_{t \to \infty} E(e^{-s_1X_1(t)}e^{-s_2X_2(t)}; I(t) = i|Z(t) = 0)\) is given by Theorem 6.5 and the limiting probability \(\gamma = \lim_{t \to \infty} P(Z(t) = 1)\) is given by Equation 3.17. Putting all these results together we get the following theorem.
Theorem 7.1 For all $i \in S$,
\[
\lim_{t \to \infty} E(e^{-s_1 X_1(t)} e^{-s_2 X_2(t)}; I(t) = i) = \gamma \bar{F}_i(s_2) + (1 - \gamma) \sum_{k \in S'_+} \bar{H}_k(s_2) \frac{\bar{G}_{k,i}(s_1, s_2)}{\pi'_k},
\]
where $\bar{F}_i(s_2)$, $\bar{H}_k(s_2)$, $\bar{G}_{k,i}(s_1, s_2)$, and $\pi'_k$ are as in Theorem 5.3, Corollary 6.4, Theorem 6.2 and Eq. (3.11), respectively.

8 Examples

8.1 Two Analytic Examples

We consider two simple fluid models with an infinite capacity buffer and a regulating two-state CTMC $\{I(t), \; t \geq 0\}$ with state space $S = \{1, 2\}$, generator matrix
\[
Q = \begin{bmatrix}
-\beta & \beta \\
\alpha & -\alpha
\end{bmatrix},
\]
and limiting distribution $\pi^I = \left[\frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta}\right]$. While $I(t) = 1$ there is no inflow of type 1 and while $I(t) = 2$ the inflow of type 1 to the buffer is $p_1 > 0$.

The difference in the two models is in the input of type 2 fluid:

Model 1: Type 2 fluid enters the buffer at a constant rate $p_2 > 0$ independent of the environment $I(t)$.

Model 2: While $I(t) = 1$ there is no inflow of type 2 and while $I(t) = 2$ the inflow of type 2 to the buffer is $p_2 > 0$.

Thus, the input rates in these two models are as follows
\[
p_1(1) = 0, \quad p_2(1) = \begin{cases} p_2 & \text{Model 1} \\ 0 & \text{Model 2} \end{cases},
\]
\[
p_1(2) = p_1, \quad p_2(2) = p_2.
\]

Clearly, all quantities that do not depend on $p_2(i)$ will be the same for the two models. Therefore, we proceed with the simultaneous analysis of the two models. When there is a difference we label the corresponding equations by (M1) for model 1 and by (M2) for model 2. To make the problem interesting we assume that $p_1 > \mu$, so that there are off-periods. Then the partitioning of the state space $S$ is given by
\[
S'_- = S_- = \{1\}, \; S'_+ = S_+ = \{2\}, \; S_0 = \emptyset.
\]

Thus, in the beginning of an on-period the state must be 1 and in the beginning of an off-period the state must be 2.
The net input rate matrix is given by

\[
R = \begin{bmatrix}
p_2 - \mu & 0 \\
0 & p_1 + p_2 - \mu \\
\end{bmatrix} \quad \text{(M1)}; \\
R = \begin{bmatrix}
-\mu & 0 \\
0 & p_1 + p_2 - \mu \\
\end{bmatrix} \quad \text{(M2)}
\]

We assume that the system is stable, i.e.,

\[
\pi^T R e = \frac{\alpha}{\alpha + \beta} (p_2 - \mu) + \frac{\beta}{\alpha + \beta} (p_1 + p_2 - \mu) < 0 \quad \text{(M1)} \quad (8.44)
\]

\[
\pi^T R e = \frac{\alpha}{\alpha + \beta} (-\mu) + \frac{\beta}{\alpha + \beta} (p_1 + p_2 - \mu) < 0 \quad \text{(M2)} \quad (8.45)
\]

Clearly, the absorption probability \( \nu_{12} \) of Theorem 3.1 is equal to 1. It is also easy to see that \( \alpha_{i1}(x) \), \( i = 1, 2 \), of Theorem 3.2 are equal 1 which follows directly from the definition of \( \alpha_{i1}(x) \). The embedded DTMCs of Theorem 3.3, \( \{I'_n, n \geq 0\} \) and \( \{I_n, n \geq 0\} \) have transition probability matrices (that are scalars and are) given by \( P' = 1, P = 1 \), and limiting distributions \( \pi' = 1, \pi = 1 \), respectively. The mean lengths of an on-period and an off-period are

\[
u_1 = 1/\beta, \quad d_2 = f_2^A(0) = \frac{p_1}{\alpha \mu + \beta (\mu - p_1)},
\]

where \( A = \{1\} \) and \( f_2^A(0) \) is found from [13] for the fluid model with only the first type of fluid. We continue with the results of Section 4 for the accumulation of type 2 during an off-period. First, we compute its mean \( y(x) \) from Theorem 4.4 to be

\[
y(x) = \frac{p_1 p_2}{(\alpha + \beta) \mu - \beta p_1} e + \frac{p_2 (\alpha + \beta)}{(\alpha + \beta) \mu - \beta p_1} x e + \left[ -\frac{p_1 p_2}{(\alpha + \beta) \mu - \beta p_1} , 0 \right]^t \quad \text{(M1)}.
\]

\[
y(x) = \frac{\mu p_2}{(\alpha + \beta) \mu - \beta p_1} e + \frac{p_2 \beta}{(\alpha + \beta) \mu - \beta p_1} x e + \left[ -\frac{\mu p_2}{(\alpha + \beta) \mu - \beta p_1} , 0 \right]^t \quad \text{(M2)},
\]

where \( e \) is a column vector of ones.

Next, we compute the LST column vector \( \tilde{\psi}_1(x, s) \) of Theorem 4.1. The characteristic equation is

\[
\det(\lambda R_1 + Q - s P_2) = 0,
\]

or equivalently

\[
A \lambda^2 + B(s) \lambda + C(s) = 0,
\]

where

\[
A := \mu (\mu - p_1) < 0,
\]

\[
B(s) := \mu (\alpha + sp_2) + (\mu - p_1)(\beta + sp_2) \quad \text{(M1)}; \quad B(s) = \mu (\alpha + sp_2) + (\mu - p_1) \beta \quad \text{(M2)},
\]

\[
C(s) := (\alpha + \beta) sp_2 + (sp_2)^2 \geq 0 \quad \text{(M1)}; \quad C(s) = \beta sp_2 \quad \text{(M2)}.
\]
It yields the following expressions (after some algebra) for \( \tilde{\psi}_{2,1}(0, s) \) necessary for the subsequent results of Sections 5 and 6.

\[
\tilde{\psi}_{2,1}(0, s) = a = \frac{\lambda_1(s) \mu + \beta + sp_2}{\beta} \quad \text{(M1)};
\]

\[
\tilde{\psi}_{2,1}(0, s) = \frac{\lambda_1(s) \mu + \beta}{\beta} \quad \text{(M2)},
\]

where \( \lambda_1(s) = \frac{B(s) - \sqrt{D(s)}}{-2A} \leq 0. \)

Now, consider the LST of \( X_2 \) over an on-period as given in Theorem 5.3. The row vector \( \tilde{F}(s) \) is now just a scalar since during an on-period the state is 1. Also, the respective matrices are simply scalars

\[
Q_{-} = q_{11} = -\beta, \quad Q_{+} = q_{12} = \beta,
\]

\[
R_{-} = r_{11} = p_2 - \mu \quad \text{(M1)}; \quad R_{-} = -\mu, \quad \text{(M2)},
\]

\[
\tilde{H}_{+}(s) = \tilde{\psi}_{21}(0, s) = \frac{\lambda_1(s) \mu + \beta + sp_2}{\beta} \quad \text{(M1)}; \quad \tilde{H}_{-}(s) = \frac{\lambda_1(s) \mu + \beta}{\beta} \quad \text{(M2)}.
\]

Note that in our models \( S'_- \cap S^- = S'_- \cap S_0 = \emptyset \) and the scalar \( F(0) \) is found from equation (5.32)

\[
F(0) \hat{R}_{-} \hat{M}^{-1} e = 1, \text{ or } F(0) = \frac{r_{11} + \beta y_2(0)}{r_{11}},
\]

\[
F(0) = 1 - \frac{\beta p_1 p_2}{(\mu - p_2)((\alpha + \beta)\mu - \beta p_1)} \quad \text{(M1)}; \quad F(0) = 1 - \frac{\beta p_2}{(\alpha + \beta)\mu - \beta p_1} \quad \text{(M2)}.
\]

Then the equation for \( \tilde{F}(s) \) becomes

\[
\tilde{F}(s)[-s\mu - \lambda_1(s)\mu] = s \left( p_2 - \mu + \frac{\beta p_1 p_2}{(\alpha + \beta)\mu - \beta p_1} \right) \quad \text{(M1)},
\]

\[
\tilde{F}(s)[-s\mu - \lambda_1(s)\mu] = s \left( -\mu + \frac{\beta p_2 \mu}{(\alpha + \beta)\mu - \beta p_1} \right) \quad \text{(M2)}.
\]

Next, we consider \( \lim_{t \to \infty} E(e^{-s_1 X_1(t)}e^{-s_2 X_2(t)}; I(t) = j\Pi(Z(t) = 0) \) as studied in Section 6. From Theorem 6.2 we have that

\[
\tilde{G}_2(s_1, s_2)(s_1 R_1 + s_2 P_2 - Q) = lLE_2 - m\tilde{B}_2(s_2),
\]

where

\[
L = \begin{bmatrix} 0 & \nu_{12} \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B}_2(s_2) = [\tilde{B}_{21}(s_2), 0],
\]

\[
l = [1/\mu_1, 0] = [1/d_2, 0], \quad m = [1/d_2, 0], \quad \tilde{B}_{21}(s_2) = \pi_2 \tilde{\psi}_{21}(0, s_2) = \tilde{\psi}_{21}(0, s_2),
\]

\[
l(0) = 1 - \frac{1}{\mu_1}, \quad m(0) = 1 - \frac{1}{d_2}.
\]
or equivalently
\[
\tilde{G}_2(s_1, s_2)(s_1R_1 + s_2P_2 - Q) = \frac{1}{d_2}[-\tilde{\psi}_{2,1}(0, s_2), 1]
\]

From Corollary 6.4 the LST of type 2 fluid in the buffer at the beginning of an arbitrary off-period is given by
\[
\tilde{H}_2(s) = \frac{\tilde{F}(s)q_{12}\pi'_2}{F(\infty)q_{12}} = \tilde{F}(s).
\]

Theorem 6.5 gives
\[
\lim_{t \to \infty} E(e^{-s_1X_1(t)}e^{-s_2X_2(t)}; I(t) = j|Z(t) = 0) = \tilde{H}_2(s_2)\tilde{G}_{2,j}(s_1, s_2), j = 1, 2.
\]

The governing CTMC with state space \( S = \{1, \ldots, 4\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \) has limiting distribution
\[
\pi = \frac{1}{(\alpha + \beta)^2} [\alpha^2, \alpha\beta, \alpha\beta, \beta^2] = [0.64, 0.16, 0.16, 0.04].
\]
Figures 4, 5 and 6 show the joint and the marginal steady-state probabilities, $P(X_2 < t, I = i), i \in S$ and $P(X_2 < t)$, for the three cases. As expected these decrease as type 2 input rate $p_2$ increases and $\lim_{t \to \infty} P(X_2 < t, I = i) = \pi_i$. Figures 7, 8 and 9 show the behavior of $P(X_2 < t)$ for each case as the server capacity $\mu$ changes. As $\mu$ increases the marginal distributions also increase for each $t$. This implies that in steady state $X_2$ decreases stochastically as $\mu$ increases. An interesting characteristic of the system is the steady-state probability of no output of type 2 (the long-run fraction of time type 2 fluid does not pass though the server), $P(X_2 = 0, I = (0, 0)) + P(X_2 = 0, I = (1, 0))$. Figure 10 gives a comparison of these as $\mu$ varies from 4.75 to 7.55 in the three cases considered. It supports the intuition that smaller values of $p_2$ lead to higher values of $P(X_2 = 0, I = (0, 0)) + P(X_2 = 0, I = (1, 0))$.

**Remark:** To obtain the numerical results above we first compute the corresponding LSTs using the developed technique as given Theorem 7.1. Then we numerically invert the LSTs by implementing the EULER algorithm of Abate and Whitt [1] which is a variant of the Fourier-series method that exploits Euler summation.
Figure 4: Steady-state distributions $P(X_2 < t, I = i)$, $i \in S$ and $P(X_2 < t)$; traffic intensity
$\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)} = 0.24$

Figure 5: Steady-state distributions $P(X_2 < t, I = i)$, $i \in S$ and $P(X_2 < t)$; traffic intensity
$\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)} = 0.37333$
Figure 6: Steady-state distributions $P(X_2 < t, I = i)$, $i \in S$ and $P(X_2 < t)$; traffic intensity

$$\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)} = 0.61333$$

Figure 7: The marginal steady-state distribution $P(X_2 < t)$ for different values of $\mu$ in the case $p = [8, 1]$; the traffic intensities $\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu(\alpha_i + \beta_i)}$ are given by util.
Figure 8: The marginal steady-state distribution $P(X_2 < t)$ for different values of $\mu$ in the case $p = [8, 6]$; the traffic intensities $\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu (\alpha_i + \beta_i)}$ are given by $\text{util}$.

Figure 9: The marginal steady-state distribution $P(X_2 < t)$ for different values of $\mu$ in the case $p = [8, 15]$; the traffic intensities $\sum_{i=1}^{2} \frac{p_i \beta_i}{\mu (\alpha_i + \beta_i)}$ are given by $\text{util}$. 
Figure 10: Comparison of the long-run probabilities of no output of type 2, $P(X_2 = 0, I = (0, 0)) + P(X_2 = 0, I = (1, 0))$, for the three cases as $\mu$ is varied between 4.75 and 7.55.
8.3 Appendix

Proof of Theorem 3.2:
The proof is based on conditioning on the state at $t = h$ and using the fact that the probabilities of two or more transitions in the CTMC $\{I(t), t \geq 0\}$ over an interval of length $h > 0$ are $o(h)$. So

$$\alpha_{ik}(x) = \sum_{j \in S, j \neq i} q_{ij} h \int_{u=0}^{h} \alpha_{jk}(x + r_1(i)u + r_1(j)(h-u))g_U(u)du + (1+q_{ii}h)\alpha_{ik}(x + r_1(i)h) + o(h),$$

where the random variable $U$ denotes the time at which the transition from the current state $i$ to some other state $j \neq i$ occurs with density $g_U(u)$ on $(0, h)$. This notation will also be used later in the derivation of similar results.

After rearranging the last equation and letting $h \to 0$ we get

$$-r_1(i) \frac{d}{dx} \alpha_{ik}(x) = \sum_{j \in S} q_{ij} \alpha_{jk}(x),$$

which can be written in matrix form to obtain (3.6). The boundary conditions (3.7) follow directly from the definition of $T$ (see Eq. (3.4) above) and the definition of $\alpha_{ik}(x)$

$$\alpha_{ik}(0) := P(I(T) = k | I(0) = i, X_1(0) = 0).$$

The solution (3.8) is obtained as usual. Also, note that by the Lemma in [13] the number of the boundary conditions is

$$|S'_-| = |\lambda_j : \text{Re}(\lambda_j) \leq 0|.$$

Proof of Theorem 3.3:
From the model description of Section 2 it is clear that the off-periods (or on-periods) depend only on $X_1(t)$ and $I(t)$. Thus at the beginning of the $n$-th off-period $T_n'$ (or on-period $T_n$) the system regenerates in the sense that $X_1(T_n') = 0$ (or $X_1(T_n) = 0$) and knowing the state of the environment $I(T_n') = k$ (or $I(T_n) = i$) is enough to determine the future off-periods (on-periods). In other words, the state of $I(t)$ at the beginning of the $n$-th off-period (or on-period) determines (independent of the history) the state at the beginning of the $(n+1)$-st off-period (or respectively on-period). Clearly, it is also independent of $n$, so

$$P(I_{n+1}' = l | I_n' = k, I_{n-1}' = k_{n-1}, \ldots, I_0' = k_0) = P(I_{n+1}' = l | I_n = k) = P(I_1' = l | I_0' = k),$$

and

$$P(I_{n+1} = j | I_n = i, I_{n-1} = i_{n-1}, \ldots, I_0 = i_0) = P(I_{n+1} = j | I_n = i) = P(I_1 = j | I_0 = i).$$

The one-step transition probabilities are found as follows

$$p'_{kl} = P(I_1' = l | I_0' = k) = \sum_{i \in S'_-} P(I_1 = i | I_0' = k)P(I_1' = l | I_1 = i), k, l \in S'_+,$$
\[ p_{ij} = P(I_1 = j|I_0 = i) = \sum_{k \in S_+'} P(I'_0 = k|I_0 = i)P(I_1 = j|I'_0 = k), \quad i, j \in S'_- \]

Note that for \( k \in S_+' \) and \( i \in S'_- \) we have
\[
P(I_1 = i|I'_0 = k) = \alpha_{ki}(0) \quad \text{and} \quad P(I'_0 = k|I_0 = i) = \nu_{ik}.
\]

This yields Equations (3.9) and (3.10). The irreducibility of \( \{I'_n, \ n \geq 0\} \) and \( \{I_n, \ n \geq 0\} \) follows from the irreducibility assumption for the governing CTMC \( \{I(t), \ t \geq 0\} \) in Section 2.

**Proof of Theorem 4.1:**

Let \( x > 0, \ j \in S \) and \( i \in S'_- \). After conditioning on a small time interval of length \( h > 0 \) and using the argument of Theorem 3.2 we have
\[
\tilde{\psi}_{ji}(x, s) = E(e^{-sA_2}; I(T) = i|I(0) = j, \ X_1(0) = x) =
\]
\[
\sum_{k \neq j} q_{jk}h \int_0^h E(e^{-s(A_2+p_2(j)u+p_2(k)(h-u))}; I(T) = i|I(0) = k, \ X_1(0) = x+r_1(j)u+r_1(k)(h-u))g_U(u)du
\]
\[
+(1 + q_{jj}h + o(h))E(e^{-s(A_2+p_2(j)h)}; I(T) = i|I(0) = j, \ X_1(0) = x + r_1(j)h) + o(h).
\]

Using the notation of (4.19) and rearranging the last equation we get
\[
\frac{e^{sp_2(j)h} \tilde{\psi}_{ji}(x, s) - \tilde{\psi}_{ji}(x + r_1(j)h, s)}{h} =
\]
\[
= \sum_{k \in S} q_{jk} \int_0^h \tilde{\psi}_{ki}(x + r_1(j)u + r_1(k)(h-u), s)g_U(u)du + \frac{o(h)}{h}.
\]

Next, we expand \( e^{sp_2(j)h} \) in Taylor series around \( h = 0, \ e^{sp_2(j)h} = 1 + sp_2(j)h + o(h) \) and let \( h \to 0 \) to get
\[
-r_1(j) \frac{\partial \tilde{\psi}_{ji}}{\partial x}(x, s) + sp_2(j)\tilde{\psi}_{ji}(x, s) = \sum_{k \in S} q_{jk} \tilde{\psi}_{ki}(x, s).
\]

The last equation can be written in vector notation to obtain Eq. (4.20),
\[
R_1 \frac{\partial \tilde{\psi}_{i}}{\partial x}(x, s) + (Q - sP_2)\tilde{\psi}_{i}(x, s) = 0, \ i \in S'_-. \]

The boundary conditions (4.21) are obtained from the definition of \( A_2 \) which is the amount of type 2 fluid that comes in the buffer during the interval \( [0, T] \). Given that \( X_1(0) = 0 \) and \( I(0) \in S'_- \) it is clear that \( T = 0 \) and therefore \( A_2 = 0 \). This yields
\[
\tilde{\psi}_{ji}(0, s) = E(e^{-sA_2}; I(T) = i|I(0) = j, \ X_1(0) = 0) = \delta_{ji}, \quad \text{if} \ i \in S'_-, \ j \in S'_-.
\]
where $\delta_{ji}$ is the Kronecker symbol,

$$
\delta_{ji} := \begin{cases} 
1, & \text{if } j = i \\
0, & \text{if } j \neq i.
\end{cases}
$$

\[\Box\]

**Proof of Theorem 5.2:**

The proof follows from a similar result of Tzenova, Adan, and Kulkarni\[20\]. We explain the main steps below.

We denote

$$M(s) := sR_{--} - Q_{--} - Q_{--}\tilde{J}_{+-}(s)$$

and its derivative $M'(s)$.

A comparison with the matrix $[sR - \tilde{Q}(s)]_{S'_-,S'_-}$ in the case of the general fluid model with jumps from \[20\], leads to the following relationships between the two models:

\begin{align}
[Q_{++}\tilde{J}_{+-}(s)]_{ij} + [Q_{--}]_{ij} &= q_{ij}\tilde{G}_{ij}(s), \ i \neq j, \ i, j \in S'_-, \quad (8.46) \\
[Q_{++}\tilde{J}_{+-}(s)]_{ii} + [Q_{--}]_{ii} &= q_{ii}\tilde{G}_{ii}(s) - q_i, \ i \in S'_-. \quad (8.47)
\end{align}

The left hand side (of (8.46) or (8.47)) consists of a first term corresponding to any possible transition from $i \in S'_-$ to some state $S'_+$ that leads to a positive jump which ends in state $j \in S'_-$ and a second term corresponding to a transition from $i$ to a state $j \in S'_-$ that represents a zero jump. Recalling that $G_{ij}(x), \ x \geq 0$ denoted the c.d.f. of the jump size in the general model of \[20\] gives the right hand sides.

Now define

$$\tilde{Q}(s) := Q_{--} + Q_{--}\tilde{J}_{+-}(s).$$

We can write it in the form

$$\tilde{Q}(s) = \bar{Q}(s) - Q_d.$$

With this representation the analysis of the zeroes of $M(s)$ proceeds along the same lines as in \[20\].

\[\Box\]

**Proof of Lemma 6.1:**

Since time is restricted to only time spent in off-periods, the Elementary Renewal Theorem gives the mean number of off-periods per time unit in steady state as $1/E(\text{Length of an off-period})$. Now, the mean length of an off-period is given by $\sum_{t \in S'_+} \pi'_td_t$. Also, the long-run fraction of off-periods that start in state $k \in S'_+$ and end in state $j \in S'_-$ in steady state is given by $\pi'_k\alpha_{kj}(0)$. Hence

$$\frac{1}{\mu_{k,j}} = \frac{\pi'_k\alpha_{kj}(0)}{\sum_{t \in S'_+} \pi'_td_t}.$$
which is equivalent to Eq. (6.35). Following the same reasoning the long-run fraction of off-periods that end in state \( j \in S'_- \) in steady state is given by \( \pi_j \) and therefore the mean number of off-periods that end in state \( j \) per time unit in steady state is

\[
\frac{1}{\mu_j} = \frac{\pi_j}{\sum_{i \in S'_-} \pi_i \lambda_i}
\]

or equivalently Eq. (6.36).

**Proof of Theorem 6.2:**

Note that while the system is in an off-period type 2 fluid can not be served, so its net input rate is \( r_2(i) = p_2(i) \). Let \( x_1 > 0, \ a_2 > 0, \ j \in S, \ k \in S'_+ \) and condition on a time interval of length \( h > 0 \) as in Theorem 3.2. Similarly, we condition on \( U \) (with density \( g_U(u), \ u \in (0, h) \)), which is the time of the transition in the CTMC \( I(t) \) in the interval \((0, h)\), to get

\[
G_{k,j}(t, x_1, a_2) = \sum_{i: r_1(i) \geq 0} (\delta_{ij} + q_{ij} h) \int_{u=0}^{h} \int_{y_1=0-}^{x_1-r_1(i)u-r_1(j)(h-u)} \int_{y_2=0-}^{a_2-r_2(i)u-r_2(j)(h-u)} dG_{k,i}(t-h, y_1, y_2) g_U(u) du
\]

\[
+ \sum_{i: r_1(i) < 0} (\delta_{ij} + q_{ij} h) \int_{u=0}^{h} \int_{y_1=-r_1(i)u-r_1(j)(h-u)}^{x_1-r_1(i)u-r_1(j)(h-u)} \int_{y_2=0-}^{a_2-r_2(i)u-r_2(j)(h-u)} dG_{k,i}(t-h, y_1, y_2) g_U(u) du
\]

\[
+ \delta_{jk} \sum_{l: r_1(l) \geq 0} \sum_{i: r_1(i) < 0} \nu_{ij} \int_{u=0}^{h} \int_{y_1=-r_1(i)u-r_1(j)(h-u)}^{x_1-r_1(i)u-r_1(j)(h-u)} \int_{y_2=0-}^{y_2=\infty} dG_{l,i}(t-h, y_1, y_2) g_U(u) du + o(h).
\]

The last term corresponds to the case when \( X_1 \) becomes zero during the interval \( h \) and the next off-period starts in \( k \), which in return automatically leads to bringing \( A_2 \) to zero by its definition. Note that if \( X_1(t-h) = 0 \) and \( i: r_1(i) < 0 \) the system is in an on-period and therefore \( G_{l,i}(t-h, 0, y_2) = 0 \) by its definition above, Eq. (6.34). Thus, \( y_1 = 0 \) is not included in the last term.

Let \( j \in S'_+ \), i.e. \( r_1(j) \geq 0 \). Then the above equation becomes

\[
G_{k,j}(t, x_1, a_2) - G_{k,j}(t-h, x_1-r_1(j)h, a_2-r_2(j)h)
\]

\[
= \sum_{i \in S} q_{ij} h \int_{u=0}^{h} G_{k,i}(t-h, x_1-r_1(i)u-r_1(j)(h-u), a_2-r_2(i)u-r_2(j)(h-u)) g_U(u) du
\]

\[
- \sum_{i: r_1(i) < 0} q_{ij} h \int_{u=0}^{h} G_{k,i}(t-h, -r_1(i)u-r_1(j)(h-u), a_2-r_2(i)u-r_2(j)(h-u)) g_U(u) du
\]

\[
+ \delta_{jk} \sum_{l: r_1(l) \geq 0} \sum_{i: r_1(i) < 0} \nu_{ij} \int_{u=0}^{h} (G_{l,i}(t-h, -r_1(i)u-r_1(j)(h-u), \infty) - G_{l,i}(t-h, 0, \infty)) g_U(u) du
\]

\[
+ o(h).
\]
After we let $t \to \infty$ and divide by $h$ we obtain
\[
\frac{G_{k,j}(x_1, a_2) - G_{k,j}(x_1 - r_1(j)h, a_2 - r_2(j)h)}{h} = \sum_{i \in S} q_{ij} \int_{u=0}^{h} G_{k,i}(x_1 - r_1(i)u - r_1(j)(h-u), a_2 - r_2(i)u - r_2(j)(h-u))g_U(u)du
\]
\[- \sum_{i : r_1(i) < 0} q_{ij} \int_{u=0}^{h} G_{k,i}(-r_1(i)u - r_1(j)(h-u), a_2 - r_2(i)u - r_2(j)(h-u))g_U(u)du
\]
\[+ \delta_{jk} \sum_{l : r_1(l) \geq 0} \sum_{i : r_1(i) < 0} \nu_{ij} \int_{u=0}^{h} G_{l,i}(-r_1(i)u - r_1(j)(h-u), \infty) - G_{l,i}(0, \infty)\frac{1}{h}g_U(u)du + o(h) + o(h) .
\]

After letting $h \to 0$ we obtain
\[
r_1(j) \frac{\partial}{\partial x_1} G_{k,j}(x_1, a_2) + r_2(j) \frac{\partial}{\partial a_2} G_{k,j}(x_1, a_2) =
\]
\[
\sum_{i \in S} q_{ij} G_{k,i}(x_1, a_2) - \sum_{i : r_1(i) < 0} q_{ij} G_{k,i}(0, a_2) - \delta_{jk} \sum_{l : r_1(l) \geq 0} \sum_{i : r_1(i) < 0} \nu_{ij} r_1(i) \frac{\partial}{\partial x_1} G_{l,i}(0, \infty).
\]
Note that during an off-period of type $k$
\[
G_{k,i}(0, a_2) = 0, \ a_2 \geq 0 \text{ for } i : r_1(i) < 0.
\]
Therefore for $j$ with $r_1(j) \geq 0$ we obtain
\[
r_1(j) \frac{\partial}{\partial x_1} G_{k,j}(x_1, a_2) + r_2(j) \frac{\partial}{\partial a_2} G_{k,j}(x_1, a_2) = \quad (8.48)
\]
\[
\sum_{i \in S} q_{ij} G_{k,i}(x_1, a_2) - \delta_{jk} \sum_{l : r_1(l) \geq 0} \sum_{i : r_1(i) < 0} \nu_{ij} r_1(i) \frac{\partial}{\partial x_1} G_{l,i}(0, \infty). \quad (8.49)
\]

In case $j \in S'_t$, i.e. $r_1(j) < 0$ we have
\[
G_{k,j}(t, x_1, a_2) - G_{k,j}(t - h, x_1 - r_1(j)h, a_2 - r_2(j)h) = G_{k,j}(t - h, -r_1(j)h, a_2 - r_2(j)h)
\]
\[
= \sum_{i \in S} q_{ij} \int_{u=0}^{h} G_{k,i}(t - h, x_1 - r_1(i)u - r_1(j)(h-u), a_2 - r_2(i)u - r_2(j)(h-u))g_U(u)du
\]
\[- \sum_{i : r_1(i) < 0} q_{ij} \int_{u=0}^{h} G_{k,i}(t - h, -r_1(i)u - r_1(j)(h-u), a_2 - r_2(i)u - r_2(j)(h-u))g_U(u)du + o(h).
\]
Note that $G_{k,j}(t - h, 0, a_2 - r_2(j)h) = 0$ for $j \in S'_t$. Hence we can subtract it from the left-hand side of the last equation. After dividing by $h$, letting $t \to \infty$, and $h \to 0$, we get
\[
r_1(j) \frac{\partial}{\partial x_1} G_{k,j}(x_1, a_2) + r_2(j) \frac{\partial}{\partial a_2} G_{k,j}(x_1, a_2) - r_1(j) \frac{\partial}{\partial x_1} G_{k,j}(0, a_2) =
\]
\[ \sum_{i \in S} q_{ij} G_{k,i}(x_1, a_2) - \sum_{i : r_1(i) < 0} q_{ij} G_{k,i}(0, a_2) = \sum_{i \in S} q_{ij} G_{k,i}(x_1, a_2), \quad j : r_1(j) < 0, \quad x_1 \geq 0, \quad a_2 \geq 0. \]  

(8.50)

Also note that over off-periods of type \( k \)
\[ G_{k,j}(0, a_2) = 0, \quad j : r_1(j) \neq 0, \]
and
\[ G_{k,j}(x_1, 0) = 0, \quad j : r_2(j) \neq 0. \]

Now, before taking the LT of both sides of the equations (8.49) and (8.50), we consider the LT of \( \frac{\partial}{\partial x_1} G_{k,j}(0, a_2), \ j \in S'_- : \)
\[ \int_{x_1=0}^{\infty} \int_{a_2=0}^{\infty} \frac{\partial}{\partial x_1} G_{k,j}(0, a_2) e^{-s_1 x_1} e^{-s_2 a_2} dx_1 da_2 = \frac{1}{s_1} \int_{0}^{\infty} \frac{\partial}{\partial x_1} G_{k,j}(0, a_2) e^{-s_2 a_2} da_2 = \]
\[ \frac{1}{s_1} \frac{\partial}{\partial x_1} G_{k,j}(0, \infty) \int_{0}^{\infty} \frac{\pi'_k \alpha_{kj}}{\pi_k \alpha_{kj}} \frac{\partial}{\partial x_1} G_{k,j}(0, \infty) e^{-s_2 a_2} da_2. \]

(8.51)

Now
\[ \pi'_k \alpha_{kj} \frac{\partial}{\partial x_1} G_{k,j}(0, a_2) \]
is equal to the c.d.f. of the total accumulation of type 2 fluid during an off-period, and the off-period is of type \( k \) and ends in state \( j \). Hence, the integral in (8.51) equals \( \tilde{B}_{k,j}(s_2) / s_2 \), and (8.51) can be written as
\[ \frac{1}{s_1 s_2} \frac{\partial}{\partial x_1} G_{k,j}(0, \infty) \tilde{B}_{k,j}(s_2) / \pi'_k \alpha_{kj}. \]

From Section 4 and Eq. (4.19) it is clear that
\[ \tilde{B}_{k,j}(s_2) = \pi'_k \tilde{\psi}_{kj}(0, s_2), \quad j \in S'_-, \]
where \( \tilde{\psi}_{kj}(0, s_2) \) are found from Theorem 4.1.

Then after we take LT of both sides of equations (8.49) and (8.50) the following equations are obtained, respectively:
\[ r_1(j) s_1 G_{k,j}^*(s_1, s_2) + r_2(j) s_2 G_{k,j}^*(s_1, s_2) \]
\[ = \sum_{i \in S} q_{ij} G_{k,i}^*(s_1, s_2) - \frac{\delta_{jk}}{s_1 s_2} \sum_{l: r_1(i) \geq 0} \sum_{i: r_1(i) < 0} \nu_{ij} r_1(i) \frac{\partial}{\partial x_1} G_{l,i}(0, \infty), \quad j \in S'_+, \]
\[ r_1(j) s_1 G_{k,j}^*(s_1, s_2) + r_2(j) s_2 G_{k,j}^*(s_1, s_2) - \frac{r_1(j)}{s_1 s_2} \frac{\partial}{\partial x_1} G_{k,j}(0, \infty) \tilde{B}_{k,j}(s_2) / \pi'_k \alpha_{kj} \]
\[ = \sum_{i \in S} q_{ij} G_{k,i}^*(s_1, s_2), \quad j \in S'_-. \]
Next, note that

\[-r_1(j) \frac{\partial}{\partial x_1} G_{k,j}(0, \infty), \ j \in S'_{-}\]

represents the mean number of type \( k \) off-periods that end in state \( j, \ j \in S'_{-} \), per time unit in steady state and therefore from the classic theory of Renewal processes, we have

\[-r_1(j) \frac{\partial}{\partial x_1} G_{k,j}(0, \infty) = \frac{1}{\mu_{k,j}},\]

where \( \mu_{k,j} \) is the expected time between two consecutive type \( k \) off-periods that end in \( j \) and is given by (6.35). Now, the above equations become

\[
\begin{align*}
    r_1(j)s_1G_{k,j}^*(s_1, s_2) + r_2(j)s_2G_{k,j}^*(s_1, s_2) = \\
    \sum_{i \in S} q_{ij}G_{k,i}^*(s_1, s_2) + \frac{\delta_{jk}}{s_1s_2} \sum_{l:r_1(l) \geq 0 \land r_1(i) < 0} \nu_{ij} \frac{1}{\mu_{l,i}}, \ j \in S'_{+},
\end{align*}
\]  

(8.52)

\[
\begin{align*}
    r_1(j)s_1G_{k,j}^*(s_1, s_2) + r_2(j)s_2G_{k,j}^*(s_1, s_2) + \frac{m\tilde{B}_{k,j}(s_2)}{s_1s_2} = \\
    \sum_{i \in S} q_{ij}G_{k,i}^*(s_1, s_2), \ j \in S'_{-},
\end{align*}
\]  

(8.53)

where in the last equation we use Lemma 6.1 to get

\[-r_1(j) \frac{\partial}{\partial x_1} G_{k,j}(0, \infty) \frac{1}{\pi_k' \alpha_{kj}} = \frac{1}{\mu_{k,j} \pi_k' \alpha_{kj}} = \frac{1}{\sum_{l \in S'_+} \pi_l'd_l} = m.
\]

Finally, using the previously introduced notation, (8.52) and (8.53) can be written in matrix form to obtain

\[
\begin{align*}
    s_1G_{k}^*(s_1, s_2)R_1 + s_2G_{k}^*(s_1, s_2)R_2 + \frac{m\tilde{B}_{k}(s_2)}{s_1s_2} = G_{k}^*(s_1, s_2)Q + \frac{lE_k}{s_1s_2},
\end{align*}
\]

which is equivalent to (6.40).

\[\Diamond\]
References


