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THE EQUATIONS OF DIRAC AND
THE $M_2($H$)$-REPRESENTATION
OF SPACE-TIME ALGEBRA STA

by

P.G. Vroegindeweij
The equations of Dirac and the $M_2(\mathbb{H})$-representation of Space-time algebra STA

P.G. Vroegindeweij

Abstract

In its original form Dirac's equation has been expressed by use of the so-called $\gamma$-matrices $\gamma^\mu$, $\mu = 0, 1, 2, 3$. They are elements of the matrix algebra $M_4(\mathbb{C})$. As emphasized by Hestenes several times the $\gamma$-matrices are merely a (faithful) matrix representation of a Lorentz basis $e^\mu$, $\mu = 0, 1, 2, 3$ for the real Clifford algebra $C_{1,3}$, also called space-time algebra STA. The use of the matrix algebra $M_4(\mathbb{C})$ to represent STA has some unsatisfactory aspects. The $\gamma$-matrices contain imaginary numbers whereas the concerned Clifford algebra is real.

Moreover the full matrix representation of $C_{1,3}$ is not $M_4(\mathbb{C})$ but $M_2(\mathbb{H})$.

For that reason we investigate in this paper several forms of Dirac's equation in terms of $M_2(\mathbb{H})$ instead of $M_4(\mathbb{C})$.

In Section 1 we give a brief summary of the original equation of Dirac, describing electrons. Section 2 contains some properties of the skew field $\mathbb{H}$, the associated linear space $\mathbb{H}^2$ and the full matrix algebra $M_2(\mathbb{H})$.

In Section 3 we describe the equation of Dirac for electrons in $M_2(\mathbb{H})$-representation.

Section 4 deals with Dirac's equations presenting the electro-weak interaction for left oriented pairs of leptons, also employing $M_2(\mathbb{H})$.

Finally in Section 5 we deal with Dirac's equation for strong interactions between quarks.

In contrast to $su(2) \times u(1)$, the Lie algebra $su(3)$ is not isomorphic to any subalgebra of $C_{1,3}$.

Therefore we do not give a description of strong interactions by use of $M_2(\mathbb{H})$. Instead of such an approach we describe these interactions using the space of quadruples of bivector fields in STA. The thus obtained description has remarkable formal resemblance to the original Dirac equations using wave functions with values in the linear space $\mathbb{C}^4$. 
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1. Preliminaries

1.1. The equation of Dirac

The equation of Dirac, describing the behaviour of electrons and photons, can (in Feynmann-slash notation) be given by

\[ i \not{D} \Psi(x) = m \Psi(x) \tag{1} \]

where \( i \in \mathbb{C}, \not{D} = \partial - iq \not{A} \), \( \Psi = \gamma^\mu D_\mu, \Theta = \gamma^\mu \partial_\mu, \not{A} = \partial^\mu A_\mu \) whence \( D_\mu = \partial_\mu - iq A_\mu \). One calls \( \Psi(x) \) the matter field, it takes its values in \( \mathbb{C}^4 \). The expression \( iq A_\mu \) is called the gauge field, it has its values in \( u(1) \). The gauge transformations can be presented by the couple

\[
\begin{align*}
\hat{\Psi}(x) &= e^{-i\alpha(x)} \Psi(x) \\
q \hat{A}_\mu &= q A_\mu - \partial_\mu \alpha(x)
\end{align*}
\tag{2}
\]

and consequently \( \hat{D}_\mu \hat{\Psi} = e^{-i\alpha} D_\mu \Psi \). Passing from \( \{\Psi(x), A_\mu\} \) to \( \{\hat{\Psi}(x), \hat{A}_\mu\} \) one has no observable effects. Both couples describe the same state.

1.2. Some remarks about the matrices \( \gamma^\mu \)

The matrices \( \gamma^\mu, \mu = 0,1,2,3 \) are fixed and traceless matrices of \( M_4(\mathbb{C}) \), usually chosen unitary whence \( \gamma^0 \) is hermitean and \( \gamma^1, \gamma^2, \gamma^3 \) are antihermitean.

The defining equations are

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I_4 \tag{3} \]

where \( \eta^{\mu\nu} = \text{diag}(1,-1,-1,-1) \).

As known these equations do not completely fix the \( \gamma^\mu \), however two unitary representations \( \{\gamma^\mu\} \) and \( \{\hat{\gamma}^\mu\} \) are related by

\[ \hat{\gamma}^\mu = U \gamma^\mu U^\dagger \text{ with } U^\dagger = U^{-1} \tag{4} \]

This is connected with the Lorentz invariance of the Dirac equation.

Remark.
Note that the \( \gamma^\mu \) are fixed matrices, \( ((\gamma^0, \gamma^1, \gamma^2, \gamma^3) \) is not a four-vector).

The so-called Pauli operators \( \sigma_1, \sigma_2, \sigma_3 \) satisfy the relations

\[ \sigma_k \sigma_l + \sigma_l \sigma_k = 2\delta_{kl}, \quad k,l = 1,2,3 \]

and
\[ [\sigma_k, \sigma_l] = -2i\epsilon_{klm} \sigma_m \quad (k, l, m = 1, 2, 3). \]  

We choose the matrix representation

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

**REMARK.**

The most usual representation for the Pauli operators is

\[ \sigma_i^T = \sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2^T = -\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]

\[ \sigma_3^T = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

where \( \sigma_k^T \) denotes the transposed of \( \sigma_k \). Instead of (5) they satisfy the relations

\[ [\sigma_i', \sigma_l'] = +2i\epsilon_{klm} \sigma_m'. \]

The reasons for our choice will be explained in Section 4.

The most used representations of the \( \gamma \)-matrices are the following: (we use the conventions \( \gamma_{\mu} = \eta_{\mu\nu} \gamma^\nu \), i.e. \( \gamma_0 = \gamma^0 \) and \( \gamma_k = -\gamma^k \), \( k = 1, 2, 3 \) throughout).

1) **The standard representation** (Dirac)

\[ \gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3. \]  

This convention is also followed by Hestenes in his papers.

2) **The Majorana representation**

\( \hat{\gamma}_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \hat{\gamma}^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} \) \]

\( \hat{\gamma}_2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \hat{\gamma}^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} \)  

In this representation the quantity \( i \hat{\theta} = i \gamma^\mu \partial_\mu \) is real.

One has \( \hat{\gamma}^\mu = U \gamma^\mu U^\dagger \) with \( U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & \sigma_2 \\ \sigma_2 & -I_2 \end{pmatrix} \).
3) The chiral representation

\[ \tilde{\gamma}^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \tilde{\gamma}^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3. \]  

(9)

This representation is employed if one wants to distinguish left and right oriented particles. For the left wave function function \( L = \frac{1}{2}(I_4 - i\gamma_0\gamma_1\gamma_2\gamma_3)\Psi \) one finds \( \tilde{\Psi} \) and similarly for the right wave function \( R = \frac{1}{2}(I_4 + i\gamma_0\gamma_1\gamma_2\gamma_3)\Psi \) the expression \( \tilde{\Psi} \).

The unitary relation between \( \tilde{\gamma}^\mu \) and \( \gamma^\mu \) can be given by \( \tilde{\gamma}^\mu = U \gamma^\mu U^\dagger \) with \( U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix} \).

4) The quaternionic representation

\[ \gamma^0_\star = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^k_\star = \begin{pmatrix} 0 & i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} \quad k = 1, 2, 3. \]  

(10)

with \( \gamma^\mu_\star = U \gamma^\mu U^\dagger, \quad U = \begin{pmatrix} iI_2 & 0 \\ 0 & I_2 \end{pmatrix}. \)

The quantities \( i\sigma_k, \quad k = 1, 2, 3 \) can be identified with the standard quaternions \( i, j \) and \( k \) (compare Section 2).
2. Quaternions

2.1. Introduction

There is a number of reasons (compare Section 3) for describing the equation of Dirac by means of the skew field (division ring) of quaternions \( \mathbb{H} \) in stead of the field \( \mathbb{C} \). In such a description the linear space \( \mathbb{H}^2 \) and the full algebra \( M_2(\mathbb{H}) \), consisting of all \((2 \times 2)\)-matrices with elements in \( \mathbb{H} \), also play a role. The elements of \( \mathbb{H} \) generally do not commute; this has a number of inconvenient consequences. For that reason, before building up Dirac’s theory furtheron, we start with a number of properties of \( \mathbb{H} \), \( \mathbb{H}^2 \) and \( M_2(\mathbb{H}) \). Of course we do not strive for completeness. We only deal with those properties that could play a role in our approach of the Dirac theory.

There are several ways to introduce the skew field \( \mathbb{H} \); here we mention three of them.

1) As pairs of complex numbers
This introduction is analogous to the introduction of complex numbers as pairs of real numbers. One can write any element \( a \) in \( \mathbb{C} \) as \( a = \alpha + i \beta \) with \( \alpha, \beta \in \mathbb{R} \) and \( i^2 = -1 \). Now define for pairs of complex numbers:

\[(a, b) (c, d) = (ac - bd, bc + ad) .\]

This yields a skew field with unit element \((1, 0)\). We write \((a, b) = a + i_2 b \) with \( i_2 = (0, 1) \).
Beyond \( i_2^2 = -1 \) we find \( i_2^3 = -1 \) and \( i_1 i_2 = -i_2 i_1 \) whence \((i_1 i_2)^2 = -1\).
We call \( i_1 i_2 = i_3 \), whence \( i_3^2 = -1 \), \( i_1 i_3 = -i_3 i_1 \), \( i_2 i_3 = -i_3 i_2 \) and finally \( i_1 i_2 i_3 = -1 \).

**Remark 1**
In retrospect \( i_1, i_2 \) and \( i_3 \) are indistinguishable (apart from orientation).

**Remark 2**
In spite of the famous inscriptions on the Hamilton bridge in Dublin we use the notation \( i_1, i_2, i_3 \) in stead of \( i, j, k \). It is more convenient and moreover the conventions \( i \in \mathbb{C} \) and also \( i \in \mathbb{H} \) sometimes can be confusing.
Every quaternion \( \mathbf{x} \) can be written as

\[ \mathbf{x} = x_0 i_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 . \]

We often write \( i_0 = 1 \) and \( x_0 i_0 = x_0 \).

2) As elements of \( M_2(\mathbb{C}) \)
This is a very straightforward representation of \( \mathbb{H} \). We identify:
1 and \( I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\( i_1 \) and \( i \sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \)

\( i_2 \) and \( i \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)

\( i_3 \) and \( i \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \) \tag{11}

**Remark**

Note that in this case the convention \( i, j, k \) instead of \( i_1, i_2, i_3 \) would be highly confusing.

In this representation every element \( x \) of \( \mathbb{H} \) can be written as

\[
x = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}.
\]

Note that \( xx^\dagger = (|a|^2 + |b|^2) I_2 \) where \( x^\dagger \) is the Hermitean conjugate of \( x \in M_2(\mathbb{C}) \). This means that

\[
\mathbb{H} / SU(2) \cong \mathbb{R}^+ \tag{12}
\]

where \( \mathbb{H}' \) is the multiplicative group of \( \mathbb{H} \setminus \{0\} \) and \( \mathbb{R}^+ \) the multiplicative group of strictly positive reals.

3) As the Clifford algebra \( C_{0,2} \)

Let \( C_{0,2} \) denote the Clifford algebra of \( \mathbb{R}^2 \) endowed with the quadratic form \( Q(x) = -x_1^2 - x_2^2 \), \( x = (x_1, x_2) \in \mathbb{R}^2 \).

The elements of \( C_{0,2} \) can be described by

\[
q = \alpha + \beta e_1 + \gamma e_2 + \delta e_1 e_2 ,
\]

\[
e_1^2 = e_2^2 = -1, \quad e_1 e_2 = -e_2 e_1
\]

that is to say \( q \in \mathbb{H} \).

The main automorphism \( e_k \mapsto -e_k \), mirroring the odd part of \( C_{0,2} \), yields

\[
\tilde{q} = \alpha - \beta e_1 - \gamma e_2 + \delta e_1 e_2
\]

while the main anti automorphism \( e_1 e_2 \mapsto e_2 e_1 \), reflecting the order of \( e_1 \) and \( e_2 \), gives

\[
\tilde{q} = \alpha + \beta e_1 + \gamma e_2 - \delta e_1 e_2.
\]
One finds for their composition
\[ q = \tilde{q} = \hat{q} = \alpha - \beta e_1 - \gamma e_2 - \delta e_1 e_2 \]
with properties \( \overline{q} = q \) and \( \overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1} \).

2.2. The anti involution \( x \mapsto \bar{x} \)

Let \( x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 \in \mathbb{H} \).
Define the quaternionic conjugate \( \bar{x} \) of \( x \) by
\[ \bar{x} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3 \]
and
the real part of \( x = \text{Re}(x) = \frac{1}{2}(x + \bar{x}) = x_0, \)
the pure part of \( x = \text{Pu}(x) = \frac{1}{2}(x - \bar{x}) = x_1 i_1 + x_2 i_2 + x_3 i_3 \).

Hence for every \( x \in \mathbb{H} \) there is a unique decomposition
\[ x = \frac{1}{2}(x + \bar{x}) + \frac{1}{2}(x - \bar{x}) = \text{Re}(x) + \text{Pu}(x). \]

It is easy to be checked that
\[ x = \bar{x} \iff \text{Pu}(x) = 0 \iff x^2 \geq 0 \]
\[ x = -\bar{x} \iff \text{Re}(x) = 0 \iff x^2 \leq 0. \]

Remark
Writing \( q = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \), \( a, b \in \mathbb{C} \), that is to say considering \( q \) as an element of \( M_2(\mathbb{C}) \) one finds the Hermitean conjugate \( q^\dagger = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \).
On the other hand, considering \( q \) as an element of \( \mathbb{H} \) one finds also for the quaternionic conjugate \( \bar{q} \) of \( q \) the expression \( \bar{q} = \begin{pmatrix} \bar{a} & \bar{b} \\ -\bar{b} & \bar{a} \end{pmatrix} \).

In fact it is confusing to use the convention \( \cdot \) in \( \mathbb{C} \) as well as in \( \mathbb{H} \).

To give \( \mathbb{H} \) more structure one introduces an anti involution of \( \mathbb{H} \) given by the map \( x \mapsto \bar{x} \), \( x \in \mathbb{H} \). It has the following properties:

a) \( \bar{x + y} = \overline{x + y} \),
b) \( \bar{z} \cdot \bar{y} = \bar{y} \bar{x} \),
c) \( \bar{z} = z \).

**Remark**
Because \([\frac{1}{2}i_k, \frac{1}{2}i_l] = \epsilon_{klm} \frac{1}{2}i_m, \ k, l, m = 1, 2, 3\) the pure quaternions generate a Lie algebra (isomorphic to \(su(2)\)). Consequently \([x, y]\) is pure, \(x \in \mathbb{H}, \ y \in \mathbb{H}\), that is to say \([x, y] = -[x, y]\) whence \([\bar{x}, \bar{y}] = [x, y]\) and that is equivalent with

\[ x\bar{y} + y\bar{x} = \bar{y}x + \bar{x}y. \]

This property brings us to the following orthogonal structure on \(\mathbb{H}\).

**Definition:**

\[ (x, y) := \frac{1}{2}(\bar{x}y + \bar{y}x), \quad (= \frac{1}{2}(\bar{x}y + \bar{y}x)) \]

with the following (immediate) properties.

0. \( (x, y) = (\overline{x, y}) \) i.e. \( (x, y) \in \mathbb{R} \)
1. \( (x, y) = (y, x) \)
2. \( (<, >) \) is bilinear
3. \( (x, x) = x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 \)

whence \( (x, x) > 0 \) for \( x \neq 0 \) and \( (x, x) = 0 \) for \( x = 0 \).
Moreover one finds the obvious orthogonality property

4. \( (\bar{x}, y) = (x, y) \).

**Definition** \( x \perp y \) iff \( (x, y) = 0 \).
Example: \( (i_k, i_l) = \delta_{kl}, \ k, l = 0, 1, 2, 3 \) hence \( \mathbb{H} \), considered as an algebra over \( \mathbb{R} \), has \( \{i_0, i_1, i_2, i_3\} \) as an orthonormal basis.
Finally we define

\[ ||x|| = \sqrt{(x, x)} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \]

with properties

1. \( ||x|| \ ||y|| = ||xy|| \),
2. \( \bar{x}x = x\bar{x} = ||x||^2 \).
Note that for \( x \neq 0 \) the inverse element \( x^{-1} \) is given by \( x^{-1} = \frac{x}{\|x\|^2} \).

**Remark**

Because \( i_3^2 = -i_1^2 = -i_2^2 = -i_3^2 = 1 \) the Lorentz structure of \( HH \) is obvious by the following definition.

\[
<x, y> := \frac{1}{4}(xy + yx + xy + yx)
\]

with properties

0. \(< x, y > = \overline{< y, x >} \), i.e. \(< x, y > \in RH \)

1. \(< x, y > = \overline{< y, x >} \)

2. \(< \cdot, \cdot > \) is bilinear

3. \(< x, x > = \frac{1}{2}(x^2 + \overline{x^2}) = x_0^2 - x_1^2 - x_2^2 - x_3^2 \)

and moreover the obvious orthogonality property

4. \(< \bar{x}, \bar{y} > = \overline{< x, y >} \).

**Remark**

Considering \( HH \) as \( C^2 \) it is also possible \( HH \) to endow with a sesquilinear complex inner product in the following way.

Let \( x' = x_0 + x_1i_1 + x_2i_2 + x_3i_3 = x_0 + x_1i_1 + (x_2 + x_3i_1)i_2 \in HH \) and associale with \( x' \) the quantity

\[
x = (x_0 + x_1i_1, x_2 + x_3i_1) \in C^2
\]

then, if \( y = (y_0 + y_1i_1, y_2 + y_3i_1) \) one finds that

\[
(x, y) = (x_0 - x_1i_1)(y_0 + y_1i_1) + (x_2 - x_3i_1)(y_2 + y_3i_1) \in C
\]

is a complex sesquilinear inner product on \( C^2 \) (dependent on the special representation above, using \( i_1 \)).

**2.3. Some remarks about \( HH^2 \) and \( M_2(HH) \)**

\( HH \) is a skew field, so there are two distinct linear spaces \( HH^2 \), the so-called left space and the right space. We choose for the latter (column space). Scalars of \( HH \) act from the right, we write
The map $A : \mathbb{H}^2 \to \mathbb{H}^2$ is called linear if $A(x + y) = Ax + Ay$ and $A(x\lambda) = (Ax)\lambda$. (The latter condition turns out to be the associativity of $\mathbb{H}$.) Note that in our right space $A(\lambda x) = \lambda(\lambda^{-1} A\lambda) x$ and hence in general not $A(\lambda x) = \lambda(Ax)$ (consider $\lambda$ as $\lambda I_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$).

After introduction of a basis in $\mathbb{H}^2$ every linear map $A : \mathbb{H}^2 \to \mathbb{H}^2$ can be described with a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with elements in $\mathbb{H}$. We want to know whether there is an inverse $A^{-1}$. Let for example $A = \begin{pmatrix} i_1 & i_2 \\ i_2 & i_1 \end{pmatrix}$ with $A^{-1} = -\frac{1}{2} A$. Calculating according to linear spaces over commutative fields one finds that $\det A = i_1^2 - i_2^2 = 0$, hence the notion of determinant has to be changed for spaces over skew fields. We want a notion of determinant such that at any rate $\det A \neq 0$ if and only if $A^{-1}$ exists and such that the theory of determinants for linear spaces over commutative fields rises again after change from a skew field to a commutative field. It is not possible to fulfil the latter condition completely. The notion of determinant that we shall introduce corresponds to the modulus of the determinant for commutative fields. First we need a short trip into the geometric algebra (all details can be found in [1]). The elements $\neq 0$ of a skew field $K$ constitute a multiplicative group $K'$. Let $C$ denote the commutator group of $K'$, that is to say the group generated by all elements $aba^{-1}b^{-1}$.

The factor group $K'/C$ is commutative and the values of $\det A$ are chosen in this commutative group.

In the underlying case $K = \mathbb{H}$ the situation is very simple.

Now we define $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$\det A = ||a_{12}a_{21}||$ if $a_{11} = 0$

$\det A = ||a_{11}a_{22} - a_{11}a_{21}a_{11}^{-1}a_{12}||$ if $a_{11} \neq 0$.

Obviously one can conclude that $A^{-1}$ exists for $a_{12}a_{21} \neq 0$ in case $a_{11} = 0$ and for $a_{22} \neq a_{21}a_{11}^{-1}a_{12}$ in case $a_{11} \neq 0$.

Example 1

$A = \begin{pmatrix} i_1 & i_2 \\ i_2 & i_1 \end{pmatrix}$, $\det A = 2 \neq 0$, $A^{-1} = -\frac{1}{2} A$. 

9
Example 2

\[ A = \begin{pmatrix} 1 & i_2 \\ i_1 & i_3 \end{pmatrix} , \quad \det A = 0 , \quad A^{-1} \text{ does not exist but notice that} \]

\[ A^T = \begin{pmatrix} 1 & i_1 \\ i_2 & i_3 \end{pmatrix} , \quad \det A^T = 2 \neq 0 , \quad (A^T)^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -i_2 \\ -i_1 & -i_3 \end{pmatrix} \]

and that not at all \((A^T)^{-1} = (A^{-1})^T\).

Example 3

If \(a_{11} \neq 0, a_{12} \neq 0, a_{21} \neq 0, a_{22} \neq 0\) one finds for the inverse \(A^{-1} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\) of

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ with } \det A \neq 0 \text{ that} \]

\[ x_{11} = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1} \]

\[ x_{12} = (a_{21} - a_{22}a_{12}^{-1}a_{11})^{-1} \]

\[ x_{21} = (a_{12} - a_{11}a_{21}^{-1}a_{22})^{-1} \]

\[ x_{22} = (a_{22} - a_{21}a_{11}^{-1}a_{12})^{-1} . \]

Note that for commutative fields one recovers Cramer's rule

\[ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} . \]

Remark 1

It is rather awkward but inevitable that for the commutative subfield \(C\) of the skew field \(JH\) we do not recapture the determinant in case of commutative fields but its absolute value.

Remark 2

In case of a commutative field the trace of \(A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\) is defined by

\[ \text{trace} (A) = a_{11} + a_{22} . \]

It has the fundamental property

\[ \text{trace} (AB) = \text{trace} (BA) . \]

Clearly (for example \(n = 1\)) this property does not hold in case of skew fields. In our case \(JH\) however we could define
Indeed it satisfies

$$\text{trace}(AB) = \text{trace}(BA)$$

but for the subfield $\mathbb{C}$ in $\mathbb{H}$ again it does not correspond to the definition

$$\text{trace}(A) = (a_{11} + a_{22})$$

although for the subfield $\mathbb{R}$ in $\mathbb{H}$ it does. Compare Remark 3 at the end of Subsection 3.3.

2.4. Final remarks

We close this second section with two remarks

A. The algebra of matrices $M_2(\mathbb{H})$ is isomorphic to the Clifford algebra $C_{1,3}$ of Minkowski space-time. One can find the details in [12].

B. A (right) linear space $V$ over $\mathbb{H}$ can be endowed with a quaternionic inner product $(\cdot, \cdot)$ with properties

0. $(x, y) \in \mathbb{H}$, $x, y \in V$
1. $(x, y) = (y, x)$
2. $(x, y_1 \alpha_1 + y_2 \alpha_2) = (x, y_1) \alpha_1 + (x, y_2) \alpha_2$ for all $\alpha_1, \alpha_2 \in \mathbb{H}$

and consequently

i) $(x, x) \in \mathbb{R}$, $x \in V$
ii) $(0, 0) = 0$
iii) $(x_1 \alpha_1 + x_2 \alpha_2, y) = \bar{\alpha}_1 (x_1, y) + \bar{\alpha}_2 (x_2, y)$

and moreover in case of definite positivity

3. $(x, x) > 0$ for $x \neq 0$.

Example

$V = \mathbb{H}^2$ and $(x, y) = \bar{x}_1 y_1 + \bar{x}_2 y_2$ where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{H}^2$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{H}^2$. 
3. The equation of Dirac by use of $\mathcal{H}$

3.1. Introductory remarks

As emphasized repeatedly by Hestenes the $\gamma$-matrices $\gamma^\mu$ are merely a representation in $M_4(C)$ of an orthonormal basis of the real Clifford algebra of Minkowski space-time. Compare e.g. [3]-[10]. Following Hestenes we call this Clifford algebra Space-time algebra STA.

In this matrix representation every element $x \in$ STA can be presented by

$$\alpha + \alpha_k \gamma^k + \alpha_{kl} \gamma^k \gamma^l + \alpha_{klm} \gamma^k \gamma^l \gamma^m + \beta \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

where the coefficients are real.

This situation has unsatisfactory aspects because the $\gamma$-matrices contain imaginary numbers whereas the coefficients are chosen real. The Majorana representation (Subsection 1.2) cannot help us here. Introduction of complex coefficients in stead of real ones would yield $M_4(C)$ but $M_4(C)$ has real dimension 32 whereas STA has dimension 16.

In the usual representations of $\gamma^\mu$ (Subsection 1.2) all $\gamma$-matrices have a $(2 \times 2)$-block structure

$$\begin{pmatrix} O_2 & A \\ B & O_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} C & O_2 \\ O_2 & D \end{pmatrix}, \quad A, B, C, D \in C(2),$$

hence the components of $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ behave like pairs.

REMARK In textbooks these pairs $\psi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$, corresponding to positive and negative energy respectively, are called the large and the small components of $\Psi$. Small components are related to Pauli spinors in the non-relativistic Pauli theory of electrons. As mentioned by Hestenes and Gurtler in [11] the Pauli equation can be written as

$$\partial \psi \cdot i_3 \hbar = \frac{\left( P_{op} - eA \right)^2}{2m} \psi + e \varphi \psi$$

with $P_{op} \psi = - \nabla \psi i_3 \hbar$, $\psi \in \mathcal{H}$, $i_3 \in \mathcal{H}$.

Notice that in (13) the usual quantities $i \in C$ and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in C^2$ are removed and replaced by $i_3 \in \mathcal{H}$ and $\psi \in \mathcal{H}$.

In Subsection 2.1 we pointed out that the quaternions can be presented by some elements of $M_2(C)$ and last but not least the theory of Clifford algebras tells us that $C_{1,3} = \text{STA}$ has matrix representation $M_2(\mathcal{H})$. So altogether we have reasons enough to investigate the equations of Dirac represented in $M_2(\mathcal{H})$ in stead of the usual representation in $M_4(C)$. Let us first give some examples of $M_2(\mathcal{H})$-representations of STA. Note that it is always possible, using $i_k = i \sigma_k$, $k = 1, 2, 3$, to switch from $M_2(\mathcal{H})$ into $M_4(C)$.

1) The most straightforward representation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & i_k \\ i_k & 0 \end{pmatrix}, \quad k = 1, 2, 3.$$
Substitution of $i_k = i \sigma_k$, $k = 1, 2, 3$ yields the $M_4(C)$-representation (10) given in Subsection 1.2.

In representation (14) even multivectors can be presented by $\begin{pmatrix} q_1 & q_2 \\ -q_2 & q_1 \end{pmatrix}$ and odd ones by $\begin{pmatrix} q_1 & q_2 \\ q_2 & -q_1 \end{pmatrix}$, $q_1, q_2 \in \mathbb{H}$.

2) Another representation

\[
\hat{h}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{h}^k = \begin{pmatrix} i_k & 0 \\ 0 & -i_k \end{pmatrix}, \quad k = 1, 2, 3
\]  
with $\hat{h}^k = U \hat{h}_k U^{-1}$, $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

**Remark**

For $U = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ one can define $U^\dagger$ in the following way. Representing $q \in \mathbb{H}$ by an element of $M_2(C)$ one finds $\bar{q} = q^\dagger$, where $\bar{q}$ is the quaternionic conjugate of $q \in \mathbb{H}$ and $q^\dagger$ the Hermitean conjugate of $q \in M_2(C)$. Hence we can write $U^\dagger = \begin{pmatrix} \bar{q}_{11} & \bar{q}_{21} \\ \bar{q}_{12} & \bar{q}_{22} \end{pmatrix}$ for $U \in M_2(\mathbb{H})$.

Indeed this corresponds to $U^\dagger$ represented in $M_4(C)$ hence in (15) one can write $\hat{h}^k = U \hat{h}_k U^\dagger$ in stead of $\hat{h}^k = U \hat{h}_k U^{-1}$.

3) A very useful representation

\[
\hat{h}_0 = \begin{pmatrix} 0 & -i_1 \\ i_1 & 0 \end{pmatrix}, \quad \hat{h}^1 = \begin{pmatrix} -i_1 & 0 \\ 0 & i_1 \end{pmatrix}, \\
\hat{h}^2 = \begin{pmatrix} i_2 & 0 \\ 0 & i_2 \end{pmatrix}, \quad \hat{h}^3 = \begin{pmatrix} 0 & i_1 \\ i_1 & 0 \end{pmatrix}
\]  
(16)

Although this representation looks rather irregular and ad hoc it is just the one that plays a very essential role in the equation of Dirac.

It satisfies the following desirable properties.

a) Even multivectors can be represented by

\[
\begin{pmatrix} a_{11} + b_{11} i_3 & a_{12} + b_{12} i_3 \\ a_{21} + b_{21} i_3 & a_{22} + b_{22} i_3 \end{pmatrix}, \quad a_{kl}, b_{kl} \in \mathbb{R}.
\]

b) Elements in the minimal left ideal, generated by the primitive idempotent $\frac{1}{2}(1 + h_3 h_0)$ can be given by $\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$, $q_1, q_2 \in \mathbb{H}$.

c) $h_5 := h_0 h_1 h_2 h_3 = i_3 i_2$.

Compare also Subsections 3.2 and 3.3.

The basic multivectors (with lower indices) are given by
\[
\begin{align*}
    h_0 &= \begin{pmatrix} 0 & -i_1 \\ i_1 & 0 \end{pmatrix}, \\
    h_1 h_2 h_3 &= h_0 h_5 = \begin{pmatrix} 0 & i_2 \\ -i_2 & 0 \end{pmatrix}, \\
    h_1 &= \begin{pmatrix} i_1 & 0 \\ 0 & -i_1 \end{pmatrix}, \\
    h_2 h_3 h_0 &= h_1 h_5 = \begin{pmatrix} -i_2 & 0 \\ 0 & i_2 \end{pmatrix}, \\
    h_2 &= \begin{pmatrix} -i_2 & 0 \\ 0 & -i_2 \end{pmatrix}, \\
    h_3 h_0 h_1 &= h_2 h_5 = \begin{pmatrix} -i_2 & 0 \\ 0 & -i_1 \end{pmatrix}, \\
    h_3 &= \begin{pmatrix} 0 & -i_1 \\ -i_1 & 0 \end{pmatrix}, \\
    h_0 h_1 h_2 &= h_3 h_5 = \begin{pmatrix} 0 & i_2 \\ i_2 & 0 \end{pmatrix}, \\
    h_0 h_1 h_2 h_3 &= h_5 = \begin{pmatrix} i_3 & 0 \\ 0 & i_3 \end{pmatrix}, \\
    h_1 h_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
    h_2 h_3 &= -h_5 h_1 h_0 = \begin{pmatrix} 0 & -i_3 \\ -i_3 & 0 \end{pmatrix}, \\
    h_2 h_0 &= \begin{pmatrix} 0 & -i_3 \\ i_3 & 0 \end{pmatrix}, \\
    h_3 h_1 &= -h_5 h_2 h_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
    h_3 h_0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
    h_1 h_2 &= -h_5 h_3 h_0 = \begin{pmatrix} -i_3 & 0 \\ 0 & i_3 \end{pmatrix}.
\end{align*}
\]

**Remarks**

1. Intentionally \( i_3 \) plays a preferential role in the representation.

2. Identification of \( i_3 \in \mathbb{H} \) and \( i \in \mathbb{C} \) yields \( h_1 h_0 = \sigma_1, h_2 h_0 = -\sigma_2, h_3 h_0 = \sigma_3 \) and \( h_2 h_3 = -i_1, h_3 h_1 = i_2, h_1 h_2 = -i_3 \).

3. The even subalgebra is spanned by 1 and \( i_3 \) and is isomorphic to \( M_2(\mathbb{C}) \).

4. \( \beta i_3 = -i_3 \beta \), only \( i_1 \) and \( i_2 \) appear in \( \beta = h^\dagger \theta_i \).

5. \( h_5 = h_0 h_1 h_2 h_3 = i_3 I_2 \) that is to say that \( i_3 \) is eigenvalue of \( h_5 \) both left and right.

6. Switching from \( M_2(\mathbb{H}) \) to \( M_4(\mathbb{C}) \) by \( i_k = i \sigma_k, k = 1, 2, 3 \) gives traceless and unitary \((4 \times 4)\)-matrices; \( h_0 \) is hermitean and \( h_1, h_2, h_3 \) are anti hermitean, as in classical representations.

7. One finds

\[
\tilde{h}_0 = h_3 h_0, \quad \tilde{h}_1 = h_3, \quad \tilde{h}_2 = h_5 h_3, \quad \tilde{h}_3 = h_2 h_3
\]

and conversely
The unitary transformation \( U \) in \( h_k = U \bar{h}_k U^\dagger \) is given by

\[
U = \pm \frac{1}{2} \begin{pmatrix}
    i_1 - i_2 & -i_1 - i_2 \\
    -1 - i_3 & -1 + i_3 \\
\end{pmatrix}
\]

and

\[
U^\dagger = U^{-1} = \pm \frac{1}{2} \begin{pmatrix}
    -i_1 + i_2 & -1 + i_3 \\
    i_1 + i_2 & -1 - i_3 \\
\end{pmatrix}.
\]

### 3.2. Space-time algebra and the equations of Dirac

In STA one can choose a Lorentz basis \( \{e_0, e_1, e_2, e_3\} \) with properties

\[
e_0 e_1 + e_1 e_0 = 2 \text{ diag}(1, -1, -1, -1).
\]

In a wellknown way one can decompose every multivector \( x \) in STA in \( k \)-grades \( x_k \), \( k = 0, 1, 2, 3, 4 \).

Hence we can write

\[
x = \alpha + \alpha^k e_k + \alpha^{kl} e_k e_l + \alpha^{klm} e_k e_l e_m + \beta e_0 e_1 e_2 e_3
\]

and also

\[
x = x_0 + x_1 + x_2 + x_3 + x_4.
\]

Having made a choice for the orientation the pseudoscalar \( e_5 = e_0 e_1 e_2 e_3 \) is an invariant, in some sense like the volume form in linear spaces. Note that \( e_5^2 = -1 \) and that

\[
e_5 e_k = -e_k e_5, \quad k = 0, 1, 2, 3.
\]

Following Hestenes we next introduce

\[
\hat{x}_k = (-1)^k x_k, \quad \text{whence} \quad \hat{x} = x_0 - x_1 + x_2 - x_3 + x_4,
\]

\[
\bar{x}_k = (-1)^{\frac{k}{2}} x_k, \quad \text{whence} \quad \bar{x} = x_0 + x_1 - x_2 - x_3 + x_4
\]

and (dependent on \( e_0 \))

\[
x^\dagger = e_0 \bar{x} e_0
\]

corresponding to Hermitean conjugation in \( M_4(\mathbb{C}) \).

Even multivectors \( x \) can be characterized by \( x = \hat{x} \) but also by \( e_5 x = x e_5 \) (and \( e_5 x = -x e_5 \) for odd multivectors). It can be taken together by \( e_5 x = \hat{x} e_5, \ x \in \text{STA} \).

In the next subsection we shall give an approach of the equation of Dirac in \( M_2(\mathbb{H}) \) language from two sides.
a) From the general point of view (without matrices) in STA and as developed by Hestenes. See the next table (column 2 and 3).
b) From the conventional representation in $M_4(C)$, summarized in Subsection 1.1 and in column 1 of the next table. Although this approach is a consequence of the general point of view in a) we give some details because it affords more insight into the structure of the Dirac equation using $M_4(C)$, respectively $M_2(III)$.

<table>
<thead>
<tr>
<th>Dirac equation</th>
<th>column 1</th>
<th>column 2</th>
<th>column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{g}\psi + qA\psi = \psi \mathfrak{g}$</td>
<td>$\partial \psi e_3 e_0 + qA\psi = \psi e_0$</td>
<td>$\partial \psi e_3 + qA\psi = m\psi$</td>
<td>$\varphi e_3 e_0 = \varphi$</td>
</tr>
<tr>
<td>current</td>
<td>$J_\mu = \Psi^\dagger \gamma_\mu \Psi$</td>
<td>$J = \psi e_0 \psi$</td>
<td>$J_\mu = (\psi^\dagger e_0 e_\mu \varphi)_0$</td>
</tr>
<tr>
<td>spin current</td>
<td>$s_\mu = -i\Psi^\dagger \gamma_\mu \gamma_5 \Psi$</td>
<td>$s = \psi e_3 \psi$</td>
<td>$s_\mu = -(\psi^\dagger e_0 e_\mu e_3 \varphi)_0$</td>
</tr>
<tr>
<td>gauge transformation</td>
<td>$\hat{\psi} = \Psi e^{-ia}$</td>
<td>$\hat{\psi} = \psi e^{-a e_3 e_0}$</td>
<td>$\hat{\varphi} = \varphi e^{-ae_3}$</td>
</tr>
<tr>
<td>$q\hat{A} = qA - \partial \alpha$</td>
<td>$q\hat{A} = qA - \partial \alpha$</td>
<td>$q\hat{A} = qA - \partial \alpha$</td>
<td></td>
</tr>
<tr>
<td>gauge invariant derivative</td>
<td>$D_\mu \Psi = \partial_\mu \Psi - qA_\mu \Psi i$</td>
<td>$D\psi = \partial\psi - qA\psi e_3 e_0$</td>
<td>$D\varphi = \partial\varphi - qA\varphi e_3$</td>
</tr>
<tr>
<td>Lagrangean density</td>
<td>$L = \Psi^\dagger \gamma_0 (\gamma^\mu D_\mu - m) \Psi$</td>
<td>$L = (\psi^\dagger e_0 (D\psi e_3 e_0 - m\psi e_0))_0$</td>
<td>$L = (\psi^\dagger e_0 (D\varphi e_3 - m\varphi))_0$</td>
</tr>
</tbody>
</table>

The above table leads us to the following remarks.

1. From a bit more general point of view one can replace $e_0$ by any unit pseudoscalar $i$, $e_0$ by any unit timelike vector $e$ and $e_3$ by any unit spacelike vector $v$ orthogonal to $e$. Compare [14]. In order to stay as close as possible to the original equation of Dirac and to the fundamental work of Hestenes we do not use these general notations.

2. Even multivectors $\psi$ (column 2) and $\varphi \in STA$ with $\varphi e_3 e_0 = \varphi$ in column 3 (i.e. $\varphi$ belongs to the minimal left ideal in STA generated by the primitive idempotent $\frac{1}{2}(1 + e_3 e_0)$) are related by

$$\varphi = \frac{1}{2}\psi(1 + e_0) (1 + e_3 e_0)$$

and conversely

$$\psi = \frac{1}{2}(\varphi - e_3 \varphi e_3) + \frac{1}{2}(\varphi + e_3 \varphi e_3) e_0 .$$

3. $\partial = e^\mu \partial_\mu$ is an invariante while $\beta = \gamma^\mu \partial_\mu$ is not. The Lorentz invariance in the columns 2 and 3 is manifest but that is not the case in column 1.

4. Clearly there is a fixed preferential direction, expressed by

i) $i \in C$ in column 1.

ii) $e_3 e_0$ in column 2.
iii) \( e_3 e_0 \) in column 3.

5. \( e_0, e_1, e_2, e_3 \) on the right side need not to be related to the vectors \( e^\mu \) in \( \partial = e^\mu \partial_\mu \).

6. Let the spinor \( \psi \) be related to the frame \( \{e_0, e_1, e_2, e_3\} \) and the spinor \( \hat{\psi} \) to the frame \( \{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\} \) with \( \hat{e}_\mu = L e^\mu L^{-1} \) (\( L \) is a Lorentz transformation) then one has the relation \( \hat{\psi} = \psi L^{-1} \).

### 3.3. The equations of Dirac in \( M_2(\mathbb{H}) \)-representation

Let us first consider the equation

\[
e^\mu D_\mu \varphi e_5 = m \varphi
\]

(table column 3).

Using representation (16) of Section 3.1 one finds for (17)

\[
h^\mu D_\mu \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix} i_3 = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix} m \ , \ \varphi_1, \varphi_2 \in \mathbb{H}
\]

or equivalently

\[
h^\mu D_\mu \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} i_3 = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} m \ , \ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{H}^2 .
\]

Next we show that the equation

\[
i\gamma^\mu D_\mu \Psi = m \Psi \ , \ \Psi \in \mathbb{C}^4
\]

(Subsection 1.1, Table column 1) is also equivalent to (18) and (18') and hence to (17). Compare especially the comment just after the introduction of (16) in Subsection 3.1.

If necessary we switch from \( M_2(\mathbb{H}) \) to \( M_4(\mathbb{C}) \) and conversely by \( i \kappa = i \sigma_k \ , \ k = 1, 2, 3 \). We write equation (1) a bit more explicitly as

\[
i\gamma^\mu \partial_\mu \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = (m - q \gamma^\mu A_\mu) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.
\]

Next we show that the equations

\[
i\gamma^\mu \partial_\mu \psi = (m - q \gamma^\mu A_\mu) \psi ,
\]

(19)
\[-i\gamma^\mu \partial_\mu (\gamma^2 \bar{\psi}) = (m - q\gamma^\mu A_\mu) (\gamma^2 \bar{\psi}) \tag{20}\]

and

\[
\gamma^\mu \partial_\mu \begin{pmatrix}
\psi_1 & -\bar{\psi}_2 \\
\psi_2 & \bar{\psi}_1 \\
\psi_3 & -\bar{\psi}_4 \\
\psi_4 & \bar{\psi}_3
\end{pmatrix} \begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix} = (m - q\gamma^\mu A_\mu) \begin{pmatrix}
\psi_1 & -\bar{\psi}_2 \\
\psi_2 & \bar{\psi}_1 \\
\psi_3 & -\bar{\psi}_4 \\
\psi_4 & \bar{\psi}_3
\end{pmatrix} \tag{21}\]

are equivalent, \(\bar{\psi}\) is the complex conjugate of \(\psi\), \(\bar{\psi} = \begin{pmatrix}
\bar{\psi}_1 \\
\bar{\psi}_2 \\
\bar{\psi}_3 \\
\bar{\psi}_4
\end{pmatrix}\).

In the proof we use the obvious relations \(\bar{\gamma}^\mu = -\gamma^2 \gamma^\mu \gamma^2, \mu = 0, 1, 2, 3\) and

\[
\gamma^2 \bar{\psi} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
\bar{\psi}_1 \\
\bar{\psi}_2 \\
\bar{\psi}_3 \\
\bar{\psi}_4
\end{pmatrix} = \begin{pmatrix}
-\bar{\psi}_2 \\
\bar{\psi}_1 \\
-\bar{\psi}_4 \\
\bar{\psi}_3
\end{pmatrix}
\]

holding especially in representation (16) i.e.

\[
\gamma^0 = \begin{pmatrix}
0 & -i\sigma_1 \\
i\sigma_1 & 0
\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}
-i\sigma_1 & 0 \\
0 & i\sigma_1
\end{pmatrix}, \quad \gamma^2 = \begin{pmatrix}
i\sigma_2 & 0 \\
0 & i\sigma_2
\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}
0 & i\sigma_1 \\
i\sigma_1 & 0
\end{pmatrix}, \tag{16'}\]

Complex conjugation of (19) gives

\[-i\bar{\gamma}^\mu \partial_\mu \bar{\psi} = (m - q\bar{\gamma}^\mu A_\mu) \bar{\psi}\]

or

\[i\gamma^2 \gamma^\mu \partial_\mu (\gamma^2 \bar{\psi}) = m \bar{\psi} + q\gamma^2 \gamma^\mu A_\mu (\gamma^2 \bar{\psi}).\]

Multiplication by \(\gamma^2\) from the left yields

\[-i\gamma^\mu \partial_\mu (\gamma^2 \bar{\psi}) = (m - q\gamma^\mu A_\mu) (\gamma^2 \bar{\psi}) \tag{20}\]

and addition of (19) and (20) yields (21). It is obvious that splitting (21) gives (19) and (20). Switching from \(M_4(C)\) to \(M_2(\mathbb{H})\) we find for (21)

\[
h^\mu \partial_\mu \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} i_3 = (m - qh^\mu A_\mu) \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} \tag{22}\]

or
\[ h^\mu D_\mu \left( \begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right)_3 = \left( \begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right)_m \] (18')

with \( D_\mu \varphi = \partial_\mu \varphi - q A_\mu \varphi i_3 \).

**Remarks**

1. The proof depends on the special representation (16') but clearly other representations yield similar results.

2. Switching from even \( \psi \) (column 2) to \( \varphi \) with \( \varphi \varepsilon \varepsilon_0 = \varphi \) (column 3) by \( \varphi = \frac{1}{2} \psi(1 + \varepsilon_0)(1 + \varepsilon \varepsilon_0) \) yields in this representation simply

\[
\psi = \left( \begin{array}{ccc}
 a + bi_3 & p + qi_3 \\
 c + di_3 & r + si_3
\end{array} \right)
\]

and

\[
\varphi = \left( \begin{array}{ccc}
 a + bi_3 + (p + qi_3) i_1 & 0 \\
 c + di_3 + (r + si_3) i_1 & 0
\end{array} \right).
\]

3. In STA one has the relation \( \varphi_0 = \frac{1}{4} \text{trace} \, M(\varphi) \), where \( \varphi_0 \) is the scalar part of \( \varphi \) and \( M(\varphi) \) the \( M_4(C) \)-representation of STA.

For the \( M_2(\mathbb{R}) \)-representation of STA the notion of trace has to be changed in

\[
\text{trace} \, A = \frac{1}{2}(a_{11} + a_{22} + \bar{a}_{11} + \bar{a}_{22})
\]

also resulting in \( \varphi_0 = \frac{1}{4} \text{trace} \, M(\varphi) \).

Compare Remark 2 at the end of Subsection 2.3.

4. The expressions

\[
J_\mu = (\varphi^t e_0 e_\mu \varphi)_0
\]

and

\[
s_\mu = -(\varphi^t e_0 e_\mu e_8 \varphi)_0
\]

(table column 3) can easily be translated into \( M_2(\mathbb{R}) \) using Remark 3.

5. The gauge transformation, given by \( \tilde{\varphi} = \varphi e^{-\alpha e_8} \)

(table column 3) reduces simply to

\[
\tilde{\varphi} = \varphi e^{-\alpha i_3}
\]

(compare \( \tilde{\varphi} = \varphi e^{-\alpha i} \) in column 1).

6. The expression \( D\varphi = \partial \varphi - q A \varphi e_8 \) (again column 3) reduces to \( D\varphi = \partial \varphi - q A \varphi i_3 \) in our \( M_2(\mathbb{R}) \)-representation (again compare \( D\tilde{\psi} = \partial \tilde{\psi} - q A \tilde{\psi} i \) in column 1 for the conventional representation).
4. The equation $D\varphi e_5 = m\varphi$

4.1. Preliminaries

In Section 1 we started with the equation of Dirac

$$(i\gamma^\mu D_\mu - m) \Psi(x) = 0$$

with $D_\mu = \partial_\mu - igA_\mu$ and $\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \in \mathbb{C}^4(x)$ describing the behaviour of electrons and photons. There are also physical phenomena, e.g. the weak interaction between leptons, that can be described by a pair of wave functions $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ with $\Psi_1, \Psi_2 \in \mathbb{C}^4$, instead of a single wave function $\Psi \in \mathbb{C}^4$. In that case the equation of Dirac is usually given by

$$(i\gamma^\mu D_\mu - m) \Psi(x) = 0$$

with

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \Psi_1, \Psi_2 \in \mathbb{C}^4$$

and

$$D_\mu = J_2 \partial_\mu - \frac{1}{2}igB^k_\mu \tau_k$$

where $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

This presentation contains an unsatisfactory ambiguity. On the one hand one considers $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \in \mathbb{C}^2$ with operators $\tau_1, \tau_2, \tau_3$ in $su(2)$ but on the other hand one considers $\Psi_1$ and $\Psi_2$ as elements of $\mathbb{C}^4$ with operator $i\gamma^\mu \partial_\mu - mI_4 \in M_4(\mathbb{C})$. A more satisfactory approach could be a description in $\mathbb{C}^4 \otimes \mathbb{C}^2$ but there is a simpler, more obvious and straightforward one in the following way.

Starting from $\Psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \\ \psi_{41} \end{pmatrix}$ and $\Psi_2 = \begin{pmatrix} \psi_{12} \\ \psi_{22} \\ \psi_{32} \\ \psi_{42} \end{pmatrix}$ one can consider the $(4 \times 2)$-matrix

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} \end{pmatrix}.$$
As usual we let act the operator \( i\gamma^\mu \partial_\mu - m \) from the left on the two columns of \( \Psi \) (and mix up the four rows of \( \Psi \)) but we let act the Lie algebra \( su(2) \) from the right on the four rows (and mix up the two columns). Indeed the ambiguity in interpretation now has been removed. The corresponding \( su(2) \)-gauge invariant operator \( D_\mu \) now can be introduced by

\[
D_\mu \Psi = \partial_\mu \Psi - \frac{ig}{2} B_k^\mu \tau_k \Psi .
\]  

(26)

Note that there is a small difference with the usual description, viz. in the operator \(-\frac{1}{2} i B_k^\mu \tau_k\) one has to replace \( B_\mu^2 \) by \(-B_\mu^2\). This is caused by the fact that \( \tau_2 = -\tau_2^T \), contrasting \( \tau_1 = +\tau_1^T \) and \( \tau_3 = +\tau_3^T \) (\( \tau_k^T \) means the transposed of \( \tau_k \)). Now it becomes also clear why in Subsection 1.2 we chose the matrices \( \sigma_k = \tau_k^T \) in stead of \( \sigma_k = \tau_k \) as usual.

Schematically one can present our approach as follows:

\[
\begin{align*}
\text{action of} & \rightarrow \quad \left( \begin{array}{cc}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22} \\
\psi_{31} & \psi_{32} \\
\psi_{41} & \psi_{42}
\end{array} \right) \\
\text{acting on} & \quad i\gamma^\mu \partial_\mu - m \\
\text{action of} & \leftarrow \quad B_k^\mu \tau_k .
\end{align*}
\]

In the next table we compare some notions, associated with the Dirac equation in the conventional description and in our one. The quantity \(-\frac{1}{2} ig B_\mu^k \tau_k\) is shortly noted by \( B_\mu \).

<table>
<thead>
<tr>
<th></th>
<th>conventional</th>
<th>here</th>
</tr>
</thead>
<tbody>
<tr>
<td>current</td>
<td>( j_\mu^k = \frac{1}{2} \bar{\Psi} \gamma^\mu \tau_k \Psi )</td>
<td>( j_\mu^k = \frac{1}{2} \text{tr}(\bar{\Psi} \gamma^\mu \tau_k \Psi) )</td>
</tr>
<tr>
<td>gauge transformation</td>
<td>( \hat{\psi} = U \psi, \quad U = e^{-\frac{1}{2} ia^k \tau_k} )</td>
<td>( \hat{\psi} = \psi U, \quad U = e^{-\frac{1}{2} ia^k \tau_k} )</td>
</tr>
<tr>
<td></td>
<td>( \hat{B}<em>\mu = U B</em>\mu U^{-1} - (\partial_\mu U) U^{-1} )</td>
<td>( \hat{B}<em>\mu = U^{-1} B</em>\mu U - U^{-1} (\partial_\mu U) )</td>
</tr>
<tr>
<td>gauge invariant derivative</td>
<td>( D_\mu = \partial_\mu + B_\mu )</td>
<td>( D_\mu \Psi = \partial_\mu \Psi + \Psi B_\mu )</td>
</tr>
<tr>
<td>Lagrangian</td>
<td>( L_1 = \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi )</td>
<td>( L_1 = \text{tr}(\bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi) )</td>
</tr>
</tbody>
</table>

Coupling of the \( u(1) \)-gauge invariant derivative \( D_\mu = \partial_\mu - iq A_\mu \) and the \( su(2) \)-gauge invariant derivative \( D_\mu = \partial_\mu - \frac{1}{2} i g B_k^\mu \tau_k \) yields (only for left oriented particles) the \( su(2) \times u(1) \)-gauge invariant derivative of the electro-weak interaction between leptons.

One can replace (25) by

\[
D_\mu = I_2 \partial_\mu - \frac{ig'}{2} A_\mu - \frac{ig}{2} B_k^\mu \tau_k .
\]  

(27)

acting on the left wave function \( L = \frac{i}{2} (1 - i\gamma_0 \gamma_1 \gamma_2 \gamma_3) \Psi \) and (26) by

\[
D_\mu L = (\partial_\mu - \frac{ig'}{2} A_\mu) L - \frac{1}{2} i g B_k^\mu L \tau_k
\]  

(28)

where again \( L = \frac{i}{2} (1 - i\gamma_0 \gamma_1 \gamma_2 \gamma_3) \Psi \).

All details of the conventional description can e.g. be found in [13].
4.2. Electro-weak interaction and STA

Let us now return to the equation

\[ D\varphi e_5 = m\varphi \]  

(29)

where \( \varphi \in \text{STA} \), \( \varphi_3 e_0 = \varphi \), \( D\varphi = e^\mu D_\mu \varphi \) and \( D_\mu \varphi = \partial_\mu \varphi - qA_\mu e_5 \) (table Subsection 3.2, column 3). It gives an equivalent description of electrons and photons as (1) does.

The condition \( \varphi_3 e_0 = \varphi \) causes an asymmetry in (29), however this condition can be presented by \( \varphi \in I^+ \), where \( I^+ \) is the minimal left ideal of STA generated by the primitive idempotent \( \frac{1}{2}(1 + e_3 e_0) \) but this presentation immediately raises the question: what about the remaining minimal left ideal \( I^- \), generated by the primitive idempotent \( \frac{1}{2}(1 - e_3 e_0) \). As argued by Hestenes in [8] the most obvious answer is that (29) in \( I^- \) (i.e. \( \varphi_3 e_0 = -\varphi \)) describes the behaviour of the neutrino (leptons appear in pairs). It turns out that one has the following unique decomposition for \( \varphi \in \text{STA} \).

\[ \varphi = \frac{1}{2}\varphi (1 + e_3 e_0) + \frac{1}{2}\varphi (1 - e_3 e_0) = \varphi_1 + \varphi_2, \varphi_1 \in I^+, \varphi_2 \in I^- \]

Hence the equation

\[ D\varphi e_5 = m\varphi , \ \varphi \in \text{STA} \]

falls to two pieces

\[ D\varphi_1 e_5 = m\varphi_1 \]

with \( \varphi_1 e_3 e_0 = \varphi_1 \) (compare (29)) and

\[ D\varphi_2 e_5 = m\varphi_2 \]

with \( \varphi_2 e_3 e_0 = -\varphi_2 \), the latter describing the neutrino. The description of the electro-weak interaction also has been considered by Hestenes in [8]. It is stated there that wave functions of left oriented particles correspond to even multivectors in STA and likewise right particles to odd multivectors. Note that the proof of this statement in [8] is based on the standard representation of the \( \gamma \)-matrices which yields the condition \( \varphi_1 = \varphi_2\gamma_5 \), \( \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \) and \( \varphi \in M_4(C) \) associated with the standard matrix representation of STA.

Elements of \( SU(2) \times U(1) \) can be considered in STA as the set

\[ \{ e^{\alpha e_5 + \beta^k e_8 e_0} \} , \ \alpha, \beta^k \in \mathbb{R} , \ k = 1,2,3 \]

The \( su(2) \times u(1) \) gauge-invariant derivative \( D_\mu \) for wave functions \( L \) of left particles can be given in STA by

\[ D_\mu L = (\partial_\mu - \frac{1}{2}e_8 g A_\mu) L - \frac{1}{2}e_8 g B^k_\mu L e_k e_0 , \]

(30)

\( g \) and \( g' \) are the coupling constants.
Note that $L$ is even, whence $L e_2 = e_2 L$. Note also the resemblance of (28) and (30). For more details we refer to [8] again.

4.3. Electro-weak interaction and the $M_2(\mathbb{H})$-representation of STA

In this subsection we express the results of Hestenes, exposed in [8] and summarized in 4.2, in the $M_2(\mathbb{H})$-representation of STA.

For the basis vectors $e_0, e_1, e_2, e_3$ we use the representation

$$
h_0 = \begin{pmatrix} 0 & -i_1 \\ i_1 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} i_1 & 0 \\ 0 & -i_1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} -i_2 & 0 \\ 0 & -i_2 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & -i_1 \\ -i_1 & 0 \end{pmatrix}
$$

(16)

throughout. Obviously one has

$$\frac{1}{2}(1 + h_3 h_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}(1 - h_3 h_0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

whence one finds for the wave functions of the electron and the neutrino the matrices

$$
\begin{pmatrix} q_{11} & 0 \\ q_{21} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & q_{12} \\ 0 & q_{22} \end{pmatrix}
$$

respectively, $q_{kl} \in \mathbb{H}$, $k, l = 1, 2$.

As pointed out in Subsection 3.3 equivalent representations in $\mathbb{H}^2$ can be given by

$$
\begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix}
$$

and

$$
\begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}.
$$

For the left part $L$ of the wave function $\Psi$, classically expressed by $L = \frac{1}{2}(1 - i\gamma_0 \gamma_1 \gamma_2 \gamma_3) \Psi$ we find after substitution of (11) in (16):

$$
L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} \\ \psi_{41} & \psi_{42} \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} \\ 0 & 0 \\ \psi_{31} & \psi_{32} \\ 0 & 0 \end{pmatrix}.
$$

This is equivalent with (compare (21))

$$
\begin{pmatrix} \psi_{11} & 0 & \bar{\psi}_{12} & 0 \\ 0 & \bar{\psi}_{11} & 0 & \bar{\psi}_{12} \\ \psi_{31} & 0 & \psi_{32} & 0 \\ 0 & \bar{\psi}_{31} & 0 & \bar{\psi}_{32} \end{pmatrix}
$$

corresponding to

$$
L = \begin{pmatrix} a_{11} + i_3 b_{11} & a_{12} + i_3 b_{12} \\ a_{21} + i_3 b_{21} & a_{22} + i_3 b_{22} \end{pmatrix}
$$

in $M_2(\mathbb{H})$.

According to Subsection 4.2 we recover (along another representation) that $L$ belongs to the even part of $M_2(\mathbb{H})$. Identification of $i_3 \in \mathbb{H}$ and $i \in C$ yields that $L \in M_2(C)$. One can also write
equivalent with $Li_3 = i_3 L$.

**Remark** One can give similar expressions for the right part of the wave function $\Psi$.

Let us now turn to the $su(2) \times u(1)$-gauge invariant derivatives

$$D_\mu L = \left( \partial_\mu - \frac{i g'}{2} A_\mu \right) L - \frac{i g}{2} B_\mu^k \tau_k$$  \hspace{1cm} (28)

and

$$D_\mu L = \left( \partial_\mu - \frac{e_5 g'}{2} A_\mu \right) L - \frac{e_5 g}{2} B_\mu^k L e_0 e_0.$$  \hspace{1cm} (30)

Note that in (30) $L$ is even, i.e. $L e_5 = e_5 L$. Translation of (30) into the $IH(2)$-representation yields after some minor manipulations.

$$D_\mu L = \left( \partial_\mu - \frac{1}{2} i_3 g' A_\mu \right) L - \frac{1}{2} i_3 g B_\mu^k L h_0 h_0$$  \hspace{1cm} (32)

with

$$h_1 h_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad h_2 h_0 = \begin{pmatrix} 0 & -i_3 \\ i_3 & 0 \end{pmatrix}, \quad h_3 h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note the striking resemblance of (28) and (32) after identification of $i \in \mathbb{C}$ and $i_3 \in IH$. 

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5. Strong interactions and STA

5.1. Preliminary remarks

Nowadays descriptions of strong interaction fields make use of the Lie algebra $su(3)$. In contrast to $su(2)$ the Lie algebra $su(3)$ cannot be faithfully represented in STA. Therefore we do not strive to give a description of strong fields expressed in terms of $M_2(H)$ like done for electro-weak fields in Section 4. Nevertheless a representation of strong fields within STA can be given. This representation uses the Lie algebra of bivectors in STA. Hestenes has shortly pointed out this possibility in [8]. In this section we give a number of details of this representation but first we give a brief summary of the $su(3)$-description of strong fields. Our treatment is not exactly the conventional one because (as in 4.1) we let act the Lie algebra $su(3)$ from the right.

5.2. Strong interaction fields

Let us first write down the equation of Dirac for triples of quarks

$$(i\gamma_\mu D_\mu - m) \Psi(x) = 0$$

where

$$\Psi = (\Psi_{\text{red}}, \Psi_{\text{blue}}, \Psi_{\text{green}}) = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \\ \psi_{41} & \psi_{42} & \psi_{43} \end{pmatrix}, \quad \psi_{kl} \in C(x)$$

$k = 1, 2, 3, 4$

$l = 1, 2, 3$

and

$$D_\mu \Psi = \partial_\mu \Psi + \Psi A_\mu$$

with $A_\mu = -\frac{1}{2} i A_k^l \lambda_k$.

The $(3 \times 3)$-matrices $\lambda_k$, $k = 1, \ldots, 8$ are the wellknown generators of $su(3)$. They can easily be found by decomposing the 8-parameter Hermitean and traceless $(3 \times 3)$-matrix

$$\begin{pmatrix} a_{11} & a_{12} - ib_{12} & a_{13} - ib_{13} \\ a_{12} + ib_{12} & a_{22} & a_{23} - ib_{23} \\ a_{13} + ib_{13} & a_{23} + ib_{23} & -a_{11} - a_{22} \end{pmatrix}$$

in

$$a_{12} \lambda_1 + b_{12} \lambda_2 + \frac{1}{2}(a_{11} - a_{22}) \lambda_3 + a_{13} \lambda_4 + b_{13} \lambda_6 + a_{23} \lambda_7 + \frac{1}{2} \sqrt{3}(a_{11} + a_{22}) \lambda_8$$
The gauge transformations can be given by
\[ \hat{A}_\mu = U^{-1} A_\mu U - U^{-1} \partial_\mu U \]
whence \[ \hat{D}_\mu \hat{\Psi} = (D_\mu \Psi) U . \]

The currents are given by
\[ j_k^\mu = \frac{1}{2} \text{tr}(\hat{\Psi} \gamma^\mu \lambda_k) \]
\[ \mu = 0, 1, 2, 3 \]
\[ k = 1, \ldots, 8 . \]

The wave function* \[ \Psi = (\Psi_1, \Psi_2, \Psi_3) \] is associated with the Lagrangian
\[ L_1 = \text{tr} \hat{\Psi} (i \gamma^\mu D_\mu - m) \hat{\Psi} . \]

5.3. Bivectors in STA

In STA the unit pseudoscalar \[ e_5 = e_0 e_1 e_2 e_3 \] is invariant up to a sign. Every bivector \[ b \in B \subset \text{STA} \] has the property \[ b^2 = \alpha + \beta e_5, \alpha, \beta \in \mathbb{R} . \]
Bivectors with \[ \beta = 0 \] are called simple. Simple bivectors can be subdivided in timelike, isotropic and spacelike ones if \[ b^2 > 0, b^2 = 0, b^2 < 0 \] respectively.

The set of bivectors \[ B \] satisfies the property \[ [B, B] \subset B \] i.e. \[ B \] is a real (6-dimensional) Lie algebra. In STA we fix now a timelike unit vector \[ e_0 \], for example the time axis of a Lorentz basis. Then (dependent on the choice of \[ e_0 \] ) there exists a unique decomposition of bivectors by

*From now on we drop the folkloristic notation \[ \Psi_{\text{red}} \] and so on.

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\[ b = \frac{1}{2}(b + e_0 be_0) + \frac{1}{2}(b - e_0 be_0), \quad b \in B. \]

We shall also write

\[ B = \frac{1}{2}(B + e_0 Be_0) + \frac{1}{2}(B - e_0 Be_0) = B_S + B_T. \]

\(B_S\) and \(B_T\) can also be characterized by \(B_S e_0 = e_0 B_S\) and \(B_T e_0 = -e_0 B_T\) respectively. After introduction of a Lorentz basis \(\{e_0, e_1, e_2, e_3\}\) we obtain immediately \(B_S^2 \leq 0\) and \(B_T^2 \geq 0\) i.e. \(B_S\) is simple and spacelike or isotropic and \(B_T\) is simple and timelike or isotropic. Note further that \(B_S = e_5 B_T = B_T e_5\) and \(B_T = e_5 B_S = B_S e_5\) and that \(b_5^I = -b^I, \quad b_5^I = b_I, \quad b_\ast \in B_S, \quad b_\ast \in B_T\).

Finally we remark that \([B_S, B_S] \subset B_S\) i.e. \(B_S\) is a sub Lie algebra of \(B\) and \([B_T, B_T] \subset B_S\) i.e. \(B_T\) is not a sub Lie algebra of \(B\).

Clearly, using the decomposition \(B = B_S + B_T\) one can consider \(B\) as a 3-dimensional "complex" Lie algebra (notice the dependence on \(e_0\)).

In this Lie algebra we shall use a basis \(\{u_1, u_2, u_3\}\) with the property

\[ [u_k, u_l] = 2e_5 \varepsilon_{kilm} u_m \quad (34) \]

for example \(u_k = e_k e_0, \quad k = 1, 2, 3.\)

Every \(b \in B\) now can be written as

\[ b = \sum_{k=1}^{3} (a_k + e_5 b_k) u_k, \quad a_k, b_k \in \mathbb{R} \]

with property

\[ b^I = \sum_{k=1}^{3} (a_k - e_5 b_k) u_k. \]

Introduction of the scalar product \((b, b) = \|b\|^2 = (b^I b)_0 = a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2\) reveals \(SU(3)\) as symmetry group of \(B\).

Next we need the map \(ad : B \rightarrow B\) given by

\[ (ad b)b_1 = [b, b_1], \quad b, b_1 \in B. \]

It has e.g. the properties

1. \(ad [b_1, b_2] = [ad b_1, ad b_2]\) (Jacobi)
2. \((ad b) [b_1, b_2] = [(ad b) b_2, b_1] + [b_1, (ad b) b_2]\) (Leibniz)

the latter being again a guise of Jacobi’s property.

Using the basis \(\{u_1, u_2, u_3\}\), as given in (34) one can write
\[ \text{ad } u_1 = 2e_\xi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{ad } u_2 = 2e_\xi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{ad } u_3 = 2e_\xi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (35) \]

Note that identification of \( e_\xi \in \text{STA} \) and \( i \in \mathbb{C} \) yields

\[ \text{ad } u_1 = 2\lambda_7, \quad \text{ad } u_2 = -2\lambda_5, \quad \text{ad } u_3 = 2\lambda_2. \]

Finally we mention that

\[ [\text{ad } u_k, \text{ad } u_l] = \text{ad}[u_k, u_l]. \]

Now we are able to express the matrices \( \lambda_1, \ldots, \lambda_8 \) in terms of bivectors of STA (again we identify \( i \in \mathbb{C} \) and \( e_\xi \in \text{STA} \)).

One finds:

\[
\begin{align*}
\lambda_1 & = -\frac{1}{4}(\text{ad } u_1 \text{ad } u_2 + \text{ad } u_2 \text{ad } u_1), \\
\lambda_2 & = \frac{1}{4}e_\xi(\text{ad } u_2 \text{ad } u_1 - \text{ad } u_1 \text{ad } u_2) \\
& = \frac{1}{2} \text{ad } u_3, \\
\lambda_3 & = -\frac{1}{4}(\text{ad } u_1 \text{ad } u_1 - \text{ad } u_2 \text{ad } u_2), \\
\lambda_4 & = -\frac{1}{4}(\text{ad } u_1 \text{ad } u_3 + \text{ad } u_3 \text{ad } u_1), \\
\lambda_5 & = \frac{1}{4}e_\xi(\text{ad } u_3 \text{ad } u_1 - \text{ad } u_1 \text{ad } u_3) \\
& = -\frac{1}{2} \text{ad } u_2, \\
\lambda_6 & = -\frac{1}{4}(\text{ad } u_2 \text{ad } u_3 + \text{ad } u_3 \text{ad } u_2), \\
\lambda_7 & = \frac{1}{4}e_\xi(\text{ad } u_3 \text{ad } u_2 - \text{ad } u_2 \text{ad } u_3) \\
& = \frac{1}{2} \text{ad } u_1, \\
\lambda_8 & = \frac{-1}{4\sqrt{3}} (\text{ad } u_1 \text{ad } u_1 + \text{ad } u_2 \text{ad } u_2 - 2\text{ad } u_3 \text{ad } u_3). 
\end{align*}
\]

Note that the given expressions are an immediate consequence of (34) and (35), using \( \{u_1, u_2, u_3\} \) as basis.

Now we return to the equation of Dirac as mentioned in 5.2.

5.4. Strong interactions expressed in STA

Let \( B^4(x) \) denote the set of quadruples of bivector fields in STA. We write them as columns
In a natural way one can introduce addition of these quadruples and also left (or right) scalar multiplication with elements \( \alpha + \beta e_5, \alpha, \beta \in \mathbb{R} \) provided with all usual properties because 
\[(\alpha + \beta e_5)b = b(\alpha + \beta e_5) \in B.
\]
We also introduce a right action of \( su(3) \) on \( B^4 \) by
\[
\begin{pmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4
\end{pmatrix}
\]
\[
\lambda_k = \begin{pmatrix}
    b_1 \lambda_k \\
    b_2 \lambda_k \\
    b_3 \lambda_k \\
    b_4 \lambda_k
\end{pmatrix}, \text{ where } b_\mu \lambda_k = \lambda_k(b_\mu)
\]
\[
\mu = 1, \ldots, 4, \quad k = 1, \ldots, 8.
\]
as introduced in 5.3.

Given the matrix
\[
\begin{pmatrix}
    \psi_{11} & \psi_{12} & \psi_{13} \\
    \psi_{21} & \psi_{22} & \psi_{23} \\
    \psi_{31} & \psi_{32} & \psi_{33} \\
    \psi_{41} & \psi_{42} & \psi_{43}
\end{pmatrix}
\]
with \( \psi_{kl} = a_{kl} + ib_{kl} \) one can consider the quadruple of bivectors in STA
\[
\varphi = \begin{pmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4
\end{pmatrix}
\]
\[
\begin{pmatrix}
    a_{11}e_1e_0 + b_{11}e_5e_1e_0 + a_{12}e_2e_0 + b_{12}e_5e_2e_0 + a_{13}e_3e_0 + b_{13}e_5e_3e_0 \\
    a_{21}e_1e_0 + b_{21}e_5e_1e_0 + a_{22}e_2e_0 + b_{22}e_5e_2e_0 + a_{23}e_3e_0 + b_{23}e_5e_3e_0 \\
    a_{31}e_1e_0 + b_{31}e_5e_1e_0 + a_{32}e_2e_0 + b_{32}e_5e_2e_0 + a_{33}e_3e_0 + b_{33}e_5e_3e_0 \\
    a_{41}e_1e_0 + b_{41}e_5e_1e_0 + a_{42}e_2e_0 + b_{42}e_5e_2e_0 + a_{43}e_3e_0 + b_{43}e_5e_3e_0
\end{pmatrix}
\]
Yet define the \((4 \times 4)\)-matrices \( \Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3 \) as \( \gamma^0, \gamma^1, \gamma^2, \gamma^0 \) but where \( i \in \mathbb{C} \) has been replaced by \( e_5 \in \text{STA} \). Now the Dirac equations (33) can be presented by
\[
(e_5\Gamma^\mu D_\mu - m)\varphi = 0
\]
with \( \varphi \in B^4 \), \( D_\mu \varphi = \partial_\mu \varphi + \varphi A_\mu \) and \( \varphi A_\mu = -\frac{1}{2} e_5 A_\mu^k \varphi \lambda_k \) as introduced above.

**Remark**

Comparing the equation \( D\varphi e_5 = m\varphi \) as dealt with in Section 4 and (36) one observes that \( D\varphi e_5 = m\varphi \) of Section 4 obeys Lorentz invariance because it is manifestly independent of coordinates, but for (36) this Lorentz invariance is not automatically satisfied. The Lorentz invariance of (36) can be proved in a similar way as done in textbooks for the conventional equation of Dirac.
For reasons of completeness we add this proof here.

We start from

\[(\varepsilon_s \Gamma^\mu D_\mu - m) \varphi(x) = 0 .\] (36)

The Lorentz transformation \( \hat{x} = Lx \) induces the transformations \( x = L^{-1}\hat{x} \) and \( \hat{D}_\mu = L^\nu_\mu D_\nu \). Substitution in (36) yields:

\[\varepsilon_s (\Gamma^\mu L^\nu_\mu) \hat{D}_\nu \varphi(L^{-1}\hat{x}) = m\varphi(L^{-1}\hat{x}) .\] (36')

For the quantities \( L^\mu_\nu \Gamma^\mu = \alpha^\nu \) one finds

\[\alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = L^\mu_\alpha \Gamma^\alpha L^\nu_\beta \Gamma^\beta + L^\mu_\beta \Gamma^\beta L^\nu_\alpha \Gamma^\alpha = \]

\[L^\mu_\alpha L^\nu_\beta (\Gamma^\alpha \Gamma^\beta + \Gamma^\beta \Gamma^\alpha) = L^\mu_\alpha L^\nu_\beta 2\eta^{\alpha\beta} = 2\eta^{\mu\nu} .\]

Using the relation \( \hat{\Gamma}^\mu = U \Gamma^\mu U^{-1} \) this yields \( \alpha^\mu = U \Gamma^\mu U^{-1} \) whence after substitution in (36'):

\[\varepsilon_s U \Gamma^\nu U^{-1} \hat{D}_\nu \varphi(L^{-1}\hat{x}) = m\varphi(L^{-1}\hat{x}) \]

or

\[\varepsilon_s \Gamma^\nu \hat{D}_\nu \varphi(L^{-1}\hat{x}) = m U^{-1} \varphi(L^{-1}\hat{x}) .\]

Defining \( \tilde{\varphi}(\hat{x}) = U^{-1} \varphi(L^{-1}\hat{x}) \) one finds

\[\varepsilon_s \Gamma^\nu \hat{D}_\nu \tilde{\varphi}(\hat{x}) = m \tilde{\varphi}(\hat{x}) \]

or

\[(\varepsilon_s \Gamma^\mu \hat{D}_\mu - m) \tilde{\varphi}(\hat{x}) = 0 \] (36'')

and the proof is complete. \( \Box \)
Final remarks

1. Let $U = e^{-\frac{1}{2}e_3 \alpha^k \lambda_k}$ and define

$$
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4
\end{pmatrix}
= 
\begin{pmatrix}
  b_1 U \\
  b_2 U \\
  b_3 U \\
  b_4 U
\end{pmatrix}.
$$

Now the gauge transformations in STA-formalism can be given by

$$
\begin{aligned}
\hat{\varphi} &= \varphi U \\
\hat{A}_\mu &= U^{-1} A_\mu U - U^{-1} (\partial_\mu U)
\end{aligned}
$$

2. Let $\varphi^I = (B_1^I, B_2^I, B_3^I, B_4^I)$ then we can define the currents by

$$
j^\mu_k = \frac{i}{2} (\varphi^I \Gamma_0 \Gamma^k \varphi \lambda_k) .
$$

This expression corresponds to

$$
j^\mu_k = \frac{i}{2} \text{tr}(\Psi^\dagger \gamma^\mu \Psi \lambda_k)
$$

mentioned in 5.2.

3. The Lagrangian $L_1$ can be defined by

$$
L_1 = (\varphi^I \Gamma_0 (e^5 \Gamma^\mu \partial_\mu - m) \varphi)_0
$$

in agreement with

$$
L_1 = \text{tr} \Psi^\dagger \gamma_0 (i\gamma^\mu \partial_\mu - m) \Psi
$$

in conventional theory.

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References


