Solution to Problem 82-3: A definite integral

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zur Algebra und Geometrie, to appear, contains the following statements:

THEOREM 1. $f_n(C) = 2^n - 1$.

THEOREM 2. $2 \leq f_2(K) \leq 12$.

Also solved by the proposer.

A Definite Integral

Problem 82-3, by A. J. JERRI (Clarkson College).

Prove that

$$\int_0^\infty y \frac{J_1(a(y, x))}{(y, x)} \cdot \frac{J_1(b(y, z))}{(y, z)} dy = \frac{J_1(b(z, x))}{(z, x)}, \quad 0 < b < a,$$

where $J_1(a(y, x))/(y, x)$ is the x -“generalized” J_0 -Hankel type translation of $J_1(ay)/y$,

$$\frac{J_1(a(y, x))}{(y, x)} \equiv \frac{ayJ_1(ay)J_0(ax) - axJ_1(ax)J_0(ay)}{y^2 - x^2}.$$

Solution by O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, the Netherlands).

From Watson [1, form. 5.11(8)] it is readily seen that

$$\frac{J_1(a(y, x))}{(y, x)} = \int_0^a sJ_0(ys)j_0(xs) ds.$$

Thus the required integral, denoted by I , can be expressed as

$$\begin{aligned} I &= \int_0^\infty y dy \int_0^a sJ_0(ys)J_0(xs) ds \int_0^b tJ_0(yt)J_0(zt) dt \\ &= \int_0^a sJ_0(xs) ds \int_0^\infty yJ_0(ys) dy \int_0^b tJ_0(yt)J_0(zt) dt. \end{aligned}$$

Here the inner double integral is evaluated by means of the inversion theorem for Hankel transforms [1, form. 14.3(2)],

$$\int_0^\infty yJ_0(ys) dy \int_0^b tJ_0(yt)J_0(zt) dt = \begin{cases} J_0(zs), & 0 < s < b, \\ 0, & s > b. \end{cases}$$

Since $0 < b < a$, we then find

$$I = \int_0^b sJ_0(xs)J_0(zs) ds = \frac{J_1(b(z, x))}{(z, x)}.$$

REFERENCE

- [1] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1958.

Solution by the proposer.

This problem is proved by utilizing the Parseval equation of the J_0 -Hankel transform and recognizing that $J_1(ay)/y$ and its “generalized” J_0 -Hankel translation $J_1(a(y, x))/(y, x)$ are the J_0 -Hankel transforms of $p_a(t)$ and $J_0(xt)p_a(t)$ respectively, where $p_a(t)$ is the gate function,

$$p_a(t) = \begin{cases} 1, & |t| < a, \\ 0, & |t| > a. \end{cases}$$

The same method can be used for proving the Hardy integral,

$$\int_{-\infty}^{\infty} \frac{\sin a(y-x)}{(y-x)} \frac{\sin b(y-z)}{(y-z)} dy = \frac{\pi \sin b(z-x)}{(z-x)}, \quad 0 < b < a,$$

where the Parseval equation for the Fourier transform is employed. It is obvious that this method can be utilized for proving more general integrals for the Hankel and for other transforms.

In a second solution, O. P. LOSSERS shows that the stated equation holds generally for complex x and z .

A Volume Problem

Problem 82-4, by M. K. LEWIS (Memorial University of Newfoundland).

An asymmetrically positioned hole of radius b is drilled at right angles to the axis of a solid right circular cylinder of radius a ($a > b$). If the distance between the axis of the drill and the axis of the cylinder is p , determine the volume of material drilled out.

Solution by J. BOERSMA, P. J. DE DOELDER and J. K. M. JANSEN (Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, the Netherlands).

Introduce Cartesian coordinates x, y, z , then the cylinder and the drill may be described by

$$(1) \quad C_a: x^2 + y^2 = a^2, \quad C_b: (x-p)^2 + z^2 = b^2.$$

The volume of the intersection of C_a and C_b is denoted by $V(a, b, p)$. Without loss of generality, we may assume $0 \leq b \leq a, p \geq 0$.

From a cross-section of C_a and C_b with the plane $z = 0$, it is readily seen that

$$(2) \quad V(a, b, p) = 2 \int \int_G [b^2 - (x-p)^2]^{1/2} dx dy,$$

where the domain of integration is

$$(3) \quad G = \{(x, y) | x^2 + y^2 \leq a^2, |x-p| \leq b\}.$$

We now distinguish three cases:

I. $0 \leq p \leq a - b$. In this case the integral (2) reduces to

$$(4) \quad V(a, b, p) = 4 \int_{p-b}^{p+b} [(a-x)(b+p-x)(x+b-p)(x+a)]^{1/2} dx.$$

The latter integral can be expressed in terms of elliptic integrals by means of Byrd and Friedman [1, form. 254.38]. Omitting the details of the tedious though straightforward calculation, we present the result

$$(5) \quad \begin{aligned} V(a, b, p) = & 4(a+b+p)^{-1/2} (a+b-p)^{-1/2} \\ & \cdot [1/6 (p^2 - 2ap - 2a^2 + 2b^2)(a+b+p)(p+a-b)K(k) \\ & + 1/6 (p^2 + 2a^2 + 2b^2)(a+b+p)(a+b-p)E(k) \\ & + (a^2 - b^2)p(p+a-b)\Pi(\alpha^2, k)], \end{aligned}$$

where

$$(6) \quad k^2 = \frac{4ab}{(a+b+p)(a+b-p)}, \quad \alpha^2 = \frac{2b}{a+b+p}.$$