Solution to Problem 94-20: Overlapping binary sequences

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Zeros in the Unit Disk

*Problem 94-19**, by M. MENEGUETTE (São Paulo State University (UNESP), Presidente Prudente, SP, Brazil).

Consider the polynomial

$$P(z) = \sum_{k=0}^n a_k z^k, \quad 0 < a_0 \leq a_1 \leq \cdots \leq a_{n-1}, \quad a_n > 0.$$

Prove or disprove that if the roots of $P(z) = 0$ lie in the unit disk and $na_n > (n-1)a_{n-1}$, then the zeros of the perturbed polynomial $P_d(z) = P(z) + dz^n$ lie in the unit disk for any $d > 0$.

This problem occurs in connection with the stability of a class of methods for the numerical solution of ordinary differential equations.

Comment by VINCENT BLONDEL (INRIA, Le Chesnay, France).

By the Eneström–Kakeya Theorem [1, Theorem 30.3] the zeros of the perturbed polynomial lie in the unit disk for any d with $d \geq a_n/(n-1)$.

REFERENCE

- [1] M. MARDEN, *Geometry of Polynomials*, Amer. Math. Soc., Providence, RI, 1966.

Overlapping Binary Sequences

Problem 94-20 by G. BLOM (Lund University and Lund Institute of Technology, Lund, Sweden).

A binary sequence $b_1, b_2, \dots, b_n$ is said to be *overlapping* if, for at least one $j \leq n-1$, the last j terms are equal to the first j terms; that is, $b_i = b_{n-j-i}$ for $i = 1, 2, \dots, j$. Let a_n be the number of overlapping binary sequences of length n . Find recurrence relations for the sequence (a_n) .

Solution by O. P. LOSSERS (Technical University Eindhoven, Eindhoven, the Netherlands).

First we prove that if there is an overlap of length j ($1 \leq j < n$), then there is an overlap of length at most $n/2$. Indeed, suppose $b_i = b_{i+(n-j)}$, for all i such that $1 \leq i < i+(n-j) \leq n$. Then $b_i = b_{i+k(n-j)}$ for all i and k such that $1 \leq i < i+k(n-j) \leq n$. This shows the existence of an overlap of length $n - k(n-j)$, which can be made smaller than $n/2 + 1$ by an appropriate choice of k .

Let $f(n)$ be the number of binary sequences of length n having no overlap. We can count the number of overlapping sequences by classifying them according to the length of the shortest overlap, where we notice that the shortest overlap by itself is a sequence without overlap. We have

$$\begin{aligned} a_{2n} &= f(1) \cdot 2^{2n-2} + f(2) \cdot 2^{2n-4} + \cdots + f(n) \cdot 2^0, \\ a_{2n+1} &= f(1) \cdot 2^{2n-1} + f(2) \cdot 2^{2n-3} + \cdots + f(n) \cdot 2^1. \end{aligned}$$

From these expressions it is easily seen that $a_{2n+1} = 2a_n$ and $a_{2n+2} = 4a_{2n} + f(n+1)$. Eliminating f results in the recurrence relations

$$\begin{aligned} a_1 &= 0, \\ a_{2n} &= 2a_{2n-1} - a_n + 2^n, \\ a_{2n+1} &= 2a_{2n}. \end{aligned}$$

Also solved by R. J. CHAPMAN (University of Exeter, Exeter, United Kingdom), H. HARBORTH (Technical University, Braunschweig, Germany), J. H. VAN LINT (Eindhoven University of Technology, Eindhoven, the Netherlands), HANS E. DE MEYER (University of Ghent, Ghent, Belgium), and the proposer.

Editorial note. H. Harborth points out that this problem is considered in [1], where the appropriate recurrence relations are given in (34). R. J. Chapman notes that the result generalizes to an alphabet of q letters: $a_1 = 0$, $a_{2n} = qa_{2n-1} - a_n + q^n$, $a_{2n+1} = qa_{2n}$. As noted by J. H. van Lint, the recurrence relations show that $p_n = a_n q^{-n}$, the overlap probability of a random sequence of length n based on an alphabet of q letters, satisfies

$$\begin{aligned} p_1 &= 0, \\ p_{2n} &= p_{2n-2} + (1 - p_n)q^{-n}, \\ p_{2n+1} &= p_{2n}, \end{aligned}$$

from which we have

$$p_{2n} = \sum_{k=1}^n (1 - p_k)q^{-k},$$

and $\lim_{n \rightarrow \infty} p_n = (q - 1)^{-1} - \sum_{k=1}^{\infty} p_k q^{-k}$. A series expansion for the limiting probability can be found by introducing the generating function $F(x) = \sum_{k=1}^{\infty} p_k x^k$ and noting that the recurrence relations imply

$$\frac{F(x)}{1+x} = \sum_{n=1}^{\infty} p_{2n} x^{2n} = \sum_{n=1}^{\infty} \left(p_{2n-2} + \frac{1-p_n}{q^n} \right) x^{2n} = \frac{x^2 F(x)}{1+x} + \frac{x^2}{q-x^2} - F\left(\frac{x^2}{q}\right),$$

and thus

$$(1-x)F(x) + F\left(\frac{x^2}{q}\right) = \frac{x^2}{q-x^2}.$$

By iteration,

$$\begin{aligned} \lim_{n \rightarrow \infty} p_n &= \frac{1}{q-1} - F(q^{-1}) = \frac{1}{q-1} - \frac{q}{(q-1)} \left[\frac{1}{q^3-1} - F(q^{-3}) \right] \\ &= \frac{1}{q-1} - \frac{q}{(q-1)(q^3-1)} + \frac{q^4}{(q-1)(q^3-1)} \left[\frac{1}{q^7-1} - F(q^{-7}) \right] = \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{1+3+\dots+(2^k-1)}}{(q-1)(q^3-1)(q^7-1)\dots(q^{2^{k+1}-1}-1)}. \end{aligned}$$

For $q = 2$ this series yields the numerical value $\lim_{n \rightarrow \infty} p_n = 0.73221315978211\dots$

REFERENCE

- [1] H. HARBORTH, *Endliche 0-1-Folgen mit gleichen Teilböcken*, J. Reine Angew. Math., 271 (1974), pp. 139-154.