Graphs related to exceptional root systems

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Published: 01/01/1976

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Graphs related to exceptional root systems

by

F.C. Bussemaker, D.M. Cvetković, J.J. Seidel

T.H.-Report 76-WSK-05

September 1976
Abstract

Let $G$ denote the set of all connected regular graphs whose adjacency matrix has least eigenvalue $-2$, and which is neither a line graph nor a cocktail-party graph. The present report contains the following results. Any graph in $G$ is switching equivalent to a line graph. Any graph in $G$, except for 5 graphs on 22 vertices with degree 9, is an induced subgraph of one of the 3 Chang graphs or of the Schlafli graph. $G$ contains 187 graphs, 68 of which are cospectral to a line graph. As a consequence of these results, known spectral characterizations of the connected regular line graphs, of the flag graphs of 2-designs, and of the line graphs of complete bipartite graphs, are made more precise. The methods consist of a mixture of mathematical reasoning and computer search, on the basis of the root system paper by Cameron c.s.
Graphs related to exceptional root systems

by

F.C. Bussemaker, D.M. Cvetković, J.J. Seidel

1. Introduction and main results

Among the regular graphs having least eigenvalue -2, are the regular line graphs and the cocktail-party graphs. It is the aim of the present paper to determine all remaining graphs.

Definition 1.1. G is the set of all connected regular graphs, whose adjacency matrix has least eigenvalue -2, and which are neither line graph nor cocktail-party graph.

Hoffman [17] posed the problem of determining G. He and Ray-Chaudhuri showed [18], that graphs in G cannot have degree ≥ 17:

Theorem 1.2. ([18], [2]). Any graph in G has at most 28 vertices and has degree at most 16.

Recently, Cameron, Goethals, Seidel and Shult [2] observed that the graphs in G correspond to sets of unit vectors at angles 60° and 90°, which are contained in well-defined sets (the root systems) in Euclidean space of dimensions 6, 7, or 8. It is on the basis of this correspondence that we will determine all graphs in G, partly by aid of a computer search. As a consequence, certain characterization theorems appearing in the literature can be made more precise. Our results are collected in the following theorems (which contain notions to be recalled later).

Theorem 1.3. For each G ∈ G there exists a graph H, with at most 8 vertices, such that G is switching equivalent to the line graph of H.

Theorem 1.4. There exist exactly 68 regular graphs which are not line graphs but which are cospectral to a line graph. These graphs, all contained in G, are displayed in table 1.5.

*) An announcement of the results of the present Report, under the same title, will appear in Proc. 5th Hungarian Colloquium on Combinatorics, Keszthely, 1976.
Table 1.5.

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<td>4</td>
<td>6</td>
<td>L(Q)</td>
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<td>10</td>
<td>153, 154, 155, 156, 157, 158, 159, 160</td>
<td>L(CP(4))</td>
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<tr>
<td>3</td>
<td>28</td>
<td>12</td>
<td>161, 162, 163</td>
<td>L(K₈)</td>
</tr>
</tbody>
</table>

(Each row of the table contains a certain number k of graphs G. The first column gives k; the second one the number n of vertices of G and the third one the degree d of G. Graphs G are given in the fourth column by their identification numbers which refer to table 9.1. In the fifth column a regular graph L(H) is given to which all graphs G from the considered row are switching equivalent and cospectral. For some graphs H we refer to fig. 1.6. The complete bipartite graph on p + q vertices is denoted by Kₚ₉. The complement of any graph H is denoted by $\overline{H}$. The cocktail-party graph CP(n) is the unique regular graph on 2n vertices of degree 2n - 2.)
Theorem 1.7. Let $L(G_1), L(G_2)$ denote cospectral, connected, regular line graphs of the connected graphs $G_1, G_2$. Then one of the following holds:

(i) $G_1$ and $G_2$ are cospectral regular graphs with the same degree,

(ii) $G_1$ and $G_2$ are cospectral semiregular bipartite graphs with the same parameters,

(iii) $\{G_1, G_2\} = \{H_1, H_2\}$, where $H_1$ is regular and $H_2$ is semiregular bipartite; in addition there exist integers $s > 1$ and $0 < t \leq \frac{1}{s}$, and reals $0 \leq \lambda_i < \frac{t\sqrt{s^2-1}}{2}$, $i = 2, 3, \ldots, \lfloor s(s-1) \rfloor$, such that $H_1$ has $s^2-1$ vertices, degree $st$, and the eigenvalues

$$st, \pm \sqrt{\lambda_i^2 + t^2}, -t \text{ (of multiplicity } s)$$
$H_2$ has $s^2$ vertices, parameters $n_1 = \frac{1}{2}s(s+1)$, $n_2 = \frac{1}{2}s(s-1)$, $d_1 = t(s-1)$, $d_2 = t(s+1)$, and the eigenvalues

$$\pm t\sqrt{s^2-1}, \pm \lambda_1, 0$$ (of multiplicity $s$).

Theorems 1.4 and 1.7 represent a spectral characterization for regular connected line graphs. Theorem 1.4 says that only 17 regular connected line graphs (those of the fifth column of Table 1.5) have cospectral mates which are not line graphs. Nonisomorphic line graphs may have the same spectrum, and theorem 1.7 classifies the possibilities. When restricted to special classes of graphs, the theorems have the following consequences.

Theorem 1.8. Let $D_1$ be a 2-design with parameters $v,k,b,r,\lambda$ and flag graph $G_1 = F(D_1)$. Let $G_2$ be a graph with the same spectrum as $G_1$. Then one of the following holds:

(i) $G_2 = F(D_2)$, where $D_2$ is a 2-design having the same parameters as $D_1$;
(ii) $(v,k,b,r,\lambda) = (3,3,6,4,2)$ and $G_2$ is graph nr. 6;
(iii) $(v,k,b,r,\lambda) = (4,3,4,3,2)$ and $G_2$ is graph nr. 9;
(iv) $(v,k,b,r,\lambda) = (4,4,4,4,4)$ and $G_2$ is graph nr. 69;
(v) $(v,k,b,r,\lambda) = (3,3,6,6,6)$ and $G_2$ is graph nr. 70;
(vi) $v = \frac{1}{2}s(s+1)$, $k = t(s-1)$, $b = \frac{1}{2}s(s+1)$, $r = t(s+1)$, $\lambda = \frac{2t(st-t-1)}{s^2}$, where $s$ and $t$ are integers with $st$ even, $t \leq \frac{1}{2}s$, $(s-2)|2t(t-1)$, and $G_2 = L(H)$ where $H$ is a regular graph on $s^2-1$ vertices with the eigenvalues

$$st, \pm \sqrt{ts(s-1-t)(s-2)^{-1}}, -t$$

of multiplicities $1, \frac{1}{2}(s-2)(s+1), \frac{1}{2}(s-2)(s+1)$, $s$, respectively.

Theorem 1.9. $L(K_{m,n})$ is characterized by its spectrum unless

(i) $m = n = 4$, where graph nr. 69 provides the only exception,
(ii) $m = 6$, $n = 3$, where graph nr. 70 provides the only exception,
(iii) $m = 2t^2 + t$, $n = 2t^2 - t$, and there exists a symmetric Hadamard matrix with constant diagonal of order $4t^2$.  

Theorem 1.10. Each graph from $G$ is an induced subgraph of one of the 3 Chang graphs or of the Schlafli graph, except for 5 graphs on 22 vertices with degree 9. These exceptions are the graphs nr. 148, nr. 149, nr. 150, nr. 151, nr. 152. They are mutually switching equivalent.

Theorem 1.11. $G$ contains 187 graphs. They are displayed in table 9.1.

Theorem 1.12. Any regular connected graph with least eigenvalue -2 is either a line graph, or a cocktail party graph, or one of the 187 graphs displayed in table 9.1.

Parts of theorem 1.7 occur in Doob [8], and Cameron c.s. [2]. The characterization problems of theorems 1.8 and 1.9 have been considered earlier by Shrikhande [28], Hoffman-Ray Chaudhuri [16], Doob [7], [10], Cvetković [4], Cameron c.s. [2].
2. Line graphs, root systems and switching

The line graph \( L(H) \) of any graph \( H \) is defined as follows. The vertices of \( L(H) \) are the edges of \( H \); two vertices of \( L(H) \) are adjacent whenever the corresponding edges of \( H \) have a vertex of \( H \) in common. Let \( N \) denote the vertex-edge \((0,1)\)-incidence matrix of \( H \). Then the \((0,1)\) adjacency matrices \( B \) of \( H \), and \( A \) of \( L(H) \) satisfy

\[
NN^T = D + B, \quad N^T N = 2I + A,
\]

where \( D \) is the diagonal matrix whose diagonal entries are the vertex degrees of \( H \). It follows (cf. [15]) that the adjacency matrix of any line graph has least eigenvalue \(-2\), provided the original graph has more edges than vertices.

Also the adjacency matrix of the complement of any regular graph of degree 1, which we shall call a cocktail-party graph, has least eigenvalue \(-2\).

Let \( G \) be any graph on \( n \) vertices whose \((0,1)\)-adjacency matrix \( A \) has least eigenvalue \(-2\), of multiplicity \( p \) (we shall say that \( G \) has least eigenvalue \(-2\)). The matrix \( I + \frac{1}{4}A \) is symmetric, has ones on the diagonal, and is positive semidefinite of rank \( n - p =: r \). Hence \( I + \frac{1}{4}A \) may be interpreted as the Gram matrix of (the inner product of) \( n \) unit vectors in Euclidean space \( \mathbb{R}^r \) of \( r \) dimensions. These vectors mutually have the angles 60° (if the corresponding vertices of \( G \) are adjacent) or 90° (otherwise). The vectors span a set of lines at 60° and 90° in \( \mathbb{R}^r \).

Now all maximal sets of lines at 60° and 90° have been determined in [2]. Before we state the corresponding theorem we need some definitions. Let \( B^r = \{e_1, \ldots, e_n\} \) be any orthonormal basis for \( \mathbb{R}^n \).

**Definition 2.1.** The root system \( A_n, n \geq 1 \), is the set of lines spanned by the vectors

\[ e_i - e_j \text{ (} i > j; \ i,j = 1, \ldots, n+1 \text{).} \]

The root system \( D_n, n \geq 4 \), is the set of lines spanned by the vectors

\[ e_i + e_j \text{ (} i > j; \ i,j = 1, \ldots, n \text{).} \]
Definition 2.2. The root system $E_8$ is the set of lines in $\mathbb{R}^8$ spanned by the vectors spanning $D_8$ and the following vectors

$$\sum_{i=1}^{8} e_i e_i = 8, \quad \sum_{i=1}^{8} e_i = 1.$$

The root system $E_7$ is the subsystem of $E_8$ consisting of the lines orthogonal to any one of its lines. The root system $E_6$ is the subsystem of $E_8$ consisting of the lines orthogonal to any two of its lines having angle 60°.

A system of lines is indecomposable if it cannot be partitioned into two disjoint, nonempty, mutually orthogonal subsystems of lines. A system of lines is maximal in a given space if it cannot be extended by new lines in the same space.

Theorem 2.3. (Cameron c.s. [2]). The only indecomposable maximal sets of lines at 60° and 90° in $\mathbb{R}^n$ are the root systems:

1) $A_n$, $D_n$ and $E_8$ for $n > 8$;
2) $E_8$ for $n = 8$;
3) $D_7$ and $E_7$ for $n = 7$;
4) $A_6$, $D_6$ and $E_6$ for $n = 6$;
5) $A_n$ and $D_n$ for $n < 6$.

Definition 2.4. A graph $G$ with the $(0,1)$-adjacency matrix $A$ is represented by a root system $X$, if there exists a set of vectors with Gram matrix $I + \frac{1}{2}A$ whose lines form a subset of $X$.

Theorem 2.5 (Cameron c.s. [2]). Any regular connected graph with least eigenvalue -2, is represented by the exceptional root system $E_8$, or/and by $D_n$, for some $n$, in which case the graph is a line graph or a cocktail-party graph.

Thus, in order to determine the set $G$ of graphs of definition 1.1, we have to investigate the root system $E_8$.

The root systems $E_i$ themselves also yield graphs, to be denoted by $G(E_i)$. Their vertices correspond to the lines of $E_i$, and two vertices are adjacent if and only if the corresponding lines are not orthogonal.
Theorem 2.6 (cf. [25]). The graphs $G(E_i)$ are strongly regular, with order $n$, eigenvalues $\lambda_j$, and multiplicities $\mu_j$, as follows:

- $G(E_6): n = 36, \lambda_1 = 20, \lambda_2 = 2, \lambda_3 = -4, \mu_1 = 1, \mu_2 = 20, \mu_3 = 15$
- $G(E_7): n = 63, \lambda_1 = 32, \lambda_2 = 4, \lambda_3 = -4, \mu_1 = 1, \mu_2 = 27, \mu_3 = 35$
- $G(E_8): n = 120, \lambda_1 = 56, \lambda_2 = 8, \lambda_3 = -4, \mu_1 = 1, \mu_2 = 35, \mu_3 = 84$

The last graph is denoted by $0^-(8,2)$.

Definition 2.7. The eigenvalues of a graph different from the least one are principal eigenvalues of the graph.

Proposition 2.8. The number of principal eigenvalues $r$ of any graph $G$ from $G$ is equal to 6, 7, or 8.

Proof. Recall that $r$ is the smallest dimension of an Euclidean space in which $G$ can be represented by a set of vectors at 60° and 90°. If $r > 8$, then $G$ cannot be represented in $E_8$. This means that it can be represented in some $D_n$. According to theorem 2.5, $G$ is then a line graph or cocktail-party graph. If $r < 6$, then $G$ can be represented in some $D_n$ with $n \leq 5$ and we have the same conclusion. Hence only the cases $r = 6, 7, 8$ remain.

This completes the proof.

The root system $E_8$ consists of 120 lines in $\mathbb{R}^8$. Along each line, in opposite directions, we take 2 vectors at length 2. If $B = \{e_1, \ldots, e_8\}$ denotes an orthonormal basis of $\mathbb{R}_8$, then the 240 vectors may be represented as follows:

- type a: 28 vectors of the form $2e_i + 2e_j; i, j = 1, \ldots, 8, i > j$;
- type a': 28 vectors opposite to those of type a;
- type b: 28 vectors of the form $-2e_i - 2e_j + \sum_{k=1}^8 e_k$;
- type b': 28 vectors opposite to those of type b;
- type c: 56 vectors of the form $2e_i - 2e_j; i, j = 1, \ldots, 8, i \neq j$;
- type d: 70 vectors of the form $-2e_i - 2e_j - 2e_k - 2e_\ell + \sum_{s=1}^8 e_s$ with distinct $i, j, k, \ell \in \{1, \ldots, 8\}$;
- type e: 2 vectors $j := \sum_{i=1}^8 e_i$, and $-j$.

Note that the root system $E_8$ is transitive on lines.

The triangular graph $T(8) := L(K_8)$ can be represented by all vectors of type a, or by all vectors of type b. Indeed, in both cases the Gram matrix of the
vectors equals $4I + 2A$, where $A$ is the adjacency matrix of $T(8)$. Replacing vectors of type $a$ by the corresponding vectors of type $b$ amounts to switching of $T(8)$ with respect to those vectors in the following sense (cf. [19], [26]).

Let the vertex set of a graph $G$ be partitioned into any disjoint subsets $U$ and $V$. Switching $G$ with respect to this partition means deleting the existing edges and adding the nonexisting edges between $U$ and $V$. Switching is an equivalence relation on the set of all graphs with a given number of vertices. Switching equivalent graphs have the same spectrum of their $(-1, 1, 0)$-adjacency matrix.

**Lemma 2.9.** (Seidel [27]). For any graph in a switching class of graphs with an even number $n$ of vertices, the dissection of $n$ into the numbers of vertices with an even and with an odd vertex degree is the same.

**Lemma 2.10.** (cf. [26]). For any graph and any of its vertices there exists a unique switching equivalent graph which has that vertex as an isolated vertex.

**Lemma 2.11.** Let $I(G)$ be the collection of graphs obtained by isolating any one vertex of the graph $G$. The graphs $G_1$ and $G_2$ are switching equivalent if and only if $I(G_1) = I(G_2)$.

**Proof.** If $I(G_1)$ and $I(G_2)$ contain a common graph, the graphs $G_1$ and $G_2$ are switching equivalent. Suppose now $G_1$ and $G_2$ are switching equivalent. Label the vertices of $G_1$ and $G_2$ by $x_1, \ldots, x_n$ in an arbitrary way. There exist a switching of $G_1$ such that the graph $G_1'$ obtained after switching and $G_2$ are isomorphic. Let $\varphi$ be any isomorphism. Consider the vertices $x_i$ and $\varphi(x_i)$. Of course, by isolating $x_i$ in $G_1'$ and $\varphi(x_i)$ in $G_2$ we get the same graphs. But also according to lemma 2.10 the graph obtained by isolating $x_i$ in $G_1$ is the same as the graph obtained by isolating $x_i$ in $G_1'$. Therefore, $I(G_1) = I(G_2)$.

We shall now describe the graphs occurring in the following theorem.
Theorem 2.12. (Seidel [24]). The only graphs having 3 distinct eigenvalues, the least one being -2, are the following:

- the cocktail-party graphs;
- $L(K_{n,n})$ and the Shrikhande graph on 16 vertices;
- $L(K_n)$ and the 3 Chang graphs on 28 vertices;
- the graphs of Petersen, Clebsch, Schlafli on 10, 16, 27 vertices.

The triangular graph $T(8) = L(K_8)$ can be represented by fig. 2.13a. The picture contains 8 lines and two vertices are adjacent if and only if they are on the same line. The three Chang graphs (pseudo-triangular graphs) $T'(8), T''(8), T'''(8)$ are represented by fig. 2.13b,c,d. They are switching equivalent (and cospectral) to $L(K_8)$. The switching sets are indicated by white circles. They represent 4 independent vertices, the vertices of an octagon, and the vertices of an independent triangle and pentagon, respectively. Moreover we have the following theorem:
Theorem 2.14. (Seidel [27]). The result of switching $T(8)$ with respect to any $4k$ ($k$ positive integer) vertices inducing the line graph of a regular graph, is again $T(8)$ or one of Chang graphs. Chang graphs can be obtained by switching $T(8)$ only in this way.

The Schlafli graph can be defined as the graph obtained by isolating any vertex of $T(8)$ (see Fig. 5.4), and the deleting that vertex. This graph is regular of degree 16, has the smallest eigenvalue $-2$, and is not a line graph.

The graph $L(K_{4,4})$, the Shrikhande graph, and the Clebsch graph are switching equivalent. They are represented in Fig. 2.15a, b, c respectively.

![Diagram](image)

Fig. 2.15.

For another definition of all these graphs the reader is referred to [24]. The Chang graphs, the Schlafli graph, the Clebsch graph and the Shrikhande graph belong to $G$. The Petersen graph is one of the 5 cubic graphs belonging to $G$, cf. the table of cubic graphs [1]. They are displayed in fig. 2.16.
Some other graphs from $G$ are known, too. One of them is the exceptional graph in characterizing symmetric block designs found by Hoffman and Ray-Chaudhuri [16] (first graph in fig. 2.17). The other two are the total graphs $T(C_4)$ and $T(C_6)$ of the circuits of length 4 and 6, respectively, as was noted by Cvetković [5].
Notice, that some graphs from $G$ have the same spectrum as a line graph, and others have not. For example, the Chang graphs are cospectral with $L(K_3)$ and the Schläfli graph is not cospectral to any line graph.
3. Additional results used in the further text

In our proofs we need several other results, which will be surveyed in this section.

**Theorem 3.1** (Interlacing theorem; cf. [20], p. 149). Let $A$ be a hermitian matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$), and let $B$ one of its principal submatrices; $B$ has the eigenvalues $\mu_1, \ldots, \mu_m$ ($\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$). Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ ($i = 1, \ldots, m$) hold.

The inequalities from this theorem are known also as the Cauchy inequalities.

**Theorem 3.2** (cf. [14]). Let $A$ be a real symmetric matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$). Given a partition \(\{1, 2, \ldots, n\} = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m\), with $|\Delta_i| = n_i > 0$, consider the corresponding blocking $A = [A_{ij}]$, so that $A_{ij}$ is an $n_i \times n_j$ block. Let $e_{ij}$ be the sum of the entries in $A_{ij}$ and put $\hat{A} = [e_{ij}/n_i]$ (i.e. $e_{ij}/n_i$ is the average row sum in $A_{ij}$), then the spectrum of $\hat{A}$ is contained in the segment $[\lambda_1, \lambda_n]$.

We shall describe now two theorems about graphs contained in the root system $E_8$, which have been proved by Cameron, c.s. [2].

**Theorem 3.3.** A graph represented by a subset of the root system $E_8$ has at most 36 vertices, and its maximum vertex degree is at most 28.

**Proof.** As mentioned earlier the graphs in $E_8$ are induced subgraphs of the graph $0^{-}(8, 2)$. According to theorem 2.6, $0^{-}(8, 2)$ has the eigenvalues $56, 8, -4$ with the multiplicities $1, 35, 84$ respectively. If a graph in $E_8$ would have more than 36 vertices then, according to theorem 3.1, it would have least eigenvalue $-4$, which is in contradiction to the fact that its least eigenvalue is not smaller than $-2$.

Because of the transitivity of $E_8$ we can represent a graph in $E_8$ such that its vertex with the maximum vertex degree is represented by the vector $j = (1, \ldots, 1)$. The vertices adjacent to that vertex are then represented by
vectors of types $a$ or $b$. There are 28 vectors of each type. Form 28 pairs of vectors each pair containing a vector of the type $a$ and the corresponding vector of the type $b$ (e.g. $(2,2,0,0,0,0,0,0)$ and $(-1,-1,1,1,1,1,1,1)$). If the maximum vertex degree were greater than 28, then there would be at least one pair in which both vectors would represent some vertices in the graph. But that is impossible since the inner product of the vectors from each pair is equal to $-4$. This completes the proof.

Theorem 3.4 (Cameron, c.s. [2]). A regular graph represented by a subset of $E_8$ has at most 28 vertices, and has degree at most 16.

A proof of this theorem will be given in section 5.

Theorem 3.5 (Doob, [9]). The line graph of a connected graph $G$ has least eigenvalue $\geq -2$. Equality holds if and only if $G$ contains an even circuit or two odd circuits.

Theorem 3.6 (Doob, [9]). The multiplicity of the eigenvalue $-2$ in the line graph of a graph $G$ is equal to the maximal number of independent even circuits in $G$.

Corollary 3.7. Let $G$ be a connected graph having $n$ vertices and $m$ edges. Then the multiplicity of the eigenvalue $-2$ in $L(G)$ equals $m-n+1$ if $G$ is bipartite, and $m-n$ otherwise.

Proof. The maximal number of independent circuits in $G$ is $m-n+1$. If $G$ is bipartite, all circuits are even and we have proved the first part. If $G$ is not bipartite, then we certainly do not have $m-n+1$ independent even circuits, and the multiplicity of $-2$ in $L(G)$ is less than $m-n+1$. On the other hand the adjacency matrix of $L(G)$ can be expressed as $N^T N - 2I$, where $N$ is the vertex edge incidence matrix of $G$. We have $m \geq n$, since otherwise $G$ would be a tree, hence bipartite. $N^T N$ has at least $m-n$ eigenvalues equal to 0, and the multiplicity of $-2$ in $L(G)$ is at least $m-n$, whence it equals $m-n$. This completes the proof.
Corollary 3.7 was noticed by Sachs for regular graphs [23].

The characteristic polynomial of the \((0,1)\)-adjacency matrix of a graph \(G\) will be denoted by \(P_G(\lambda)\).

**Lemma 3.8** (Sachs, [23]). If \(G\) is a regular graph of degree \(d\) with \(n\) vertices and \(m\) (\(= \frac{1}{2} nd\)) edges, then

\[
P_L(G)(\lambda) = (\lambda + 2)^{m-n} P_G(\lambda - d + 2).
\]

A graph is called semiregular bipartite with parameters \(n_1, n_2, d_1, d_2\) if it is bipartite on \(n_1 + n_2\) vertices, with vertex degrees \(d_1\) in the first part and \(d_2\) in the second one.

**Lemma 3.9** (Cvetković, [4]). If \(G\) is a semiregular bipartite graph with the parameters \(n_1, n_2, d_1, d_2\) (\(n_1 \geq n_2\)), then

\[
P_L(G)(\lambda) = (\lambda + 2)^{\beta} \sqrt{-\frac{\alpha_1}{\alpha_2}}^{n_1-n_2} P_G(\sqrt{\alpha_1} \alpha_2) P_G(-\sqrt{\alpha_1} \alpha_2),
\]

where \(\alpha_i = \lambda - d_i + 2\) (\(i = 1, 2\)) and \(\beta = n_1 d_1 - n_1 - n_2\).

**Lemma 3.10** (Doob, [7]). If \(m = 2\) or \(m = n = 1\), then the graph \(L(K_m, n)\) is characterized by its spectrum.

**Lemma 3.11** (Finck, Grohman, [11]). Let \(G_1 \vee G_2\) (the \(\vee\)-product of graph \(G_1\) and \(G_2\)) be the graph obtained by joining by edges each vertex of \(G_1\) and each vertex of \(G_2\). If \(G_1\) and \(G_2\) are regular of degrees \(d_1\) and \(d_2\), and if they have \(n_1\) and \(n_2\) vertices, respectively, then

\[
P_{G_1 \vee G_2}(\lambda) = \frac{P_{G_1}(\lambda) P_{G_2}(\lambda)}{(\lambda - d_1)(\lambda - d_2)(\lambda - d_1 - n_1)(\lambda - d_2 - n_2)}.
\]

**Lemma 3.12** (Sachs, [22]). Let \(\lambda_1 = d, \lambda_2, \ldots, \lambda_n\) be the eigenvalues of a regular graph of degree \(d\). Then the complement \(\overline{G}\) has the eigenvalues \(n - 1 - d, -\lambda_2 - 1, \ldots, -\lambda_n - 1\).
Lemma 3.13 (Heilbronner, [13]). Let \( x \) be a vertex in a graph \( G \) which is adjacent only to the vertex \( y \). Then
\[
P_G(\lambda) = \lambda p_{G-\{x\}}(\lambda) - p_{G-\{x,y\}}(\lambda).
\]

Lemma 3.14 (Collatz, Sinogowitz, [3]). If \( \lambda \) is an eigenvalue of a bipartite graph \( G \), then \(-\lambda\) is also an eigenvalue of \( G \), with the same multiplicity.

Two graphs are cospectral if they have the same spectrum of the \((0,1)\)-adjacency matrix.

Lemma 3.15. Two regular graphs of the same degree are cospectral if and only if their \((-1,1,0)\)-adjacency matrices are cospectral.

Lemma 3.16. If \( G \) is a semiregular bipartite graph with parameters \( n_1, n_2, d_1, d_2 \) \((n_1 > n_2)\) and if \( \lambda_1, \lambda_2, \ldots, \lambda_{n_2} \) are first \( n_2 \) largest eigenvalues of \( G \), then
\[
P_{L(G)}(\lambda) = (\lambda - d_1 + 2)^{n_1 - n_2} (\lambda + 2)^{n_2} \prod_{i=2}^{n_2-1} ((\lambda - d_1 + 2)^i - (\lambda - d_2 + 2) - \lambda_2^i) .
\]

Proof. It is easy to see that \( \lambda_1 = \sqrt{d_1 d_2} \) and that the spectrum of \( G \) contains at least \( n_1 - n_2 \) eigenvalues equal to 0. Having in mind that by lemma 3.14 the spectrum of a bipartite graph is symmetric with respect to 0, we get lemma 3.16 from lemma 3.9 by straightforward calculation.

Lemma 3.17. If \( G \) is connected semiregular bipartite then the parameters of \( G \) are determined from the spectrum of \( L(G) \).

Proof. Let \( n_1, n_2, d_1, d_2 \) be parameters of \( G \). Then \( L(G) \) has \( n_1 d_1 = n_2 d_2 \) vertices, is regular of degree \( d_1 + d_2 - 2 \) and has the number -2 in the spectrum with the multiplicity \( n_1 d_1 - n_1 - n_2 + 1 \). Since all mentioned quantities can be determined from the spectrum of \( L(G) \), we have a system of equations from which the parameters \( n_1, n_2, d_1, d_2 \) can be determined uniquely. This proves the lemma.
4. Graphs from $G$ with $n > 2(d + 2)$

Using theorem 3.2 we shall first prove a general theorem which provides some bounds for graphs contained in $E_8$. The theorem also has some further consequences not directly related to our problem.

**Theorem 4.1.** Let $G$ be a regular graph of degree $d$ with $n$ vertices and with the eigenvalues $\lambda_1 = d, \lambda_2, \ldots, \lambda_n$. Let $G_I$ be an induced (not necessarily regular) subgraph of $G$ having $n_I$ vertices and average (arithmetical) value of vertex degrees $d_I$. Then

$$
\lambda_n \leq d_I - \frac{(d - d_I)n_I}{n - n_I} \leq \frac{n_I(d - \lambda_2)}{n} + \lambda_2.
$$

**Proof.** Partition the vertex set of $G$ into set of vertices of $G_I$ and the set of remaining vertices. Divide the adjacency matrix of $G$ into blocks according to that partition of vertices. The average values of row sums of blocks form the following matrix

$$
\begin{bmatrix}
    d & d - d_I \\
    (d - d_I)n_I & d - \frac{(d - d_I)n_I}{n - n_I}
\end{bmatrix}.
$$

The eigenvalues of this matrix are $d$ and $d_I - \frac{(d - d_I)n_I}{n - n_I}$. According to theorem 3.2 we have $\lambda_n \leq d_I - \frac{(d - d_I)n_I}{n - n_I}$ and the left-hand inequality in (4.2) is proved.

In order to prove the right-hand side inequality, we consider the complements $\overline{G}, \overline{G}_I$ of $G, G_I$. $\overline{G}$ is a regular graph on $n$ vertices of degree $n - 1 - d$ and according to lemma 3.12 its least eigenvalue is $-\lambda_2 - 1$. $\overline{G}_I$ is an induced subgraph of $\overline{G}$, has $n_I$ vertices and the average vertex degree $n - 1 - d_I$. Applying the left-hand side inequality of (4.2) to $\overline{G}$ and $\overline{G}_I$ we obtain the right-hand inequality. This completes the proof.

**Proposition 4.3.** Let $n$ and $d$ be the number of vertices and degree of a regular graph contained in the root system $E_8$. Then $n \leq 2d + 8$.

**Proof.** Each regular graph contained in $E_8$ is an induced subgraph of the graph $\overline{G}(8,2)$ which is described in theorem 2.6 (the converse, of course, does not hold). Theorems 4.1 and 2.6 yield $n \leq 2d + 8$. This completes the proof.
Let us note that \( n = 2d + 8 \) holds for the void graph on 8 vertices. In order to find all regular graphs in \( E_8 \) with \( 2d + 4 < n \leq 2d + 8 \) we shall first derive a restriction on the possible pairs \((n,d)\). In the remaining cases we used a computer to find the graphs.

As mentioned earlier, a regular graph of degree \( d \) with \( n \) vertices in \( E_8 \) has the eigenvalues \(-2\) with the multiplicity \( n - 8 \), \( d \) with the multiplicity 1, and 7 other eigenvalues, say, \( x_1, \ldots, x_7 \) with \(-2 \leq x_i \leq d \) \((i = 1, \ldots, 7)\). The sum of eigenvalues is zero and the sum of their squares is twice the number of edges. Therefore we have

\[
\sum_{i=1}^{7} x_i = 2n - d - 16, \quad \sum_{i=1}^{7} x_i^2 = nd - d^2 - 4n + 32.
\]

We shall use the following well-known inequality

\[
\left( \sum_{i=1}^{m} \alpha_i \right)^2 \leq m \sum_{i=1}^{m} \alpha_i^2,
\]

equality holding if and only if the real numbers \( \alpha_i \) \((i = 1, \ldots, m)\) are all equal (cf. [21]). Taking \( m = 7 \) and \( \alpha_i = x_i \) \((i = 1, \ldots, 7)\) we obtain

\[
(2n - d - 16)^2 \leq 7(nd - d^2 - 4n + 32).
\]

For \( n = 2d + 8 \), (4.4) cannot be satisfied by any \( d \neq 0 \). The case \( n = 2d + 7 \) together with (4.4) implies \( d \leq 4 \). In the case \( n = 2d + 6 \) we get \( d \leq 7 \), and in the case \( n = 2d + 5 \) we have \( d \leq 9 \).

Regular graphs of degree \( d \leq 2 \) are line graphs. Cubic graphs \((d = 3)\) in \( E_8 \) with \( n > 2d + 4 \) have exactly 12 vertices. From the table of cubic graphs [11] one can see that all cubic graphs on 12 vertices having the least eigenvalue \(-2\) are line graphs. So we are interested in graphs with \( d \geq 4 \). The relations mentioned above yield the following table of possible values \( d \) and \( n \) for graphs in \( G \) with \( n > 2d + 4 \):

<table>
<thead>
<tr>
<th>( d )</th>
<th>4</th>
<th>4</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>20</td>
<td>21</td>
</tr>
</tbody>
</table>

Let us describe now the computer search.

Because of the transitivity of \( E_8 \) we may assume that the representation \( S \) of a graph \( G \) contains the vector \( j \). If \( G \) is regular of degree \( d \) with \( n \) vertices, then the representation \( S \) of \( G \) contains the vector \( j \), \( d \) vectors of the type \( a \) or \( b \) and \( n - 1 - d \) vectors of the type \( c \) or \( d \).
Let $s_i$ be the sum of $i$-th coordinates of all vectors in $S$ ($i = 1, \ldots, 8$). The sum of coordinates of a vector of the type $a$ or $b$ is 4, while this sum is 0 for the types $c$ and $d$. Therefore

$$\sum_{i=1}^{8} s_i = 8 + d \cdot 4 + (n-1-d)0 = 4(d + 2).$$

In order to find $\sum_{i=1}^{8} s_i^2$ notice that this sum equals the sum of squares of all coordinates of all vectors plus the twofold sum of all possible products of two different vectors. The first summand is equal to $8n$ (since the sum of squares of coordinates is equal to 8 for all vectors in $F_8$). The second summand is twice the 4-fold number of edges of $G$, so $4nd$. Therefore

$$\sum_{i=1}^{8} s_i^2 = 4n(d + 2).$$

The vectors from $S$ determine, of course, the quantities $s_i$ ($i = 1, \ldots, 8$). It is important to notice that the quantities $s_i$ determine a set of vectors to which all vectors from $S$ belong.

Let $x$ be a vector of the type $a$ belonging to $S$. Let the $i$-th and $j$-th coordinate of $x$ be equal to 2. The sum of the inner products of $x$ with all other vectors from $S$ is equal to

$$2(s_i - 2) + 2(s_j - 2) + 0 \sum_{k=1}^{8} s_k = 2(s_i + s_j - 4).$$

On the other hand this sum must be $4d$ because $G$ is regular of degree $d$. So we have $s_i + s_j = 2(d + 2)$.

For a vector of the type $b$ having $(-1)'s$ in coordinates $i$ and $j$ we get in the same way (using also $\sum_{i=1}^{8} s_i = 4(d + 2)$) the relation $s_i + s_j = 0$.

For the vector of the type $c$ having $i$-th coordinate equal to 2 and $j$-th coordinate equal to $-2$ we get $s_i - s_j = 2(d + 2)$.

Finally, for a type $d$ vector having $(-1)'s$ in the positions $i,j,k,\ell$ we have $s_i + s_j + s_k + s_{\ell} = 0$. 

Therefore, if some of these relations are fulfilled for the given \( s_i \) (\( i = 1, \ldots, 8 \)), we can construct the corresponding vectors. Each such vector has to be considered as a possible candidate for \( S \). Let \( T \) be the set of vectors constructed in this manner.

We have to search for all sets of integers \( s_i \) (\( i = 1, \ldots, 8 \)), satisfying the relations

\[
\sum_{i=1}^{8} s_i = 4(d + 2), \quad \sum_{i=1}^{8} s_i^2 = 4n(d + 2),
\]

for which there are at least \( d \) relations of the form \( s_i + s_j = 0 \) or \( 2(d + 2) \) and at least \( n - d - 1 \) relations of the form \( s_i - s_j = 2(d + 2) \) or \( s_i + s_j + s_k + s_\ell = 0 \). Then we have to construct the corresponding sets of vectors \( T \). If for given \( s_i \)'s a graph \( G \) exists, then \( S \in T \), where \( S \) is the representation of \( G \).

A computer program based on the procedure described above has been constructed and all possible pairs \( (n,d) \) with \( n > 2(d + 2) \) were checked. The program provides for each \( (n,d) \) the possible sets \( T \) and the corresponding Gram matrices. There were a few sets \( T \) which could be treated without computer.

The only graph we found in this way is the graph \( L(K_5,4) \) with \( n = 20 \) and \( d = 7 \). So, there is no graph in \( G \) with \( n > 2(d + 2) \).

Now we can formulate the following proposition.

**Proposition 4.5.** For any graph from \( G \) we have \( n \leq 2(d + 2) \).
5. Some results for graphs with $n \leq 2d + 4$

We shall prove now some useful results for regular graphs contained in $E_8$ with $n \leq 2d + 4$.

**Proposition 5.1.** For each graph $G$ from $G$ with $n \leq 2d + 4$ there exists a graph $H$ with at most 8 vertices such that $G$ is switching equivalent to $L(H)$. All graphs $G$ with $n < 2d + 4$ have at most 7 principal eigenvalues.

**Proof.** Let $G \in G$ with $n \leq 2d + 4$ and consider the graph $G' = G \lor K_1$. First we shall show that $G'$ can be represented in $E_8$. This will be done by proving that $G'$ has at most 8 principal eigenvalues. According to lemma 3.11, the spectrum of $G'$ contains all eigenvalues of $G$ different from $d$ and the roots of the equation $\lambda^2 - d\lambda - n = 0$. If $n = 2d + 4$, one root of this equation is $-2$ and the other one is larger than $-2$. In the case $n < 2d + 4$ both roots are larger than $-2$. Hence, if $n = 2d + 4$, $G'$ has the same number of principal eigenvalues as $G$ and it can be represented in $E_8$ since $G$ does. If $n < 2d + 4$, $G$ has at most 7 principal eigenvalues. Indeed, suppose it has 8 principal eigenvalues. Then $G'$ has 9 such eigenvalues and can be represented in the root system $A_9$ or $D_9$. This means that $G$ can also be represented in $A_9$ or $D_9$. According to theorem 2.5, $G$ then is a line graph of a cocktail-party graph, which contradicts $G \in G$. Hence, $G$ has at most 7 distinct eigenvalues and therefore $G'$ has at most 8 such eigenvalues and can be represented in $E_8$.

Because of the transitivity of $E_8$ we can represent $G' = G \lor K_1$ such that the vertex corresponding to $K_1$ is represented by the vector $j = (1, \ldots, 1)$. Then the vertices of $G$ are represented by vectors of the types $a$ and $b$. Let us switch now the graph $G$ with respect to vertices represented by vectors of the type $b$. The obtained graph is the line graph of a graph $H$ having at most 8 vertices. This completes the proof.

**Proposition 5.2.** If $G \in G$ and if $n \leq \frac{3}{2}(d + 2)$, then $G$ is an induced subgraph of the Schlafli graph. Any regular graph with $n > \frac{3}{2}(d + 2)$ is not an induced subgraph of the Schlafli graph. If $G \in G$ and if $n < \frac{3}{2}(d + 2)$ then $G$ has at most 6 principal eigenvalues.
Proof. Let the vertices of a graph $K_2$ be denoted by $x$ and $y$. For any $G \in G$ define the graph $G''$ as the graph obtained by joining with edges all the vertices of $G$ to the vertex $x$ of $K_2$. Similar as in proposition 5.1 we shall prove that, under assumption $n \leq \frac{3}{2}(d + 2)$, $G''$ can be represented in $E_8$. The eigenvalues of $G''$ can be computed by lemmas 3.11 and 3.13. All eigenvalues of $G$ different from $d$ are also eigenvalues of $G''$. The remaining eigenvalues of $G''$ are the roots of the equation

$$\lambda^3 - d\lambda^2 - (n + 1)\lambda + d = 0.$$  

(5.3)

If $n \leq \frac{3}{2}(d + 2)$, all roots of (5.3) are not smaller than $-2$. (5.3) has a root equal to $-2$ just in the case $n = \frac{3}{2}(d + 2)$.

According to proposition 5.1 $G$ has at most 7 principal eigenvalues. If $n = \frac{3}{2}(d + 2)$, $G''$ has at most 8 principal eigenvalues and can be represented in $E_8$. In the case $n < \frac{3}{2}(d + 2)$ the graph $G$ has at most 6 principal eigenvalues, since otherwise $G''$ would have 9 principal eigenvalues and $G$ would not be an element of $G$. Hence, again $G''$ can be represented in $E_8$.

Let us take the representation of $G''$ such that the vertex $x$ is represented by the vector $j$. Then the vertices of $G$ and the vertex $y$ are represented by vectors of types $a$ and $b$. Without loss of generality we can represent the vertex $y$ by the vector $(2,2,0,0,0,0,0,0)$. Since $y$ is not adjacent to any vertex of $G$, vectors representing vertices of $G$ must be of the type as given in Fig. 5.4.

![Fig. 5.4.](image)

(* denotes vectors of the type $a$, and o means a vector of the type $b$). Hence, $G$ is an induced subgraph of the Schlafli graph.
In order to prove that regular graphs with \( n > \frac{3}{2}(d + 2) \) are not induced subgraphs of the Schlafli graph, apply theorem 4.1 to the Schlafli graph. Since the distinct eigenvalues of the Schlafli graph are 16, 4, -2, any induced regular subgraph of the Schlafli graph of degree \( d \) with \( n \) vertices satisfies \( n \leq \frac{3}{2}(d + 2) \). This completes the proof.

Proposition 5.5. If \( G \in G \) and if \( n \leq \frac{4}{3}(d + 2) \), then \( G \) is an induced subgraph of the Clebsch graph. Any regular graph with \( n > \frac{4}{3}(d + 2) \) is not an induced subgraph of the Clebsch graph. If \( G \in G \) and if \( n < \frac{4}{3}(d + 2) \) then \( G \) has at most 5 principal eigenvalues.

Proof. Let the vertices of the graph \( K_{1,2} \) be denoted by \( x, y, z \) such that \( y \) is the vertex of degree 2. For any \( G \in G \) define the graph \( G'' \) as the graph obtained by joining with edges all the vertices of \( G \) to the vertex \( x \) of \( K_{1,2} \). As earlier we shall prove that \( G'' \), under assumption \( n \leq \frac{4}{3}(d + 2) \), can be represented in \( E_8 \). By the use of lemmas 3.11 and 3.13 we readily get that all eigenvalues of \( G \) different from \( d \) are the eigenvalues of \( G'' \), and that \( G'' \) has 4 additional eigenvalues which are the roots of the equation

\[
\lambda^4 - d\lambda^3 - (n + 2)\lambda^2 + 2d\lambda + n = 0.
\]

For \( n = \frac{4}{3}(d + 2) \) the equation (5.6) has a root -2 and the other roots are > -2. For \( n < \frac{4}{3}(d + 2) \) all roots of (5.6) are > -2. Hence, in the first case \( G \) must have at most 6 principal eigenvalues and in the second one at most 5. Anyway, \( G'' \) can be represented in \( E_8 \).

Like in the previous case, let \( x \) be represented by \( j \) and \( y \) by \((2,2,0,0,0,0,0,0)\), and we again have the situation from Fig. 5.4.

Now consider the vertex \( z \). It is nonadjacent to \( x \) and, hence, orthogonal to \( j \), i.e. \( z \) must be represented by a vector of the types \( c \) or \( d \). Also, this vector must have the inner product 4 with \((2,2,0,0,0,0,0,0)\). If \( z \) is of type \( c \), then we take \( z = (2,0,-2,0,0,0,0,0) \), without loss of generality. We conclude that the only possible vectors in a representation of \( G \) are as indicated in Fig. 5.7a)
If \( z \) is represented by a vector of type \( d \), we can take \( z = (1,1,1,1,-1,-1,-1,-1) \). Then we have the situation in Fig. 5.7b). But the graphs in fig. 5.7a) and b) are both isomorphic to the Clebsch graph. Hence \( G \) is an induced subgraph of the Clebsch graph.

To prove the rest, recall that the distinct eigenvalues of the Clebsch graph are 10, 2, -2. By theorem 4.1 we get that for an induced subgraph of the Clebsch graph the relation \( n \leq \frac{4}{3}(d + 2) \) holds. This completes the proof.

We shall give now a proof for a theorem given in section 3.

Proof of theorem 3.4. As shown in section 4, regular graphs in \( E_8 \) with \( n > 2(d + 2) \) have at most 21 vertices. By proposition 5.1 graphs with \( n \leq 2(d + 2) \) are switching equivalent to some \( L(H) \) where \( H \) has at most 8 vertices and, hence, \( G \) has at most 28 vertices. This also holds for line graphs and cocktail-party graphs with less than 8 principal eigenvalues. The only cocktail-party graph with 8 principal eigenvalues has 18 vertices. By corollary 3.7, connected line graphs with 8 principal eigenvalues are line graphs of nonbipartite graphs on 8 vertices or of bipartite graphs on 9 vertices. In the first case the conclusion is clear. In the second case, a bipartite graph on 9 vertices has at most 20 edges and we again have the same conclusion.
Let us prove now the second part of the theorem. For line graphs and cocktail-party graphs in $E_8$ the degree is obviously less than 16. The graphs from $G$ with $n \leq \frac{3}{2}(d + 2)$ are induced subgraphs of the Schlafli graph by proposition 5.2, and hence their degree is not greater than 16. For the remaining graphs we have $28 \geq n > \frac{3}{2}(d + 2)$, which implies $d \leq 16$. This completes the proof.

We shall summarize the results about the graphs from $G$ in the following proposition.

**Proposition 5.8.** The possible values of $d$ and $n$ for graphs from $G$ are displayed in the following table:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n = n_3$</th>
<th>$n_3 &lt; n &lt; n_2$</th>
<th>$n = n_2$</th>
<th>$n_2 &lt; n &lt; n_1$</th>
<th>$n = n_1$</th>
</tr>
</thead>
<tbody>
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<td>3</td>
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<td>12</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>10</td>
<td>12</td>
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</tr>
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<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>28</td>
</tr>
</tbody>
</table>

where $n_1 = 2(d + 2)$, $n_2 = \frac{3}{2}(d + 2)$, $n_3 = \frac{4}{3}(d + 2)$.

**Proof.** In section 4 we have seen that there are no graphs in $G$ with $n > 2(d + 2)$. By proposition 5.5 the graphs from $G$ with $n < \frac{4}{3}(d + 2)$ have at most 5 principal eigenvalues. By proposition 2.8 they are line graphs or cocktail-party graphs. Hence, for graphs from $G$ we have $\frac{4}{3}(d + 2) \leq n \leq 2(d + 2)$. Of course, we also have $3 \leq d \leq 16$. Some pairs $d, n$, satisfying these inequalities, have been excluded from the table by the following reasons. From the
table of cubic graphs up to 14 vertices [1], we observe that cubic graphs in G have ten vertices. Some other pairs are excluded by the inequality we are going to derive.

Graphs from G with n < 2d + 4 have at most 7 principal eigenvalues (by proposition 5.1) and hence, they can be represented in the root system E_7. That means that these graphs are induced subgraphs of the graph G(E_7) described in theorem 2.6. Applying theorem 4.1 to this graph we get the inequality \( n \geq \frac{9}{4}(d - 4) \). This completes the proof.

In the last table we have separated some values of n and d in special columns. For the regular graphs in E_8 with n = 2(d + 2) we shall say that they are in the first layer. Those with n = \( \frac{3}{2}(d + 2) \) are in the second layer and those with n = \( \frac{4}{3}(d + 2) \) are in the third layer.

All pairs n,d from proposition 5.8 have been treated by a computer search. The procedure was similar to that described in section 4. The difference is that we now know by proposition 5.1 that all graphs can be represented only by vectors of the types a and b. For the quantities \( s_i \), defined in section 4, we now have the relations

\[
(5.9) \quad \sum_{i=1}^{8} s_i = 4n, \quad \sum_{i=1}^{8} s_i^2 = 4n(d + 2).
\]

For each vector of the type a we have a relation of the form \( s_i + s_j = 2(d + 2) \). For each vector of the type b we have \( s_i + s_j = 2(n - d - 2) \). So we have to search for all integral solutions of (5.9) in \( s_i \)'s for which there are at least n relations of the forms \( s_i + s_j = 2(d + 2) \) or \( 2(n - d - 2) \).

Using this procedure we found that there are no graphs from G between the layers; we found only a few line graphs between the layers.

Proposition 5.10. There are no graphs from G between the layers.

Remark. In certain cases the nonexistence of graphs from G between the layers can be proved without references to a computer search. For example, it was noticed in [2], that no graph in E_8 with n = 28, d = 16 exists.
We shall give now the non-existence proofs for 31 out of 48 cases from the table of proposition 5.8 in the form of lemmas.

**Lemma 5.11.** There are no graphs in $G$ with the following values of $d$ and $n$:

<table>
<thead>
<tr>
<th>$d$</th>
<th>12</th>
<th>12</th>
<th>13</th>
<th>13</th>
<th>14</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>26</td>
<td>27</td>
<td>26</td>
<td>28</td>
<td>27</td>
<td>28</td>
<td>28</td>
<td>28</td>
</tr>
</tbody>
</table>

**Proof.** According to proposition 5.1, the graphs $G$ from $G$ with the mentioned values of $n$ and $d$ have at most 7 principal eigenvalues $d, x_1, \ldots, x_6$. From the trace of the adjacency matrix of $G$ we get

$$\frac{1}{6} \sum_{i=1}^{6} x_i = \frac{1}{6}(2n - 14 - d).$$

$G$ is an induced subgraph of $G(E_7)$ by theorem 2.6. According to theorem 3.1 we then have $x_i \leq 4$, $i = 1, \ldots, 6$. Therefore $\frac{1}{6}(n - 14 - d) \leq 4$, and the values of $n$ and $d$ from the lemma do not satisfy this inequality. This completes the proof.

**Lemma 5.12.** There are no graphs in $G$ with the following values of $d$ and $n$:

<table>
<thead>
<tr>
<th>$d$</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>22</td>
<td>23</td>
</tr>
</tbody>
</table>

**Proof.** Now, according to proposition 5.2, the corresponding graphs have at most 6 principal eigenvalues: $d$ and $x_1, \ldots, x_5$. The graphs are subgraphs of $G(E_6)$ by theorem 2.6 and as earlier we have

$$\frac{1}{5} \sum_{i=1}^{5} x_i = \frac{1}{5}(2n - 12 - d) \leq 2.$$

The values $n, d$ from the lemma do not satisfy this inequality. This completes the proof.

**Lemma 5.13.** There are no graphs in $G$ with the following values of $d$ and $n$:

<table>
<thead>
<tr>
<th>$d$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>19</td>
<td>20</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
</tr>
</tbody>
</table>
Proof. For the quantities \( x_1, \ldots, x_6 \) defined in the proof of lemma 5.11 we have

\[
6 \sum_{i=1}^{6} x_i = 2n - 14 - d, \quad 6 \sum_{i=1}^{6} x_i^2 = 2n - d^2 - 4n + 28
\]

and we get the inequality

\[
(2n - 14 - d)^2 \leq 6(nd - d^2 - 4n + 28),
\]

which is not satisfied by the values from the lemma. This completes the proof.

Lemma 5.14. There are no graphs in \( G \) with 26 vertices.

Proof. According to proposition 6.6, such graphs would be switching equivalent to some \( L(H) \) where \( H \) has 8 vertices and 26 edges and either each component of \( H \) is Eulerian or anti-Eulerian or \( H \) is even-odd bipartite. Obviously, such graphs \( H \) do not exist. This completes the proof.

Lemma 5.15. There are no graphs \( G \) in \( G \) with the following values of \( d \) and \( n \):

\[
\begin{array}{ccc}
  d & 11 & 12 & 13 \\
  n & 20 & 24 & 24 \\
\end{array}
\]

Proof. \( G \) is switching equivalent to some \( L(H) \). Since \( G \) has the largest eigenvalue \( d \), \( L(H) \) must contain the eigenvalue \( n - 1 - 2d \) in the spectrum of its \((-1,1,0)\)-adjacency matrix. By inspection through the table 7.7 we see that for the given values \( d, n \) no \( H \) exists. This completes the proof.

Lemma 5.16. There are no graphs \( G \) in \( G \) with \( n = 25, d = 14 \).

Proof. According to proposition 5.1 \( G \) is switching equivalent to some \( L(H) \), where \( H \) is one of 5 possible graphs on 8 vertices with 25 edges. The graphs \( G \) are Eulerian on an even number of vertices. In order to get \( G \) from \( L(H) \) we must switch \( L(H) \) w.r.t. the set of vertices of the same parity. So for each \( \tilde{H} \) the switching set for \( L(H) \) is uniquely determined. By inspection we see that there are no graphs \( G \). This completes the proof.
Lemma 5.17. There are no graphs $G$ in $G$ with $d = 8$, $n = 14$.

Proof. We have to consider graphs $H$ with 14 edges from table 7.7. For all such $H$ the least eigenvalue of $L(H)$ is less than $-6$ (in $(-1,1,0)$-language). That means that $G$ would have the second largest eigenvalue $> \frac{5}{2}$. According to the interlacing theorem that is impossible since $G$ is an induced subgraph of $G(E_6)$. This completes the proof.

Lemma 5.18. There are no graphs $G$ in $G$ with the following values of $d$ and $n$:

\[
\begin{array}{ccc}
d & 8 & 9 & 11 \\
n & 18 & 18 & 22 \\
\end{array}
\]

Proof. If $G$ exists, it is switching equivalent to an Eulerian graph $H$ on 8 vertices with $n$ edges. Therefore the spectrum of $L(H)$ (in the $(-1,1,0)$-language) contains the eigenvalue $n - 1 - 2d$. For the given values of $n$ and $d$ only the following line switching equivalence classes of table 7.7 contain this eigenvalue: 37, 29, 32. The corresponding possible spectra of $G$ (in $(0,1)$-language) are then:

\[
11, 4, 3^5, -2^{15}; 9, 3^3, 1^3, -2^{11}; 8, 3^3, 2^3, 1, -2^{11}.
\]

If a regular graph on $n$ vertices has distinct eigenvalues $\lambda_1 = d, \lambda_2, \ldots, \lambda_5$ then $(d - \lambda_2)(d - \lambda_3) \ldots (d - \lambda_5)$ is divisible by $n$ (cf. [29]). For the spectra given above this divisibility condition is not fulfilled and the graphs $G$ do not exist. This completes the proof.

The remaining cases can also be treated without computer. In almost all of these cases the equations (5.9) can be solved by hand and pencil in a reasonable time, but for the convenience we did the search by computer.

The procedure described above is quite inefficient for graphs in the layers. In the first layer we have $2(d + 2) = 2(n - d - 2)$ and we cannot distinguish between the vectors of the types $a$ and $b$. Besides, all $s_1$'s are equal in that case (see lemma 7.1). In the second layer the sets $T$ (defined in section 4) always represent the Schlafli graph. That means that the graphs are induced subgraphs of the Schlafli graph, but we already know this fact by proposition 5.2. Similarly, the procedure gives that the graphs in the third layer are subgraphs of the Clebsch graph; again nothing new. The only useful fact obtained by the computer search for graphs in layers is formulated in the following proposition.
Proposition 5.19. All graphs in the second layer can be obtained if we start with the following solution of (5.9): \( s_1 = s_2 = n, \)
\[ s_3 = s_4 = \ldots = s_8 = \frac{1}{3} n. \]

Proof. Let \( G \) be a graph from the second layer. According to proposition 5.2, \( G \) is an induced subgraph of the Schlafli graph. Therefore, \( G + K_1 \) can be represented by vectors of the types \( a \) and \( b \). Let us represent the isolated vertex by the vector \( y = j - 2e_1 - 2e_2 \). Consider the column sums \( s_i \) for the representation \( S \) of \( G \) defined in this way. The sum of inner products of \( y \) with all vectors from \( S \) is
\[ -s_1 - s_2 + \sum_{i=3}^{8} s_i = 0, \]
since \( y \) represents an isolated vertex. This relation, together with (5.9) gives \( s_1 + s_2 = 2n \). Suppose now \( s_1 = n + \alpha, s_2 = n - \alpha \) for some \( \alpha \). We shall first show that \( \alpha = 0 \). Using (5.9) we get
\[ \sum_{i=3}^{8} s_i = 2n \quad \text{and} \quad \sum_{i=3}^{8} s_i^2 = 4n(d + 2) - 2n^2 - 2\alpha^2. \]

For any numbers \( s_i \) we have
\[ (\sum_{i=3}^{8} s_i)^2 \leq 6 \sum_{i=3}^{8} s_i^2, \]
and therefore we get
\[ 4n^2 \leq 6(4n(d + 2) - 2n^2 - 2\alpha^2), \]
which implies
\[ n \leq \frac{3}{2}(d + 2) - \frac{3\alpha^2}{4n}. \]

Since \( G \) is in the second layer, we have \( n = \frac{3}{2}(d + 2) \) and \( \alpha = 0 \). Hence, \( s_1 = s_2 = n \) and
\[ \sum_{i=3}^{8} s_i = 6(\frac{n}{3}), \quad \sum_{i=3}^{8} s_i^2 = 6(\frac{n}{3})^2 \]
which implies \( s_3 = s_4 = \ldots = s_8 = \frac{n}{3} \). This completes the proof. \( \square \)

All graphs in layers will be found in sections 7 and 8.
6. Regular factorization and Eulerian graphs

Consider an $H$ from proposition 5.1. There exists a partition of the vertex set of $L(H)$ into two subsets having, say, $p$ and $q$ vertices ($p + q = n$, $p \geq q$), such that the switching with respect to the $q$ vertices yields a regular graph in $E_8$. To the described partition of vertices of $L(H)$ there corresponds a partition of the $n$ edges of $H$ into two parts having $p$ and $q$ edges. Let us denote the spanning subgraphs of $H$ defined by these $p$ and $q$ edges by $H_1$ and $H_2$, respectively. So the graph $H$ is factorized into two factors $H_1$ and $H_2$.

**Definition 6.1.** The factorization of a graph $H$ on 8 vertices into factors $H_1$ and $H_2$ is regular if $L(H)$ after switching with respect to $L(H_2)$ yields a regular graph in $E_8$.

We shall describe now regular factorization in terms of the column sums $s_i$ $(i = 1, \ldots, 8)$ defined in section 4.

**Proposition 6.2.** Let $n \leq 2(d+2)$ and let $H$ be any graph on 8 vertices having $n$ edges and vertex degrees $d_1, \ldots, d_8$. Let, further, $H$ be factorized into spanning subgraphs $H_1$ and $H_2$ having $p$ and $q$ ($p + q = n$, $p \geq q$) edges, respectively, and vertex degrees $x_1, \ldots, x_8$ and $y_1, \ldots, y_8$, respectively. The factorization of $H$ is regular and yields a regular graph of degree $d$ in $E_8$ if and only if the quantities $s_i$ $(i = 1, \ldots, 8)$ defined by $s_i = q + 2(x_i - y_i)$ satisfy the following conditions:

1) $\sum_{i=1}^{8} s_i = 4n$, $\sum_{i=1}^{8} s_i^2 = 4n(d + 2)$;

2) $s_1 + s_j = 2(d + 2)$ for each edge $(i,j)$ of $H_1$;

3) $s_1 + s_j = 2(n - d - 2)$ for each edge $(i,j)$ of $H_2$. 
Proof. Suppose first the factorization is regular. Represent each edge \((i,j)\) of \(H_1\) by the vector \(2e_i + 2e_j\) and each edge \((i,j)\) of \(H_2\) by \(j - 2e_i - 2e_j\). The set \(S\) of these vectors is the representation of a regular graph in \(E_8\) of degree \(d\) and with \(n\) vertices. The column sums \(s_i\) of this representation satisfy the conditions 1), 2) and 3) of the theorem, as explained in section 5.

On the other hand, the number of 2's in the \(i\)-th column of \(S\) is \(x_i\) and the number of \((-1)\)'s is \(y_i\). Obviously, \(s_i = 2x_i + q - 2y_i\) and the theorem is proved in one direction.

Suppose now the \(s_i\) satisfy 1), 2) and 3). From \(x_i + y_i = d_i\) and \(x_i - y_i = \frac{1}{2}(s_i - q)\) we get the formulas

\[
(6.3) \quad x_i = \frac{1}{2}d_i + \frac{1}{2}(s_i - q), \quad y_i = \frac{1}{2}d_i - \frac{1}{2}(s_i - q).\]

Consider an edge \((i,j)\) of \(H_1\). The corresponding vertex of \(L(H)\) will have the following degree after switching \(L(H)\) w.r.t. \(L(H_2)\)

\[
x_i + x_j - 2 + q - y_i - y_j = (x_i - y_i) + (x_j - y_j) + q - 2 = \\
\frac{1}{2}(s_i - q) + \frac{1}{2}(s_j - q) + q - 2 = \frac{1}{2}(s_i + s_j) - 2 = d.
\]

For an edge \((i,j)\) in \(H_2\) we have similarly

\[
y_i + y_j - 2 + p - x_i - x_j = p - 2 - (x_i - y_i) - (x_j - y_j) = \\
p - 2 - \frac{1}{2}(s_i - q) - \frac{1}{2}(s_j - q) = n - \frac{1}{2}(s_i + s_j) = d.
\]

Hence, the factorization is regular and yields a regular graph of degree \(d\) in \(E_8\). This completes the proof.

Definition 6.4. A graph is called Eulerian (anti-Eulerian) if all its vertex degrees are even (odd).

Definition 6.5. A graph is called even-odd bipartite if it is bipartite and the vertex degrees are even in one part and odd in the other one.

Recall that by proposition 5.1, each graph in \(G\) with \(n \leq 2(d+2)\) is switching equivalent to \(L(H)\) for some graphs \(H\) on 8 vertices. In some special cases we shall make this statement more precise.
Proposition 6.6. Let $G$ be a regular graph in $E_8$ with $n \leq 2(d + 2)$ and $n$ even. For any graph $H$ on 8 vertices, such that $G$ is switching equivalent to $L(H)$, one of the following holds:
1) Each component of $H$ is Eulerian or anti-Eulerian;
2) $H$ is even-odd bipartite.

Proof. By lemma 2.9, the graph $L(H)$ is Eulerian or anti-Eulerian. In the first case the vertex degrees of any two adjacent vertices of $H$ must be of the same parity and we have 1). In the second case adjacent vertices of $H$ have vertex degrees of different parities, and $H$ is bipartite. Hence, we have 2). This completes the proof.

We shall prove now a proposition related to regular factorization of Eulerian graphs which will be used later. For convenience, we shall indicate a factorization of a graph by colouring the edges of each factor, say red and blue.

Proposition 6.7. Let $H$ be any Eulerian graph on 8 vertices, with an even number of edges, different from graphs a), b), c) in fig. 6.8. The edges of $H$ can be coloured by two colours (say, red and blue) so that for each vertex the number of red edges incident to that vertex equals the number of blue edges incident to that vertex.

Proof. Suppose first $H$ is connected. Find an even circuit in $H$ and throw it out. Continue this procedure as long as possible. If all edges of $H$ can be thrown out in this way, colour the edges of each circuit by two colours alternatively. So we have the desired colouring.

In another case the procedure will give a non-avoid Eulerian graph having no even circuits. It can not be a tree (forest) and therefore there are some circuits which are odd. Two odd circuits cannot have an common edge since then an even circuit would exist. If two odd circuits have a common vertex then the edges of these two circuits can be coloured by two colours so that the colours are balanced at each vertex. Hence, these two odd circuits can be treated as one even circuit and can also be thrown out. Suppose therefore that all odd circuits are mutually disjoint. There are no edges outside the circuits since the graph is Eulerian. Since $H$ has an even number
of edges and only 8 vertices, the number of odd circuits is exactly 2 and they will be denoted by $O_1$ and $O_2$.

Since $H$ is connected we can find among the removed even circuits a sequence $E_1, E_2, \ldots, E_k$ of even circuits such that $O_1$ has a common vertex $x_1$ with $E_1$, $E_1$ has a common vertex $x_2$ with $E_2$, etc., $E_k$ has a common vertex $x_{k+1}$ with $O_2$. Colour the two edges of $O_1$, incident with $x_1$ in red and two edges from $E_1$ incident to $x_1$ in blue. Starting from these edges in both directions around $E_1$, colour the edges of $E_1$ alternatively in red and blue. Finally, the two edges of $E_1$ incident to $x_2$ will be coloured by the same colour, say red. The edges from $E_2$ incident to $x_2$ will then get the blue colour. Continuing in this way we can colour the edges contained in the circuit sequence $O_1, E_1, \ldots, E_k, O_2$ by two colours so that at each vertex the colours are balanced. Hence, the colouring of $H$ is complete.

Suppose now, $H$ is disconnected. We may ignore isolated vertices and so the proof is valid for graphs with only one nontrivial component. Since the graph is Eulerian, any nontrivial component has at least 3 vertices. Hence, the number of nontrivial components is at most 2. The only Eulerian graphs with 8 vertices, with an even number of edges and with exactly 2 nontrivial components are represented in fig. 6.8.

\begin{figure}[h]
\begin{center}
\includegraphics[width=\textwidth]{fig6.8.png}
\end{center}
\caption{Fig. 6.8.}
\end{figure}

The graph under d) can be coloured in the desired manner and others cannot. This completes the proof.
Remark 6.9. The graph under $c$) still has a regular factorization, where the edge $u$ belongs to one factor, and all others to the other one.
7. First layer

We shall find now all graphs in the first layer. If we restrict ourselves to the first layer, the statements from section 6 can be made more precise.

Lemma 7.1. The column sums $s_1, \ldots, s_8$ for a regular graph in $E_8$ with $n = 2(d+2)$ are all equal to $d+2$.

Proof. $n = 2(d+2)$ together with (5.9) implies

$$\left(\sum_{i=1}^{8} s_i\right)^2 = 8 \sum_{i=1}^{8} s_i^2 = 64(d+2)^2$$

and we get $s_1 = s_2 = \ldots = s_8 = d+2$. This completes the proof.

Proposition 7.2. For each regular graph $G$ in $E_8$ with $n = 2(d+2)$ there exists an Eulerian or an anti-Eulerian graph $H$ on $8$ vertices such that $G$ is switching equivalent to $L(H)$.

Proof. Since all $s_i$'s are equal, the relation $x_i - y_i = \frac{1}{4}(s_i - q)$ (from proposition 6.2) implies that all differences $x_i - y_i$ are equal. Since $x_i - y_i$ and $x_i + y_i$ are of the same parity and since $d_i = x_i + y_i$, all vertex degrees of $H$ are of the same parity. This completes the proof.

Proposition 7.3. For any graph $H$ on $8$ vertices with $n = 2(d+2)$ edges, the number $q$ of edges of the subgraph $H_2$ in a regular factorization of $H$ yielding a regular graph of degree $d$, satisfies

1) $q \equiv (d + 2) \pmod{4}$ if $H$ is Eulerian;
2) $q \equiv d \pmod{4}$ if $H$ is anti-Eulerian.

Proof. By (6.3) and lemma 7.1 the vertex degrees of $H_1$ and $H_2$ are given by

$$x_i = \frac{1}{4}d_i + \frac{1}{4}(d+2-q), y_i = \frac{1}{4}d_i - \frac{1}{4}(d+2-q)$$

and the proposition follows.

Proposition 6.2, lemma 7.1 and relation (7.4) immediately give the following proposition.
Proposition 7.5. Let G be an Eulerian or anti-Eulerian graph on 8 vertices with vertex degrees $d_1, \ldots, d_8$ and with $n = 2(d+2)$. A factorization of H into spanning subgraphs $H_1$ and $H_2$ is regular and yields a regular graph of degree d with n vertices if and only if the vertex degrees $x_1, \ldots, x_8$ of $H_1$ and the vertex degrees $y_1, \ldots, y_8$ of $H_2$ satisfy the relations

$$x_i = \frac{1}{4}d_i + \alpha, \quad y_i = \frac{1}{4}d_i - \alpha \quad (i = 1, \ldots, 8),$$

for some $\alpha$.

According to proposition 7.2, in order to get all graphs in the first layer, we should start with all Eulerian and anti-Eulerian graphs on 8 vertices and with an even number of edges, and look for all their regular factorizations. Since we know all cubic graphs in the first layer it is sufficient to consider Eulerian and anti-Eulerian graphs with 12 and more edges. Obviously, there are no Eulerian or anti-Eulerian graphs with 26 edges (since such a graph should be obtained by removing two edges from $K_8$) and, hence, there are no graphs in the first layer with $n = 26$. The case $n = 28$ is also excluded since the only graphs in the first layer with $n = 28$ are the Chang graphs. Therefore, a survey of all Eulerian or anti-Eulerian graphs on 8 vertices with an even number $n$ ($12 \leq n \leq 24$) edges is given in the following table 7.7. The set of these graphs will be denoted by $EA$. 
### Table 7.7.

Eulerian and Anti-Eulerian Graphs on \( n \) Vertices with \( m \) Edges (\( n = 12, 14, \ldots, 24 \)).

<table>
<thead>
<tr>
<th>Number of Edges ( m )</th>
<th>Spectrum of ( L(G) ) in ([-1.0, 1.0]) Language</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 1 )</td>
<td>( {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23} )</td>
<td>([-1.0, 1.0])</td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>( {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23} )</td>
<td>([-1.0, 1.0])</td>
</tr>
<tr>
<td>( m = 3 )</td>
<td>( {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23} )</td>
<td>([-1.0, 1.0])</td>
</tr>
<tr>
<td>( m = 4 )</td>
<td>( {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23} )</td>
<td>([-1.0, 1.0])</td>
</tr>
</tbody>
</table>

Note: The spectrum of the Laplacian matrix \( L(G) \) of a graph \( G \) is the set of its eigenvalues, which are used in graph theory to study various properties of the graph.

The language \([-1.0, 1.0]\) refers to the interval of real numbers from \(-1.0\) to \(1.0\), which is a common domain for studying the spectra of graphs in spectral graph theory.

### Table 7.8.

Classification of Eulerian and Anti-Eulerian Graphs

<table>
<thead>
<tr>
<th>Graph Type</th>
<th>Class Number</th>
<th>Spectral Radius</th>
<th>Adjacency Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eulerian</td>
<td>( 1 )</td>
<td>7.7</td>
<td>( {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23} )</td>
</tr>
<tr>
<td>Anti-Eulerian</td>
<td>( 2 )</td>
<td>6.5</td>
<td>( {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23} )</td>
</tr>
<tr>
<td>Class Nr. 19</td>
<td>Spectrum of L(G) in ${\pm 1, \pm i}$-Language</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------------</td>
<td>-------------------------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.000</td>
<td>1.000</td>
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NUMBER OF EDGES IS 24
Of course, different graphs from table 7.7 can give the same graph in the first layer. Therefore the following definition is useful.

**Definition 7.8.** Graphs $G_1$ and $G_2$ are line-switching equivalent if their line graphs $L(G_1)$ and $L(G_2)$ are switching equivalent.

Obviously, line switching equivalent graphs give the same graphs in the first layer. Therefore the line switching equivalence classes of the set EA have been found and a representative for each class has been chosen (the first quoted graph in table 7.7). The search for line switching equivalent graphs has been performed by the use of lemma 2.11 with the aid of the computer.

The following two useful propositions can be stated by inspection through table 7.7.

**Proposition 7.9.** For each Eulerian (anti-Eulerian) graph $H$ from EA there exists an anti-Eulerian (Eulerian) graph $H'$ in EA such that $H$ and $H'$ are line switching equivalent.

This means that we can start with only Eulerian or only anti-Eulerian graphs in searching for the graphs in the first layer. But the representatives of classes in table 7.7 have been chosen following another criterion, which will be explained later.

Line graphs of line switching equivalent graphs have of course, the same spectrum of their $(-1,1,0)$-adjacency matrix. But the converse also holds in the set EA.

**Proposition 7.10.** Each line switching equivalence class of graphs in EA is uniquely determined by the spectrum of the $(-1,1,0)$-adjacency matrix of the line graph of any graph from the considered class.

**Remark 7.11.** This proposition, together with lemma 3.15, is very useful in proving theorem 1.4. Let $H$ be a regular graph on 8 vertices. Then the graph $L(H)$ is in the first layer. In order to find all graphs from $G$ which are cospectral with $L(H)$, it is sufficient to start with $H$ itself and find all its regular factorizations. But now all $d_i$'s from (7.6) are constant. This means that the factors $H_1$ and $H_2$ are regular. Hence to each factorization of $H$ into two regular factors a graph in the first layer corresponds! Therefore, for
any class in table 7.7 (if possible) a regular representative has been chosen irrespective of the fact whether it is Eulerian or anti-Eulerian.

According to formulas (7.4), only one value of $q$ is possible if one of vertex degrees $d_v$ of $H$ is 0 or 1. In this case the number of regular factorizations of $H$ is smaller, and this was another criterion for selection of representatives of line switching equivalence classes of the set $EA$.

Algorithm 7.12 (for finding the graphs in the first layer). Find any set of representatives of line switching equivalence classes of the $EA$. For any $H$ from that set, and for any possible $a$, find all factorizations of $H$ into factors $H_1$ and $H_2$, whose vertex degrees are given by (7.6). Switch $L(H)$ w.r.t. the vertices corresponding to edges of $H_2$. The obtained graph is regular and it is in the first layer. All graphs from the first layer can be obtained in this way.

In many cases the algorithm can be performed very quickly by hand and pencil. For example, all nonisomorphic factorizations of cubic graphs on 8 vertices into one 2-factor and one 1-factor are displayed in fig. 7.13.
The complete search has been performed by computer, and so we arrived at the following proposition.

Proposition 7.14. There are exactly 163 graphs from $G$ in the first layer. They are displayed in table 9.1.
8. The second and the third layer

First we shall find all graphs from $G$ in the second layer. Now we have $n = \frac{3}{2}(d + 2)$.

According to proposition 1.19, for each graph in the second layer there is a representation by vectors of types a and b such that the corresponding column sums $s_i$ are given by $s_1 = s_2 = n$, $s_3 = s_4 = \ldots = s_8 = \frac{1}{3}n$. The corresponding set $T$, defined in section 4, consists of vectors $2e_i + 2e_i$ ($i = 3, 4, \ldots, 8$); $2e_i + 2e_i$ ($i = 3, 4, \ldots, 8$) and $j - 2e_i - 2e_j$ ($i < j$; $i, j = 3, 4, \ldots, 8$). This set of vectors represents the Schlafli graph and we are looking for subsets $S$ having the given column sums $s_i$ ($i = 1, \ldots, 8$).

As earlier, let $p$ be the number of vectors of type a in $S$ and $q$ the number of vectors of type b in $S$. Since $s_1 = s_2$, $p$ is even and $S$ contains $\frac{1}{p}$ vectors having first coordinate equal to 2 and $\frac{1}{p}$ vectors having the second coordinate equal to 2.

Consider the last 6 columns of the representation $S$. Let $r, s, t$ be the number of these columns having two 2's, one number 2 and no number 2, respectively. Let $j, k, \ell$ be the number of $(-1)$'s in the column of each of mentioned types, respectively. Putting $n = 3v$ ($3 \leq v \leq 9$) we have the following relations

\[ p \equiv 0 \pmod{2}, \quad p + q = 3v, \quad r + s + t = 6, \quad 2r + s = p, \]

\[ j = v - \frac{1}{p} + 2, \quad k = v - \frac{1}{p} + 1, \quad \ell = v - \frac{1}{p}. \]

As pointed out earlier, representation $S$ defines a graph $H$ on 8 vertices and a regular factorization $H = H_1 \cup H_2$. Consider the subgraph $F$ of $H_2$ obtained by removal of vertices 1 and 2. The vertex degrees of this subgraph are just the quantities $j, k, \ell$ which appear with the multiplicities $r, s, t$.

Of course, we have $j = k + 1 = \ell + 2 \leq 7$. Now it is easy to obtain all possible sets of the mentioned parameters. Given $v$, we can choose some $p$ and then we also know $j, k$ and $\ell$. Then we can choose some $r$ which yields $s$ and $t$. All possible sets of parameters are given in table 8.1 together with the corresponding vertex degree sequence of $F$. The value $n = 27$ is excluded since in this case the only graph in the second layer is, of course, the Schlafli graph.
Table 8.1.

I. \( v = 3 \), \( n = 9 \), \( d = 4 \).

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</table>
V. \( v = 7, n = 21, d = 12 \).

\[
\begin{array}{cccccccccc}
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<th>k</th>
<th>s</th>
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<th>t</th>
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<th>\text{number of graphs } F</th>
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<td>0</td>
<td>(3,3,3,3,3,3)</td>
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\end{array}
\]

VI. \( v = 8, n = 24, d = 14 \).

\[
\begin{array}{cccccccccc}
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<th>\text{degree sequence}</th>
<th>\text{number of graphs } F</th>
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<td></td>
<td>(5,5,5,5,3,3)</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
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</table>
\end{array}
\]

Using vertex degree sequences it is easy to construct all possible graphs \( F \) on 6 vertices. (One can also use the table of graphs on 6 vertices from [12].) Given a graph \( F \) the corresponding graph \( H \) can easily be completed and the corresponding regular factorization determined. Different \( F \)'s can yield the same graph in the second layer. Therefore all obtained graphs have been checked for isomorphism by the use of computer. In this way we arrived at the following proposition.

**Proposition 8.2.** There are exactly 21 graphs from \( G \) in the second layer. They are displayed in table 9.1.
Let us find now all graphs from \( G \) in the third layer. According to proposition 5.5 all such graphs are induced subgraphs of the Clebsch graph. The only possible pairs of parameters are \( n = 16, d = 10, n = 12, d = 7, n = 8, d = 4 \).

In the first case the only solution is the Clebsch graph. In the third case we have the complement of a cubic graph on 8 vertices. Using the table of cubic graphs [1] we easily find that the only solution is the second graph in fig. 2.17. (It is the complement of the Möbius ladder on 8 vertices, in other words, the total graph of the circuit of length 4.)

For the remaining case consider the Clebsch graph as represented in fig. 8.3.

![Fig. 8.3.](image)

We have to remove 4 vertices. From the graph, we are looking for, exactly 36 edges go toward the 4 vertices. Hence, the subgraph induced by these 4 vertices contains exactly 2 edges and, hence, it is \( K_{1,2} + K_1 \) or \( 2K_2 \). The first case does not work and, in the second case, two independent edges in the Clebsch graphs can essentially be chosen in only one way, as indicated in fig. 8.3. After removing \( 2K_2 \) we get the unique solution.
Proposition 8.4. There are exactly 3 graphs from $G$ in the third layer: the Clebsch graph, graph from fig. 8.3 and the total graph of a circuit of length 4.
9. Tables for the graphs from \( G \)

The 187 graphs from \( G \) are displayed in table 9.1. Each graph is given by the list of its edges, and the edges are represented as pairs of vertices. The graphs are classified in layers, and then arranged according to the number of vertices. For the first layer the graphs are divided into switching equivalence classes, which correspond to the line switching equivalence classes of Eulerian and anti-Eulerian graphs on 8 vertices. The class numbers refer to table 7.7 except for the cubic graphs (nr. 1 - nr. 5) and the Chang graphs (nr. 161 - nr. 163).

For each graph the spectrum is calculated. In the first layer the common spectrum for the graphs in each switching equivalence class is given.

In table 9.2 the switching sets for constructing graphs in \( G \) from the corresponding line graphs are given. For each graph in the first layer with \( 12 \leq n \leq 24 \) a switching set is given which corresponds to the line graph of the representative (Eulerian or anti-Eulerian) graph \( H \) of the corresponding class from table 7.7. Each switching set is represented as a subset of the set of edges of \( H \).

For each graphs \( G \) in the second layer with \( 9 \leq n \leq 24 \) a graph \( H \) on 8 vertices is given by its edges, and the set of edges is partitioned into two subsets. Switching \( L(H) \) w.r.t. the vertices corresponding to any of these two subsets of edges yields \( G \).
### Table 9.1

**Graphs $G$ from $G$**

**First Layer**

<table>
<thead>
<tr>
<th>Class Nr.</th>
<th>Spectrum of $G$ in (0-1)-Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3.000$ $2.000$ $1.000$ $1.000$ $-1.000$ $-1.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$</td>
</tr>
<tr>
<td>Graph Nr. 1</td>
<td>Edges: $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, $10$</td>
</tr>
<tr>
<td>2</td>
<td>$3.000$ $1.000$ $1.000$ $0.000$ $0.000$ $0.347$ $-0.347$ $-1.532$ $-1.532$ $-2.000$ $-1.000$ $-2.000$ $-2.000$</td>
</tr>
<tr>
<td>Graph Nr. 2</td>
<td>Edges: $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, $10$, $11$</td>
</tr>
<tr>
<td>3</td>
<td>$3.000$ $2.000$ $1.000$ $1.000$ $0.000$ $-1.000$ $-1.000$ $1.000$ $-1.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$</td>
</tr>
<tr>
<td>Graph Nr. 3</td>
<td>Edges: $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, $10$, $11$, $12$</td>
</tr>
</tbody>
</table>

**Number of Vertices is 12, Degree is 4**

<table>
<thead>
<tr>
<th>Class Nr.</th>
<th>Spectrum of $G$ in (0-1)-Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$4.000$ $3.000$ $2.000$ $2.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$</td>
</tr>
<tr>
<td>Graph Nr. 4</td>
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<td>2</td>
<td>$4.000$ $3.000$ $2.000$ $2.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$</td>
</tr>
<tr>
<td>Graph Nr. 5</td>
<td>Edges: $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, $10$, $11$, $12$</td>
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<td>3</td>
<td>$4.000$ $3.000$ $2.000$ $2.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$ $1.000$</td>
</tr>
<tr>
<td>Graph Nr. 6</td>
<td>Edges: $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, $10$, $11$, $12$</td>
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</tbody>
</table>

**Number of Vertices is 10, Degree is 3**

<table>
<thead>
<tr>
<th>Class Nr.</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3.000$ $2.000$ $1.000$ $1.000$ $0.000$ $0.000$ $-1.000$ $-1.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$</td>
</tr>
<tr>
<td>Graph Nr. 7</td>
<td>Edges: $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, $10$, $11$, $12$</td>
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<tr>
<td>2</td>
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</tr>
<tr>
<td>Graph Nr. 8</td>
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</tr>
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<td>$3.000$ $2.000$ $1.000$ $1.000$ $0.000$ $-1.000$ $-1.000$ $1.000$ $-1.000$ $-2.000$ $-2.000$ $-2.000$ $-2.000$</td>
</tr>
<tr>
<td>Graph Nr. 9</td>
<td>Edges: $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, $10$, $11$, $12$</td>
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</tbody>
</table>
### Second Layer

**Number of Vertices is 9, Degree is 4**

**Graph NR. 104**
- Edges: 12, 3, 11, 6, 7, 10, 12, 1, 5, 3, 4, 8, 9
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

**Graph NR. 105**
- Edges: 12, 3, 11, 6, 7, 10, 12, 1, 5, 3, 4, 8, 9
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

---

**Number of Vertices is 12, Degree is 6**

**Graph NR. 164**
- Edges: 10, 9, 7, 6, 5, 4, 3, 2, 1, 1, 2, 3, 4, 5
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

**Graph NR. 165**
- Edges: 10, 9, 7, 6, 5, 4, 3, 2, 1, 1, 2, 3, 4, 5
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

---

**Number of Vertices is 15, Degree is 8**

**Graph NR. 171**
- Edges: 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

**Graph NR. 172**
- Edges: 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

**Graph NR. 173**
- Edges: 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

---

**Graph NR. 174**
- Edges: 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000

**Graph NR. 175**
- Edges: 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0
- Spectrum of G in (0, +1)-Language:
  - Degree 0: 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
  - Degree 1: 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000
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<th>SPECTRUM OF G IN (0,1)-LANGUAGE</th>
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SWITCHING SETS

FIRST LAYER

NUMBER OF VERTICES IS 12, DEGREE IS 4

CLASS NR. 2
GRAPH NR. 6
SWITCHING SET: 1 3 1 6 2 7 3 5

CLASS NR. 3
GRAPH NR. 7
SWITCHING SET: 3 4 1 4 3 6 2 5

CLASS NR. 4
GRAPH NR. 8
SWITCHING SET: 1 2 1 4 2 5 3 4 3 5 6 7

CLASS NR. 5
GRAPH NR. 9
SWITCHING SET: 1 6 2 7 3 8 4 5

CLASS NR. 6
GRAPH NR. 10
SWITCHING SET: 1 6 2 7 3 8 4 5

CLASS NR. 7
GRAPH NR. 11
SWITCHING SET: 1 6 2 7 3 8 4 5

CLASS NR. 8
GRAPH NR. 12
SWITCHING SET: 1 6 2 7 3 8 4 5

NUMBER OF VERTICES IS 14, DEGREE IS 5

CLASS NR. 10
GRAPH NR. 14
SWITCHING SET: 1 2 1 3 1 6 2 4 3 5

CLASS NR. 11
GRAPH NR. 15
SWITCHING SET: 1 2 1 3 2 3 4 6 4 7 5 6 5 7

GRAPH NR. 16
SWITCHING SET: 1 2 1 3 2 4 3 6 4 7 5 6 5 7

CLASS NR. 12
GRAPH NR. 17
SWITCHING SET: 1 2 1 3 2 3 2 5 3 7

GRAPH NR. 18
SWITCHING SET: 1 2 1 3 2 3 2 5 3 7

CLASS NR. 13
GRAPH NR. 19
SWITCHING SET: 1 2 1 3 2 3 4 6 4 7 5 6 5 7

GRAPH NR. 20
SWITCHING SET: 1 2 1 3 2 3 4 6 4 7 5 6 5 7

CLASS NR. 14
GRAPH NR. 21
SWITCHING SET: 1 2 1 3 2 4 3 5 4 6 5 7 6 8

CLASS NR. 15
GRAPH NR. 22
SWITCHING SET: 1 2 1 3 2 4 3 5 4 6 5 7 6 8

GRAPH NR. 23
SWITCHING SET: 1 2 1 3 2 4 3 5 4 6 5 7 6 8
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<td>1. 23 1. 44 1. 63 2. 51 3. 73</td>
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NUMBER OF VERTICES IS 16, DEGREE IS 6
CLASS NR. 22
GRAPH NR. 51
SWITCHING SET: 1 2 3 4 5 6 7 8
GRAPH NR. 52
SWITCHING SET: 1 2 3 4 5 6 7 8

CLASS NR. 23
GRAPH NR. 53
SWITCHING SET: 1 2 3 4 5 6 7 8
GRAPH NR. 54
SWITCHING SET: 1 2 3 4 5 6 7 8

CLASS NR. 24
GRAPH NR. 55
SWITCHING SET: 1 2 3 4 5 6 7 8
GRAPH NR. 56
SWITCHING SET: 1 2 3 4 5 6 7 8

CLASS NR. 25
GRAPH NR. 57
SWITCHING SET: 1 2 3 4 5 6 7 8
GRAPH NR. 58
SWITCHING SET: 1 2 3 4 5 6 7 8

CLASS NR. 26
GRAPH NR. 59
SWITCHING SET: 1 2 3 4 5 6 7 8
GRAPH NR. 60
SWITCHING SET: 1 2 3 4 5 6 7 8

NUMBER OF VERTICES IS 18. DEGREE IS 7
CLASS NR. 29

GRAPH NR. 01  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 02  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 03  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 04  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 05  SWITCHING SET: 1 2 3 3 2 3 3 7

CLASS NR. 30

GRAPH NR. 06  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 07  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 08  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 09  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 10  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 11  SWITCHING SET: 1 2 3 3 2 3 3 7

CLASS NR. 31

GRAPH NR. 96  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 97  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 98  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 99  SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 100 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 101 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 102 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 103 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 104 SWITCHING SET: 1 2 3 3 2 3 3 7

CLASS NR. 32

GRAPH NR. 105 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 106 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 107 SWITCHING SET: 1 2 3 3 2 3 3 7

NUMBER OF VERTICES IS 10, DEGREE IS 0

CLASS NR. 33

GRAPH NR. 108 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 109 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 110 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 111 SWITCHING SET: 1 2 3 3 2 3 3 7
GRAPH NR. 112 SWITCHING SET: 1 2 3 3 2 3 3 7
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<td>GRAPH NR. 114</td>
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</tr>
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| NUMBER OF VERTICES IS 12, DEGREE IS 9 |

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<td>GRAPH NR. 144</td>
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<td>GRAPH NR. 146</td>
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GRAPH NR. 147
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

CLASS NR. 37

GRAPH NR. 148
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 149
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 150
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 151
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 152
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

NUMBER OF VERTICES IS 24, DEGREE IS 10

CLASS NR. 38

GRAPH NR. 153
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 154
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 155
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 156
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 157
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 158
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 159
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 160
SWITCHING SET: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

SECOND LAYER

NUMBER OF VERTICES IS 19, DEGREE IS 6

CLASS NR. 39

GRAPH NR. 161
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 162
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 163
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

NUMBER OF VERTICES IS 12, DEGREE IS 6

CLASS NR. 40

GRAPH NR. 164
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 165
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 166
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

NUMBER OF VERTICES IS 15, DEGREE IS 8

CLASS NR. 41

GRAPH NR. 167
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 168
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 169
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 170
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

NUMBER OF VERTICES IS 18, DEGREE IS 8

CLASS NR. 42

GRAPH NR. 171
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 172
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 173
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

GRAPH NR. 174
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19
NUMBER OF VERTICES IS 18, DEGREE IS 10

GRAPH NR. 175
SUBSET 1: 1, 3, 5, 7, 9, 11, 13, 15, 17
SUBSET 2: 2, 4, 6, 8, 10, 12, 14, 16, 18

GRAPH NR. 176
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

GRAPH NR. 177
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

NUMBER OF VERTICES IS 21, DEGREE IS 12

GRAPH NR. 178
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

GRAPH NR. 179
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

GRAPH NR. 180
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

NUMBER OF VERTICES IS 24, DEGREE IS 14

GRAPH NR. 181
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

GRAPH NR. 182
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

GRAPH NR. 183
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18

GRAPH NR. 184
SUBSET 1: 1, 2, 3, 4, 5, 6, 7, 8, 9
SUBSET 2: 10, 11, 12, 13, 14, 15, 16, 17, 18
10. Proving the main theorems

Using the results from sections 4 through 9, we shall now prove the theorems given in section 1. Each proof is formulated in such a way that the references to computer search are reduced as much as possible.

Proof of theorem 1.3. By propositions 4.5 and 5.1.

Proof of theorem 1.4. According to theorem 3.5, all line graphs have the least eigenvalue $\geq -2$. Let us discuss first the case of strict inequality.

According to the same theorem, the line graph of a connected graph $H$ has the least eigenvalue $>-2$ if and only if $H$ has no circuits or exactly one circuit which is odd. The only graphs $H$ which satisfy this condition and which have a regular line graph are the complete bipartite graphs $K_{1,s}$ ($s = 1,2,\ldots$) and the circuits of odd length. Hence the only connected regular line graphs having the least eigenvalue $>-2$ are complete graphs and odd circuits. It is well known (see, for example, [4]) that these graphs are characterized by their spectra.

If the least eigenvalue is $-2$, then, according to theorem 2.5, all connected graphs which are not a line graph but which are cospectral with a line graph, are contained in $E_8$ or are cocktail-party graphs. The second possibility cannot appear, since the cocktail-party graphs are characterized by their spectra (see, for example, [4]). Concerning $E_8$ we have, according to proposition 2.8, to look only at line graphs with 6, 7, or 8 principal eigenvalues. Therefore, a survey of such line graphs will be given now.

According to corollary 3.7, we must consider line graphs of connected regular graphs on 8 and 7 vertices and of nonbipartite connected regular graphs on 6 vertices. In addition, the line graphs of connected semiregular bipartite graphs on 9, 8, and 7 vertices come into consideration. We shall exclude regular line graphs of degree 0,1,2, since they are characterized by their spectra (see, for example, [4]).

Regular line graphs are arranged in certain layers. Indeed, consider regular graphs on $v$ vertices of degree $k$. The corresponding line graphs have $n = \frac{1}{2}vk$ vertices and are regular of degree $d = 2k - 2$. Eliminating $k$ we get $n = \frac{1}{2}v(d + 2)$. For a fixed $v$ we have a layer.

Consider now semiregular bipartite graphs with parameters $v_1, v_2, k_1, k_2$. The line graphs have $n = v_1k_1 = v_2k_2$ vertices and are regular of degree
\[ d = k_1 + k_2 - 2. \] Again we have \( n = \frac{v_1 v_2}{v_1 + v_2} (d + 2) \), hence, a layer with fixed \( \frac{v_1 v_2}{v_1 + v_2} \).

We need the "intersection" of these layers with \( E_8 \). Without difficulties we can find the following table of regular connected line graphs of degree \( \geq 3 \), having 6, 7 or 8 principal eigenvalues.

<table>
<thead>
<tr>
<th>graphs</th>
<th>n</th>
<th>d</th>
<th>( d + 2 )</th>
<th>( \frac{n}{d + 2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L(K_5,4) )</td>
<td>20</td>
<td>7</td>
<td>8</td>
<td>20/9 ( \approx 2.22 )</td>
</tr>
<tr>
<td>( L(K_8) )</td>
<td>28</td>
<td>12</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>( L(CP(4)) )</td>
<td>24</td>
<td>10</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>3 line graph of regular graphs</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>of degree 5 on 8 vertices</td>
<td>20</td>
<td>8</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>6 line graphs of regular graphs</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>of degree 4 on 8 vertices</td>
<td>16</td>
<td>6</td>
<td>7 or 8</td>
<td>2</td>
</tr>
<tr>
<td>5 line graphs of connected cubic graphs on 8 vertices</td>
<td>12</td>
<td>4</td>
<td>7 or 8</td>
<td>2</td>
</tr>
<tr>
<td>( L(K_6,3) )</td>
<td>18</td>
<td>7</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>( L(Q) ) (see Fig. 1.6)</td>
<td>12</td>
<td>4</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>( L(K_5,3) )</td>
<td>15</td>
<td>6</td>
<td>7</td>
<td>15/8 ( \approx 1.87 )</td>
</tr>
<tr>
<td>( L(K_7) )</td>
<td>21</td>
<td>10</td>
<td>7</td>
<td>7/4 ( = 1.75 )</td>
</tr>
<tr>
<td>2 line graphs of regular graphs</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>of degree 4 on 7 vertices</td>
<td>14</td>
<td>6</td>
<td>7</td>
<td>7/4 ( = 1.75 )</td>
</tr>
<tr>
<td>( L(K_4,3) )</td>
<td>12</td>
<td>5</td>
<td>6</td>
<td>12/7 ( \approx 1.71 )</td>
</tr>
<tr>
<td>( L(K_7,2) )</td>
<td>14</td>
<td>7</td>
<td>8</td>
<td>14/9 ( \approx 1.55 )</td>
</tr>
<tr>
<td>( L(K_6) )</td>
<td>15</td>
<td>8</td>
<td>6</td>
<td>3/2 ( = 1.50 )</td>
</tr>
<tr>
<td>( L(CP(3)) )</td>
<td>12</td>
<td>6</td>
<td>6</td>
<td>3/2 ( = 1.50 )</td>
</tr>
<tr>
<td>( L(\tilde{C}_6) )</td>
<td>9</td>
<td>4</td>
<td>6</td>
<td>3/2 ( = 1.50 )</td>
</tr>
<tr>
<td>( L(K_6,2) )</td>
<td>12</td>
<td>6</td>
<td>7</td>
<td>3/2 ( = 1.50 )</td>
</tr>
<tr>
<td>( L(K_5,2) )</td>
<td>10</td>
<td>5</td>
<td>6</td>
<td>10/7 ( \approx 1.43 )</td>
</tr>
</tbody>
</table>
The graphs $L(K_{5,4})$, $L(K_{6,3})$ and $L(K_{7,2})$ have a specific position. The first and the third cannot be represented by vectors of types $a$ and $b$. However, $L(K_{6,3})$ can be represented by such vectors, as is shown in fig. 10.2.

![Fig. 10.2.](image)

According to lemma 3.10 and theorem 2.12 all line graphs from table 10.1, outside the layers, are characterized by their spectra, except for $L(K_{3,5})$ and for line graphs on 14 vertices of degree 6. $L(K_{3,5})$ is characterized by its spectrum, as proved in [6]. For $n = 14$, $d = 6$ there are no graphs in $G$ as we have proved by the computer search described in section 5.

Since we have found all graphs from $G$ in the second layer it can easily be seen by inspection that the line graphs in the second layer have no cospectral non-line-graphs. So only connected regular line graphs $L(H)$ in the first layer remain. But now we always have cospectral "mates". By remark 7.11, for regular $H$ we have to factorize $H$ into two regular factors in all possible ways and we obtain the exceptional graphs as stated in the theorem and explained in section 7. If $H = Q$ or $H = K_{6,3}$ we have to start with graphs nr. 3 or nr. 159 from table 7.7 and the exceptions can easily be constructed. This proves the theorem.

**Proof of theorem 1.7.** It is known that if a connected line graph $L(G)$ is regular then $G$ (assume $G$ is connected) is regular or semiregular bipartite. If $G_1$ and $G_2$ from the theorem are both regular or both semiregular bipartite we have the cases (i) and (ii) from the theorem; this readily follows from lemmas 3.8, 3.17 and 3.9.
Suppose therefore that \( \{G_1, G_2\} = \{H_1, H_2\} \) where \( H_2 \) is semiregular bipartite with parameters \( n_1, n_2, d_1, d_2 \) \((n_1 > n_2)\) and where \( H_1 \) is regular nonbipartite of degree \( d \) with \( n \) vertices. Since \( L(H_1) \) and \( L(H_2) \) are cospectral they must have the same degree, the same number of vertices and the same multiplicity of the eigenvalue \(-2\). This yields the following relations

\[
d_1 + d_2 - 2 = 2d - 2, \quad n_1 d_1 = \frac{nd}{2} (= n_2 d_2),
\]

i.e.

\[
(10.3) \quad d = \frac{d_1 + d_2}{2},
\]

\[
(10.4) \quad nd = 2n_1 d_1 = 2n_2 d_2,
\]

\[
(10.5) \quad n = n_1 + n_2 - 1.
\]

Substituting (10.3) and (10.5) in (10.4) we obtain

\[
(10.6) \quad n_1 - n_2 = \frac{d_1 + d_2}{d_2 - d_1}.
\]

Let \( \lambda_1, \lambda_2, \ldots, \lambda_{n_2} \) be the first \( n_2 \) largest eigenvalues of \( H_2 \). As mentioned in the proof of lemma 3.16, \( H_2 \) has also the eigenvalues \(-\lambda_1, -\lambda_2, \ldots, -\lambda_{n_2}\) and \( n_1 - n_2 \) eigenvalues equal to 0, where \( \lambda_1 = \sqrt{d_1 d_2} \). Since the sum of squares of eigenvalues is twice the number of edges we have

\[
2d_1 d_2 + 2 \sum_{i=2}^{n_2} \lambda_i^2 = 2n_1 d_1,
\]

\[
(10.7) \quad \sum_{i=2}^{n_2} \lambda_i^2 = n_1 d_1 - d_1 d_2.
\]

According to lemmas 3.8 and 3.17, the eigenvalues of \( H_1 \) are \( \frac{1}{2}(d_1 + d_2) \) with the multiplicity 1 (largest eigenvalue), \( \frac{1}{2}(d_1 - d_2) \) with the multiplicity \( n_1 - n_2 \) and \( \pm \sqrt{\lambda_i^2 + \frac{1}{4}(d_1 - d_2)^2} \), \( i = 2, 3, \ldots, n_2 \). The sum of eigenvalues must be 0 and this yields again the relation (10.6). Considering the sum of squares we have

\[
\left(\frac{d_1 + d_2}{2}\right)^2 + (n_1 - n_2) \left(\frac{d_1 - d_2}{2}\right)^2 + 2 \sum_{i=2}^{n_2} \left(\lambda_i^2 + \frac{(d_1 - d_2)^2}{4}\right) = 2n_1 d_1.
\]
Using (10.7) we get
\[ n_1 + n_2 = \left( \frac{d_1 + d_2}{d_2 - d_1} \right)^2. \]

Taking \( d_1 + d_2 = s(d_2 - d_1) \), \( s \) being an integer greater than 1, relations (10.6) and (10.8) yield
\[ n_1 = \frac{s^2 + s}{2} \quad \text{and} \quad n_2 = \frac{s^2 - s}{2}. \]

According to (10.3), \( d_1 \) and \( d_2 \) are of the same parity and since \( d_2 > d_1 \), we can take \( d_2 = d_1 + 2t \), \( t \) being a positive integer. Then
\[ d_1 = t(s - 1) \quad \text{and} \quad d_2 = t(s + 1). \]

Since \( d_1 \leq n_2 \) and \( d_2 \leq n_1 \) we see that \( t \leq s/2 \). The spectra of \( H_1 \) and \( H_2 \) can now easily be determined and the proof of the theorem is complete. \( \square \)

Remark 10.9. With \( s = 2 \) we have \( H_1 = K_3 \) and \( H_2 = K_{1,3} \), but then \( L(H_1) \) and \( L(H_2) \) are not only cospectral, but also isomorphic. (This is, by a theorem of Whitney, the only exception for \( L(G_1) = L(G_2) \Rightarrow (G_1 = G_2, G_1 \) and \( G_2 \) being connected). For \( s = 3 \) we have \( H_2 = Q \) (see fig. 1.6), but \( H_1 \) does not exist. For \( s = 4 \) and \( t = 2 \) we have \( H_2 = K_{10,6} \) and \( H_1 = L(K_6) \) and, of course, \( L(K_{10,6}) \) and \( L(L(K_6)) \) are cospectral but not isomorphic. In the case \( s = 4, \; t = 1, \)
\( H_2 \) belongs to the design with the parameters \( v = 6, \; b = 10, \; r = 5, \; k = 3, \)
\( \lambda = 2, \) and \( H_1 \) is the Petersen graph. For higher values of \( s \) in the known examples, \( H_2 \) is a 2-design. It would be interesting to know whether a pair of graphs \( H_1, H_2 \) from the theorem exist such that \( H_2 \) is not the graph of a 2-design.

Proof of theorem 1.8. Under the assumptions of the theorem \( G_2 \) can be a line graph or not.

In the first case we apply theorem 1.7. Then the case (ii) of theorem 1.7 yields the case (i) of theorem 1.8. In the case (iii) of theorem 1.7 we have \( n_1 = n, \; n_2 = v, \; d_1 = k, \; d_2 = r \) and the well-known relation \( \lambda(v-1) = r(k-1) \) yields \( (s-2)\lambda = 2t(st-t-1) \). Of course, \( \lambda \) must be an integer and we get the condition \( (s-2)|2t(t-1) \). We also have \( \lambda_i = \sqrt{r - \lambda} \) \((i = 2, 3, \ldots, |s(s-1)|)\). The spectrum of the regular graph \( H_1 \) from theorem 1.7 can easily be calculated and we obtain the case (vi) of theorem 1.8.
Only the fact that $st$ is even must still be proved. This can be done, like in [2], by calculating a diagonal entry of $A^3$, where $A$ is the adjacency matrix of $H_1$.

If $G_2$ is not a line graph then we apply theorem 1.4 and the cases (ii)-(v) from theorem 1.8 readily follow. This completes the proof.

**Proof of theorem 1.9.** The exceptional graphs again can be line graphs or not. If they are not line graphs, the cases (i) and (ii) of theorem 1.9 can immediately be found by theorem 1.4. The exceptions which are line graphs can be described by theorem 1.7. From $n_1 = d_2 = m$ and $n_2 = d_1 = n$ we have $t = s/2$ and $n_1 = 2t^2 + t$, $n = 2t^2 - t$. Since the eigenvalues of $K_{m,n}$ are $\pm \sqrt{mn}$ and 0, the spectrum of the graph $H_1$ from theorem 1.7 consists of eigenvalues $2t^2$, $t$ and its adjacency matrix $A$ satisfies $A^2 = t^2(I + J)$. Replacing the zeros of $A$ by $(-1)'s$, and bordering the matrix with $(-1)'s$, we obtain a symmetric Hadamard matrix with diagonal $-I$. This completes the proof.

**Proof of theorem 1.10.** By propositions 4.5 and 5.10 there are no graphs from $G$ outside of the layers. By proposition 5.2 all graphs from $G$ in the second and in the third layer are induced subgraphs of the Schlafli graph. We shall now prove that all graphs from $G$ in the first layer with the mentioned exceptions are induced subgraphs of some Chang graphs.

For $n = 28$, this trivially holds. There is no graph with $n = 26$. The 5 cubic graphs from the first layer are induced subgraphs of some Chang graphs which can be checked directly. Other graphs $G$ from $G$ in the first layer can be obtained by applying Algorithm 7.12 to some anti-Eulerian graphs $H$ from $EA$ (according to proposition 7.9). Let $H_1, H_2$ represent a regular factorization of $H$. Of course, $H$ is an induced subgraph of $K_8$. Let $\overline{H}$ be complement of $H$, hence $H_1 \cup H_2 \cup \overline{H} = K_8$.

If $H_2$ is regular (i.e. if $H$ is regular), the switching of $L(K_8)$ with respect to $L(H_2)$ yields, according to theorem 2.14, a Chang graph. This implies that the graph $G$ from $G$, obtained by switching $L(H_1 \cup H_2)$ w.r.t. $L(H_2)$, is an induced subgraph of that Chang graph.

Now let $H_2$ be not regular and suppose that $G$ is a subgraph of some Chang graph. Then, according to theorem 2.14, there exists a factorization of $\overline{H}$ into, say, the graphs $H_1'$ and $H_2'$ such that $H_2 \cup H_2'$ is regular (then, namely, the switching of $L(K_8)$ w.r.t. $L(H_2 \cup H_2')$ yields a Chang graph and $G$ is an induced subgraph of it). We shall show that the factorization $\overline{H} = H_1' \cup H_2'$ of $\overline{H}$ is a regular factorization.
If \( d_1, \ldots, d_8 \) are the vertex degrees of \( H \), then the vertex degrees of \( \bar{H} \) are \( d'_1 = 7 - d_1, \ldots, d'_8 = 7 - d_8 \). Vertex degrees \( y_1 \) of \( H_2 \) are given by (7.4). Suppose \( H_2 \cup H'_2 \) is regular of degree \( f \). Then the vertex degrees \( y'_1 \) of \( H'_2 \) are

\[
y'_1 = f - y_1 = f - \left( \frac{1}{2} d_1 - \frac{1}{4} (d + 2 - q) \right) = \frac{1}{2} (7 - d_1) + \frac{1}{4} (d + 2 - q) + f - \frac{7}{2}
\]

where \( \alpha \) does not depend on \( i \). Hence, \( \bar{H} \) is Eulerian and the factorization \( \bar{H} = H'_1 \cup H'_2 \) is regular.

According to proposition 6.7 and remark 6.9, all nonregular Eulerian graphs on 8 vertices (with an even number of edges) really have at least one regular factorization, except for the graph in fig. 6.8a). Hence, all graphs \( G \) are induced subgraphs of some Chang graphs except in the last case when \( H \) is the graph nr. 207 from table 7.7 and we get 5 exceptional graphs as mentioned in theorem 1.10. This completes the proof.

**Proof of theorem 1.11.** According to the propositions 4.5 and 5.10 there are no graphs from \( G \) outside the layers. The graphs in the layers have been found in sections 7 and 8 and they are displayed in table 9.1. This completes the proof.

**Proof of theorem 1.12.** This theorem is an immediate consequence of theorems 2.5 and 1.11. This completes the proof.
II. References


