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Part 1: Theoretical analysis

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ANALYSIS OF THE ASYMMETRIC SHORTEST QUEUE PROBLEM
PART I: THEORETICAL ANALYSIS

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Abstract. In this paper we study a system consisting of two parallel servers with different service rates. Jobs arrive according to a Poisson stream and generate an exponentially distributed workload. On arrival a job joins the shortest queue and in case both queues have equal lengths, he joins the first queue with probability $q$ and the second one with probability $1 - q$, where $q$ is an arbitrary number between 0 and 1. In a previous paper we showed that for the symmetric problem, that is for equal service rates and $q = \frac{1}{2}$, the equilibrium distribution of the lengths of the two queues can be represented by an infinite sum of product form solutions by using an elementary compensation procedure. The main purpose of this paper is to prove this product form result for the asymmetric problem by using a generalization of the compensation procedure. In a companion paper we show that the product form representation leads to a numerically highly attractive algorithm.

Keywords: difference equation, product form, queues in parallel, stationary queue length distribution, shortest queue problem.

Introduction

Consider a queueing system consisting of two parallel servers with different service rates. Jobs arrive according to a Poisson stream and generate an exponentially distributed workload. On arrival a job joins the shortest queue and in case both queues have equal lengths, he joins the first queue with probability $1 - q$ and the second one with probability $q$, where $q$ is an arbitrary number between 0 and 1. This problem is known as the asymmetric shortest queue problem, that is, the asymmetric variant of the symmetric problem with identical servers and routing probability $q = \frac{1}{2}$. Haight [7] originally introduced the problem. Kingman [9] and Flatto and

McKean [4] treated the symmetric problem by using a generating function analysis. They showed that the generating function for the equilibrium distribution of the lengths of the two queues is a meromorphic function. Then by the decomposition of the generating function into partial fractions, it follows that the equilibrium probabilities can be represented by an infinite sum of product form solutions. However, the decomposition leads to cumbersome formulae for the equilibrium probabilities and the method does not seem to be generalizable to the asymmetric problem. So far no useful theoretical results for the asymmetric problem are available in the literature.

In a previous paper [1] we showed that for the symmetric shortest queue problem, the equilibrium distribution of the lengths of the two queues can be found in an elementary way directly from the equilibrium equations. The key idea was a compensation procedure: the queue length distribution could be represented by an infinite sum of product form solutions, which was generated term by term, such that each term compensates the error, introduced by its preceding term, on one of the boundaries of the state space. In [2] we showed that this compensation procedure could be easily extended to the "simple" asymmetric shortest queue problem. By "simple" we mean a system with identical servers and routing probability \( q \neq \frac{1}{2} \). The purpose of this paper is to generalize the compensation procedure to the "hard" asymmetric problem in which the service rates are different as well. This generalization is not straightforward. For the symmetric problem we had to construct a solution of the equilibrium equations on essentially one region. For the asymmetric problem we distinguish between two different regions and the basic problem for the construction of the solutions on both regions, is how to compensate the errors introduced by both solutions on the common boundary of the two regions. Only for the simple asymmetric case this problem can be solved easily, because in that case the equilibrium equations in the interior of these two regions are still a mirror image of each other.

The main result is that, also for the hard asymmetric problem, the equilibrium distribution of the lengths of both queues can be exactly represented by an infinite sum of product form solutions. Because of the interaction of the solutions on both regions, this sum does not have a simple linear structure (as in the symmetric case), but a binary tree structure. Further, the compensation procedure yields nice recurrence relations for the successive terms in the infinite sum and we show that the terms decrease exponentially fast.

As we already mentioned, so far a theoretical study of the asymmetric shortest queue problem is lacking, but also useful numerical studies of this model are scarce in the literature. Grassmann [6] performed a numerical study of the steady state as well as the transient behaviour of the asymmetric system, but to restrict the state space he treated both queues as bounded. Treating one queue as bounded, Rao and Posner [10] showed that the equilibrium distribution can be expressed in a modified matrix-geometric form and they developed an efficient computational procedure. Hooghiemstra, Keane and Van de Ree [8] proposed a power series
method to calculate the stationary queue length distribution for fairly general multidimensional exponential queueing systems. Although the power series method works numerically satisfactory for the shortest queue problem, see e.g. Blanc [3], the theoretical basis for the application to this problem is still incomplete. A common disadvantage of these numerical approximations is that in general no bounds can be given for the error of the numerical results.

In a companion paper (part II) we will treat the numerical benefits of the product form structure. Due to the recurrence relations, the successive terms of the infinite sum of product form solutions can be easily calculated and the convergence of this infinite sum is exponentially fast. In part II we will derive bounds for the error of each partial sum. Based on these properties a numerically highly attractive algorithm is obtained.

The present paper is organized as follows. In section 1 we will present the equilibrium equations. Then, in the next section, we will derive the main result, which states that the equilibrium probabilities can be represented by an infinite sum of product form solutions. In the following three sections we complete the proof of the main result, particularly treating the convergence of the infinite sum of product form solutions. A summary of the results and some conclusions can be found in the final section.

1. Equilibrium equations

For simplicity of notation we suppose that the servers have service rates \( \gamma_1 \) and \( \gamma_2 \) respectively with \( \gamma_1 > 0, \gamma_2 > 0 \) and \( \gamma_1 + \gamma_2 = 2 \), the Poisson arrival process has a rate \( 2\rho \) with \( 0 < \rho < 1 \) and on arrival each job generates an exponentially distributed workload with unit mean. The parameter \( q \) denotes the probability that an arriving job is sent to the second queue in case both queues have equal lengths. The parallel queue system can be represented by a continuous time Markov process, with a state space consisting of the pairs \((m, n)\), \(m, n = 0, 1, \ldots\) where \(m\) and \(n\) are the lengths of the two queues. The transition rates are illustrated in figure 1.

Let \( \{p_{m,n}\} \) denote the equilibrium distribution of the lengths of the two queues. The equilibrium equations become for all \(n > m\)

\[
\begin{align*}
p_{m,n} (2\rho + 1) &= p_{m-1,n} 2\rho + p_{m,n+1} \gamma_2 + p_{m+1,n} \gamma_1 & \text{if } m > 0, n > m+1 \\
p_{m,m+1} (2\rho + 1) &= p_{m-1,m+1} 2\rho + p_{m,m+2} \gamma_2 + p_{m+1,m+1} \gamma_1 + p_{m,m} 2q\rho & \text{if } m > 0 \\
p_{0,n} (2\rho + \gamma_2) &= p_{0,n+1} \gamma_2 + p_{1,n} \gamma_1 & \text{if } n > 1 \\
p_{0,1} (2\rho + \gamma_2) &= p_{0,2} \gamma_2 + p_{1,1} \gamma_1 + p_{0,0} 2q\rho
\end{align*}
\]

and for all \(n < m\),
Figure 1: $(m, n)$ transition rate diagram

\[
\begin{align*}
    p_{m,n} &\; 2(p + 1) = p_{m,n-1,1} 2\rho + p_{m+1,n} \gamma_1 + p_{m,n+1} \gamma_2 & \text{if } n > 0, m > n+1 \\
    p_{n+1,n} &\; 2(p + 1) = p_{n+1,n-1} 2\rho + p_{n+2,n} \gamma_1 + p_{n+1,n+1} \gamma_2 + p_{n,n} 2(1-q)\rho & \text{if } n > 0 \\
    p_{m,0} &\; (2\rho + \gamma_1) = p_{m+1,0} \gamma_1 + p_{m,1} \gamma_2 & \text{if } m > 1 \\
    p_{1,0} &\; (2\rho + \gamma_1) = p_{2,0} \gamma_1 + p_{1,1} \gamma_2 + p_{0,0} 2(1-q)\rho \\
\end{align*}
\]

and on the diagonal,

\[
\begin{align*}
    p_{m,m} &\; 2(p + 1) = p_{m-1,m} 2\rho + p_{m,m+1} \gamma_2 + p_{m,m-1} 2\rho + p_{m+1,m} \gamma_1 & \text{if } m > 0 \\
    p_{0,0} &\; 2\rho = p_{0,1} \gamma_2 + p_{1,0} \gamma_1 \\
\end{align*}
\]

For the symmetric problem, that is $\gamma_1 = \gamma_2 = 1$ and $q = \frac{1}{2}$, we have $p_{m,n} = p_{n,m}$. Hence, for this special case the analysis can be restricted to the upper triangle and in [1] we proved that the equilibrium probabilities $p_{m,n}$ can be represented by an infinite sum of product form solutions. That is, there exist parameters $\zeta_i$ and $\theta_i$ and coefficients $c_i$ such that for all $n > m$

\[
p_{m,n} = p_{n,m} = \sum_{i=0}^{\infty} c_i \zeta_i^m \eta_i^n .
\]

For the asymmetric problem we can’t restrict the analysis to the upper triangle, but we have to construct a solution on the lower triangle as well, such that both solutions interact on the diagonal. In view of the product form results for the symmetric problem, we expect that there exist parameters $\zeta_i$, $\theta_i$, $\eta_i$ and $\xi_i$ and coefficients $c_i$ and $d_i$ such that for all $n > m$

\[
p_{m,n} = \sum_{i=0}^{\infty} c_i \zeta_i^m \eta_i^n
\]

and for all $m > n$
The main purpose of this paper is to prove these product form representations. As for the symmetric problem, we apply a coordinate transformation in the upper triangle as well as in the lower triangle. In the upper triangle \( n \geq m \) we apply the transformation \( s = m \) and \( r = n - m \) and in the lower triangle \( m \geq n \) the transformation \( s = n \) and \( r = m - n \). Then both upper and lower triangle in the \((m, n)\) plane are transformed into the first quadrant of the \((s, r)\) plane. In figure 2 we display the transformation of the transition diagram in the upper triangle \( n \geq m \) to the \((s, r)\) plane. An intuitive reason to work with these coordinates is that the solutions in the upper and lower triangle behave similar in the \(s\)-direction (that is, in the direction of the diagonal).

\[
p_{m,n} = \sum_{i=0}^{\infty} d_i \theta_i^m \xi_i^n.
\]

Define for all \( s \geq 0 \) and \( r \geq 0 \) the probabilities

\[
Q_{s,r} = p_{s,s+r},
\]

\[
q_{s,r} = p_{s+r,s}.
\]

Hence \( Q_{s,r} \) lives in the upper triangle and \( q_{s,r} \) in the lower triangle. By definition \( Q_{s,0} = q_{s,0} \).

From now on we make the convention that any upper case solution lives in the upper triangle and any lower case solution lives in the lower triangle. For convenience we give the set of equilibrium equations in the new coordinates \( s \) and \( r \). The set of equations for the probabilities \( p_{m,n} \) in the upper triangle \( n > m \) is transformed into the following set of equations for the probabilities \( Q_{s,r} \).

\[
Q_{s,r} (2p + 1) = Q_{s-1,r+1} 2p + Q_{s,r+1} y_2 + Q_{s+1,r-1} y_1 \quad \text{if } s > 0, r > 1 \quad (1)
\]

\[
Q_{s,1} (2p + 1) = Q_{s-1,2} 2p + Q_{s,2} y_2 + Q_{s+1,0} y_1 + Q_{s,0} 2q \rho \quad \text{if } s > 0 \quad (2)
\]

\[
Q_{0,r} (2p + y_2) = Q_{0,r+1} y_2 + Q_{1,r-1} y_1 \quad \text{if } r > 1 \quad (3)
\]
\[ Q_{0,1}(2p + \gamma_2) = Q_{0,2} \gamma_2 + Q_{1,0} \gamma_1 + Q_{0,0} 2q \rho \]  
and the set of equations in the lower triangle \( m > n \) is transformed into the following set of equations for the probabilities \( q_{s,r} \),

\[ q_{s,r} (2p + 1) = q_{s-1,r+1} 2p + q_{s,r+1} \gamma_1 + q_{s+1,r-1} \gamma_2 \]  
if \( s > 0, r > 1 \) \hspace{1cm} (5)

\[ q_{s,1} (2p + 1) = q_{s-1,2} 2p + q_{s,2} \gamma_1 + q_{s+1,0} \gamma_2 + q_{s,0} 2(1-q) \rho \]  
if \( s > 0 \) \hspace{1cm} (6)

\[ q_{0,r} (2p + \gamma_1) = q_{0,r+1} \gamma_1 + q_{1,r-1} \gamma_2 \]  
if \( r > 1 \) \hspace{1cm} (7)

\[ q_{0,1} (2p + \gamma_1) = q_{0,2} \gamma_1 + q_{1,0} \gamma_2 + q_{0,0} 2(1-q) \rho \]  
(8)

The equations on the diagonal become

\[ Q_{s,0} (2p + 1) = Q_{s-1,1} 2p + Q_{s,1} \gamma_2 + \gamma_1 2p + q_{s-1,1} \gamma_1 \]  
if \( s > 0 \) \hspace{1cm} (9)

\[ Q_{0,0} 2p = Q_{0,1} \gamma_2 + q_{0,1} \gamma_1 \]  
(10)

We will try to prove that there exist parameters \( \alpha_i, \beta_i \) and \( \delta_i \) and coefficients \( c_i \) and \( d_i \) such that for all \( s \geq 0, r \geq 1 , \)

\[ Q_{s,r} = \sum_{i=0}^{\infty} c_i \alpha_i^s \eta_i^r \]  
and

\[ q_{s,r} = \sum_{i=0}^{\infty} d_i \alpha_i^s \xi_i^r . \]  

Clearly the forms for \( p_{m,n}, Q_{s,r} \) and \( q_{s,r} \) are equivalent, with \( \alpha_i = \xi_i \eta_i = \theta_i \xi_i \). The forms for \( Q_{s,r} \) and \( q_{s,r} \) express that the behaviour of both solutions in the \( s \) direction is similar, because they use the same basis of \( \alpha \) factors, but the behaviour in the \( r \) direction differs essentially. Throughout the analysis we will use the trivial, but vital property that the equations, on which the analysis is based, are linear, i.e. if two functions satisfy an equation, then any linear combination also satisfies the equation.

2. Derivation of the main result

The objective in this section is to study the structure of the equilibrium probabilities. In particular we will investigate whether the equilibrium probabilities have some kind of separable structure. Let us start to forget about the equations in the corner of the state space, that is in \((0,0)\) and \((0,1)\). At the end of this section we pay attention to these equations. Inserting the equations on the line \( r = 0 \) we can eliminate the probabilities \( Q_{s,0} = q_{s,0} \) in the equations on the line \( r = 1 \). Then we obtain the following set of equations for the probabilities \( Q_{s,r} \).
This set of equations is the basis for the analysis of the probabilities $Q_{s,r}$ and $q_{s,r}$, $s \geq 0$, $r \geq 1$. The equations on the line $r = 0$ can be used as definition for the probabilities $Q_{s,0}$. Obviously, the above set of equations does not allow a separable solution of the form $Q_{s,r} = \alpha^s \beta^r$ and $q_{s,r} = \alpha^s \delta^r$. However, numerical experiments suggest that there exist $\alpha$, $\beta$, and $\delta$ such that for some constants $K$ and $L$,

$$Q_{s,r} - K \alpha^s \beta^r$$
$$q_{s,r} - L \alpha^s \delta^r$$

as $s \to \infty$ and $r \geq 1$. These asymptotic formulas again illustrate that the behaviour of $Q_{s,r}$ and $q_{s,r}$ is similar in the $s$ direction, but different in the $r$ direction. In figure 3 we illustrated the asymptotic behaviour for the special case $\rho = 0.5$, $\gamma_1 = 0.7$ and $\eta = 0.4$. The probabilities $Q_{s,r}$ and $q_{s,r}$ were computed by solving a finite capacity shortest queue system exactly, i.e. by means of a Markov chain analysis. In the example we computed the equilibrium distribution for a system where each queue has a maximal capacity of 15 jobs, which approximates well the infinite capacity system in case $\rho = 0.5$, $\gamma_1 = 0.7$ and $\eta = 0.4$. In figure 3a we displayed the ratios of $Q_{s,r}$ in the $s$ direction. In figure 3b the same is done for $q_{s,r}$. This yields, at least for large $s$, the parameter $\alpha$. In figure 3c and 3d we displayed the ratios of $Q_{s,r}$ and $q_{s,r}$ in the $r$ direction, yielding the parameters $\beta$ and $\delta$. 
Clearly, for the case $\rho = 0.5$, $\gamma_1 = 0.7$ and $q = 0.4$ we have that $Q_{s,r} \sim K 0.250^s 0.066^r$ and $q_{s,r} \sim L 0.250^s 0.138^r$ for some $K$ and $L$, which holds even for moderate $s$. The question is, what are in general the parameters $\alpha$, $\beta$ and $\delta$ and what is the relation between the constants $K$ and $L$?

Intuitively, $\alpha$ stands for the ratio of the probability that there are $m+2$ and $m$ jobs in the system. So a reasonable choice seems $\alpha = \rho^2$, which is supported by the numerical example. The parameters $\beta$ and $\delta$ follow by observing that the form $\alpha^s \beta^r$ has to satisfy equation (1) and $\alpha^s \delta^r$ has to satisfy equation (5). Inserting the form $\alpha^s \beta^r$ in (1) and dividing both sides by the common term $\alpha^{-1} \beta^{-1}$ we get a quadratic form for the unknown $\beta$. Similarly we obtain a quadratic form for the parameter $\delta$ by inserting $\alpha^s \delta^r$ in equation (5). This is stated in the following lemma.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$Q_{s,r+1}/Q_{s,r}$</th>
<th>$q_{s,r+1}/q_{s,r}$</th>
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<tbody>
<tr>
<td>0</td>
<td>0.066 0.066 0.066 0.066 0.066 0.066</td>
<td>0.138 0.138 0.138 0.138 0.138 0.138</td>
</tr>
<tr>
<td>1</td>
<td>0.068 0.067 0.066 0.066 0.066 0.066</td>
<td>0.139 0.138 0.138 0.138 0.138 0.138</td>
</tr>
<tr>
<td>2</td>
<td>0.074 0.068 0.066 0.066 0.066 0.066</td>
<td>0.142 0.139 0.139 0.138 0.138 0.138</td>
</tr>
<tr>
<td>3</td>
<td>0.105 0.074 0.068 0.067 0.066 0.066</td>
<td>0.157 0.142 0.139 0.139 0.138 0.138</td>
</tr>
<tr>
<td>4</td>
<td>0.110 0.074 0.068 0.067 0.066 0.066</td>
<td>0.233 0.158 0.143 0.139 0.139 0.138</td>
</tr>
<tr>
<td>5</td>
<td>0.110 0.074 0.068 0.067 0.066 0.066</td>
<td>0.138 0.138 0.138 0.138 0.138 0.138</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$s$</th>
<th>$Q_{s+1,r}/Q_{s,r}$</th>
<th>$q_{s+1,r}/q_{s,r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.209 0.245 0.250 0.250 0.250 0.250</td>
<td>0.171 0.223 0.243 0.248 0.250 0.250</td>
</tr>
<tr>
<td>1</td>
<td>0.210 0.245 0.250 0.250 0.250 0.250</td>
<td>0.172 0.223 0.243 0.248 0.250 0.250</td>
</tr>
<tr>
<td>2</td>
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<td>0.175 0.224 0.243 0.248 0.250 0.250</td>
</tr>
<tr>
<td>3</td>
<td>0.233 0.252 0.251 0.250 0.250 0.250</td>
<td>0.193 0.229 0.244 0.249 0.250 0.250</td>
</tr>
<tr>
<td>4</td>
<td>0.232 0.274 0.256 0.251 0.250 0.250</td>
<td>0.285 0.253 0.250 0.250 0.250 0.250</td>
</tr>
<tr>
<td>5</td>
<td>0.232 0.274 0.256 0.251 0.250 0.250</td>
<td>0.285 0.253 0.250 0.250 0.250 0.250</td>
</tr>
</tbody>
</table>
Lemma 1.

(i) The form $\alpha^2 \beta'$ is a solution of (1) if and only if $\alpha$ and $\beta$ satisfy the quadratic form

$$\alpha \beta 2(p + 1) = \beta^2 2p + \alpha \beta^2 \gamma_2 + \alpha^2 \gamma_1.$$  

(ii) The form $\alpha^2 \beta'$ is a solution of (5) if and only if $\alpha$ and $\beta$ satisfy the quadratic form

$$\alpha \beta 2(p + 1) = \beta^2 2p + \alpha \beta^2 \gamma_1 + \alpha^2 \gamma_2.$$  

For fixed $\alpha$ the quadratic form (14) is solved for

$$X_+ (\alpha) = \alpha \frac{p + 1 + \sqrt{(p + 1)^2 - (2p + \alpha \gamma_2) \gamma_1}}{2p + \alpha \gamma_2}$$

and for fixed $\beta$,

$$Y_+ (\beta) = \beta \frac{2(p + 1) - \beta \gamma_2 + \sqrt{(2(p + 1) - \beta \gamma_2)^2 - 8p \gamma_1}}{2 \gamma_1}.$$  

In the same way $x_+ (\alpha)$ are defined as the roots of (15) for fixed $\alpha$ and $y_+ (\beta)$ as the roots for fixed $\beta$. Putting $\alpha = \rho^2$ in (14) we obtain two roots $\beta = X_+ (\rho^2) = \rho$ and $\beta = X_- (\rho^2) = \rho^2 \gamma_1 / (2 + \rho \gamma_2)$. The root $\beta = \rho$ yields the asymptotic solution $Q_{s,t} \sim K \rho^{2t} \rho'$ for some $K$, which corresponds to the equilibrium distribution of two independent $M | M | 1$ queues, each with a workload $p$. It is very unlikely that the equilibrium distribution of the shortest queue problem behaves asymptotically like this distribution. Therefore the only reasonable choice is $\beta = \rho^2 \gamma_1 / (2 + \rho \gamma_2)$, which is also supported by the numerical example. In the same way we obtain $\delta = x_- (\rho^2) = \rho^2 \gamma_2 / (2 + \rho \gamma_1)$. Finally, what is the relation between the constants $K$ and $L$? Figure 3 shows that there is no boundary effect near $r = 1$. This suggests that the asymptotic solutions (13) satisfy the equations for $r = 1$. Inserting these asymptotic forms in equation (11) (or (12), which yields the same result) and using equation (14) we obtain the following relation between the constants $K$ and $L$,

$$K \rho \gamma_2 + 2(1 - q) \gamma_2 = L \rho \gamma_1 + 2q \gamma_1.$$  

Thus for some constant $M$,

$$K = M \frac{\rho \gamma_1 + 2q}{\gamma_1} \quad \text{and} \quad L = M \frac{\rho \gamma_2 + 2(1 - q)}{\gamma_2}.$$  

Hence we empirically found that for some $M$,

$$Q_{s,t} \sim M \frac{\rho \gamma_1 + 2q}{\gamma_1} \rho^{2t} \left[ \frac{\rho^2 \gamma_1}{2 + \rho \gamma_2} \right]^t.$$
Define the constants $a_0 = p^2$, $a_1 = x_-(a_0)$ and $f_u = x_+(a_0)$ and the coefficients $d_1 = (p \gamma_1 + 2q)/\gamma_1$ and $d_2 = (p \gamma_2 + 2(1-q))/\gamma_2$. As is illustrated in figure 3 for the special case $p = 0.5$, $\gamma_1 = 0.7$ and $q = 0.4$, the behaviour of the equilibrium probabilities in the interior of the set $\{(s, r), s \geq 0, r \geq 1\}$ as well as near the boundary $r = 1$ is perfectly described by the asymptotic solutions $X_{s,r} = d_1 a_0 a_1$ and $x_{s,r} = d_2 a_0 a_1$, but the behaviour near the boundary $s = 0$ is not captured by them. We have chosen the constants $a_0$, $a_1$ and $f_u$ and the coefficients $d_1$ and $d_2$ such that the asymptotic solutions $X_{s,r} = d_1 a_0 a_1$ and $x_{s,r} = d_2 a_0 a_1$ satisfy the equations (1), (11), (5) and (12), Further, one easily verifies that the solution $X_{s,r}$ violates equation (3) and $x_{s,r}$ equation (7) on the boundary $s = 0$. Obviously we can further improve these asymptotic solutions by adding a term to correct the error on the boundary $s = 0$.

0.1. Compensation on the boundaries $s = 0$

Form the linear combination $X_{s,r} = d_1 a_0 a_1 + d_1 c_1 a_0 a_1'$. We will try to choose $c_1$, $a_0$ and $a_1'$ such that this linear combination satisfies equation (3) and (1). Inserting $X_{s,r}$ in equation (3) and dividing both sides by $d_1$, gives for all $r > 1$

$$\left(\beta_1' + c_1 a_0 a_1'\right) (2p + \gamma_2) = \left(\beta_1'^{+1} + c_1 a_0 a_1'^{+1}\right) \gamma_2 + (\alpha_0 a_1^{-1} + c_1 a_0 a_1'^{-1}) \gamma_1.$$  

Since this must hold for all $r > 1$, we have to put $\beta = \beta_1$. Further we want $\alpha a_0 a_1'$ to satisfy equation (1). By virtue of lemma 1 there are two $\alpha$'s such that $\alpha a_0 a_1'$ satisfies equation (1), namely $\alpha_0 = Y_+(\beta_1) = p^2$ and $\alpha_1 = Y_-(\beta_1) = 2p^3 \gamma_1/(2 + p \gamma_2)$. So we have to put $\alpha = \alpha_1$. Then for any $c_1$, the linear combination $X_{s,r} = d_1 a_0 a_1' + d_1 c_1 a_0 a_1'$ satisfies equation (1), because equation (1) is linear. Finally, dividing the above equation by the common term $\beta_1'^{-1}$ gives an equation for the unknown $c_1$. Hence we can choose the coefficient $c_1$ such that the linear combination $X_{s,r}$ also satisfies equation (3). In exactly the same way we can choose a coefficient $c_2$ such that the linear combination $x_{s,r} = d_2 a_0 a_1' + d_2 c_2 a_0 a_1'$ satisfies the equations (5) and (7), where $\alpha_0 = y_- (\beta_2)$.

In general, the result of this procedure can be stated as

Lemma 2.

(i) The linear combination $X_{s,r} = k_1 Y_+ (\beta)^f \beta' + k_2 Y_- (\beta)^f \beta'$ satisfies the equations (1) and (3) if $k_1$ and $k_2$ satisfy

$$k_2 = -\frac{Y_- (\beta) - \beta}{Y_+ (\beta) - \beta} \cdot k_1.$$  

(16)
(ii) The linear combination \( x_{s,r} = k_1 y_+ (\beta)' \beta' + k_2 y_-(\beta)' \beta' \) satisfies the equations (5) and (7) if \( k_1 \) and \( k_2 \) satisfy (16).

Proof.

We prove (i); the proof of (ii) is similar. By virtue of lemma 1 the forms \( Y_+ (\beta)' \beta' \) and \( Y_- (\beta)' \beta' \) both satisfy equation (1). Since equation (1) is linear, any linear combination also satisfies (1). Inserting the linear combination \( x_{s,r} = k_1 Y_+ (\beta)' \beta' + k_2 Y_- (\beta)' \beta' \) in (3) and dividing by the common term \( \beta'^{-1} \) gives

\[
(k_1 + k_2) \beta (2p + \gamma_2) = (k_1 + k_2) \beta^2 \gamma_2 + k_1 Y_+ (\beta) \gamma_1 + k_2 Y_- (\beta) \gamma_1,
\]

which can be rewritten as

\[
k_2 = \frac{\beta (2p + \gamma_2) - \beta^2 \gamma_2 - Y_+ (\beta) \gamma_1}{\beta (2p + \gamma_2) - \beta^2 \gamma_2 - Y_- (\beta) \gamma_1}, \tag{17}
\]

Since \( Y_- (\beta) \) and \( Y_+ (\beta) \) are the roots of (14),

\[
Y_- (\beta) \gamma_1 + Y_+ (\beta) \gamma_1 = \beta 2(p + 1) - \beta^2 \gamma_2.
\]

Substituting in (17) gives relation (16).

Applying lemma 2(i) with \( \beta = \beta_1 \), \( k_1 = d_1 \) and \( k_2 = d_1 c_1 \), yields

\[
c_1 = \frac{\alpha_1 - \beta_1}{\alpha_0 - \beta_1},
\]

and applying lemma 2(ii) with \( \beta = \beta_2 \), \( k_1 = d_2 \) and \( k_2 = d_2 c_2 \),

\[
c_2 = \frac{\alpha_2 - \beta_2}{\alpha_0 - \beta_2}.
\]

Then the linear combination \( x_{s,r} = d_1 \alpha_0 \beta_1 + d_1 c_1 \alpha_1 \beta_1 \) satisfies equation (1) and (3) and \( x_{s,r} = d_2 \alpha_0 \beta_2 + d_2 c_2 \alpha_2 \beta_2 \) satisfies equation (5) and (7). For the special case of \( \rho = 0.5 \), \( \gamma_1 = 0.7 \) and \( q = 0.4 \) we displayed in figure 4 the same ratios as in figure 3 for the asymptotic solutions \( x_{s,r} \) and \( x_{s,r} \).

Comparing figure 3 and 4, we see that this refinement also captures the behaviour of the equilibrium probabilities near the boundary \( s = 0 \). We conclude that for some \( M \)

\[
Q_{m,r} \sim M \frac{d_1 (\alpha_0 \beta_1' + d_1 c_1 \alpha_1' \beta_1')}{d_1 (\alpha_0 \beta_1' + d_1 c_1 \alpha_1' \beta_1')}, \quad q_{m,r} \sim M \frac{d_2 (\alpha_0 \beta_2' + d_2 c_2 \alpha_2' \beta_2')}{d_2 (\alpha_0 \beta_2' + d_2 c_2 \alpha_2' \beta_2')}
\]

as \( s + r \to \infty \) and \( r \geq 1 \).
We added an extra term to $X_{s,r}$ and $x_{s,r}$ to compensate the error on the boundary $s = 0$. This compensation is identical to the one in [1] for the symmetric case. On the other hand we introduced a new error on the boundary $r = 1$, since the extra terms violate the equations (11) and (12). The compensation of the errors on this boundary forms the key problem in this paper. As we showed in [1], for the symmetric problem the compensation on the boundary $r = 1$ is essentially the same as the compensation on the boundary $s = 0$, but for the asymmetric problem it is different, because the solutions in the upper and lower triangle interact on the boundary $r = 1$.

Because $\alpha_1 < \alpha_0$ the term $\alpha_1 \beta^r_1$ is very small compared to $\alpha_0 \beta^r_1$ even for moderate $s$. Therefore its disturbing effect on the boundary $r = 1$ is practically negligible. However this second order error can be compensated by adding an extra correction term to the solution in the

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<tr>
<th>Figure 4a: The ratios $X_{s+1,r} / X_{s,r}$</th>
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<th>Figure 4c: The ratios $X_{s,r+1} / X_{s,r}$</th>
<th>Figure 4d: The ratios $x_{s,r+1} / x_{s,r}$</th>
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| 0.138 | 0.138 | 0.138 | 0.138 | 0.138 | 0.138 |

| 0.066 | 0.066 | 0.066 | 0.066 | 0.066 | 0.066 |
| 0.138 | 0.138 | 0.138 | 0.138 | 0.138 | 0.138 |
upper triangle and an extra one to the solution in the lower triangle, that is, to $X_{s,r}$ and $x_{s,r}$. Hence, the compensation of the term $\alpha_1 \beta_1^r$, which lives in the upper triangle, does effect the solution in the lower triangle! In the same way we can correct the error introduced by the term $\alpha_2 \beta_2^r$. So we have to add a total of four extra terms. The reason why a total of two extra terms, one in upper and the other in the lower triangle, is not enough will become clear from the discussion below.

2.2. Compensation on the boundaries $r = 1$

Let us start to compensate the error introduced by the term $d_1 c_1 \alpha_1 \beta_1^r$. Form the linear combinations $X_{s,r} = d_1 c_1 \alpha_1 \beta_1^r + d_3 \alpha_2 \beta^r$ and $x_{s,r} = d_4 \delta^r \eta^r$. We will try to choose the parameters $d_3$, $\alpha$, $\beta$, $d_4$, $\delta$ and $\eta$ such that $X_{s,r}$ and $x_{s,r}$ satisfy the equations (1), (11), (5) and (12). For the time being, we forget about the term $d_1 \alpha_1 \beta_1^r$ in $X_{s,r}$ and $d_2 \alpha_2 \beta_2^r$ in $x_{s,r}$. These terms already satisfy the equations (1), (11), (5) and (12). When they are added to $X_{s,r}$ and $x_{s,r}$ afterwards, then $X_{s,r}$ and $x_{s,r}$ still satisfy the equations (1), (11), (5) and (12), because of the linearity of these equations. Inserting $X_{s,r}$ and $x_{s,r}$ in (11) and (12) gives for all $s > 0$

\[(d_1 c_1 \alpha_1 \beta_1 + d_3 \alpha_2 \beta^r) 2(p + 1) = d_1 c_1 \alpha_1 \beta_1^r (2p + \alpha_1 \gamma_2) + d_3 \alpha_2 \beta^r (2p + \alpha_2 \gamma_2) + \frac{\gamma_1}{2(p + 1)} \]

\[+ (d_1 c_1 \alpha_1 \beta_1 (2p + \alpha_1 \gamma_2) + d_3 \alpha_2 \beta (2p + \alpha_2 \gamma_2) + d_4 \delta^r \eta (2p + \delta \gamma_1)) \frac{2qp}{2(p + 1)} \]

and

\[d_4 \delta^r \eta 2(p + 1) = d_4 \delta^r \eta (2p + \gamma_1) + \]

\[+ (d_1 c_1 \alpha_1 \beta_1 (2p + \alpha_1 \gamma_2) + d_3 \alpha_2 \beta (2p + \alpha_2 \gamma_2) + d_4 \delta^r \eta (2p + \delta \gamma_1)) \frac{\gamma_2}{2(p + 1)} \]

\[+ (d_1 c_1 \alpha_1 \beta_1 (2p + \alpha_1 \gamma_2) + d_3 \alpha_2 \beta (2p + \alpha_2 \gamma_2) + d_4 \delta^r \eta (2p + \delta \gamma_1)) \frac{2(1-q)p}{2(p + 1)} \]

Since this must hold for all $s > 0$, we have to put $\alpha = \delta = \alpha_1$. Hence, the correction terms must have the same $\alpha$ factor as the error term $\alpha_1 \beta_1^r$. This is exactly the reason why we can not compensate the errors introduced by the terms $\alpha_1 \beta_1^r$ and $\alpha_2 \beta_2^r$ simultaneously by adding only two extra terms, because both terms have a different $\alpha$ factor! Further we want $\alpha_1 \beta^r$ to satisfy equation (1). By virtue of lemma 1 there are there are two $\beta$'s such that $\alpha_1 \beta^r$ satisfies equation (1), namely $\beta_1 = X_+ (\alpha_1)$ and $\beta_2 = X_- (\alpha_1)$. So we have to put $\beta = \beta_3$. We also want $\alpha_1 \eta^r$ to satisfy equation (5). Again by virtue of lemma 1 there are two $\eta$'s such that $\alpha_1 \eta^r$ satisfies equation (5). Since we want the correction term $d_4 \alpha_1 \eta^r$ to be as small as possible, we put $\eta = \beta_4 = X_- (\alpha_1)$. Then for any $d_3$ and $d_4$, the linear combinations $X_{s,r}$ and $x_{s,r}$ satisfy the equations (1) and (5). Finally, dividing the above equations by the common term $\alpha_1 \beta_1^r$ gives two
equations for the unknowns \(d_3\) and \(d_4\). Hence we can choose the coefficients \(d_3\) and \(d_4\) such that the linear combinations \(X_{s,r}\) and \(x_{s,r}\) satisfy all the equations (1), (11), (5) and (12). In exactly the same way we can compensate the error introduced by the term \(d_2 c_2 \alpha \beta_2^2\): we can choose coefficients \(d_5\) and \(d_6\) such that the linear combinations \(X_{s,r}\) and \(x_{s,r}\) satisfy the equations (1), (11), (5) and (12), where \(\beta_6 = x_{-}(\alpha_2)\) and \(\beta_5 = X_{-}(\alpha_2)\).

In general, the result can be stated as

Lemma 3.

(i) The linear combinations \(X_{s,r} = k_1 \alpha \beta_1 X_{+}(\alpha)^r + k_2 \alpha \beta_2 X_{-}(\alpha)^r\) and \(x_{s,r} = k_3 \alpha \beta_3 x_{-}(\alpha)^r\) satisfy the equations (1), (11), (5) and (12) if \(k_1, k_2\) and \(k_3\) satisfy

\[
\begin{align*}
k_2 &= - \frac{(\alpha \gamma_1 + 2qp)/X_{-}(\alpha) + (\alpha \gamma_2 + 2(1-q)p)/X_{+}(\alpha) - 2(p+1)}{(\alpha \gamma_1 + 2qp)/X_{+}(\alpha) + (\alpha \gamma_2 + 2(1-q)p)/X_{-}(\alpha) - 2(p+1)} k_1, \\
k_3 &= - \frac{\gamma_1 (\alpha \gamma_2 + 2(1-q)p) (1/X_{-}(\alpha) - 1/X_{+}(\alpha))}{\gamma_2 ((\alpha \gamma_1 + 2qp)/X_{+}(\alpha) + (\alpha \gamma_2 + 2(1-q)p)/X_{-}(\alpha) - 2(p+1))} k_1.
\end{align*}
\]

(ii) The linear combinations \(X_{s,r} = k_3 \alpha \beta_3 X_{-}(\alpha)^r\) and \(x_{s,r} = k_1 \alpha \beta_1 X_{+}(\alpha)^r + k_2 \alpha \beta_2 X_{-}(\alpha)^r\) satisfy the equations (1), (11), (5) and (12) if \(k_1, k_2\) and \(k_3\) satisfy

\[
\begin{align*}
k_2 &= - \frac{(\alpha \gamma_1 + 2qp)/X_{+}(\alpha) + (\alpha \gamma_2 + 2(1-q)p)/X_{-}(\alpha) - 2(p+1)}{(\alpha \gamma_1 + 2qp)/X_{+}(\alpha) + (\alpha \gamma_2 + 2(1-q)p)/X_{-}(\alpha) - 2(p+1)} k_1, \\
k_3 &= - \frac{\gamma_1 (\alpha \gamma_1 + 2qp) (1/X_{-}(\alpha) - 1/X_{+}(\alpha))}{\gamma_2 ((\alpha \gamma_1 + 2qp)/X_{+}(\alpha) + (\alpha \gamma_2 + 2(1-q)p)/X_{-}(\alpha) - 2(p+1))} k_1.
\end{align*}
\]

Proof.

We prove (i); the proof of (ii) is similar. By virtue of lemma 1(i) both \(\alpha \beta_1 X_{+}(\alpha)^r\) and \(\alpha \beta_2 X_{-}(\alpha)^r\) satisfy (1) and by linearity, also \(X_{s,r}\). Similarly, \(x_{s,r}\) satisfies equation (5). Inserting \(X_{s,r}\) and \(x_{s,r}\) in (11) and (12) and dividing both sides by the common term \(\alpha \beta_1^{r-1}\) yields

\[
(k_1 \alpha X_{+}(\alpha) + k_2 \alpha X_{-}(\alpha)) 2(p+1) = (k_1 \alpha X_{+}(\alpha)^2 + k_2 X_{-}(\alpha)^2) (2p + \alpha \gamma_2) +
\]

\[
+ ((k_1 X_{+}(\alpha) + k_2 X_{-}(\alpha)) (2p + \alpha \gamma_2) + k_3 X_{-}(\alpha) (2p + \alpha \gamma_1)) \frac{\alpha \gamma_1}{2(p+1)}
\]

\[
+ ((k_1 X_{+}(\alpha) + k_2 X_{-}(\alpha)) (2p + \alpha \gamma_2) + k_3 X_{-}(\alpha) (2p + \alpha \gamma_1)) \frac{2qp}{2(p+1)}
\]

and

\[
k_4 \alpha x_{-}(\alpha) 2(p+1) = k_4 x_{-}(\alpha)^2 (2p + \alpha \gamma_1) +
\]
By inserting equation (14) and (15) and multiplying both sides with $2(p + 1)$ this reduces to

\[(k_1 + k_2) \alpha^2 \gamma_1 \, 2(p + 1) = (k_1 X_+(\alpha) + k_2 X_-(\alpha)) (2p + \alpha \gamma_2) (\alpha \gamma_1 + 2qp) + k_3 x_-(\alpha) (2p + \alpha \gamma_1) (\alpha \gamma_1 + 2qp)\]

and

\[k_3 \alpha^2 \gamma_2 \, 2(p + 1) = (k_1 X_+(\alpha) + k_2 X_-(\alpha)) (2p + \alpha \gamma_2) (\alpha \gamma_2 + 2(1 - q)p) + k_3 x_-(\alpha) (2p + \alpha \gamma_1) (\alpha \gamma_2 + 2(1 - q)p)\]

The equations (20) and (21) immediately yield

\[k_3 \gamma_2 (\alpha \gamma_1 + 2qp) = (k_1 + k_2) \gamma_1 (\alpha \gamma_2 + 2(1 - q)p) .\]

Multiplying both sides of (20) by $\gamma_2$ and inserting (22) in (20) gives

\[k_2 = -k_1 \times \frac{X_+(\alpha) \gamma_2 (2p + \alpha \gamma_2) (\alpha \gamma_1 + 2qp) + x_-(\alpha) \gamma_1 (2p + \alpha \gamma_1) (\alpha \gamma_2 + 2(1 - q)p) - \alpha^2 \gamma_1 \gamma_2 \, 2(p + 1)}{X_-(\alpha) \gamma_2 (2p + \alpha \gamma_2) (\alpha \gamma_1 + 2qp) + x_-(\alpha) \gamma_1 (2p + \alpha \gamma_1) (\alpha \gamma_2 + 2(1 - q)p) - \alpha^2 \gamma_1 \gamma_2 \, 2(p + 1)}\]

Since $X_+(\alpha)$ and $X_-(\alpha)$ are the roots of the quadratic form (14), and $x_+(\alpha)$ and $x_-(\alpha)$ of (15),

\[X_+(\alpha) X_-(\alpha) (2p + \alpha \gamma_2) = \alpha^2 \gamma_1 , \quad x_+(\alpha) x_-(\alpha) (2p + \alpha \gamma_1) = \alpha^2 \gamma_2 .\]

Using the relations (23) to rewrite $X_+(\alpha), X_-(\alpha)$ and $x_-(\alpha)$ and dividing the numerator and denominator by $\alpha^2 \gamma_1 \gamma_2$ yields relation (18). Relation (19) follows by inserting (18) in (22).

Recall that at the beginning of this section we eliminated the numbers $X_{s,0} = x_{s,0}$ by inserting the equations on the line $r = 0$ in the equations on the line $r = 1$. The reason was to simplify the compensation on the common boundary of the upper and lower triangle. Using the equations on the line $r = 0$ as definition for $X_{s,0}$ we obtain

**Corollary**

(i) For all $s \geq 0$ and $r \geq 1$, let $X_{s,r} = k_1 \alpha^r X_+(\alpha)^r + k_2 \alpha^r X_-(\alpha)^r$ and $x_{s,r} = k_3 \alpha^r x_-(\alpha)^r$, where the coefficients $k_1, k_2$ and $k_3$ satisfy the relations of lemma 3(i). Then for all $s > 0$

\[X_{s,0} = (k_1 + k_2) \frac{\alpha^{s+1} \gamma_1}{\alpha \gamma_1 + 2qp} = k_3 \frac{\alpha^{s+1} \gamma_2}{\alpha \gamma_2 + 2(1 - q)p} .\]
For all $s \geq 0$ and $r \geq 1$, let $X_{s,r} = k_3 \alpha^s X_\gamma(\alpha^r)$ and $x_{s,r} = k_1 \alpha^s x_\gamma(\alpha^r)$, where the coefficients $k_1$, $k_2$ and $k_3$ satisfy the relations of lemma 3(ii). Then for all $s > 0$

$$X_{s,0} = k_3 \frac{\alpha^{s+1} \gamma_1}{\alpha \gamma_1 + 2q \rho} = (k_1 + k_2) \frac{\alpha^{s+1} \gamma_2}{\alpha \gamma_2 + 2(1 - q) \rho}.$$

Proof.

We prove (i); the proof of (ii) is similar. Inserting $X_{s,r}$ and $x_{s,r}$ in equation (9), we obtain

$$X_{s,0} = ((k_1 X_\gamma(\alpha) + k_2 X_\gamma(\alpha))(2 \rho + \alpha \gamma_2) + k_3 x_\gamma(\alpha)(2 \rho + \alpha \gamma_1)) \frac{\alpha^{s-1}}{2(\rho + 1)}.$$

The first equality follows by inserting (20) and the second one by inserting (21) or from the first equality by inserting (22). □

Applying lemma 3(i) with $\alpha = \alpha_1$, $k_1 = d_1 c_1$, $k_2 = d_3$ and $k_3 = d_4$, yields

$$d_3 = -\frac{(\alpha_1 \gamma_1 + 2q \rho)/X_\gamma(\alpha_1) + (\alpha_1 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_1) - 2(\rho + 1)}{(\alpha_1 \gamma_1 + 2q \rho)/X_\gamma(\alpha_1) + (\alpha_1 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_1) - 2(\rho + 1)} d_1 c_1,$$

and

$$d_4 = -\frac{\gamma_2 ((\alpha_1 \gamma_1 + 2q \rho)/X_\gamma(\alpha_1) + (\alpha_1 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_1) - 2(\rho + 1))}{\gamma_2 ((\alpha_1 \gamma_1 + 2q \rho)/X_\gamma(\alpha_1) + (\alpha_1 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_1) - 2(\rho + 1))} d_1 c_1.$$

Applying lemma 3(ii) with $\alpha = \alpha_2$, $k_1 = d_5$, $k_3 = d_2 c_2$, and $k_4 = d_6$ yields

$$d_5 = -\frac{\gamma_2 ((\alpha_2 \gamma_1 + 2q \rho)/X_\gamma(\alpha_2) + (\alpha_2 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_2) - 2(\rho + 1))}{\gamma_2 ((\alpha_2 \gamma_1 + 2q \rho)/X_\gamma(\alpha_2) + (\alpha_2 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_2) - 2(\rho + 1))} d_2 c_2,$$

and

$$d_6 = -\frac{(\alpha_2 \gamma_1 + 2q \rho)/X_\gamma(\alpha_2) + (\alpha_2 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_2) - 2(\rho + 1)}{(\alpha_2 \gamma_1 + 2q \rho)/X_\gamma(\alpha_2) + (\alpha_2 \gamma_2 + 2(1 - q) \rho)/x_\gamma(\alpha_2) - 2(\rho + 1))} d_2 c_2.$$

Then the total linear combinations $X_{s,r} = d_1 \alpha_0 \beta_1 + d_1 c_1 \alpha_1 \beta_1 + d_3 \alpha_1 \beta_2 + d_5 \alpha_1 \beta_5$ and $x_{s,r} = d_2 \alpha_2 \beta_2 + d_2 c_2 \alpha_2 \beta_5 + d_4 \alpha_2 \beta_5 + d_6 \alpha_2 \beta_6$ satisfy the equations (1), (11), (5) and (12). Now we compensated the errors on the boundary $r = 1$, but we introduced new errors on the boundary $s = 0$, since the extra compensation terms in the upper triangle, that is $d_3 \alpha_1 \beta_2$ and $d_5 \alpha_1 \beta_5$, violate equation (3) and the extra compensation terms in the lower triangle violate equation (7). But it is clear how to continue this compensation procedure, which consists of two basic compensation steps:

(i) A term which lives in the upper triangle say, can be compensated on the boundary $s = 0$ by adding an extra compensation term in the upper triangle according to lemma 2(i).

(ii) A term which lives in the upper triangle say, can be compensated on the boundary $r = 1$ by adding two extra compensation terms, one in the upper triangle and the other in the...
lower triangle, according to lemma 3(i).
In this way we produce a complete binary tree of compensation terms. Below we will formalize this compensation procedure.

2.3. The compensation procedure
Generate a tree of numbers $\alpha_0, \alpha_1, \ldots$ and $\beta_1, \beta_2, \ldots$ of a structure as depicted in figure 5.

![Figure 5: the binary tree structure of the numbers $\alpha_i$ and $\beta_i$](image)

In figure 5 the $\alpha$'s and $\beta$'s are numbered from the root and from left to right. Actually, the numbering of the $\alpha$'s and $\beta$'s is irrelevant: we will only use that the $\alpha$-root has index 0 and that the other $\alpha$'s have the same index as their $\beta$-parent. For specifying the recursion relations to generate the tree, we need the following notation.

- $\beta_{l(i)}$ the left descendant of $\alpha_i$,
- $\beta_{r(i)}$ the right descendant of $\alpha_i$,
- $\alpha_p(i)$ the $\alpha$-parent of $\beta_i$.

Further define $L$ as the set of indices $i$ of $\beta_i$'s which are a left descendant and $R$ as the set of indices $i$ of $\beta_i$'s which are a right descendant, that is,

$$L = \{l(i), \ i = 0, 1, 2, \ldots\},$$
\[ R = \{ r(i), i = 0, 1, 2, \ldots \} \] .

The numbers \( \alpha_i \) and \( \beta_i \) are defined as roots of the quadratic forms (14) and (15). More precisely, for the initial value \( \alpha_0 = \rho^2 \), generate \( \alpha_1, \alpha_2, \ldots \) and \( \beta_1, \beta_2, \ldots \) such that for all \( i = 0, 1, \ldots \)

\[
\begin{align*}
\beta_{l(i)} &= X_-(\alpha_i), \\
\beta_{r(i)} &= X_-(\alpha_i), \\
\alpha_l(i) &= Y_-(\beta_{l(i)}), \\
\alpha_r(i) &= Y_-(\beta_{r(i)}).
\end{align*}
\]

By virtue of lemma 1 all the solutions \( \alpha_{p(i)} \beta_i \) and \( \alpha_i \beta_i \) satisfy equation (1) if \( i \in L \) and equation (5) if \( i \in R \). Because the equations (1) and (5) are linear, any linear combination of solutions \( \alpha_{p(i)} \beta_i \) with \( i \in L \) satisfies equation (1) and any linear combination of solutions \( \alpha_{p(i)} \beta_i \) with \( i \in R \) satisfies equation (5). Now form the linear combinations \( X_{s,r} \) and \( x_{s,r}, \) for all \( s \geq 0 \) and \( r \geq 1 \) defined as

\[ X_{s,r} = \sum_{i \in L} d_i \left( \alpha_{p(i)} + c_i \alpha_i \right) \beta_i \] (24)

\[ = d_1 \beta_i \alpha_0 + \sum_{i \in L} (d_i c_i \beta_i + d_{l(i)} \beta_{l(i)}) \alpha_i + \sum_{i \in R} d_i \left( \beta_{l(i)} \alpha_i \right) \] (25)

and

\[ x_{s,r} = \sum_{i \in R} d_i \left( \alpha_{p(i)} + c_i \alpha_i \right) \beta_i \] (26)

\[ = d_2 \beta_i \alpha_0 + \sum_{i \in R} (d_i c_i \beta_i + d_{r(i)} \beta_{r(i)}) \alpha_i + \sum_{i \in L} d_i \left( \beta_{r(i)} \alpha_i \right) \] (27)

where in the first sums we formed pairs with a common factor \( \beta_i \) and in the second ones with a common factor \( \alpha_i \). The sum (24) lives in the upper triangle and the sum (26) in the lower triangle and they are graphically depicted in figure 6. The sum (24) is the sum of all terms, which are a left descendant in figure 6 and sum (26) is the sum over all terms, which are a right descendant.

The coefficients \( c_i \) are generated such that the terms \( \left( \alpha_{p(i)} + c_i \alpha_i \right) \beta_i \) satisfy equation (3) for all \( i \in L \) and equation (7) for all \( i \in R \). By virtue of lemma 2 this yields for all \( i \in L \),

\[ c_i = \frac{Y_-(\beta_i) - \beta_i}{Y_+(\beta_i) - \beta_i} \]

and for all \( i \in R \),

\[ c_i = \frac{Y_-(\beta_i) - \beta_i}{Y_+(\beta_i) - \beta_i}. \]
Figure 6: the binary tree structure of the terms $d_i \left( \alpha_p \gamma_1 + c_1 \alpha_1 \right) \beta_1$

Since the equations (3) and (7) are linear, the linear combination (24) satisfies equation (3) and the linear combination (26) equation (7). The coefficients $d_i$ are generated such that the linear combinations (25) and (27) satisfy the equations (11) and (12). Initially put $d_1 = (p \gamma_1 + 2q)/\gamma_1$ and $d_2 = (p \gamma_2 + 2(1 - q))/\gamma_2$ and generate the coefficients $d_{i(l)}$ and $d_{i(r)}$ such that for all $i \in L$ the terms $(d_i c_i + d_{i(l)} \beta_{i(l)}(i)) \alpha_i$ and $d_{i(r)} \beta_{i(r)}(i) \alpha_i$ satisfy equation (11) and (12) and for all $i \in R$ the terms $(d_i c_i + d_{i(r)} \beta_{i(r)}(i)) \alpha_i$ and $d_{i(l)} \beta_{i(l)}(i) \alpha_i$ satisfy equation (11) and (12). By virtue of lemma 3 this yields for all $i \in L$

$$d_{i(l)} = \frac{(\alpha_i \gamma_1 + 2qp)/\gamma_1 \left( (\alpha_i \gamma_1 + 2qp) \gamma_1 + \gamma_2 + 2(1 - q)p \right)}{(\alpha_i \gamma_1 + 2qp) \gamma_1 + \gamma_2 + 2(1 - q)p \right)} \frac{d_i c_i}{d_i}$$

and for all $i \in R$

$$d_{i(r)} = \frac{\gamma_1 \left( (\alpha_i \gamma_1 + 2qp) \gamma_1 + \gamma_2 + 2(1 - q)p \right)}{\gamma_1 \left( (\alpha_i \gamma_1 + 2qp) \gamma_1 + \gamma_2 + 2(1 - q)p \right) \gamma_1 - 2(1 - q)p \right)} \frac{d_i c_i}{d_i}.$$

Since the equations (11) and (12) are linear, the linear combinations (25) and (27) satisfy equation (11) and (12). This completes the definition of the numbers $X_{s,r}$ for all $s \geq 0$ and $r \geq 1$. The numbers $X_{s,0} = x_{s,0}$ are defined by the equations on the line $r = 0$. According to the corollary of lemma 3 we obtain for all $s > 0$

$$X_{s,0} = \sum_{i \in L} d_i \left( \frac{\alpha_p \gamma_1}{\alpha_p} + c_i \frac{\alpha_1 \gamma_1}{\alpha_1} \right) = \sum_{i \in R} d_i \left( \frac{\alpha_p \gamma_2}{\alpha_p} + c_i \frac{\alpha_1 \gamma_2}{\alpha_1} \right).$$
Finally we define \( X_{0,0} \) by equation (4) or (8). As we will show, both definitions yield the same result. Inserting the product form (24) and (28) in equation (4) yields

\[
X_{0,0} = \frac{1}{2q\rho} \sum_{i \in L} \big( (1 + c_i) \beta_i (2\rho + \gamma_2) - (1 + c_i) \beta_i^2 \gamma_2 - \frac{\alpha_p^2 \gamma_1}{\alpha_p(i) + 2q\rho} - c_i \frac{\alpha^2 \gamma_1}{\alpha_i + 2q\rho} \big) \tag{30}
\]

Recall that we have chosen the coefficients \( c_i \) such that for all \( i \in L \) the terms \( (\alpha^2_p(i) + c_i \alpha^2_i) \beta_i^s \) satisfy equation (3), that is,

\[
(1 + c_i) \beta_i^s (2\rho + \gamma_2) - (1 + c_i) \beta_i^{s+1} \gamma_2 = (\alpha_p(i) + c_i \alpha_i) \beta_i^{s-1} \gamma_1 .
\]

Dividing both sides by \( \beta_i^{s-1} \) and then inserting in (30) we obtain that representation (28) also holds for \( s = 0 \). On the other hand, using equation (8) as definition for \( X_{0,0} \) we obtain representation (29) for \( s = 0 \). As follows from the corollary of lemma 3, representation (28) and (29) are also equivalent for \( s = 0 \). This completes the definition of all the numbers \( X_{s,r} \) and \( x_{s,r} \). Obviously, these numbers formally satisfy all equilibrium equations, except for the one in the origin of the state space. But, summing over all equilibrium equations, except for the one in the origin exactly yields the equation in the origin. Hence we constructed a formal solution for the complete set of equilibrium equations. It is still a formal solution, because we have not proved yet that the infinite sums of product form solution converge absolutely for all \( s \geq 0 \) and \( r \geq 0 \) (we need absolute convergence, because we have to reorder terms in order to show that all equations are satisfied).

2.4. Main theorem

To prove that the numbers \( X_{s,r} \) and \( x_{s,r} \) satisfy all the equilibrium equations, we have to show that the infinite sums of product form solutions converge absolutely. Unfortunately, in general these infinite sums do not converge absolutely for all \( s \geq 0 \) and \( r \geq 0 \), but from a certain index \( N \), depending on the model parameters \( \rho, \gamma_1 \) and \( q \). In section 4 we will give a constructive definition for the index \( N \). Hence we are not able to prove that the product form representation for the equilibrium distribution holds for all \( s \) and \( r \), as announced in section 1. Instead we will prove

Theorem 1.  

There exists an index \( N \) such that for all \( s \geq 0, r \geq 1 \) and \( s + r > N \),

\[
Q_{s,r} = C^{-1} X_{s,r} ,
\]

\[
q_{s,r} = C^{-1} x_{s,r} ,
\]

where \( C \) is a normalizing constant.
To prove the theorem we will first show that for all \( s \geq 0, r \geq 1 \) and \( s + r > N \), the infinite sums (24) and (26) converge absolutely and for all \( s \geq N \) the infinite sum (28) converges absolutely. Next we show that the total absolute weight of the numbers \( X_{s,r} \) and \( x_{s,r} \) is finite, that is,

\[
\sum_{s+r \geq N} |X_{s,r}| + |x_{s,r}| < \infty .
\]

Then it is obvious that the solutions \( X_{s,r} \) and \( x_{s,r} \) solve all the equilibrium equations for \( s + r > N \). Because the Markov process is irreducible, we can uniquely extend these solutions to the complete complete first quadrant of the \((s, r)\) plane such that the equilibrium equations for all \( s + r \leq N \) are also solved. For this, we actually have to solve \( 2(N + 1) \) equilibrium equations for only \( 2N + 1 \) unknowns. So it seems that we have too many equations, but because the equilibrium equations are dependent, we may omit one arbitrarily chosen equation. Finally we have to show that the solutions \( X_{s,r} \) and \( x_{s,r} \) are not trivial, that is,

\[
X_{s,r} \neq 0 \quad \text{and} \quad x_{s,r} \neq 0 .
\]

Hence we constructed a nontrivial and convergent solution of the equilibrium equations. By a result of Foster ([5], Theorem 1), this proves that the asymmetric shortest queue problem is ergodic. Since the equilibrium distribution of an ergodic system is unique, the numbers \( X_{s,r} \) and \( x_{s,r} \) can be normalized to produce the equilibrium distribution.

3. Preliminary results

To prove that the infinite sums of product form solutions converge absolutely we first have to obtain some information about the behaviour of the numbers \( \alpha_i, \beta_i, c_i \) and \( d_i \). We will start to prove that the numbers \( \alpha_i \) and \( \beta_i \) converge exponentially fast, for which we need the following monotonicity result.

Lemma 4.

(i) Let \( 0 < \alpha \leq \alpha_0 \), then \( X_+ (\alpha) > X_+ (\alpha) > 0 \) and \( x_+ (\alpha) > x_+ (\alpha) > 0 \) and the ratios \( X_+ (\alpha) / \alpha \) and \( x_+ (\alpha) / \alpha \) are decreasing and \( X_- (\alpha) / \alpha \) and \( x_- (\alpha) / \alpha \) are increasing in \( \alpha \).

(ii) Let \( 0 < \beta \leq \beta_0 = \rho \), then \( Y_+ (\beta) > Y_+ (\beta) > 0 \) and \( y_+ (\beta) > y_+ (\beta) > 0 \) and the ratios \( Y_+ (\beta) / \beta \) and \( y_+ (\beta) / \beta \) are decreasing and \( Y_- (\beta) / \beta \) and \( y_- (\beta) / \beta \) are increasing in \( \alpha \).

Proof.

(i) We prove the results for \( X_+ (\alpha) \); the proofs for \( x_+ (\alpha) \) are similar. Since \( \alpha \leq \alpha_0 = \rho^2 \),

\[
(p + 1)^2 - (2p + \alpha \gamma_2) \gamma_1 \geq (p + 1)^2 - (2p + p^2 \gamma_2) \gamma_1 = (1 + p (1 - \gamma_1))^2 > 0 ,
\]

so the discriminant of the quadratic form (14) for fixed \( \alpha \) is strictly positive and since \( \alpha > 0 \),
this implies that $X_+ (\alpha)$ and $X_- (\alpha)$ are two distinct positive roots. Immediately from its definition, it follows that $X_+ (\alpha) / \alpha$ is decreasing. Further,

$$\frac{X_- (\alpha)}{\alpha} = \frac{\gamma_1}{2 \rho + \alpha \gamma_2} \frac{\alpha}{X_+ (\alpha)} = \frac{\gamma_1}{\rho + 1 + \sqrt{(\rho + 1)^2 - (2 \rho + \alpha \gamma_2 \gamma_1)},}$$

which is increasing in $\alpha$.

(ii) We only prove the results for $Y_\pm (\beta)$; the proofs for $y_\pm (\beta)$ are similar. Since $\beta \leq \beta_0 = \rho$,

$$(2 (\rho + 1) - \gamma_2 \gamma_2)^2 - 8 \rho \gamma_1 \geq (2 (\rho + 1) - \rho \gamma_2)^2 - 8 \rho \gamma_1$$

$$= (2 + \rho \gamma_1)^2 - 8 \rho \gamma_1 = (2 - \rho \gamma_1)^2 > 0,$$

so the discriminant of the quadratic form (14) for fixed $\beta$ is strictly positive and since $\beta > 0$ and $2 (\rho + 1) - \gamma_2 \gamma_2 > 0$, this implies that $Y_+ (\beta)$ and $Y_- (\beta)$ are two distinct positive roots. Immediately from its definition, it follows that $Y_+ (\beta) / \beta$ is decreasing in $\beta$. Further,

$$\frac{Y_- (\beta)}{\beta} = \frac{2 \rho}{\gamma_1} \frac{\beta}{Y_+ (\beta)},$$

which is increasing in $\beta$. \hfill \Box

From this lemma we immediately obtain

**Corollary.**

For each left branch,

$$\alpha_0 \geq \alpha_i > \beta_{l(i)} > \alpha_{l(i)} > 0,$$

where the decrease of $\beta_{l(i)}$ and $\alpha_{l(i)}$ is at least

$$\beta_{l(i)} \leq \frac{\gamma_1}{2 + \rho \gamma_2} \alpha_i, \quad \alpha_{l(i)} \leq \frac{2 \rho}{2 + \rho \gamma_2} \beta_{l(i)}$$

and similarly, for each right branch,

$$\alpha_0 \geq \alpha_i > \beta_{r(i)} > \alpha_{r(i)} > 0,$$

where the decrease of $\beta_{r(i)}$ and $\alpha_{r(i)}$ is at least

$$\beta_{r(i)} \leq \frac{\gamma_2}{2 + \rho \gamma_1} \alpha_i, \quad \alpha_{r(i)} \leq \frac{2 \rho}{2 + \rho \gamma_1} \beta_{r(i)}.$$
Proof.
The lemma is proved by induction: we descend the tree by starting in the root.
Assume $0 < \alpha_i \leq \alpha_0$, which trivially holds for $i = 0$. Then by lemma 4(i),

$$0 < \beta_{l(i)} = X_-(\alpha_i) \leq \frac{X_-(\alpha_0)}{\alpha_0} \alpha_i = \frac{\gamma_1}{2 + \rho \gamma_2} \alpha_i$$

and $\beta_{l(i)} = X_-(\alpha_i) \leq X_-(\alpha_0) = \beta_1$, thus by lemma 4(ii),

$$\alpha_{l(i)} = Y_-(\beta_{l(i)}) \leq \frac{Y_-(\beta_1)}{\beta_1} \beta_{l(i)} = \frac{2\rho}{2 + \rho \gamma_2} \beta_{l(i)} \cdot$$

The inequalities for $\beta_{r(i)}$ and $\alpha_{r(i)}$ can be proved similarly.

The corollary implies that the numbers $\alpha_i$ and $\beta_i$ decrease exponentially fast and uniformly in the depth of the tree. For let $R$ be the maximum of the rates $\gamma_1 / (2 + \rho \gamma_2)$, $2\rho / (2 + \rho \gamma_2)$, $\gamma_2 / (2 + \rho \gamma_1)$ and $2\rho / (2 + \rho \gamma_1)$ and define the function

$$depth(x) \quad \text{the depth of the number } x \text{ in the tree of figure 5.}$$

For example, $depth(\alpha_1) = depth(\alpha_2) = 2$ and $depth(\beta_7) = depth(\beta_8) = \ldots = depth(\beta_{14}) = 5$.

Then for all $i = 1, 2, \ldots$

$$\beta_i < R^{depth(\beta_i)} \alpha_0 \quad \alpha_i < R^{depth(\alpha_i)} \alpha_0 \cdot (31)$$

These are rough bounds on the convergence of the numbers $\alpha_i$ and $\beta_i$. The exact asymptotic behaviour of these numbers is stated in

Lemma 5.

As $depth(\alpha_i) \to \infty$, then

$$\frac{\beta_{l(i)}}{\alpha_i} \to \frac{1}{A_2} \quad \text{and as } depth(\beta_{r(i)}) \to \infty, \text{ then}$$

$$\frac{\alpha_{r(i)}}{\beta_{r(i)}} \to A_1 \cdot \frac{\alpha_{l(i)}}{\beta_{l(i)}} \to a_1 \cdot$$

where

$$A_1 = \frac{\rho + 1 - \sqrt{(\rho + 1)^2 - 2\rho \gamma_1}}{\gamma_1}, \quad A_2 = \frac{\rho + 1 + \sqrt{(\rho + 1)^2 - 2\rho \gamma_1}}{\gamma_1}$$

$$a_1 = \frac{\rho + 1 - \sqrt{(\rho + 1)^2 - 2\rho \gamma_2}}{\gamma_2}, \quad a_2 = \frac{\rho + 1 + \sqrt{(\rho + 1)^2 - 2\rho \gamma_2}}{\gamma_2} \cdot$$
Proof.
We prove the first limit; the others are proved similarly. Let $\text{depth}(\alpha_i) \to \infty$, then, by virtue of (31), $\alpha_i \to 0$, so
\[
\frac{\beta_i}{\alpha_i} = \frac{X_-(\alpha_i)}{\alpha_i} \to \frac{A_1 \gamma_1}{2\rho} = \frac{1}{A_2}.
\]
Note that by virtue of lemma 4 and its corollary, it follows that the ratios in lemma 5 decrease monotonously along any path in the tree. Now the asymptotic behaviour of the numbers $\alpha_i$ and $\beta_i$ is known, we will investigate the behaviour of the coefficients $c_i$ and $d_i$. The behaviour of the coefficients $c_i$ is stated in

Lemma 6.
For all $i = 1, 2, \ldots$, the coefficients $c_i$ are strictly positive, and as $\text{depth}(\beta_i) \to \infty$, then
\[
c_i \to \frac{1-A_1}{A_2-1} \text{ if } i \in L, \quad c_i \to \frac{1-a_1}{a_2-1} \text{ if } i \in R.
\]

Proof.
We prove the lemma for $i \in L$; the case $i \in R$ is similar. Since $\beta_i \leq \beta_0$, we obtain by lemma 4
\[
\frac{Y_-(\beta_0)}{\beta_0} < \frac{Y_-(\beta_i)}{\beta_i} = \rho < 1, \quad \frac{Y_+(\beta_0)}{\beta_0} > \frac{Y_+(\beta_i)}{\beta_i} = \frac{1}{\rho} > 1,
\]
which proves that $c_i$ is strictly positive. Let $\text{depth}(\beta_i) \to \infty$, then by (31), $\beta_i \to 0$, so
\[
c_i = \frac{1-Y_-(\beta_i)/\beta_i}{Y_+(\beta_i)/\beta_i-1} \to \frac{1-A_1}{A_2-1}.
\]

It is more complicated to obtain the behaviour of the coefficients $d_i$. We first have to prove that the denominator in the definition of $d_i$ does not equal zero. For this we need the following monotonicity result.

Lemma 7.
For $0 < \alpha \leq \alpha_0$,
\[
(\alpha \gamma_1 + 2\rho \alpha) / X_+(\alpha) + (\alpha \gamma_2 + 2(1-q)\rho) / x_+(\alpha) - 2(\rho + 1) \alpha
\]
is monotonously decreasing.
Proof.

We show that the derivative is negative for all $0 < \alpha \leq \alpha_0$. Since

\[ \frac{d}{d\alpha} \frac{(\alpha \gamma_1 + 2qp) \alpha / X_+ (\alpha)}{X_+ (\alpha) + \frac{(\alpha \gamma_1 + 2qp) \gamma_2}{2 \sqrt{(\rho + 1)^2 - (2p + \alpha \gamma_2) \gamma_1}} = \gamma_1 \alpha / X_+ (\alpha) + \frac{(\alpha \gamma_1 + 2qp) \gamma_2}{2 \sqrt{(\rho + 1)^2 - (2p + \alpha \gamma_2) \gamma_1}} \]

and using that both terms are increasing in $\alpha$, yields

\[ \frac{d}{d\alpha} (\alpha \gamma_1 + 2qp) \alpha / X_+ (\alpha) < \gamma_1 \alpha_0 / X_+ (\alpha_0) + \frac{(\alpha_0 \gamma_1 + 2qp) \gamma_2}{2 \sqrt{(\rho + 1)^2 - (2p + \alpha_0 \gamma_2) \gamma_1}} = \rho \gamma_1 + \frac{(\rho \gamma_1 + 2q) \rho \gamma_2}{2 (1 + \rho (1 - \gamma_1))} < \rho \gamma_1 3/2 + q \]

where the latter inequality follows from the monotonicity of $\rho / (1 + \rho (1 - \gamma_1))$. Similarly,

\[ \frac{d}{d\alpha} (\alpha \gamma_1 + 2qp) \alpha / x_+ (\alpha) < \rho \gamma_2 3/2 + 1 - q \]

Hence, the total derivative is less than

\[ \rho \gamma_1 3/2 + q + \rho \gamma_2 3/2 + 1 - q - 2(p + 1) = \rho - 1 < 0 \]

By virtue of this lemma we immediately obtain that the denominator in the definition of $d_i$ is strictly positive, i.e.

Corollary.

For $0 < \alpha < \alpha_0$,

\[ (\alpha \gamma_1 + 2qp) / X_+ (\alpha) + (\alpha \gamma_2 + 2(1 - q)p) / x_+ (\alpha) - 2(p + 1) > 0 \]

Proof.

Multiplying the left hand side of the above inequality by $\alpha_i$ and applying lemma 7 yields

\[ (\alpha_i \gamma_1 + 2qp) \alpha_i / X_+ (\alpha_i) + (\alpha_i \gamma_2 + 2(1 - q)p) \alpha_i / x_+ (\alpha_i) - 2(p + 1) \alpha_i > (\alpha_0 \gamma_1 + 2qp) \alpha_0 / X_+ (\alpha_0) + (\alpha_0 \gamma_2 + 2(1 - q)p) \alpha_0 / x_+ (\alpha_0) - 2(p + 1) \alpha_0 = 0. \]

Since the numerator in the definition of $d_i$ is bounded below by the denominator, it follows by virtue of lemma 6 and the corollary of lemma 7 that the coefficients $d_i$ are alternating, that is, for all $i = 1, 2, \ldots$

\[ \frac{d_{i(1)}}{d_i} < 0 , \quad \frac{d_{i(2)}}{d_i} < 0 . \]
The next lemma describes the asymptotic behaviour of the coefficients \( d_i \).

**Lemma 8.**

For all \( i \in L \),

\[
\frac{d_{i(I)}}{d_i} \to -\frac{q A_2 + (1-q) a_1}{q A_1 + (1-q) a_1} \frac{1-A_1}{A_2 - 1}
\]

\[
\frac{d_{r(i)}}{d_i} \to -\frac{(1-q) (A_2 - A_1) \gamma_1}{(q A_1 + (1-q) a_1) \gamma_2} \frac{1-A_1}{A_2 - 1}
\]

as \( \text{depth}(\alpha_i) \to \infty \) and for all \( i \in R \),

\[
\frac{d_{i(I)}}{d_i} \to -\frac{q (a_2 - a_1) \gamma_2}{((1-q) a_1 + q A_1) \gamma_1} \frac{1-a_1}{a_2 - 1}
\]

\[
\frac{d_{r(i)}}{d_i} \to -\frac{(1-q) a_2 + q A_1}{(1-q) a_1 + q A_1} \frac{1-a_1}{a_2 - 1}
\]

as \( \text{depth}(\alpha_i) \to \infty \).

**Proof.**

We prove the first limit; the others are proved similarly. Multiplying the numerator and denominator in the definition of \( d_{i(I)} \) by \( a_i \) and letting \( \text{depth}(\alpha_i) \to \infty \), so by (31) \( \alpha_i \to 0 \), we obtain

\[
\frac{d_{i(I)}}{d_i} \to -\frac{q A_2 + (1-q) a_1}{q A_1 + (1-q) a_1} \frac{1-A_1}{A_2 - 1}
\]

\[
\frac{d_{r(i)}}{d_i} \to -\frac{(1-q) (A_2 - A_1) \gamma_1}{(q A_1 + (1-q) a_1) \gamma_2} \frac{1-A_1}{A_2 - 1}
\]

Summarizing, the lemmas 5, 6 and 8 give a complete description of asymptotic behaviour of the numbers \( \alpha_i, \beta_i, c_i \) and \( d_i \), and this is exactly what we need in order to prove that the infinite sums of product form solutions converge absolutely.

4. On the convergence

Now we are in the position to prove that the numbers \( X_{s,r} \) and \( x_{s,r} \) converge absolutely from a certain index \( N \). Consider a fixed \( s \geq 0 \) and \( r \geq 1 \). Since for all \( i \) the constants \( \alpha_i, \beta_i \) and \( c_i \) are positive, it follows that \( X_{s,r} \) and \( x_{s,r} \) converge absolutely if and only if

\[
\sum_{i=1}^{\infty} |d_i| (\alpha_{p(i)}^s + c_i \alpha_i^s) \beta_i^s < \infty.
\]

From the lemmas 5, 6 and 8 we obtain for all \( i \in L \).
as \( \text{depth}(\alpha_i) \to \infty \) and for all \( i \in R \)

\[
\frac{1}{d_l(\iota)} (\alpha_{l(\iota)} + c_{l(\iota)} \alpha_{r(\iota)}) \beta_{l(\iota)} \to D_l (A_1/A_2)^{x+y} \\
\frac{1}{d_r(\iota)} (\alpha_{r(\iota)} + c_{r(\iota)} \alpha_{l(\iota)}) \beta_{r(\iota)} \to D_r (A_1/A_2)^{x+y} \\
\frac{1}{d_l(\iota)} (\alpha_{l(\iota)} + c_{l(\iota)} \alpha_{r(\iota)}) \beta_{l(\iota)} \to d_l (A_1/a_2)^{x+y} \\
\frac{1}{d_r(\iota)} (\alpha_{r(\iota)} + c_{r(\iota)} \alpha_{l(\iota)}) \beta_{r(\iota)} \to d_r (A_1/a_2)^{x+y}
\]

as \( \text{depth}(\alpha_i) \to \infty \), where \( D_l, D_r, d_l \) and \( d_r \) denote the limits of \( |d_l(\iota)| / |d_l| \) and \( |d_r(\iota)| / |d_l| \), that is, by virtue of lemma 8,

\[
D_l = \frac{q A_2 + (1-q) a_1}{q A_1 + (1-q) a_1} \frac{1-A_1}{A_2 - 1} \\
D_r = \frac{(1-q) (A_2 - A_1) \gamma_1}{(q A_1 + (1-q) a_1) \gamma_2} \frac{1-A_1}{A_2 - 1} \\
d_l = \frac{q (a_2 - a_1) \gamma_2}{(1-q) a_1 + q A_1 \gamma_1} \frac{1-a_1}{a_2 - 1} \\
d_r = \frac{(1-q) a_2 + q A_1}{(1-q) a_1 + q A_1} \frac{1-a_1}{a_2 - 1} .
\]

Hence in the limit the terms behave geometrically. The problem is to formulate conditions, in terms of these limiting rates, which guarantee the convergence of the infinite sum. For this we need the notion of a positive geometrical binary tree.

**Definition 1.**

The numbers \( n_1, n_2, n_3, \ldots \) form a positive geometrical binary tree if

(i) the numbers \( n_i \) have a binary tree structure as depicted in figure 7,

(ii) the initial values \( n_1 \) and \( n_2 \) are positive,

(iii) the geometrical behaviour is determined by the nonnegative matrix \( \begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix} \) such that

\[
n_{l(\iota)} = R_l n_i, \quad n_{r(\iota)} = R_r n_i \quad \text{if } n_i \text{ is a left descendant and}
\]

\[
n_{l(\iota)} = r_l n_i, \quad n_{r(\iota)} = r_r n_i \quad \text{if } n_i \text{ is a right descendant.}
\]
Figure 7: the binary tree structure of the numbers $n_i$

Note that the tree of numbers $n_1$, $n_2$, $n_3$, ... is of the same structure as the tree of terms $1d_1 (\alpha_p^2) + c_1, \alpha_p^2$, which is depicted in figure 6. Let $\sigma(A)$ denote the spectral radius of the matrix $A$, then in particular,

$$\sigma\left[ \begin{array}{cc} R_l & r_l \\ R_r & r_r \end{array} \right] = \frac{R_l + r_r + \sqrt{(R_l - r_r)^2 + 4 R_r r_l}}{2}.$$

The following lemma provides a necessary and sufficient condition for the convergence of the infinite sum $\sum_{i=1}^{\infty} n_i$.

Lemma 9.

$\sum_{i=1}^{\infty} n_i$ converges if and only if $\sigma\left[ \begin{array}{cc} R_l & r_l \\ R_r & r_r \end{array} \right] < 1$.

Proof.

Define

$W_m$ the sum of all numbers $n_i$ at depth $m$, which are a left descendant,

$w_m$ the sum of all numbers $n_i$ at depth $m$, which are a right descendant,

then for all $m = 0, 1, ...$

$$\begin{bmatrix} W_{m+1} \\ w_{m+1} \end{bmatrix} = \begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix} \begin{bmatrix} W_m \\ w_m \end{bmatrix} = \begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix}^m \begin{bmatrix} W_1 \\ w_1 \end{bmatrix},$$

where $W_1 = n_1$ and $w_1 = n_2$. Hence

$$\sum_{i=1}^{\infty} n_i = \sum_{m=0}^{\infty} \left( W_{m+1} + w_{m+1} \right) = (1, 1) \sum_{m=0}^{\infty} \begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix}^m \begin{bmatrix} W_1 \\ w_1 \end{bmatrix}. $$
If \( \sigma < 1 \), then \( \begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix}^m \) converges exponentially fast, so \( \sum_{i=1}^{\infty} n_i \) converges.

If on the other hand, \( \sum_{i=1}^{\infty} n_i \) converges, then, since \( W_1 \) and \( w_1 \) are positive and \( \begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix} \geq 0, \)
\[
\begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix} \to 0 \quad \text{as} \quad m \to \infty,
\]
which holds if and only if \( \sigma < 1 \).

Hence, the convergence of a positive geometrical binary tree is completely determined by the spectral radius of the matrix of rates. Since the tree of terms \( |d_i| \left( \alpha_{p(i)}^\delta + c_i \alpha_{f(i)}^\delta \right) \beta_i \) behaves asymptotically as a positive geometric binary tree with rates
\[
\begin{bmatrix} R_l & r_l \\ R_r & r_r \end{bmatrix} = \begin{bmatrix} D_l (A_1/A_2)^{t+r} & d_l (a_1/a_2)^r (A_1/A_2)^{f+r} \\ D_r (A_1/A_2)^{t+r} & d_r (a_1/a_2)^{t+r} \end{bmatrix},
\]
we expect that the convergence is also determined by the spectral radius of this matrix. First, let us define

**Definition 2.**

For all \( n = 0, 1, 2, \ldots \), \( \sigma_n \) is defined by the following equation

\[
\sigma_n = \frac{1}{2} \left( D_l (A_1/A_2)^n + d_r (a_1/a_2)^n + \sqrt{(D_l (A_1/A_2)^n - d_r (a_1/a_2)^n)^2 + 4 D_r d_l (A_1/A_2)^n (a_1/a_2)^n} \right).
\]

From this definition we conclude that

\[
\sigma = \begin{bmatrix} D_l (A_1/A_2)^{t+r} & d_l (a_1/a_2)^r (A_1/A_2)^{f+r} \\ D_r (A_1/A_2)^{t+r} & d_r (a_1/a_2)^{t+r} \end{bmatrix} = \sigma_{t+r}
\]

and if \( \sigma_{t+r} < 1 \), then we will prove that the infinite sum

\[
\sum_{i=1}^{\infty} |d_i| \left( \alpha_{p(i)}^\delta + c_i \alpha_{f(i)}^\delta \right) \beta_i
\]
converges, and otherwise, if \( \sigma_{t+r} > 1 \), then it diverges.

Assume that \( \sigma_{t+r} < 1. \) Then there exist rates \( R_r, R_l, r_l \) and \( r_r \) such that
\[ R_l > D_l (A_1 / A_2)^{s+r}, \quad r_l > d_l (a_1 / a_2)^{s} (a_1 / A_2)^{r} \]
\[ R_r > D_r (A_1 / A_2)^{s} (A_1 / a_2)^{r}, \quad r_r > d_r (a_1 / a_2)^{s+r} \]

and
\[
\sigma \left[ \begin{array}{c} R_l \ r_l \\ R_r \ r_r \end{array} \right] < 1.
\]

Then, for depth \((\alpha_i)\) large enough, depth \((\alpha_i) \geq M\) say, we obtain for all \(i \in L\)
\[
\frac{|d_{l}(i)| (\alpha_{l}^{(i)} + c_{l}(i) \alpha_{l}^{(i)}) \beta_{l}^{(i)}}{|d_{l}| (\alpha_{l}^{(i)} + c_{l} \alpha_{l}^{(i)}) \beta_{l}^{(i)}} < R_l, \quad \frac{|d_{r}(i)| (\alpha_{r}^{(i)} + c_{r}(i) \alpha_{r}^{(i)}) \beta_{r}^{(i)}}{|d_{r}| (\alpha_{r}^{(i)} + c_{r} \alpha_{r}^{(i)}) \beta_{r}^{(i)}} < R_r.
\]  (32)

and for all \(i \in R\)
\[
\frac{|d_{l}(i)| (\alpha_{l}^{(i)} + c_{l}(i) \alpha_{l}^{(i)}) \beta_{l}^{(i)}}{|d_{l}| (\alpha_{l}^{(i)} + c_{l} \alpha_{l}^{(i)}) \beta_{l}^{(i)}} < r_l, \quad \frac{|d_{r}(i)| (\alpha_{r}^{(i)} + c_{r}(i) \alpha_{r}^{(i)}) \beta_{r}^{(i)}}{|d_{r}| (\alpha_{r}^{(i)} + c_{r} \alpha_{r}^{(i)}) \beta_{r}^{(i)}} < r_r.
\]  (33)

Note that we really use here that the ratios converge uniformly in the depth of the tree. Consider the positive geometrical binary tree of numbers \(n_i\) with rates \([R_l \ r_l]\) and initial values \(n_1 = n_2 = K\), where the constant \(K\) is large enough such that for all depth \((\alpha_i) \leq M\),
\[
|d_{l}| (\alpha_{l}^{(i)} + c_{l} \alpha_{l}^{(i)}) \beta_{l}^{(i)} \leq n_i.
\]

From the relations (32) and (33) it follows that all the terms are bounded by the numbers \(n_i\).

Hence, the tree of numbers \(|d_{l}| (\alpha_{l}^{(i)} + c_{l} \alpha_{l}^{(i)}) \beta_{l}^{(i)}\) can be bounded by a positive geometrical tree of numbers \(n_i\), which is convergent by virtue of lemma 9 and thus
\[
\sum_{i=1}^{\infty} |d_{l}| (\alpha_{l}^{(i)} + c_{l} \alpha_{l}^{(i)}) \beta_{l}^{(i)} \leq \sum_{i=1}^{\infty} n_i < \infty.
\]

If on the other hand \(\sigma_{s+r} > 1\), then the tree of numbers \(|d_{l}| (\alpha_{l}^{(i)} + c_{l} \alpha_{l}^{(i)}) \beta_{l}^{(i)}\) can be bounded below by a divergent geometrical tree, and thus
\[
\sum_{i=1}^{\infty} |d_{l}| (\alpha_{l}^{(i)} + c_{l} \alpha_{l}^{(i)}) \beta_{l}^{(i)} = \infty.
\]

In case the spectral radius \(\sigma_{s+r}\) equals unity, nothing can be said in general. In the same way, one can show that the numbers \(X_{s,0}\) converge absolutely whenever \(\sigma_{s+1} < 1\).

Define \(N\) as the smallest index such that \(\sigma_{N+1} < 1\). Then the results can be summarized as
Theorem 2.

For all $s \geq 0$, $r \geq 1$ and $s + r > N$, the numbers $X_{s,r}$ and $x_{s,r}$ converge absolutely, and for all $s \geq N$, the numbers $X_{s,0}$ converge absolutely.

Finally we will show that the total absolute weight of the numbers $X_{s,r}$ and $x_{s,r}$ is finite. That is,

$$\sum_{s \geq 0, r \geq 0} (|X_{s,r}| + |x_{s,r}|) < \infty.$$  

By equation (9), it follows that

$$\sum_{s > N, r > N} (|X_{s-1,r}| 2(p + 1) \leq \sum_{s > N} (|X_{s-1,1}| 2p + |X_{s,1}| \gamma_2 + |x_{s-1,1}| 2p + |x_{s,1}| \gamma_1).$$

Hence, it suffices to prove that

Theorem 3.

$$\sum_{s \geq 0, r \geq 1} (|X_{s,r}| + |x_{s,r}|) < \infty.$$  

Proof.

By (24) and (26),

$$\sum_{s \geq 0, r \geq 1} (|X_{s,r}| + |x_{s,r}|) \leq \sum_{s \geq 0, r \geq 1} \sum_{i = 1}^{\infty} |d_i| (\alpha_{p(i)}^s + c_i \alpha_i^s) \beta_i^s,$$

$$= \sum_{s = 0}^{N-1} \sum_{i = 1}^{\infty} |d_i| (\alpha_{p(i)}^s + c_i \alpha_i^s) \frac{\beta_i^{s+1}}{1 - \beta_i} + \sum_{i = 1}^{\infty} |d_i| (\alpha_{p(i)}^N + c_i \alpha_i^N) \frac{\beta_i}{1 - \beta_i} < \infty,$$

since the spectral radius of the matrix of limiting rates for each infinite sum equals $\sigma_{N+1} < 1$.  

As argued at the end of subsection 2.4, the theorems 2 and 3 nearly complete the proof of theorem 1. We finally have to show that $X_{s,r}$ and $x_{s,r}$ are nonnull solutions. In the next section we prove that the first term in the product form representations (24) and (26) is dominating as $s + r \to \infty$. This implies that $X_{s,r}$ and $x_{s,r}$ are indeed nonnull solutions, and thus completes the proof of theorem 1.
5. Asymptotic expansion

In this section we will show that the product form representations (24) and (26) for the numbers $X_{s,r}$ and $x_{s,r}$ yield a complete asymptotic expansion as $s + r \to \infty$ and $r \geq 1$. In particular we will prove the asymptotic equivalences

$$X_{s,r} \sim d_1 (\alpha_0^1 + c_1 \alpha_1^1) \beta_1^1,$$  \hspace{1cm} (34)

$$x_{s,r} \sim d_2 (\alpha_0^2 + c_2 \alpha_2^2) \beta_2^2$$  \hspace{1cm} (35)

as $s + r \to \infty$ and $r \geq 1$, which imply that $X_{s,r}$ and $x_{s,r}$ are nonnull solutions.

First we have to reorder the terms in the representations (24) and (26) in a nonincreasing order. Since the numbers $\alpha_i$ decrease to zero, uniformly in the depth of the tree, we can order the tree of numbers $\alpha_i$ in a nonincreasing sequence

$$\alpha_{i_0} \geq \alpha_{i_1} \geq \alpha_{i_2} \geq \ldots \downarrow 0$$

and thus we can reorder (24) and (26) as

$$X_{s,r} = \sum_{n=0}^{\infty} d_{l(i_n)} (\alpha_{i_n}^l + c_{l(i_n)} \alpha_{i_n}^l) \beta_{l(i_n)},$$ \hspace{1cm} (36)

$$x_{s,r} = \sum_{n=0}^{\infty} d_{r(i_n)} (\alpha_{i_n}^r + c_{r(i_n)} \alpha_{i_n}^r) \beta_{r(i_n)}.$$ \hspace{1cm} (37)

By virtue of the corollary of lemma 4, it follows that $\alpha_{i_n} > \alpha_{i_{n+1}}$ and $\alpha_{i_n} > \alpha_{r(i_n)}$, and by virtue of the monotonicity stated in lemma 4, $\beta_{l(i_n)} > \beta_{l(i_{n+1})}$ and $\beta_{r(i_n)} > \beta_{r(i_{n+1})}$ whenever $\alpha_{i_n} > \alpha_{i_{n+1}}$.

Hence we obtain

**Lemma 12.**

If $\alpha_{i_n} > \alpha_{i_{n+1}}$ then

$$d_{l(i_{n+1})} (\alpha_{i_n}^l + c_{l(i_{n+1})} \alpha_{i_n}^l) \beta_{l(i_{n+1})} = o \left[ d_{l(i_n)} (\alpha_{i_n}^l + c_{l(i_n)} \alpha_{i_n}^l) \beta_{l(i_n)} \right],$$

$$d_{r(i_{n+1})} (\alpha_{i_n}^r + c_{r(i_{n+1})} \alpha_{i_n}^r) \beta_{r(i_{n+1})} = o \left[ d_{r(i_n)} (\alpha_{i_n}^r + c_{r(i_n)} \alpha_{i_n}^r) \beta_{r(i_n)} \right]$$

as $s + r \to \infty$ and $r \geq 1$.

Thus successive terms in (36) and (37) are a refinement whenever $\alpha_{i_n} > \alpha_{i_{n+1}}$.

Further, we have the following set of $O$-formulas.
Lemma 13.

For all $m = 1, 2, ...$

$$X_{s,r} = \sum_{n=0}^{m-1} d_i(l_n) \left( \alpha^{t_i}_n + c_i(l_n) \alpha^{f_i}(l_n) \right) \beta^r_i(l_n) + O \left[ \sum_{n=0}^{m} d_i(l_n) \left( \alpha^{t_i}_n + c_i(l_n) \alpha^{f_i}(l_n) \right) \beta^r_i(l_n) \right],$$

$$x_{s,r} = \sum_{n=0}^{m-1} d_r(l_n) \left( \alpha^{t_r}_n + c_r(l_n) \alpha^{f_r}(l_n) \right) \beta^r_i(l_n) + O \left[ \sum_{n=0}^{m} d_r(l_n) \left( \alpha^{t_r}_n + c_r(l_n) \alpha^{f_r}_n \right) \beta^r_i(l_n) \right]$$

as $s + r \to \infty$ and $r \geq 1$.

Proof.

Let $0 \leq u \leq s$ and $1 \leq v \leq r$ and $u + v = N + 1$, then

$$\sum_{n=m}^{\infty} \left| d_i(l_n) \right| \left( \alpha^{t_i}_n + c_i(l_n) \alpha^{f_i}(l_n) \right) \beta^r_i(l_n) \leq \alpha^{t_i}_{N-1} \beta^{r}_{N-1} \sum_{n=m}^{\infty} \left| d_i(l_n) \right| \left( \alpha^{t_i}_n + c_i(l_n) \alpha^{f_i}_n \right) \beta^r_i(l_n)$$

$$\leq \alpha^{t_i}_{N-1} \beta^{r}_{N-1} M,$$

where $M = \max_{u+v=N+1} \sum_{n=m}^{\infty} \left| d_i(l_n) \right| \left( \alpha^{t_i}_n + c_i(l_n) \alpha^{f_i}_n \right) \beta^r_i(l_n)$.

The proof of the second $O$-formula is identical. \qed

Since $\alpha_0 = \alpha_s > \alpha_1$, the $O$-formulas for $m = 1$ improve (34) and (35), as

$$X_{s,r} = d_i(l_0) \left( \alpha^{t_i}_0 + c_i(l_0) \alpha^{f_i}(l_0) \right) \beta^r_i(l_0) + O \left[ d_i(l_1) \left( \alpha^{t_i}_1 + c_i(l_0) \alpha^{f_i}(l_1) \right) \beta^r_i(l_1) \right]$$

as $s + r \to \infty$ and $r \geq 1$, and similarly,

$$x_{s,r} = d_r(l_0) \left( \alpha^{t_r}_0 + c_r(l_0) \alpha^{f_r}(l_0) \right) \beta^r_i(l_0) + O \left[ d_r(l_1) \left( \alpha^{t_r}_1 + c_r(l_0) \alpha^{f_r}_1 \right) \beta^r_i(l_1) \right]$$

as $s + r \to \infty$ and $r \geq 1$. Accordingly, if $\alpha_i > \alpha_j$, then the $O$-formulas for $m = 2$ improve the ones for $m = 1$, and so on. If on the other hand some $\alpha_{im}$'s are identical, $\alpha_i > \alpha_{i+1} = ... = \alpha_{i+k-1} > \alpha_{i+k}$ say, then the $O$-formulas for $m = n+k$ improve the ones for $m = n+1$. 
6. Conclusions

In the previous sections we proved that for all \( s \geq 0, r \geq 1 \) and \( s + r > N \) the numbers \( X_{s,r} \) and \( x_{s,r} \), as defined by (24) and (26), converge absolutely and for all \( s \geq N \) the numbers \( X_{s,0} \), as defined by (28), converge absolutely. Further, the solutions \( X_{s,r} \) and \( x_{s,r} \) are not trivial and their total absolute weight is finite. It follows from their construction that they satisfy all the equilibrium equations for \( s + r > N \). We argued in subsection 2.4 that these solutions can be extended to the complete first quadrant of the \((s, r)\) plane such that the remaining equations are also satisfied. By a result of Foster ([5], Theorem 1), this proves that the asymmetric shortest queue system is ergodic. Since the equilibrium distribution of an ergodic system is unique, the solutions \( X_{s,r} \) and \( x_{s,r} \) can be normalized to produce the equilibrium distribution. Let us summarize the

**Main result.**

*Let \( N \) denote the smallest index, such that \( \sigma_{N+1} < 1 \), then for all \( s \geq 0, r \geq 1 \) and \( s + r > N \),*

\[
Q_{s,r} = C^{-1} X_{s,r} = C^{-1} \sum_{i \in L} d_i (\alpha_p^i + c_i \alpha_f^i) \beta_i^r,
\]

\[
q_{s,r} = C^{-1} x_{s,r} = C^{-1} \sum_{i \in R} d_i (\alpha_p^i + c_i \alpha_f^i) \beta_i^r,
\]

*where \( C \) is a normalizing constant.*

**Remark.**

In general the index \( N \) will be quit small, or even zero. In the special case that \( \gamma_1 = \gamma_2 = 1 \), it follows from the results in [1] that \( N = 0 \). Only for very unbalanced systems, that is, as \( \gamma_1 \to 0 \) or \( \gamma_1 \to 2 \), then the index \( N \) becomes (somewhat) larger. But there, the shortest queue discipline is not reasonable any more. In table 1 below we computed the index \( N \) for fixed \( q = \frac{1}{2} \) and increasing values of \( \rho \) and \( \gamma_1 \).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$\rho$ & $\eta$ & $N$ \\
\hline
0.1 & 0.2 & 1 \\
0.1 & 0.5 & 1 \\
0.1 & 0.8 & 0 \\
0.5 & 0.2 & 1 \\
0.5 & 0.5 & 1 \\
0.5 & 0.8 & 0 \\
0.9 & 0.2 & 1 \\
0.9 & 0.5 & 0 \\
0.9 & 0.8 & 0 \\
\hline
\end{tabular}
\caption{Table 1}
\end{table}

References


