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Infinite divisibility
and the waiting-time paradox

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INFINITE DIVISIBILITY AND THE WAITING-TIME PARADOX

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Abstract. It is shown that the interarrival time \( Z \) covering the point zero in a stationary renewal process generated by \( X \) has the form \( Z \overset{d}{=} X + Y \) with \( Y \) nonnegative and independent of \( X \), if and only if \( X \) is infinitely divisible. In the special case that \( X \) has a compound-exponential distribution there is a similar decomposition of the stationary waiting time. These results shed some new light on the waiting-time paradox.

1. Introduction and summary

The \textit{waiting-time paradox} consists of the fact that a passenger arriving at a bus stop will probably have to wait considerably longer than about half the interarrival time, \( X \) say, of two buses. The paradox is resolved by the observation that a passenger is more likely to arrive in a long interval than in a short one. Alternatively, the length \( Z \) of the interval covering the arrival time of the passenger tends to be longer than \( X \), i.e., we have

\[
Z \geq X
\]

in distribution. On an appropriate sample space (1.1) can be written as

\[
Z \overset{d}{=} X + Y,
\]

where \( Y \) is nonnegative and, in general, \( X \) and \( Y \) are dependent.

In this note we consider the case where \( X \) and \( Y \) in (1.2) are \textit{independent}. It turns out that this happens if and only if \( X \) is \textit{infinitely divisible}, and this result sheds some new light on the waiting-time paradox and on the behaviour of nonnegative processes with stationary, independent increments [sii-processes]. In the special case that \( X \) is \textit{compound-exponential} [see Definition 2.5], also the (stationary) waiting-time \( W \) of a bus passenger admits of a decomposition similar to (1.2):

\[
W \overset{d}{=} X + A,
\]

where \( A \) is nonnegative and independent of \( X \).

In Section 2 we collect some results about renewal theory and infinite divisibility. Section 3 contains a characterization of infinite divisibility for nonnegative random variables by means of (1.2), and some of its consequences. In Section 4 equation (1.3) is considered for non-lattice distributions. In Section 5 we briefly present the solutions of (1.2) and (1.3) for distributions on the non-negative integers. We shall mostly use the renewal (life-time) terminology rather than the waiting-time terminology.
2. Preliminaries

We need some information on renewal theory and on infinite divisibility for non-negative random variables. Some of the results hold only for non-lattice distributions, and have analogues for lattice distributions. We shall only consider lattice random variables with values in $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Sometimes it will be essential that these random variables have positive probability at 0.

2.1. Renewal theory

We consider a renewal process generated by a sequence $(X_n)_{n \in \mathbb{N}}$ of independent, non-negative random variables distributed as $X$ with distribution function $F$ and expectation $\mathbb{E}X = \mu \in (0, \infty)$; these conditions will be assumed to hold throughout unless otherwise stated. We write $S_n = \sum_{k=1}^n X_k$, and define the number $N_t$ of renewals in $(0, t]$, the age $V_t$ of the unit in service at time $t$, the remaining lifetime $W_t$ of this unit (the waiting time for the next bus), and the total life-time $Z_t$ as follows:

$$N_t := \#\{n \in \mathbb{N} : S_n \leq t\},$$
$$V_t := t - S_{N_t}, \quad W_t := S_{N_t+1} - t,$$
$$Z_t = X_{N_t+1} = V_t + W_t.$$ 

The following result is well known; see Feller (1971).

Lemma 2.1. If $F$ is non-lattice, then

$$V_t \xrightarrow{d} V, \quad W_t \xrightarrow{d} W \xrightarrow{d} V, \quad Z_t \xrightarrow{d} Z \quad [t \to \infty],$$

where $Z$ and $W$ have distribution functions given by

\begin{align}
F_Z(z) &= \frac{1}{\mu} \int_{0,z} x \, dF(x) \quad [z \geq 0], \\
F_W(w) &= \frac{1}{\mu} \int_0^w \{1 - F(x)\} \, dx \quad [w \geq 0].
\end{align}

Alternatively, $V$, $W$ and $Z$ can be regarded as quantities in a stationary renewal process (started at $-\infty$); then $Z$ is the life time straddling 0, say, and $V$ and $W$ are the lengths of the parts into which $Z$ is divided by 0. The random variables $V$, $W$ and $Z$ will be used in this sense, sometimes without comment. They satisfy

$$Z = V + W \quad \text{and} \quad V \overset{d}{=} W \overset{d}{=} U Z,$$

where $U$ is uniformly distributed on $(0, 1)$ and independent of $Z$; see Winter (1989).
There is an analogue to Lemma 2.1 for lattice distributions. Now $X$ is distributed on $\mathbb{Z}^+$, and the quantities $N_n$, $V_n$, $W_n$ and $Z_n$ are defined for $n \in \mathbb{N}$ in the same way as $N_t$, $V_t$, $W_t$ and $Z_t$. We then have the following well-known result; see Feller (1968). We use the notations $V$, $W$ and $Z$ also for the limits of $V_n$, $W_n$ and $Z_n$.

**Lemma 2.2.** If $X$ has an aperiodic distribution $(p_k)_{k \in \mathbb{Z}^+}$ on $\mathbb{Z}^+$, then

$$
V_n \xrightarrow{d} V, \quad W_n \xrightarrow{d} W \xrightarrow{d} V + 1, \quad Z_n \xrightarrow{d} Z \quad [n \to \infty],
$$

where $Z$ and $W$ have distributions given by

\begin{align*}
\text{(2.4)} \quad \mathbb{P}(Z = k) &= \frac{1}{\mu} k p_k \quad [k \in \mathbb{N}], \\
\text{(2.5)} \quad \mathbb{P}(W = k) &= \frac{1}{\mu} \sum_{j=k}^{\infty} p_j \quad [k \in \mathbb{N}].
\end{align*}

**Remark.** Comparison with (2.4) learns that (2.1) also holds for distributions on $\mathbb{Z}^+$. Since the span of the lattice is of no real importance, (2.1) holds for all distributions on $\mathbb{R}^+$ with a finite moment. Since $W$ in (2.2) is absolutely continuous, the non-lattice condition there is essential. 

### 2.2. Infinite divisibility

We recall the definition of infinite divisibility: a random variable $X$ (or its distribution function $F$) is said to be *infinitely divisible* if for any $n \in \mathbb{N}$ iid random variables $X_1^{(n)}, \ldots, X_n^{(n)}$ exist such that

$$
X \overset{d}{=} X_1^{(n)} + \cdots + X_n^{(n)}.
$$

In what follows the Laplace-Stieltjes transform [LSt] of a function $H$ will be denoted by $\hat{H}$. We take the following result from Feller (1971).

**Lemma 2.3.** A distribution function $F$ on $\mathbb{R}^+$ is infinitely divisible iff its LSt satisfies

\begin{equation}
\log \hat{F}(s) = \int_0^\infty (e^{-sx} - 1) x^{-1} dK(x),
\end{equation}

where $K$ is a nondecreasing function on $\mathbb{R}^+$ with (necessarily) $\int_1^\infty x^{-1} dK(x) < \infty$.

$K$ will be called the *canonical function* of $F$, or of the corresponding random variable. We note that the random variables $X^{(n)}$ in (2.6) are infinitely divisible with distribution function $F^{*(1/n)}$, the $n$-th convolution root of $F$, and with canonical function $K^{(n)} = (1/n) K$. Differentiation of (2.7) yields

\begin{equation}
- \frac{\hat{F}'(s)}{\hat{F}(s)} = \hat{K}(s),
\end{equation}

and inversion of this leads to the following equivalent relation [see Steutel (1970)]; let $s \downarrow 0$ for the final statement.
Corollary 2.4. A distribution function $F$ on $\mathbb{R}_+$ is infinitely divisible iff $F$ satisfies
\begin{equation}
(2.9) \quad \int_{(0,x]} y \, dF(y) = (F * K)(x) \quad [x > 0],
\end{equation}
where $K$ is nondecreasing on $\mathbb{R}_+$ and $*$ denotes convolution. The, possibly infinite, first moment $\mu$ of $F$ is given by
\begin{equation}
(2.10) \quad \mu = \int_{[0,\infty)} dK(x) = K(\infty).
\end{equation}

For Section 4 we need the following subclass of infinitely divisible distributions.

Definition 2.5. Let $S(\cdot)$ be a process with stationary, independent increments and let $T$ be exponentially distributed with $\mathbb{E}T = 1$, independent of $S(\cdot)$. Then the random variable $X := S(T)$ is said to have a compound-exponential distribution.

The following lemma is immediate if we denote the (infinitely divisible) distribution function of $S(1)$ by $F_0$.

Lemma 2.6. A distribution function $F$ on $\mathbb{R}_+$ is compound-exponential iff $\widehat{F}$ has the form
\begin{equation}
(2.11) \quad \widehat{F}(s) = \frac{1}{1 - \log \widehat{F_0}(s)},
\end{equation}
where $F_0$ is an infinitely divisible distribution function on $\mathbb{R}_+$. The distribution functions $F$ and $F_0$ have the same first moment.

We conclude this section with analogues (and also special cases) of Corollary 2.4 and Lemma 2.6 for infinitely divisible distributions on $\mathbb{Z}_+$. Here we use probability generating functions [pgf's] rather than LSt's. We stress that the factors $X_j^{(n)}$ in (2.3) are $\mathbb{Z}_+$-valued iff $P(X = 0) > 0$; for details we refer to Steutel (1970).

Lemma 2.7. A distribution $(p_k)_{k \in \mathbb{Z}_+}$ on $\mathbb{Z}_+$ with $p_0 > 0$ is infinitely divisible iff the quantities $r_k$ with $k \in \mathbb{Z}_+$ defined by
\begin{equation}
(2.12) \quad (k + 1) p_{k+1} = \sum_{j=0}^{k} p_j r_{k-j} \quad [k \in \mathbb{Z}_+],
\end{equation}
are nonnegative, i.e., iff the coefficients of $R(z) := P'(z)/P(z)$ are nonnegative.

Lemma 2.8. A distribution on $\mathbb{Z}_+$ with pgf $P$ is compound-exponential iff $P$ has the form
\begin{equation}
(2.13) \quad P(z) = \frac{1}{1 - \log Q(z)} = \frac{1}{1 - \alpha(G(z) - 1)},
\end{equation}
where $Q$ is the pgf of an infinitely divisible distribution on $\mathbb{Z}_+$ with $Q'/Q = \alpha G'$, $\alpha > 0$ and $G$ a pgf.
3. An extreme case of the waiting-time paradox

We return to the renewal process generated by a nonnegative random variable $X$ as described in Section 2.1. It is well known [see e.g. Ross (1970)] that for the total life-time $Z$ with distribution function given by (2.1) we have

$$Z \overset{d}{=} X,$$

i.e., $\mathbb{P}(Z > x) \geq \mathbb{P}(X > x)$ for $x \in \mathbb{R}_+$, or on a suitable sample space,

(3.1) \hspace{1em} Z \overset{d}{=} X + Y,

where $Y$ is nonnegative and, in general, not independent of $X$. Here we are interested in the situation where $X$ and $Y$ are independent. The following theorem can also be read as a characterization theorem for infinitely divisible random variables on $\mathbb{R}_+$ with a finite first moment [compare Remark following Lemma 2.2].

**Theorem 3.1.** Let $Z$ be the life time covering the point $0$ in a stationary renewal process generated by $X$. Then $Z$ can be written as in (3.1) with $X$ and $Y$ independent iff $X$ is infinitely divisible. The distribution function of $Y$ is given by $F_Y = (1/\mu)K$, where $K$ is the canonical function of $X$.

**Proof.** Let $Z$ satisfy (3.1) with $X$ and $Y$ independent. Then by (2.1) we have

(3.2) \hspace{1em} F_Z(z) = \frac{1}{\mu} \int_{(0,z]} x \, dF(x) = (F * F_Y)(z) \quad [z \geq 0].

Multiplying by $\mu$ we see that $F$ satisfies the functional equation (2.9) with $K = \mu F_Y$, hence by Corollary 2.4 $F$ is infinitely divisible. Conversely, if $F$ is infinitely divisible, then by (2.10) the functional equation (2.9) can be written in the form (3.2) for some distribution function $F_Y$, which means that in (3.1) $X$ and $Y$ can be taken independent. \qed

**Remark.** By a suitable choice of $K$, for $F_Y = (1/\mu)K$ any distribution function is possible. If $F(0) > 0$, then $X$ is compound-Poisson [cf. van Harn (1978)], i.e.,

$$X \overset{d}{=} \tilde{X}_1 + \cdots + \tilde{X}_N,$$

where $\tilde{X}_1, \tilde{X}_2, \ldots$ are independent and distributed as $\tilde{X}$ with distribution function $G$, say, and $N$ is Poisson distributed, independent of the $\tilde{X}_j$'s. Now $F_Y$ takes the form

$$F_Y(y) = \frac{1}{\mathbb{E}X} \int_{(0,y]} u \, dG(u),$$

i.e., $Y$ has the same distribution as the total life-time $\tilde{Z}$ in the renewal process generated by $\tilde{X}$. This can, roughly, be explained as follows. Since $X$ is a Poisson sum
of $\tilde{X}_j$'s, the (positive) $X$-interval covering a fixed point can be regarded as the sum of the $\tilde{X}$-interval covering this point and a (Poisson) number of independent $\tilde{X}_j$'s, i.e., as $X + \tilde{Z} \overset{d}{=} X + Y$.

The question remaining is: what is the meaning of the random variable $Y$ in (3.2). An answer can be obtained by considering the renewal process generated by $X^{(n)}$ with distribution function $F^{*}(1/n)$. Denoting the corresponding $Z$-random variable by $Z^{(n)}$, and using the fact that $X^{(n)}$ is infinitely divisible with canonical function $K^{(n)} = (1/n)K$, by Theorem 3.1 we have

$$Z^{(n)} \overset{d}{=} X^{(n)} + Y^{(n)},$$

where the distribution function of $Y^{(n)}$ is given by

$$F_{Y^{(n)}} = \frac{1}{\mathbb{E}X^{(n)}} K^{(n)} = \frac{n}{\mu} K^{(n)} = \frac{1}{\mu} K = F_Y,$$

independent of $n$. More generally, considering the sii-process $X(\cdot)$ with $X(1) \overset{d}{=} X$ and $X(1/n) \overset{d}{=} X^{(n)}$, we obtain the following result.

**Theorem 3.2.** Let $X$ be a nonnegative infinitely divisible random variable with finite first moment $\mu$ and with canonical function $K$. Let $X(\cdot)$ denote the sii-process with $X(1) \overset{d}{=} X$, and for $t > 0$ let $Z(t)$ be the $Z$-random variable in a stationary renewal process generated by $X(t)$. Then

$$(3.3) \quad Z(t) \overset{d}{=} X(t) + Y \quad [t > 0],$$

where $X(t)$ and $Y$ are independent and $F_Y = (1/\mu)K$, independent of $t$.

Equation (3.3) tells us that, if we approach the point 0 with small steps, i.e., if $t$ is small, then this point is passed with a non-small step of length at least $Y$, in distribution. Also, though $X(t) \overset{d}{\rightarrow} 0$, we have $Z(t) \overset{d}{\rightarrow} Y \neq 0$ as $t \downarrow 0$: we get an extreme case of the waiting-time paradox. For large $t$, on the other hand, the term $X(t)$ will dominate in the right-hand side of (3.3): the paradox, almost, disappears. Combining (2.3) with (3.3) yields the following result for the remaining life-time $W$.

**Corollary 3.3.** Let $X(\cdot)$ be as in Theorem 3.2, and for $t > 0$ let $W(t)$ be the $W$-random variable in a stationary renewal process generated by $X(t)$. Then

$$W(t) \overset{d}{=} UX(t) + UY \quad [t > 0],$$

where $U$, $X(t)$ and $Y$ are independent, $U$ is uniform on $(0,1)$, and $Y$ is as in Theorem 3.2. In particular,

$$(3.5) \quad W(t) \overset{d}{\rightarrow} UY \quad [t \downarrow 0].$$
4. Compound-exponential life times

Since the compound-exponential distributions [cf. Definition 2.5] are a special subclass of the infinitely divisible distributions, one may expect the random variable $Y$ in (3.1) to have special properties, if $X$ is taken compound-exponential. Indeed we have the following result.

**Theorem 4.1.** If $X$ is a nonnegative, compound-exponential random variable with a finite first moment, then the life time $Z$ covering 0 in a stationary renewal process generated by $X$ satisfies

$$(4.1) \quad Z \overset{d}{=} X + X' + Y_0,$$

where $X' \overset{d}{=} X$, $Y_0$ is nonnegative, and $X$, $X'$ and $Y_0$ are independent. Equivalently, the random variable $Y$ in (3.1) satisfies $Y \overset{d}{=} X' + Y_0$ with $X'$ and $Y_0$ independent and independent of $X$.

**Proof.** By Lemma 2.6 the LSt $\widehat{F}$ of $X$ has the form $\widehat{F}(s) = 1/(1 - \log \widehat{F}_0(s))$, where $F_0$ is an infinitely divisible distribution function on $\mathbb{R}_+$. From this equation and (2.8) it follows that the canonical function $K$ of $F$ is related to the canonical function $K_0$ of $F_0$ by $K = F * K_0$:

$$\widehat{K}(s) = -\frac{d}{ds} \log \widehat{F}(s) = \widehat{F}(s) \left\{ -\frac{d}{ds} \log \widehat{F}_0(s) \right\} = \widehat{F}(s) \widehat{K}_0(s).$$

Dividing by $\mu$, the first moment of both $F$ and $F_0$, and letting $Y_0$ be a random variable with distribution function $(1/\mu)K_0$, we see that $Y$ in (3.1) satisfies

$$F_Y = F * F_{Y_0},$$

so $Y$ can be obtained as $Y \overset{d}{=} X' + Y_0$ with $X'$ and $Y_0$ independent. \(\square\)

**Remark.** As in Theorem 3.2 one can consider the renewal process generated by $X(t)$. When $X = X(1)$ is compound-exponential, then so is $X(t)$ for $0 < t < 1$. Repeating the calculations for Theorem 4.1 with $X$ replaced by $X(t)$, we obtain

$$(4.2) \quad Z(t) \overset{d}{=} X(t) + X'(t) + Y_0(t) \quad [0 < t \leq 1],$$

where $Y_0(t) \overset{d}{=} X(1 - t) + Y_0$ with $Y_0$ as in (4.1) independent of $X(1 - t)$. So, we recover (3.3) with $Y \overset{d}{=} X + Y_0$ and $Y_0$ as in (4.1). \(\square\)

Choosing a uniformly distributed random variable $U$ independent of $X$, $X'$ and $Y_0$, we can rewrite (4.1) as

$$Z \overset{d}{=} (X + UY_0) + (X' + UY_0),$$
where $\bar{U} = 1 - U \overset{d}{=} U$. Since we also have $Z = V + W$ with $V \overset{d}{=} W$ [cf. Lemma 2.1 and (2.3)], this suggests that in this case

\[(4.3) \quad W \overset{d}{=} X + A,\]

where $X$ and $A$ are independent with $A$ of the form $A = U Y_0$. This is the content of the next theorem. Since apart from some mass at zero the random variable $A$, and hence $W$, has an absolutely continuous distribution, we consider the non-lattice case here; cf. Remark following Lemma 2.2.

**Theorem 4.2.** If $X$ is compound-exponential and non-lattice with finite first moment $\mu$, then the stationary remaining life-time $W$ satisfies (4.3) with $X$ and $A$ independent, and $A$ of the form $A \overset{d}{=} U Y_0$, where $Y_0$ is nonnegative, $U$ is uniform on $(0,1)$ and independent of $Y_0$.

**Proof.** Use Lemmas 2.1, 2.3 and 2.6. Since $X$ has LST $\hat{F} = 1/(1 - \log \hat{F}_0)$ with $F_0$ infinitely divisible, the LST of $W$ can be written as

\[\hat{F}_W(s) = \frac{1 - \hat{F}(s)}{\mu s} = \hat{F}(s) \frac{-\log \hat{F}_0(s)}{\mu s} = \hat{F}(s) \int_0^{\infty} \frac{1 - e^{-sx}}{sx} dF_{Y_0}(x),\]

where, as in the proof of Theorem 4.1, $F_{Y_0} = (1/\mu) K_0$ with $K_0$ the canonical function of $F_0$. The equality in the extreme members of this equation easily translates in (4.3) with $A$ of the form $A = U Y_0$ as desired.

It is unclear whether (4.3) with $X$ and $A$ independent implies that $X$ has a compound-exponential distribution. The fact that $A$ is of the form $A \overset{d}{=} U Y_0$ implies that $A$ has a density on $(0, \infty)$ which is nonincreasing.

**Remark.** Proceeding as in the Remark following Theorem 4.1, for $W(t)$ we obtain

\[(4.4) \quad W(t) \overset{d}{=} X(t) + U Y_0(t) \quad [0 < t \leq 1],\]

which can be written as

\[(4.5) \quad W(t) \overset{d}{=} (1 - U)X(t) + U\{X(t) + Y_0(t)\},\]

where $X(t) + Y_0(t) \overset{d}{=} Y$, independent of $t$; compare Corollary 3.3.

5. Analogues for lattice distributions

In this section we briefly present the analogues of Theorems 3.1, 4.1 and 4.2 for $Z_+\text{-valued}$ random variables $X$. We consider the discrete-time renewal process generated by $X$ with $\mathbb{P}(X = k) = p_k$ for $k \in \mathbb{Z}_+$ and $\mathbb{E}X = \mu \in (0, \infty)$, and use the
§5: Analogues for lattice distributions

notation as established in Lemma 2.2. We restrict attention to \( X \) with \( p_0 > 0 \). This leads to the clearest analogues, and it is no real restriction: The results for \( Z \) hold for arbitrary \( X \); for the result on \( W \) the restriction \( \mathbb{P}(X \leq 1) > 0 \) is necessary, and the case \( p_0 = 0, p_1 > 0 \) can be reduced to the case \( p_0 > 0 \) by a shift. The proofs differ only slightly from those in Sections 3 and 4.

**Theorem 5.1.** Let \( X \) have an aperiodic distribution \((p_k)_{k \in \mathbb{Z}^+}\) on \( \mathbb{Z}^+ \) with \( p_0 > 0 \). Then the random variable \( Z \) with distribution given by (2.4) satisfies

\[
Z \overset{d}{=} X + Y,
\]

where \( Y \) is a (necessarily \( \mathbb{N} \)-valued) random variable independent of \( X \), iff \( X \) is infinitely divisible.

**Proof.** By (2.4), saying that \( Z \) satisfies (5.1) with \( X \) and \( Y \) independent, is equivalent to the following assertion:

\[
\mathbb{P}(Z - 1 = k) = \frac{1}{\mu} (k + 1)p_{k+1} = \sum_{j=0}^{k} p_j \mathbb{P}(Y - 1 = k - j) \quad [k \in \mathbb{Z}^+].
\]

An appeal to Lemma 2.7 finishes the proof; note that necessarily \( \sum_{k=0}^{\infty} \tau_k = \mu \). \(\square\)

From the proof it will be clear that \( Y \) in (5.1) has distribution \((r_{k-1}/\mu)_{k \in \mathbb{N}}\). Of course, also Theorem 3.2 has a lattice analogue; we don't spell it out.

The analogues of Theorems 4.1 and 4.2 are combined in the following theorem. Note that a compound-exponential distribution on \( \mathbb{Z}^+ \) necessarily has positive mass at 0.

**Theorem 5.2.** Let \( X \) have an aperiodic, compound-exponential distribution \((p_k)\) on \( \mathbb{Z}^+ \). Then the random variable \( Z \) with distribution given by (2.4) satisfies

\[
Z \overset{d}{=} X + X' + Y_0,
\]

where \( X' \overset{d}{=} X, Y_0 \) is \( \mathbb{N} \)-valued, and \( X, X' \) and \( Y_0 \) are independent. For the random variable \( W \) with distribution given by (2.5) one has

\[
W \overset{d}{=} X + A,
\]

where \( X \) and \( A \) are independent and \( A \) has a nonincreasing distribution on \( \mathbb{N} \).

**Proof.** By Lemma 2.8 the pgf \( P \) of \( X \) has the form \( P = 1/(1 - \log Q) \) with \( Q \) an infinitely divisible pgf; hence differentiation of \( 1/P \) leads to the equation \( P'/P^2 = Q'/Q \). Since the pgf \( P_{Z-1} \) of \( Z - 1 \) is given by \((1/\mu) P'\), it follows that

\[
P_{Z-1}(z) = \left\{ P(z) \right\}^2 (1/\mu) Q'(z)/Q(z).
\]
Now, \( (1/\mu) Q'/Q \) can be read as the pgf of \( Y_0 - 1 \) for some \( \mathbb{N} \)-valued random variable \( Y_0 \); cf. the remark following Theorem 5.1. We conclude that \( Z \) satisfies (5.2).

With regard to (5.3) it is sufficient to note that

\[
P_{W-1}(z) = \frac{1 - P(z)}{\mu(1 - z)} = P(z) \frac{-\log Q(z)}{\mu(1 - z)} = P(z) \frac{1 - G(z)}{(\mu/\alpha)(1 - z)};
\]

cf. the second representation of (2.13) in Lemma 2.8.

\[\square\]

### 6. An example

For \( X(\cdot) \) in Theorem 3.2 we take a Gamma process: \( X(t) \) has a gamma distribution with parameters \( t \) and \( \lambda \), so with density \( x \mapsto \lambda^t x^{t-1} e^{-\lambda x} / \Gamma(t) \). The canonical function of \( X(t) \) has density \( x \mapsto \lambda x e^{-\lambda x} \), and it follows that for the total life-time \( Z(t) \) in the stationary renewal process generated by \( X(t) \) \([t > 0, \text{fixed}]\) we have

\[
Z(t) \overset{d}{=} X(t) + Y,
\]

where \( X(t) \) and \( Y \) are independent and \( Y \) has an exponential distribution with parameter \( \lambda \), independent of \( t \).

For \( 0 < t \leq 1 \) the random variable \( X(t) \) is compound-exponential: its LST can be represented as \( 1/(1 - \log F_0) \), where \( F_0 \) is infinitely divisible with canonical function \( K_0 \) satisfying

\[
\hat{K}_0(s) = -\frac{\hat{F}_0(s)}{\hat{F}_0(s)} = -\frac{d}{ds} \log \hat{F}_0(s) = \frac{t}{\lambda} \left( \frac{\lambda}{\lambda + s} \right)^{1-t}.
\]

From Theorem 4.1 it follows that for \( 0 < t \leq 1 \)

\[
Z(t) \overset{d}{=} X(t) + X'(t) + Y_0(t),
\]

where \( Y_0(t) \) has a gamma distribution with parameters \( 1 - t \) and \( \lambda \). The special case \( t = 1 \), i.e., \( X(1) \) exponential, gives \( Y_0(1) = 0 \) with probability 1, as is well known. In general, \( Z(t) \) has a gamma distribution with shape parameter \( t + t + (1 - t) = t + 1 \);

for \( t = \frac{1}{2} \) the random variable \( Z(t) \) is the sum of three iid random variables.

By Theorem 4.2, for the remaining life-time \( W(t) \) in the renewal process generated by \( X(t) \) we find

\[
W(t) \overset{d}{=} X(t) + A(t),
\]

with

\[
\hat{F}_A(t)(s) = \frac{(1 + s/\lambda)^t - 1}{ts/\lambda}.
\]

For \( t = 1 \) we obtain the well-known result that \( W(1) \overset{d}{=} X(1) \), i.e., \( \mathbb{P}(A(1) = 0) = 1 \).
References


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