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Abstract: Johnson [1954] gave an efficient algorithm for minimizing makespan in a two-machine flow shop; there is no advantage to preemption in this case. McNaughton's wrap-around rule [1959] finds a shortest preemptive schedule on identical parallel machines in linear time. A similarly efficient algorithm is unlikely to exist for the simplest common generalization of these problems. We show that preemptive scheduling in a two-stage flow shop with at least two identical parallel machines in one of the stages so as to minimize makespan is NP-hard in the strong sense.

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1. Introduction

Many manufacturing systems have a flow shop architecture. A flow shop is a multi-stage production process with the property that all products have to pass through the stages in the same order. More precisely, there are n jobs J_j ($j = 1, \ldots, n$), each consisting of a chain of m operations $(O_{1j}, \ldots, O_{mj})$ that have to be executed in this order. Operation O_{ij} has to spend a time p_{ij} at stage i of the process. In the classical flow shop, each stage consists of a single machine, which can handle at most one operation at a time. It is more realistic to assume that, at every stage, a number of identical machines is available that can operate in parallel. This model, which is known as the m-stage 'multiprocessor', 'hybrid' or 'flexible' flow shop, is receiving an increasing amount of attention in the literature.

Most papers on multiprocessor flow shop scheduling deal with the minimization of makespan under the assumption that preemption is not allowed. That is, once an operation is started, it must be completed without interruption. Buten and Shen [1973], Sriskandarajah and Sethi [1989] and Blazewicz, Dror, Pawlak, and Stecke [1992] designed approximation algorithms for the two-stage case and analyzed their performance by empirical and theoretical means. Arthanari [1974] and Brah and Hunsucker [1991] proposed branch-and-bound algorithms for finding optimal solutions to the two-stage and general case, respectively.

We are interested in the computational complexity of the problem of minimizing makespan in a multiprocessor flow shop. In 1954, Johnson [1954] gave his classical $O(n \log n)$ algorithm for the two-machine flow shop, i.e., the two-stage case with a single machine per stage. The question is whether this celebrated result can be extended to obtain polynomial-time algorithms for more general cases.

Obviously, as long as preemption is not allowed, the problem becomes NP-hard as soon as there are at least two machines at any stage, since the single-stage problem of finding a shortest nonpreemptive schedule on two parallel machines is already NP-hard in itself. We therefore assume that preemption is permitted. In this case, the two-machine flow shop is still optimally scheduled by

Johnson's algorithm, and a shortest schedule on any number of identical parallel machines is found in $O(n)$ time by McNaughton's wrap-around rule [1959]. However, from a complexity point of view, preemption is not going to be of much help, unless P = NP. In this note, we will show that preemptive scheduling in a two-stage flow shop with at least two identical parallel machines in one of the stages so as to minimize makespan is NP-hard in the strong sense.

Extending standard notation [Graham, Lawler, Lenstra and Rinnooy Kan, 1979], we denote the problem with two machines at stage 1 and one machine at stage 2 by $F2(P2,1)|pmtn|C_{\max}$. In $F2(1,P2)|pmtn|C_{\max}$, stage 1 has one machine and stage 2 has two. In $F2(P2,1)||C_{\max}$ and $F2(1,P2)||C_{\max}$, preemption is not permitted.

This note is organized as follows. To simplify the exposition, we first show that our problem is NP-hard in the ordinary sense in Section 2. The construction is then extended to establish NP-hardness in the strong sense in Section 3.

2. NP-hardness in the ordinary sense

We first show that $F2(P2,1)|pmtn|C_{\max}$ is NP-hard in the ordinary sense. Our reduction starts from the Partition problem, which is NP-complete in the ordinary sense:

Partition: Given an integer b and a multiset $N=\{a_1,\ldots,a_n\}$ of n integers with $\Sigma_{j=1}^{n} a_j = 2b$, is it possible to partition N into two disjoint subsets of equal sum b?

Given an instance of Partition, we construct the following instance of $F2(P2,1)|pmtn|C_{\max}$. We choose a number α with $\alpha > 1$. For each $j=1,\ldots,n$ we define a *partition job* $J_j=(O_{1j},O_{2j})$ with $p_{1j}=\alpha a_j$ and $p_{2j}=a_j$. In addition, we define four *separation jobs*, which have to create time slots for the execution of the partition jobs; their processing times are given in Table 1. Note that no schedule can be shorter than $2(\alpha b+b)$ and that a schedule of that length contains no idle time.

J_j	J_{n+1}	J_{n+2}	J_{n+3}	J_{n+4}
p_{1j}	0	$\alpha b+b$	$\alpha b+2b$	b
p_{2j}	αb	αb	0	0

Table 1. Processing times of the separation jobs.

Suppose S is a subset of N with sum equal to b. We construct a schedule of length $2(\alpha b+b)$ as follows (cf. Figure 1). Let M_1 and M_2 denote the first-stage machines, and let M_3 be the second-stage machine. The processing of the first-stage operations corresponding to the elements of S starts at time 0 on M_1; they are executed consecutively during a period of length αb. The processing of the corresponding second-stage operations starts at time αb on M_3 and is preceded by the execution of $O_{2,n+1}$. Operation $O_{1,n+2}$ starts at time 0 on M_2 and is executed without interruption; operation $O_{2,n+2}$ is processed without delay on M_3. The execution of the remaining partition jobs starts at time $\alpha b+b$ on M_2 and at time $2\alpha b+b$ on M_3. Finally, the operations $O_{1,n+3}$ and $O_{1,n+4}$ are scheduled on M_1 and M_2, respectively, to complete at time $2(\alpha b+b)$.

Conversely, suppose that a schedule σ of length $2(\alpha b+b)$ exists. Without loss of generality, we assume that the second-stage operations are executed without preemption in order of the completion times of the corresponding first-stage operations. Since σ contains no machine idle time and J_{n+1} is the only job with first-stage processing time equal to 0, operation $O_{2,n+1}$ completes at time αb. For

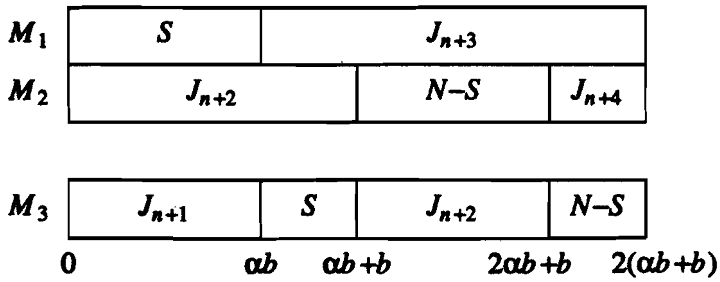

Figure 1. A schedule with partition sets S and $N-S$.

similar reasons, operations $O_{1,n+3}$ and $O_{1,n+4}$ complete at time $2(\alpha b+b)$. Let $O_{1,n+2}$ complete at time $\alpha b+b+\Delta$, with $\Delta \geq 0$. Machine M_3 has to perform partition jobs from time αb to time $\alpha b+b+\Delta$. Their first-stage operations must be performed by M_1 and M_2 before J_{n+3} and J_{n+2} start. The execution of these operations takes at least time $\alpha(b+\Delta)$, and the amount of time available is $\alpha b+\Delta$. From $\alpha(b+\Delta) \leq \alpha b+\Delta$ and $\alpha > 1$, it follows that $\Delta = 0$, so that the total processing time of the partition jobs in question is equal to αb on M_1 and to b on M_3. Hence, a schedule of length $2(\alpha b+b)$ certifies that we have a yes-instance of Partition.

This completes the proof that $F2(P2,1) \mid pmtn \mid C_{\max}$ is NP-hard in the ordinary sense.

3. NP-hardness in the strong sense

We now use the idea of the previous section to prove that $F2(P2,1) \mid pmtn \mid C_{\max}$ is even NP-hard in the strong sense. We start from the 3-Partition problem, which is known to be NP-complete in the strong sense:

3-Partition: Given an integer b and a multiset $N=\{a_1,\ldots,a_{3n}\}$ of $3n$ positive integers with $b/4 < a_j < b/2$ and $\Sigma_{j=1}^{3n} a_j = nb$, is there a partition of N into n mutually disjoint subsets $N_1,\ldots,N_n$ such that the elements in N_j add up to b, for $j=1,\ldots,n$?

J_j	J_{3n+1}	J_{3n+2}	J_{3n+3}	$\cdots$	J_{4n}	J_{4n+1}	J_{4n+2}
p_{1j}	0	$\alpha b+b$	$\alpha b+2b$	$\cdots$	$\alpha b+2b$	$\alpha b+2b$	b
p_{2j}	αb	αb	αb	$\cdots$	αb	0	0

Table 2. Processing times of the separation jobs.

Given an instance of 3-Partition, we construct the following instance of $F2(P2,1) \mid pmtn \mid C_{\max}$. As in Section 2, for each $j \in N$ we define a *partition job* $J_j=(O_{1j},O_{2j})$ with $p_{1j}=\alpha a_j$ and $p_{2j}=a_j$, where $\alpha > 1$. In addition, we define $n+2$ *separation jobs*, which have to create time slots for the execution of the partition jobs; their processing times are given in Table 2. Note that no schedule can be shorter than $n(\alpha b+b)$ and that a schedule of that length contains no idle time. As before, M_1 and M_2 are the first-stage machines and M_3 is the second-stage machine.

Proposition 1. *If the 3-Partition instance is a yes-instance, then there exists a schedule of length $n(\alpha b+b)$.*

Proof. Let $N_1, \ldots, N_n$ constitute the yes-answer to the 3-Partition instance. Consider the schedule given in Figure 2. It is easily checked that it is feasible and that its length is equal to $n(\alpha b + b)$. $\square$

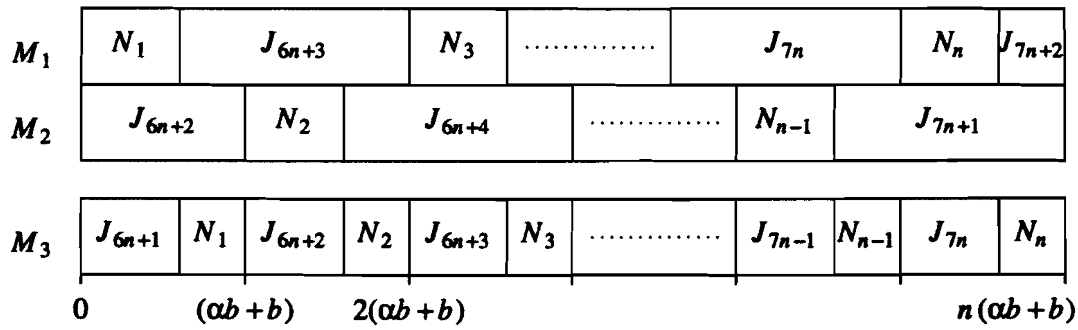

Figure 2. A schedule with partition sets $N_1, \ldots, N_n$.

In order to show that, conversely, a schedule of length $n(\alpha b + b)$ certifies that we have a yes-instance of 3-Partition, we need the following propositions. Their proofs follow from the same arguments as used in Section 2 and are therefore omitted.

Proposition 2. *There exists an optimal schedule such that* $C_{1j} \le C_{1,j+1}$ *and* $C_{2j} \le C_{2,j+1}$, *for* $j = 3n+1, \ldots, 4n+2$. $\square$

Proposition 3. *A schedule of length* $n(\alpha b + b)$ *satisfies the following properties:*
(a) *it contains no machine idle time;*
(b) $C_{1j} = (j - 3n - 1)(\alpha b + b)$, *for* $j = 3n+1, \ldots, 4n+1$;
(c) O_{2j} *is started immediately after* O_{1j} *is completed, for* $j = 3n+1, \ldots, 4n+1$. $\square$

Proposition 4. *A schedule of length* $n(\alpha b + b)$ *that satisfies Proposition 2 certifies that the 3-Partition instance is a yes-instance.* $\square$

Since the reduction requires polynomial time, we have proven the following theorem.

Theorem. *The problem* $F2(P2,1) \mid pmtn \mid C_{\max}$ *is* NP-*hard in the strong sense.* $\square$

Corollary. *The problems* $F2(1,P2) \mid pmtn \mid C_{\max}$, $F2(P2,1) \mid \mid C_{\max}$, *and* $F2(1,P2) \mid \mid C_{\max}$ *are* NP-*hard in the strong sense.* $\square$

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