A Note on the Excess Heat Generated by Right-Angled Thermode

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Introduction

Experiments have shown that the generation of heat by an electric current in a $U$-shaped thermode is not uniform. Near the corners the source strength of heat appears to be higher than in the straight parts of the thermode, and also larger than if the same thermode were straight.

In the present note we investigate this observation theoretically, by calculating in a $2-D$ model of an doubly infinite $L$-shaped thermode the source strength, and comparing the total (integrated) source strength with that of a straight thermode.

Apart from direct practical relevance, this result bears upon the problem of the equivalent source strength to be applied in a $1-D$ model approximating a slender $L$-shaped thermode. Therefore, we will consider right from the beginning a general geometry with non-equal legs.

The model

Consider in the $(x,y)$-plane the $L$-shaped region $\Omega_L$ and the straight region $\tilde{\Omega}_L$ of the same surface $(a+b)L$:

$$\Omega_L = \{0 \leq x \leq L + \tfrac{1}{2}a, \ 0 \leq y \leq b\} \cup \{0 \leq x \leq a, \ 0 \leq y \leq L + \tfrac{1}{2}b\}$$

$$\tilde{\Omega}_L = \{-L \leq x \leq 0, \ -\tfrac{1}{2}a \leq y \leq \tfrac{1}{2}a\} \cup \{0 \leq x \leq L, \ -\tfrac{1}{2}b \leq y \leq \tfrac{1}{2}b\}$$

and ends I and II as depicted in figure 1.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{L-shaped thermode and 1-D equivalent}
\end{figure}
\[ \Omega_L \text{ models the } L\text{-shaped thermode, and } \bar{\Omega}_L \text{ the } 1\text{-}D \text{ equivalent made of the same amount of material. The regions are electrically conducting with conductivity } \sigma \text{ and electric field potential } \phi. \text{ They are heat conducting with conductivity } k \text{ and temperature distribution } T. \text{ The electric field is stationary, satisfying (using Ohm's law [1])} \]

\[ \nabla \cdot (\sigma(T)\nabla \phi) = 0 \text{ in } \Omega_L, \bar{\Omega}_L \tag{3} \]

\[ \nabla \phi \cdot \mathbf{n} = 0 \text{ at } \partial \Omega_L, \partial \bar{\Omega}_L \tag{4} \]

\[ \phi = -V \text{ at end I} \tag{5} \]

\[ \phi = V \text{ at end II} \tag{6} \]

In \( \bar{\Omega}_L \) the field is independent of \( y \); across \( x = 0 \) \( \phi \) is assumed to be continuous, and from conservation of charge the total current must be continuous, which is:

\[ a\sigma \phi_x(x = 0-) = b\sigma \phi_x(x = 0+) \tag{7} \]

To make progress the electric conductivity \( \sigma \) will be taken constant, so that \( \phi \) satisfies Laplace's equation

\[ \nabla^2 \phi = 0 \tag{8} \]

According to Joule's law [1], the electric field in the presence of a current does work which is dissipated as heat by the amount (per unit time and volume) of

\[ \sigma|\nabla \phi|^2 \tag{9} \]

With material density \( \rho \) and specific heat \( c \), the equation for the temperature distribution is then given by [2]

\[ \rho c \frac{\partial T}{\partial t} = \nabla \cdot (k(T)\nabla T) + \sigma|\nabla \phi|^2 \tag{10} \]

and suitable boundary conditions. The temperature problem, however, will not be considered here further.

**Solution**

(i) **1-D equivalent thermode**

The solution in the straight thermode is easily found to be

\[ \phi = \tilde{Q}x/a - \tilde{\tau}, \quad -L \leq x \leq 0 \]

\[ = \tilde{Q}x/b - \tilde{\tau}, \quad 0 \leq x \leq L \tag{11} \]
where

\[ \tilde{Q} = 2 \frac{V}{L} \frac{ab}{a + b} \]  \hfill (12)

\[ \tilde{\varepsilon} = V \frac{a - b}{a + b}. \]  \hfill (13)

The local source strength is evidently

\[ |\nabla \phi|^2 = \tilde{Q}^2/a^2, \quad -L \leq x \leq 0 \]
\[ = \tilde{Q}^2/b^2, \quad 0 \leq x \leq L \]  \hfill (14)

(we drop the factor \( \sigma \) from here on). The total amount of heat generated in \( \Omega_L \) is

\[ aL \frac{\tilde{Q}^2}{a^2} + bL \frac{\tilde{Q}^2}{b^2} = 4 \frac{V^2}{L} \frac{ab}{a + b}. \]  \hfill (15)

(ii) \( L \)-shaped thermode

The solution for \( \Omega_L \) (with \( L \) finite) is difficult. However, we will in the end consider the limit for \( L \to \infty \) and \( V/L \) fixed, and in that case the solution can be given by means of conformal mapping [3,4].

Using standard techniques (see Appendix) the following analytic function

\[ G(w) = \frac{2}{\pi} i \left\{ b \arctan \frac{\sqrt{b^2 - w}}{\sqrt{a^2 + w}} + a \log \frac{a\sqrt{b^2 - w} + b\sqrt{a^2 + w}}{\sqrt{w}\sqrt{a^2 + b^2}} \right\} \]  \hfill (16)

defining the mapping \( z = G(w) \) may be found, that maps \( \Omega_\infty \) in the \( z \)-plane (via \( z = x + iy \)) to the upper half \( w \)-plane (figure 2), such that the boundary \( \partial \Omega_\infty \) corresponds with the real axis \( \text{Im}(w) = 0 \), \( G(b^2) = 0 \), \( G(-a^2) = a + ib \), \( \text{Im}(G) \to \infty \) if \( w \to 0 \), and \( \text{Re}(G) \to \infty \) if \( |w| \to \infty \). \( G \) has branch cuts \((-\infty, -a^2]\) and \([b^2, \infty)\). The definitions of \( \log \) and \( \arctan \) are the principal values, with

\[ \arctan(x) = \frac{1}{2} i \log \frac{i + x}{i - x}. \]
The most important limiting behaviour of $G$ is:

$$G(w) \approx \frac{b}{\pi} \log \frac{4w}{a^2 + b^2} + a\left(1 - \frac{2}{\pi} \arctan \frac{b}{a}\right) \quad (w \to \infty), \quad (17)$$

$$G(w) \approx \frac{2i}{\pi} a \log \frac{2ab}{\sqrt{w} \sqrt{a^2 + b^2}} + \frac{2}{\pi} ib \arctan \frac{b}{a} \quad (w \to 0), \quad (18)$$

$$G(w) \approx \frac{2i}{\pi b} \sqrt{a^2 + b^2} \sqrt{b^2 - w} \quad (w \to b^2), \quad (19)$$

$$G(w) \approx a + ib + \frac{2i}{\pi} \frac{b}{3a^2} \frac{(a^2 + w)\sqrt{a^2 + w}}{\sqrt{a^2 + b^2}} \quad (w \to -a^2). \quad (20)$$

Finally, it is to be noted that

$$\frac{dG}{dw} = \frac{b}{\pi i} \frac{\sqrt{a^2 + w}}{w \sqrt{b^2 - w}}. \quad (21)$$

The final solution is obtained by positioning a point source of strength $Q$ (to be determined) in $w = 0$

$$F = \frac{Q}{\pi} \log w. \quad (22)$$
The complex potential \( F = F(z) \) is then implicitly given by

\[
z = G(\exp(\frac{\pi F}{Q})) .
\]  

(23)

The physical potential \( \phi \) is, up to a constant, given by the real part of \( F \)

\[
F(z) = \phi(x, y) + i\psi(x, y) + \text{constant} .
\]  

(24)

The field lines are given by \( \psi = \text{constant} \), and the equipotential lines by \( \phi = \text{constant} \). Both \( \phi \) and \( \psi \) satisfy Laplace's equation, and are related by the Cauchy–Riemann conditions. As a result we have

\[
F'(z) = \phi_x + i\psi_x = -i\phi_y + \psi_y
\]  

(25)

\[
|F'(z)|^2 = |\nabla \phi|^2 .
\]  

(26)

Furthermore, the Jacobian of the mapping \((x, y) \mapsto (\phi, \psi)\) is \( |\nabla \phi|^2 \), i.e.

\[
|\nabla \phi|^2 dxdy = d\phi d\psi .
\]  

(27)

So the integral \( \iint_A |\nabla \phi|^2 dxdy = \text{surface}(\hat{A}) \), where \( \hat{A} \) is the image in \((\phi, \psi)\)-space of region \( A \).

For given constants \( Q \) and \( c \) we are now able to calculate any solution \( \phi \). An example (with \( a = 2, b = 1 \)) is depicted in figure 3 and 4 where field lines and equipotential lines are drawn, and contour lines of constant source strength \( |\nabla \phi|^2 \). For practical evaluation it may be noted that \( |\nabla \phi|^2 \) may be expressed, after differentiating (23) and then using (26), as

\[
|\nabla \phi|^2 = \frac{Q^2}{b^2} \frac{\sqrt{b^4 - 2b^2e^u \cos v + e^{2u}}}{\sqrt{a^4 + 2a^2e^u \cos v + e^{2u}}}
\]  

(28)

where \( \pi F/Q = u + iv \).
Equipotential and field lines for $a=2.0$, $b=1.0$

Figure 3.

Contours of constant source strength for $a=2.0$, $b=1.0$

Figure 4.
To obtain some symmetry it will be shown to be convenient to select the following constant in the definition of $\phi$ (24)

$$\text{constant} = \frac{Q}{\pi} \left( \log(ab) + \frac{b}{a} \arctan \frac{b}{a} - \frac{a}{b} \arctan \frac{a}{b} \right) + c$$

where $c$ is a new constant, necessary for the boundary conditions still to be fulfilled.

Using the asymptotic formula's for $G$ given above (17-20), we have now for

$x \to \infty, \ 0 \leq y \leq b$:

$$\phi \simeq \frac{Q}{b} x - \frac{Q}{\pi} \left( \log 2 - \log \frac{1}{2} \left( \frac{a}{b} + \frac{b}{a} \right) + \frac{b}{a} \arctan \frac{b}{a} - \frac{a}{b} \arctan \frac{a}{b} \right) - c,$$

(30)

$$\psi \simeq \frac{Q}{b} y ;$$

(31)

$y \to \infty, \ 0 \leq x \leq a$:

$$\phi \simeq -\frac{Q}{a} y + \frac{Q}{\pi} \left( \log 2 - \log \frac{1}{2} \left( \frac{a}{b} + \frac{b}{a} \right) + \frac{b}{a} \arctan \frac{b}{a} - \frac{a}{b} \arctan \frac{a}{b} \right) - c,$$

(32)

$$\psi \simeq \frac{Q}{a} x ;$$

(33)

$x \to 0, \ y \to 0$:

$$\phi \simeq \text{constant} + \frac{\pi Q}{4(a^2 + b^2)} (x^2 - y^2);$$

(34)

$x \to a, \ y \to b \ (x = a + r \cos \theta, \ y = b + r \sin \theta)$:

$$\phi \simeq \text{constant} - \frac{Q}{\pi} \left( 1 + \frac{b^2}{a^2} \right)^{\frac{1}{3}} \left( \frac{3\pi}{2b} \right)^{\frac{2}{3}} r^\frac{3}{2} \cos \left( \frac{2}{3} \theta - \frac{1}{3} \pi \right).$$

(35)

We see that in the legs the field $\nabla \phi$ becomes constant, is singular at the (inner) corner $(x, y) = (a, b)$, and vanishes at the (outer) corner $(x, y) = (0, 0)$ such, that the field lines are hyperbola shaped near $(0,0)$ (compare with figure 3 and 4).

Indeed, the field of $\Omega_L$ may be approximated for large $L$ by the field of $\Omega_\infty$ (for suitable $Q$). Inspecting the asymptotic behaviour of $\phi$ (30,32), we see that for given potential $\pm V$ at the ends I and II and large $L$, the constants $Q$ and $c$ are to be selected as

$$Q = \frac{2V}{a+b} \frac{a+b}{ab} L - g(a/b)$$

(36)

$$c = \frac{a-b}{ab} \frac{V(L + \frac{1}{2}(a+b))}{a+b} \frac{a+b}{ab} L - g(a/b)$$

(37)
where

\[
g(\lambda) = \frac{2}{\pi} \left( \log 2 - \log \frac{1}{2} (\lambda + \lambda^{-1}) + \lambda \arctan \lambda \right. \\
\left. + \lambda^{-1} \arctan \lambda^{-1} \right) - \frac{1}{2} (\lambda + \lambda^{-1}) .
\]

(38)

Note that

\[
g(\lambda) = g(\lambda^{-1}) = \frac{2}{\pi} \log 2 .
\]

(39)

Comparing \(Q\) and \(\bar{Q}\) we observe, with the same potential difference \(2V\), a slightly higher source strength \(|\nabla \phi|^2\) in the legs of the \(L\)-shaped thermode, since \(Q > \bar{Q}\). This is because the field lines cut the corner \((x, y) = (a, b)\) and are therefore shorter. This effect, however, disappears for \(L \to \infty\) (\(V/L\) constant), and so does not explain the excess heat. Also the higher intensity of the field near \((x, y) = (a, b)\) is not an evident candidate to explain the higher source strength because it is compensated by a lower intensity near \((x, y) = (0, 0)\).

To strictly compare the \(L\)-shaped thermode with the \(1-D\) equivalent (straight) thermode we will consider the total (integrated) source strength for \(L \to \infty\) and \(V/L\) fixed. Of course, this would tend to infinity for each case, so we will consider the limit of the difference. Since the fields differ only in the corner neighbourhood, this difference quantifies the difference near the corner.

**Total source strength difference**

We are interested in the total excess heat source strength

\[
\varepsilon = \lim_{L \to \infty} \left\{ \iint_{\Omega_L} |\nabla \phi|^2 dx dy - \iint_{\bar{\Omega}_L} |\nabla \bar{\phi}|^2 dx dy \right\}
\]

with \(V/L\) fixed.

The second integral is given by (15). The first integral is easily found by using (27), and noting that for large \(L\) the equipotential lines become practically parallel to the ends I and II.

So we have

\[
\varepsilon = \lim_{L \to \infty} \left\{ \int_{-V}^{V} \int_{0}^{Q} d\psi d\phi - aL \frac{\bar{Q}^2}{a^2} - bL \frac{\bar{Q}^2}{b^2} \right\}
\]

\[
\varepsilon = \lim_{L \to \infty} \left\{ 2VQ - 4 \frac{V^2}{L} \frac{ab}{a+b} \right\}
\]
\[ \varepsilon = \frac{V^2}{L^2} ab \frac{4}{(\frac{a}{b} + 1)(\frac{b}{a} + 1)} g(\frac{a}{b}) \]  
(41)

\((V/L \text{ fixed})\).

We see that indeed \(\varepsilon\) is positive. The shape factor

\[ \varepsilon_0(\frac{a}{b}) = \frac{4}{(\frac{a}{b} + 1)(\frac{b}{a} + 1)} g(\frac{a}{b}) \]

(42)

is given in figure 5.

\[ \varepsilon_0 \text{ varies between } 2 \text{ (at 0 and } \infty) \text{ and } 2\pi^{-1} \log 2 \text{ (at } a/b = 1), \text{ or:} \]

\[ \frac{2}{\pi} \log 2 \frac{abV^2}{L^2} \leq \varepsilon \leq 2 \frac{abV^2}{L^2}. \]

(43)

Conclusion

An \(L\)-shaped thermode with leg sizes typically \(L \times a\) and \(L \times b\), a potential difference \(2V\), electric conductivity \(\sigma\) and small thickness \(d\), generates near its corner more heat than elsewhere; typically of the order of \(\sigma abV^2/L^2\).
Appendix

Conformal mapping of polygonal boundaries

According to Schwarz–Christoffel’s theorem ([3,4]), a polygonal boundary in the $z$-plane with interior angles $\alpha_1, \alpha_2, \alpha_3, \ldots$ is mapped on to the real axis $\text{Im}(w) = 0$ of the $w$-plane by the transformation given by

$$
\frac{dw}{dz} = K(w - p_1)^{1-\frac{\alpha_1}{\pi}}(w - p_2)^{1-\frac{\alpha_2}{\pi}}(w - p_3)^{1-\frac{\alpha_3}{\pi}} \ldots
$$

where $K$ is a constant and $p_1, p_2, p_3, \ldots$ are the (real) values of $w$ corresponding to the vertices of the polygon. The corresponding region in the $w$-plane is the half-plane $\text{Im}(w) > 0$. The interior angle of a vertex at infinity is zero. One such point may be mapped to infinity in the $w$-plane, with the corresponding factor $(w - w_1)$ effectively constant.

In the present problem we selected the equation

$$
\frac{dw}{dz} = \frac{\pi i}{b} \frac{w\sqrt{b^2 - w}}{\sqrt{a^2 + w}}.
$$

The interior angles to be dealt with are 0 and 0 for infinity, and $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. One infinity in the $z$-plane is mapped to infinity in the $w$-plane, the other to $w = 0$. The real values corresponding to the other vertices are most conveniently taken to be $b^2$ and $-a^2$. Finally, the constant $K$ is taken here equal to $\pi/b$.

References


