On a heating problem in miniature soldering

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Published: 01/01/1994

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):
On a heating problem in miniature soldering

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November 1994
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IWDE Report 94-06
(rev. December 1995)

Abstract

The soldering of small, delicate electronic devices by means of a blade thermode (a small, thin, rectangular U- or W-shaped soldering iron) requires the lower side of the thermode to have a uniform temperature distribution. This is not easily obtained: during the start-up the corners tend to be too hot, in the stationary phase too cold. In the present study the various aspects are analysed that determine the heat flow and the temperature distribution, both for the dynamic and the stationary case. Assuming temperature-independent material parameters, approximate solutions for the dynamic problem and an exact solution for the stationary problem are obtained, yielding practical design rules. The analysis compares very well with a numerical finite element simulation.
1. Introduction

The soldering of electronic devices, like integrated circuits ("chips"), on a print board is known as miniature, subminiature, and micro soldering. It is usually done by application of a U- or W-shaped piece of metal, called a thermode. This thermode is electrically heated up to about 300°C (this varies with the application), while it is positioned with its lower side onto the pins or contacts to be soldered. Consider, for clarity, the following U-shaped geometry with ends indicated by A and D and corners indicated by B and C:

An electric potential difference is applied at the ends A and D. This induces a current that heats up the thermode according to Joule's law. The soldering takes place along side B-C. The basic question motivating the present study is:

How to keep the temperature along the side B-C as uniform as possible, in order to avoid damage of the electronic circuits from excessive heating on the one hand, as well as incomplete soldering of the connectors by insufficient temperature on the other hand.

To answer this question we have to solve the combined problem of Ohmic heating and heat conduction. This problem is non-stationary since the usual procedure involves in each soldering cycle:

- a cool thermode at the start,
- a high current for the initial heating to quickly attain the operational temperature,
- a low current for the final stage when this temperature is to be retained.

Experience has shown that it is difficult to obtain a uniform temperature profile. In the high current stage the corners B and C are heated up much more than the rest, leading to temperatures on the order of 350°C, which is 50°C too high. In the low current stage the corners B and C cool off to a value of the order of 250°C, which is 50°C too low. This change is due to a change in the relative importance of heat generation compared to heat conduction:

- at the high current stage the generation dominates, especially at the inner corners of B and C where the electric field has a singularity,
- at the low current stage the conduction dominates, in which case the cool ends A and D pull down the temperature at B and C.
To quantify these observations and to show the role of the parameters involved, we will analyse a mathematical model of a thermode, utilizing the symmetry and slenderness of the geometry, and assuming linear physical relations and ideally isolated boundaries. The results relate the amount of excess heat due to the corner singularities, the initial value problem, and an exact solution for the stationary problem, leading to possible rules for thermode design.

2. The model

An industrial thermode is typically made of molybdenum, a Hastelloy alloy, tungsten (wolfram) or nichrome, which are all electrically good conductors (homogeneous and isotropic), with the electric current density \( \vec{j} \) and the electric field \( \vec{E} \) satisfying Ohm’s law

\[
\vec{j} = \sigma \vec{E}, \tag{2.1}
\]

(ref. [1], [2], [3]) where \( \sigma \) is the electric conductivity, the inverse of the specific electric resistance \( 1/\sigma \). For the effectively stationary current flow, as we will have here, the conservation of electric charge leads to the following equation for the electric current density \( \vec{j} \)

\[
\nabla \cdot \vec{j} = 0, \tag{2.2}
\]

while the electric field \( \vec{E} \) satisfies

\[
\nabla \times \vec{E} = 0, \tag{2.3}
\]

and therefore has a potential \( \phi \):

\[
\vec{E} = -\nabla \phi. \tag{2.4}
\]

The electric conductivity \( \sigma \) is a material parameter, quite strongly dependent on the temperature. For example, for molybdenum it drops from \( \sigma = 19 \times 10^6 \text{ (}\Omega\text{m})^{-1} \) at 21°C to \( 8.0 \times 10^6 \text{ (}\Omega\text{m})^{-1} \) at 300°C (ref. [4], [5]). Nevertheless, to make progress and considering the fact that we are finally in the main part of the thermode interested in a uniform temperature anyway, we will assume a constant \( \sigma \), independent of \( T \). This then leads to the Laplace equation for \( \phi \)

\[
\nabla^2 \phi = 0. \tag{2.5}
\]

At the isolated boundaries the normal component of \( \vec{E} \) vanishes, so at all boundaries, except for the ends, we have

\[
\nabla \phi \cdot \vec{n} = 0. \tag{2.6}
\]
The dissipated heat by the work done by the field per unit time and volume is given by Joule's law, and leads to the heat source distribution

$$\vec{J} \cdot \vec{E} = \sigma|\nabla \phi|^2.$$  \hspace{1cm} (2.7)

Since energy is conserved, the net rate of heat conduction and the rate of increase of internal energy are balanced by the heat source, which yields (ref. [6],[7]) the equation

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T + \sigma|\nabla \phi|^2.$$  \hspace{1cm} (2.8)

where $T$ is the temperature, $k$ is the thermal conductivity, $\rho$ is the density and $c$ the specific heat of the material. The conductivity is mildly dependent on the temperature. For molybdenum it varies from $k = 142 \text{ W/mK}$ at $21^\circ\text{C}$ to $k = 120 \text{ W/mK}$ at $300^\circ\text{C}$, so we further assume here also that $k$ is a constant. $\rho$ is practically constant and $c$ varies slightly. Typically (for molybdenum) from $c = 250 \text{ J/kg K}$ at $21^\circ\text{C}$ to $275 \text{ J/kg K}$ at $300^\circ\text{C}$, and therefore it will also further be taken to be constant.

Assuming that conduction and convection by the surrounding air and the soldering material is negligible, the normal heat flux vanishes at isolated boundaries, similar to the electric field. So we have at all boundaries other than the ends

$$\nabla T \cdot \vec{n} = 0.$$  \hspace{1cm} (2.9)

The typical thickness of a thermode is $d = 0.5 \text{ mm}$ which is small compared with a leg width of (the order of) $2 \text{ mm}$ and length of $20 \text{ mm}$. In view of equations (2.6) and (2.9) we may therefore assume that both $\phi$ and $T$ are constant in cross-wise direction, making the problem geometrically two-dimensional. The geometry will be further simplified by using the inherent symmetry, so that we end with the L-shaped region

$$\Omega_L = \{(x, y) \mid (0 \leq x \leq a, \ 0 \leq y \leq L_a) \cup (0 \leq x \leq L_b, \ 0 \leq y \leq b)\},$$  \hspace{1cm} (2.10)

(Figure 1) with the cool end I and the hot end II.
The typical leg lengths $L_a$ and $L_b$ are sufficiently larger than the widths $a$ and $b$ to warrant the assumption that the electric field and the temperature field behave practically one-dimensionally near the ends I and II, fitting with the boundary conditions

\[
\begin{align*}
\text{I: } & T = 0, \phi = 0 \quad \text{at } y = L_a, \\
\text{II: } & \frac{\partial T}{\partial x} = 0, \phi = \frac{1}{2}V \quad \text{at } x = L_b.
\end{align*}
\]

(2.11)

Both $T$ and $\phi$ are defined up to an arbitrary constant. If required, any suitable constant may be added.

We end with table (1) where typical values of the physical quantities are summarized. Among other things, this will be useful later to check the typical order of magnitude of various dimensionless groups of parameters.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>thickness $d$</td>
<td>0.5 mm</td>
</tr>
<tr>
<td>length $L_a, L_b$</td>
<td>15 - 30 mm</td>
</tr>
<tr>
<td>width $a, b$</td>
<td>1.5 - 2 mm</td>
</tr>
<tr>
<td>temperature $T$</td>
<td>25 - 300°C, (corners 250 - 350°C)</td>
</tr>
<tr>
<td>voltage $V$</td>
<td>0.1 - 2 V</td>
</tr>
<tr>
<td>density $\rho$</td>
<td>molybdenum: 10 200 kg/m$^3$</td>
</tr>
<tr>
<td></td>
<td>tungsten: 19 300 kg/m$^3$</td>
</tr>
<tr>
<td></td>
<td>Hastelloy (typ.): 9 000 kg/m$^3$</td>
</tr>
<tr>
<td>specific heat $c$</td>
<td>275 J/kg K</td>
</tr>
<tr>
<td></td>
<td>141 J/kg K</td>
</tr>
<tr>
<td></td>
<td>375 J/kg K</td>
</tr>
<tr>
<td>thermal conductivity $k$</td>
<td>120 W/mK</td>
</tr>
<tr>
<td></td>
<td>142 W/mK</td>
</tr>
<tr>
<td></td>
<td>13 W/mK</td>
</tr>
<tr>
<td>electric conductivity $\sigma$</td>
<td>$1 \times 10^7 (\Omega m)^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$8 \times 10^6 (\Omega m)^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$7 \times 10^5 (\Omega m)^{-1}$</td>
</tr>
</tbody>
</table>

Table 1: Typical values of the problem parameters.

3. The electric field

The solution for $\phi$ in $\Omega_L$, with $L_a$ and $L_b$ finite, is difficult. However, since $L_a/a$ and $L_b/b$ are sufficiently larger than 1, the solution is practically indistinguishable from a solution in

\[\Omega_\infty = \{(x, y) \mid (0 \leq x \leq a, 0 \leq y < \infty) \cup (0 \leq x < \infty, 0 \leq y \leq b)\},\]

(3.1)

with the behaviour that

\[
\begin{align*}
\frac{\partial \phi}{\partial y} & \rightarrow \text{constant} \quad \text{for } y \rightarrow \infty, 0 \leq x \leq a, \\
\frac{\partial \phi}{\partial x} & \rightarrow \text{constant} \quad \text{for } x \rightarrow \infty, 0 \leq y \leq b.
\end{align*}
\]

(3.2)
Figure 2: Complex $z = x + iy$-plane mapped to the $\phi + i\psi$-plane via the $w$-plane.

This is because the lines $\phi = \text{constant}$ (the equipotential lines) in the legs are quickly perpendicular to the boundaries, and thus allow the application of the condition $\phi = 0$ at end I and $\phi = \frac{1}{2}V$ at end II.

Now the solution can be given by means of a relatively simple conformal mapping ([8],[9]), where we identify the physical $(x, y)$-plane with the complex $z$-plane via $z = x + iy$. Using standard techniques (see Appendix A) the following function, analytic in the upper complex half plane,

$$G(w) = \frac{2ia}{\pi} \arctan\left( \frac{a\sqrt{b^2 - w}}{b\sqrt{a^2 + w}} \right) + \frac{2ib}{\pi} \arctan\left( \frac{\sqrt{b^2 - w}}{\sqrt{a^2 + w}} \right)$$  \hspace{1cm} (3.3)

defines the mapping

$$z = G(w)$$  \hspace{1cm} (3.4)

that maps the upper half $w$-plane to $\Omega_\infty$ in the $z$-plane (Figure 2)*. Further details about its limiting behaviour, etc., may be found in the Appendix A.

The final solution is obtained by positioning a point source of strength $Q$ (to be determined!) in $w = 0$, corresponding to a source (or sink, if we adopt a different sign convention) of electricity in $0 \leq x \leq a, y \rightarrow \infty$. The complex potential of this source in the $w$-plane is given by

$$F = \frac{Q}{\pi} \log(w/b^2) + \phi_0$$  \hspace{1cm} (3.5)

*Perhaps not elegant, we leave the physical dimensions of length in $z$ and $(\text{length})^2$ in $w$.  

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where $\phi_0$ is a constant to be determined. It is necessary for the problem as posed, but with no direct physical relevance. (Finally, it will appear that $\phi_0 = \phi(0,0)$.) The complex potential $F = F(z)$ is now implicitly given by

$$z = G \left( b^2 e^{\pi Q^{-1}(F-\phi_0)} \right).$$

(3.6)

The physical potential $\phi$ is given by the real part of $F$:

$$F(z) = \phi(x,y) + i\psi(x,y).$$

(3.7)

$\psi$, the imaginary part of $F$, is the conjugate of $\phi$. The lines given by $\phi(x,y) = \text{constant}$ are called the equipotential lines. The applied potential is constant along the ends I and II, i.e. these ends coincide with equipotential lines. The lines given by $\psi(x,y) = \text{constant}$ are called the flux lines: the flux between two flux lines is constant. Isolated boundaries always coincide with flux lines. Thus we can choose $\text{Im}(\phi_0) = 0$, so that

$$\psi = 0 \quad \text{along} \quad \{x = 0, \; 0 \leq y < \infty \} \cup \{0 \leq x < \infty, \; y = 0\},$$

$$\psi = Q \quad \text{along} \quad \{x = a, \; b \leq y < \infty \} \cup \{a \leq x < \infty, \; y = b\},$$

(3.8)

while indeed $\phi(0,0) = \phi_0$. Furthermore, we can choose $Q$ and $\phi_0$ such that

$$\phi = 0 \quad \text{along} \quad \{0 \leq x \leq a, \; y = L_a\},$$

$$\phi = \frac{1}{2}V \quad \text{along} \quad \{x = L_b, \; 0 \leq y \leq b\},$$

(3.9)

(in the aforementioned approximate sense, for large enough $L_a/a$ and $L_b/b$). Using the asymptotic results (A.4), we obtain then

$$Q = \frac{1}{2}V \left( \frac{L_a}{a} + \frac{L_b}{b} - \frac{2}{\pi} \left( \frac{a}{b} \arctan \frac{b}{a} + \arctan \frac{a}{b} + \log 4 - \log \left( \frac{a + b}{a} \right) \right) \right)$$

(3.10)

$$\approx \frac{1}{2}V \left( \frac{L_a}{a} + \frac{L_b}{b} \right)$$

$$\phi_0 = \frac{1}{2}V \left( \frac{L_a}{a} - \frac{2}{\pi} \left( \frac{b}{a} \arctan \frac{b}{a} + \log 2 - \frac{1}{2} \log \left( \frac{a}{b} + \frac{b}{a} \right) \right) \right)$$

(3.11)

$$\approx \frac{1}{2}V \left( \frac{L_a}{a} / \frac{L_a}{a} + \frac{L_b}{b} \right)$$

An example of the electric field thus obtained is depicted in the Figure 3. The corresponding heat source distribution is given by Figure 4. For later use, we summarize
Figure 3: Equipotential and flux lines for $a=1.0$, $b=2.0$

the resulting behaviour of $\phi$ in the various regions of interest:

$$\phi = \phi_0 - \frac{Q}{a} y$$
$$+ \frac{Q}{\pi} \left( \log \frac{4a^2}{a^2 + b^2} + \frac{2b}{a} \arctan \frac{b}{a} \right)$$
$$y \to \infty, \ 0 \leq x \leq a \quad (3.12)$$

$$|\nabla \phi|^2 = \left( \frac{Q}{a} \right)^2$$

$$\phi = \phi_0 + \frac{Q}{b} x$$
$$- \frac{Q}{\pi} \left( \log \frac{4b^2}{a^2 + b^2} + \frac{2a}{b} \arctan \frac{a}{b} \right)$$
$$x \to \infty, \ 0 \leq y \leq b \quad (3.13)$$

$$|\nabla \phi|^2 = \left( \frac{Q}{b} \right)^2$$

$$\phi = \phi_0 + \frac{2Q}{\pi} \log \frac{a}{b}$$
$$- \frac{Q}{\pi} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^{1/3} \left( \frac{3}{2} \pi r \right)^{2/3} \cos \left( \frac{2}{3} \theta - \frac{1}{3} \pi \right)$$
$$x + iy = a + ib + re^{i\theta}, \quad r \to 0, \ \frac{1}{2} \pi \leq \theta \leq 2\pi \quad (3.14)$$

$$|\nabla \phi|^2 = Q^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^{2/3} \left( \frac{3}{2} \pi r \right)^{-2/3}$$
Figure 4: Contours of constant source strength for $a=1.0$, $b=2.0$

\[
\phi = \phi_0 + \frac{\pi Q r^2 \cos 2\theta}{4 a^2 + b^2}, \quad \frac{r}{r_0}, 0 \leq \theta \leq \frac{1}{2}\pi
\]

(3.15)

All together, we have now a conformal mapping from the physical $(x,y)$-plane to the electrostatic field $(\psi, \theta)$-plane (Figure 2). Since $\phi$ and $\psi$ both satisfy Laplace’s equation, and are related by the Cauchy-Riemann equations

\[
F'(z) = \psi_x + i\psi_y = -i\phi_y + \psi_y
\]

(3.16)

we have

\[
|F'(z)|^2 = |\nabla \phi|^2
\]

(3.17)

which is not only $\sigma^{-1}$ times the heat source distribution, but also just the inverse of the Jacobian of the mapping $(x,y) \rightarrow (\phi, \psi)$, i.e.

\[
|\nabla \phi|^2 dx dy = d\phi d\psi.
\]

(3.18)

When practical evaluation is needed, it may be noted that

\[
|\nabla \phi|^2 = \frac{Q^2}{ab} \sqrt{\frac{\cosh(u - \log b^2) - \cos v}{\cosh(u - \log a^2) + \cos v}}
\]

(3.19)

where $u + iv = \pi Q^{-1}(F - \phi_0) + \log b^2$. 

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4. The initial value problem

At the start of the soldering process the electric heat source dominates the effect of thermal diffusion in most of the thermode. Hence the source is only balanced by the energy storage term $\rho c \frac{\partial T}{\partial t}$. This is because a smooth behaviour of $|\nabla \phi|$ results (except for the vicinity of the cold end $I$) in a smooth temperature profile with a small diffusion term $\nabla^2 T$. This is especially the case in the legs where $\nabla \phi$ is constant. So initially we have four domains to consider:

a) The region with a smoothly behaving $\nabla \phi$, not near the cold end, where the temperature increase is directly coupled to the source with only a secondary role for the diffusion.

b) The cold end $I$ where the temperature boundary condition $T = 0$ creates a steep temperature gradient, giving a heat diffusion of equal importance as the heat-up and source terms. It is of practical interest to know after what time the diffusion of the low end-temperatures reduces the temperature increase in the thermode.

c) At the inner corner $(a, b)$ the source $|\nabla \phi|^2$ has a singularity of the order $O(r^{-2/3})$ (where $r$ is the distance to the corner) which creates locally an intense temperature rise, so that diffusion becomes of equal importance as the other terms. The initial temperature over-shoot at the corner diffuses away after some time. It is of practical interest to know if this time is less than the total heat-up time.

d) At the outer corner $(0, 0)$ the source $|\nabla \phi|^2$ has a behaviour of the type $O(r^2)$ which corresponds to a vanishing heat source and therefore also in this case a diffusion term which is of equal importance as the the other terms. So we may expect that at the very beginning of the process this corner is slightly colder than the rest of the thermode, then the hot inner corner becomes effective and heats up the whole corner region, until this is again reduced by the cooling effect of the end.

a) The main domain

The fact that the diffusion term in equation (2.8) is of secondary importance suggests an iterative process to construct a solution. Assuming a vanishing zeroth order solution $T_0(x, y, t) \equiv 0$, we can construct formally higher order approximations by:

$$\rho c \frac{\partial T_n}{\partial t} = k \nabla^2 T_{n-1} + \sigma |\nabla \phi|^2$$

(4.1)

which, in view of the source being independent of time, readily results into the following (formal) solution

$$T(x, y, t) = \frac{\sigma}{k} \sum_{n=1} \frac{1}{n!} \left( \frac{kt}{\rho c} \right)^n \nabla^{2(n-1)}(|\nabla \phi|^2)$$

(4.2)

$$= \frac{\sigma}{\rho c} |\nabla \phi|^2 t + \cdots$$
In general, this result is only to be interpreted in some asymptotic sense for small time. Obviously, it is neither valid for all \( t \) nor for all \( x \) since none of the boundary conditions is explicitly applied. However, in a practical situation of heating up a thermode as quickly as possible, we are at start-up usually in this “small-time” regime, at least near end II, the middle of the side used for soldering, which is the region of most interest. Using (3.13) we have

\[
T(L_b, 0, t) = \frac{\sigma}{\rho c} \left( \frac{Q}{b} \right)^2 t. \tag{4.3}
\]

This may be compared with an “exact” numerical solution of the present problem, (2.8) in (2.10) with (2.9, 2.11). This solution has been generated by the finite element package SEPRAN ([10]), as described in Appendix C. In Figure 5 the temperature is shown at \( x = L_b \) and \( y = 0 \) for a thermode of molybdenum with \( V = 0.4 \) V, \( L_a = 20 \) mm, \( L_b = 10 \) mm and \( a = b = 2 \) mm. The similarity between the analytical and numerical solutions is very good.

Under the assumption that the influence of cold end I has not yet reached the end II, we can use the above result to estimate the total heat-up time. It is found that the temperature \( T_\infty \) is reached after the total heat-up time

\[
t_{\text{heat-up}} = \frac{\rho c}{\sigma} \left( \frac{b}{Q} \right)^2 T_\infty. \tag{4.4}
\]

This is for the above example, with \( T_\infty = 275^\circ \) C, for molybdenum equal to 1.42 s (numerically: 1.48 s), for tungsten 1.72 s, and for Hastelloy 24.4 s.

Note that in leg I and leg II the source strength is not equal but has a ratio of \( a^2/b^2 \). This means that when \( a \) and \( b \) differ in any substantial way, a temperature gradient across the corner region is built up, which after some time will be levelled by a flow of heat from one leg to the other. This effect will not be considered here.
b) The cold end

Near the end I the heat source is constant: \(|\nabla \phi|^2 = (Q/a)^2\) (eq.(3.12)). Also the boundary and initial value condition are uniform: \(T = 0\) at \(y = L_a\) and \(t = 0\). This implies that the solution is only a function of \(y\) and \(t\), so that equation (2.8) reduces to

\[
\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} + \sigma \left(\frac{Q}{a}\right)^2
\]

(4.5)

This equation may be solved by the similarity solution

\[
T(x, y, t) = \frac{\sigma}{\rho c} \left(\frac{Q}{a}\right)^2 t f(\eta), \\
\eta^2 = \frac{\rho c}{4kt} (L_a - y)^2 \\
f(\eta) = (1 + 2\eta^2) \text{erf}(\eta) + \frac{2}{\sqrt{\pi}} \eta e^{-\eta^2} - 2\eta^2 \\
\text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2} d\xi
\]

(4.6)

The behaviour of \(f\) is graphically described by Figure 6. For small time \(t\) and large \(\eta\), the shape function \(f \simeq 1\), so that the temperature behaves like in the “main domain” as \(T = (\sigma/\rho c)|\nabla \phi|^2 t\). When at a fixed position \(y\) the time \(t\) increases, the similarity variable \(\eta\) decreases. For \(t\) so large that \(\eta \leq O(1)\), \(f\) reduces to values smaller than 1, diminishing the growth of \(T\). So in this way the effect of the cold end I is felt at the corner \((y \sim 0)\) after a diffusion time

\[
t_{\text{cold-end}} \geq \frac{\rho c L_a^2}{4k}.
\]

(4.7)

For example \((L_a = 20\text{ mm})\), this is for molybdenum 2.34 s, for tungsten 1.92 s, and for Hastelloy 26.0 s.
c) The singular inner corner \((a,b)\).

It is seen from eq.(3.14) that near the corner \((a,b)\) the heat source is a function of \(r\) (the distance to the corner) only, and behaves like \(|\nabla \phi|^2 = O(r^{-2/3})\). Also the boundary conditions \(\partial T/\partial \theta = 0\) at \(\theta = \frac{1}{2}\pi\) and \(2\pi\) do not invoke any dependence of \(T\) on \(\theta\), the angular coordinate in the corner. Therefore, \(T\) is a function of \(r\) and \(t\) only, so that equation (2.8) reduces to

\[
\rho c \frac{\partial T}{\partial t} = k \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \sigma Q^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^{2/3} \left( \frac{3}{2\pi r} \right)^{-2/3}.
\]

(4.8)

As explained in more detail in Appendix B, this equation has the following similarity solution

\[
T(x,y,t) = \frac{\sigma}{k} Q^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^{2/3} \left( \frac{9kt}{8\pi \rho c} \right)^{2/3} \left[ \Gamma \left( \frac{5}{3} \right) \frac{1}{1F1} \left( \frac{-2}{3}; 1; -\eta^2 \right) - \eta^{4/3} \right]
\]

(4.9)

where \(1F1(a; b; z)\) is the confluent hypergeometric function. A graphical description of the shape function \(\Gamma \left( \frac{5}{3} \right) \frac{1}{1F1} \left( \frac{-2}{3}; 1; -\eta^2 \right) - \eta^{4/3}\) as a function of \(\eta\) is given in Figure 7. As is shown in Appendix B, this solution perfectly matches to the far field “main

![Figure 7: Scaled temperature \(\Gamma \left( \frac{5}{3} \right) \frac{1}{1F1} \left( \frac{-2}{3}; 1; -\eta^2 \right) - \eta^{4/3}\) near inner corner \(a, b\)](image-url)
domain” behaviour for large \( r \) (large \( \eta \)). The most important property of (4.9) is the value at the corner:

\[
T(a, b, t) = \Gamma\left(\frac{5}{3}\right) \left(\frac{9}{8\pi}\right)^{2/3} \frac{Q^2}{k} \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{2/3} \left(\frac{kt}{\rho c}\right)^{2/3}
\]  

(4.10)

When we compare this value with the far field (the average value between leg I and leg II, for subjective reasons of symmetry) we may obtain an estimate for the time it takes before the temperature overshoot at the corner is dissipated away. The ratio

\[
\frac{T(a, b, t)}{\frac{1}{2}(T(I) + T(II))} = \frac{\Gamma\left(\frac{5}{3}\right)}{2} \left(\frac{9}{\pi}\right)^{2/3} \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-1/3} \left(\frac{kt}{\rho c}\right)^{-1/3}
\]

is numerically smaller than 1 when \( t \) is larger than the corner excess diffusion time

\[
t_{ced} = \frac{81\Gamma\left(\frac{5}{3}\right)^3 \rho c}{8\pi^2 k} ab \left(\frac{a}{b} + \frac{b}{a}\right)^{-1} = 0.7547 \frac{pc}{k} ab \left(\frac{a}{b} + \frac{b}{a}\right)^{-1}
\]  

(4.11)

For example \((a = b = 2 \text{ mm})\), this is for molybdenum 0.035 s, for tungsten 0.029 s, and for Hastelloy 0.39 s. Apparently, for the values here selected the temperature overshoot does not exist long enough to be of importance.

This is very well confirmed by the same finite element “exact” solution of above (molybdenum with \( V = 0.4 \text{ V}, L_a = 20 \text{ mm}, L_b = 10 \text{ mm} \) and \( a = b = 2 \text{ mm} \); Appendix C). In Figure 8 the temperature near the corner \( x = a, y = b, \) (both the numerical

![Figure 8: Temperature at corner (a,b) and at end II](image)

and the analytical) is compared with the far field (at end II). After about 0.04 seconds the corner temperature does not dominate anymore over the far field temperature.
d) The smooth outer corner (0,0).

In a way, this is rather similar to the inner corner. Again the source is a function of \( r \) only, although now a very smooth one: \(|\nabla \phi|^2 = O(r^2)\), as given by equation (3.15). And again the boundary conditions are such that \( T \) is a function of \( r \) and \( t \) only, so that equation (2.8) reduces to

\[
\rho c \frac{\partial T}{\partial t} = k \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\sigma \pi^2 Q^2}{4(a^2 + b^2)^2} r^2. \tag{4.12}
\]

This equation is also of the type solved in Appendix B, however, the confluent hypergeometric function simplifies now greatly, since \( _1F_1(-2;1;z) = 1 - 2z + \frac{1}{2}z^2 \), and we obtain

\[
T(x,y,t) = \frac{\sigma \pi^2 Q^2}{4(\rho c)^2(a^2 + b^2)^2}(2kt^2 + \rho c r^2 t). \tag{4.13}
\]

Due to its zero source strength the corner is only heated by conduction from its neighbourhood. If the distance between corner (0,0) and \((a,b)\) is large enough (i.e., if \( \eta^2 = \rho c(a^2 + b^2)/4k\tau_{ced} \) (see eq. 4.9) is large enough), the corner (0,0) is never heated by corner \((a,b)\) more than by the far fields \( T(I) \) or \( T(II) \). In that case, the corner temperature that starts as a quadratic function in time (equation (4.13)), will after some time (the corner shortage diffusion time) \( \tau_{ced} \) become comparable with the far field, and then gradually change into some linear-type growth that follows the far field. As a result, not only the temperature \( T(0,0,t) \) itself, but also its slope \( \partial T(0,0,t)/\partial t \) will never exceed the far field counterparts. An estimate for this corner shortage diffusion time is therefore the time it takes before approximation (4.13) and the far field have the same slope:

\[
\frac{\partial T(0,0,t)}{\partial t} = \frac{1}{2} \frac{\partial(T(I) + T(II))}{\partial t}
\]

which results into

\[
\tau_{ced} = \frac{1}{2\pi^2 k} \frac{\rho c}{a} b \left( \frac{a}{b} + \frac{b}{a} \right)^3.	ag{4.14}
\]

This is equal to or larger than \( \tau_{ced} \), as we have the ratio

\[
\frac{\tau_{ced}}{\tau_{ced}} = \frac{4}{81\Gamma(\frac{3}{2})^3} \left( \frac{a}{b} + \frac{b}{a} \right)^4 = \frac{1}{14.90} \left( \frac{a}{b} + \frac{b}{a} \right)^4,
\]

which is for \( a/b = 1 \) equal to 1.07, and larger otherwise.

These observations may be confirmed by comparison with the numerical FEM-solution, introduced earlier (molybdenum with \( V = 0.4 \) V, \( L_a = 20 \) mm, \( L_b = 10 \) mm and \( a = b = 2 \) mm). In Figure 9 the analytical expression (4.13) for the temperature at the corner \( x = 0, y = 0 \) is compared with its numerical counterpart, and the "far field" temperature at end II. We see indeed that after about 0.04 sec. the corner temperature tends to follow the far field.
5. The stationary problem

In the previous section we have dealt with the first question, i.e. how to keep the excess temperature in the corner below the end temperature. Now we will consider the second question of a uniform temperature distribution in the stationary problem.

After the final temperature \( T_\infty \) at end II is reached, the heating will be reduced to just the amount necessary to maintain this temperature (i.e. the electric potential difference \( V \) is switched down to a small value). After a short time the temperature is stationary, and equation (2.8) reduces to

\[
  k\nabla^2 T + \sigma |\nabla \phi|^2 = 0. \tag{5.1}
\]

The second question with respect to the uniformity of temperature can now be answered by an exact solution. Before we do this, however, we will present some preliminary considerations.

If the thermode were straight, the temperature would be non-uniform. It would be a quadratic function in the axial coordinate, and the cold ends I would pull down the temperature from its maximum value in II to zero at the ends. Now there are two ways to obtain a uniform temperature in leg II. Near its maximum the temperature varies only little, so one way is to position the corners relatively close to the end II. Another possibility is to use the potentially compensating effect of the intense source strength at the corners, since near the inner corner \((a, b)\) the heat source is much higher than elsewhere. However, near the outer corner \((0, 0)\) this is not the case and it is much lower. The question is therefore: is the effect of the corner a net heat source strength excess or is it only a relocation. The answer is that there is indeed a net gain, which indeed can be utilized for our purposes. This is seen as follows.
a) The total excess source strength

Suppose that we break the L-shaped thermode into two parts (I' and II') of size \( a \times (L_a - \frac{1}{2}b) \) and \( b \times (L_b - \frac{1}{2}a) \).

If these parts are then ideally connected together such that in each part the field is one-dimensional, the constant source strength in either part is given by

\[
|\nabla \phi|^2 = \left( \frac{1}{2} \frac{V}{a} \right)^2 \frac{L_a}{a + L_b} - \frac{1}{2} \frac{b}{a + a/b} \quad \text{in I'}
\]

\[
|\nabla \phi|^2 = \left( \frac{1}{2} \frac{V}{b} \right)^2 \frac{L_a}{a + L_b} - \frac{1}{2} \frac{a}{b + a/b} \quad \text{in II'}
\]

The total excess heat source (that is: the heat source due to the presence of the corner) is then the difference between the net source of the original and the straightened shape:

\[ 
\epsilon = \int \int_{\Omega_{L}} |\nabla \phi|^2 \, dx \, dy - \int \int_{I' + II'} |\nabla \phi|^2 \, dx \, dy. \]  

The first integral is most easily obtained by using the fact that \( |\nabla \phi|^2 \) is the inverse of the Jacobian of the transformation of \((x, y)\) to \((\phi, \psi)\). The second integral is almost trivial. For clarity we make the result independent of the lengths \(L_a\) and \(L_b\) by taking the limits \(V, L_a, L_b \to \infty\), while keeping \(V/(L_a/a + L_b/b)\) constant. Then we have

\[
\epsilon = \lim_{V,L_a,L_b \to \infty} \left\{ \int_0^{\frac{1}{2}V} \int_0^Q d\psi \, d\phi - a(L_a - \frac{1}{2}b)|\nabla \phi_I|^2 - b(L_b - \frac{1}{2}a)|\nabla \phi_{II}|^2 \right\} 
\]

\[ = \left( \frac{1}{2} \frac{V}{L_a/a + L_b/b} \right)^2 \left( a(b + a/b) \right) \]

where

\[ g(\lambda) = \frac{2}{\pi} \left( \lambda \arctan \lambda + \lambda^{-1} \arctan \lambda^{-1} + \log 4 - \log(\lambda + \lambda^{-1}) \right) - \frac{1}{2}(\lambda + \lambda^{-1}) \]

Since \( g \) is a positive function with

\[ g(\lambda) = g(\lambda^{-1}) \geq g(1) = \frac{2}{\pi} \log 2 \simeq 0.4413, \]

the excess heat \( \epsilon \) (5.4) is positive. This implies that indeed the presence of the corner yields a higher net heat production.
b) The exact solution

Since the transformation \((x, y) \rightarrow (\phi, \psi)\) has a Jacobian \(|\nabla \phi|^{-2}\), the Laplacian is transformed into

\[
\nabla^2_{x, y} T = |\nabla \phi|^2 \nabla^2_{\phi, \psi} T. \tag{5.5}
\]

In our stationary equation (5.1) the term \(|\nabla \phi|^2\) can now be divided out, and the transformed "source" is just a constant

\[
k \nabla^2_{\phi, \psi} T + \sigma = 0. \tag{5.6}
\]

Furthermore, the region in the \((\phi, \psi)\)-plane is now the rectangle \((0, \frac{1}{2}V) \times (0, Q)\), where \(T = 0\) along the side \(\phi = 0\) and has vanishing normal derivative along the others: \(\partial T/\partial \phi = 0\) at \(\phi = \frac{1}{2}V\), \(\partial T/\partial \psi = 0\) along \(\psi = 0\) and \(\psi = Q\). As a result, the temperature is a function of only \(\phi\), which makes the solution of (5.6)

\[
T(x, y) = \frac{\sigma (\frac{1}{2}V)^2}{2k} \left(1 - \left(1 - \frac{\phi}{\frac{1}{2}V}\right)^2\right). \tag{5.7}
\]

Note that this solution is valid for any geometry, with isolated boundaries and two sides with a given constant potential and temperature. Our result can be considered...
an extension of the result given in Carslaw and Jaeger (ref. [7, ch. 4.10]) for a thin wire heated by an electric current. A graphical description of the temperature field is given in the Figure 10. Now we are able to compare the temperature at the center of the lower bar (end point II) with the outer corner (0,0). These points correspond with, respectively, $\phi = \frac{1}{2}V$ and $\phi = \phi_0$. So we obtain for the ratio

$$\frac{T(0,0)}{T(L_b,0)} = \frac{T(\phi = \phi_0)}{T(\phi = \frac{1}{2}V)} = 1 - \left(1 - \frac{\phi_0}{\frac{1}{2}V}\right)^2$$

which is only a function of the geometrical factor $\phi_0/\frac{1}{2}V$. For example for the geometry $L_a = 20\text{ mm}, L_b = 10\text{ mm}, a = b = 2\text{ mm}$ with $T_\infty = 275^\circ\text{ C}$, this yields a corner temperature of $247.6^\circ\text{ C}$. A more qualitative description, given in Figure 11, shows the dependence on $a$ and $b$ for $L_a = 20$ and $L_b = 10$. Under the usual approximation of large $L_a/a$ and $L_b/b$ the above expression becomes

$$\frac{T(0,0)}{T(L_b,0)} \approx 1 - \left(\frac{L_b/b}{L_a/a + L_b/b}\right)^2.$$

This shows that a constant temperature along leg II requires $\phi_0 = \frac{1}{2}V$, which is not possible. However, we can make the difference as small as we like:

![Graph](image-url)

Figure 11: Corner/middle temperature ratio for $L_a = 20, L_b = 10$
The temperature ratio (5.8) along leg II can be made larger than \(1 - \delta^2\), for any \(\delta\), if a geometry is selected such that
\[
\phi_0/\frac{1}{2}V = 1 - \delta.
\] (5.10)

For large \(L_a/a\) and \(L_b/b\) this condition becomes
\[
\frac{a}{L_a} = \frac{\delta}{1 - \delta} \frac{b}{L_b}.
\] (5.11)

6. Conclusions

The nonuniform heating problem of a U-shaped thermode may be split up into the following parts. During the initial dynamic phase the material heats up following the (nonuniform) source distribution. This leads to a uniform temperature in the straight parts, an overshoot near the inner corners where the electric field is singular, and a lower temperature near the outer corners where the electric field vanishes. This distribution, together with the cold boundary condition at the cool ends, induces a local heat flux which is not in balance with the source, so the temperature will change in time to its stationary equilibrium. The times involved with this redistribution may be used to select the problem parameters such that the undesired overshoot is controlled:

the corner excess diffusion time
\[
t_{ced} = \frac{81\Gamma(\frac{5}{2})^3}{8\pi^2} \frac{\rho c}{k} ab\left(\frac{a}{b} + \frac{b}{a}\right)^{-1} = 0.7547 \frac{\rho c}{k} ab\left(\frac{a}{b} + \frac{b}{a}\right)^{-1}
\]
should be smaller than the total heat-up time
\[
t_{heat-up} = \frac{\rho c}{\sigma} \left(\frac{b}{Q}\right)^2 T_\infty
\]
or larger than the cold-end diffusion time
\[
t_{cold-end} \geq \frac{\rho c}{4k} L_a^2
\]
for the corner overshoot to be of no importance.

The other part of the nonuniform heating problem is the stationary problem. The cold ends cool off the thermode in such a way that the corners may become too cold. On the other hand the same corners induce a locally intense electric field that may compensate for this effect, so as to keep the corners and the rest of the bar meant for
soldering at an almost uniform temperature. It is noted that this is only a geometric property. This almost uniform temperature is obtained as follows. (For large legs) a variation in temperature of at most $\delta^2$ is obtained by selecting the geometry such that

$$\frac{a}{L_a} = \frac{b}{1 - \delta L_b}.$$ 

Appendix A. Conformal mapping of polygonal boundaries

According to the Schwarz-Christoffel theorem ([8],[9]), a polygonal boundary in the complex $z$-plane with interior angles $\alpha_1, \alpha_2, \alpha_3, ...$ is mapped on to the real axis $\text{Im}(w) = 0$ of the complex $w$-plane by the transformation $z = z(w)$, given by

$$\frac{dw}{dz} = K(w - p_1)^{1-\alpha_1/\pi} (w - p_2)^{1-\alpha_2/\pi} (w - p_3)^{1-\alpha_3/\pi} \ldots$$

(A.1)

where $K$ is a constant and $p_1, p_2, p_3, ...$ are the real values of $w$ corresponding to the vertices of the polygon. The region in the $w$-plane corresponding to the polygonal interior is the half-plane $\text{Im}(w) > 0$. The interior angle of a vertex at infinity is zero. One such point may be mapped to infinity in the $w$-plane, with at the same time $K$ tending to zero such that the factor $K(w - w_1)$ is effectively constant.

In the present problem for $\Omega_\infty$ the following equation is selected

$$\frac{dw}{dz} = \frac{\pi i w \sqrt{b^2 - w}}{b \sqrt{a^2 + w}}.$$  

(A.2)

with solution $z = G(w)$ given by

$$G(w) = \frac{2ia}{\pi} \text{artanh} \left( \frac{a \sqrt{b^2 - w}}{b \sqrt{a^2 + w}} \right) + \frac{2ib}{\pi} \arctan \left( \frac{\sqrt{b^2 - w}}{\sqrt{a^2 + w}} \right)$$

$$= \frac{ai}{\pi} \log \left( \frac{b \sqrt{a^2 + w} + a \sqrt{b^2 - w}}{b \sqrt{a^2 + w} - a \sqrt{b^2 - w}} \right) - \frac{b}{\pi} \log \left( \frac{i \sqrt{a^2 + w} + \sqrt{b^2 - w}}{i \sqrt{a^2 + w} - \sqrt{b^2 - w}} \right)$$

(A.3)

The interior angles to be dealt with are $0$ and $0$ in the legs at infinity, $\frac{1}{2} \pi$ at $(0,0)$, and $\frac{3}{2} \pi$ at $(a,b)$. The infinite vertex $y \to \infty$, $0 \leq x \leq a$ is mapped to $w = 0$, the infinite vertex $x \to \infty$, $0 \leq y \leq b$ to $w \to \infty$. The real values corresponding to the other vertices are most conveniently taken to be $b^2$ and $-a^2$, so that

$$G(b^2) = 0 \text{ and } G(-a^2) = a + ib$$

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where the limit \( \text{Im}(w) \downarrow 0 \) is understood. The definitions for log, square root, arctan, and artanh are the principal values, with

\[
\arctan(x) = \frac{i}{2} \log \frac{i + x}{i - x}, \quad \text{artanh}(x) = \frac{1}{2} \log \frac{1 + x}{1 - x}.
\]

The combined branch cuts of the square roots and the logarithms give \( G \) branch cuts along \((-\infty, 0]\) and \([b^2, \infty)\). The boundary \( \partial \Omega_{\infty} \) is mapped to the real \( w \)-axis as follows:

- \( x > a, \ y = b \) for \( -\infty \leq w \leq -a^2 \),
- \( x = a, \ y > b \) for \( -a^2 \leq w \leq 0 \),
- \( x = 0, \ y > 0 \) for \( 0 \leq w \leq b^2 \),
- \( x > 0, \ y = 0 \) for \( b^2 \leq w \leq \infty \).

The behaviour of \( G \) near the vertices is given by the asymptotic expressions

\[
\begin{align*}
G(w) &\sim \frac{b}{\pi} \left( \log \frac{w}{ab} + \log \frac{4ab}{a^2 + b^2} + \frac{2a}{b} \arctan \frac{a}{b} \right) \quad (w \to \infty; x \to \infty) \\
G(w) &\sim \frac{ia}{\pi} \left( \log \frac{w}{ab} + \log \frac{4ab}{a^2 + b^2} + \frac{2b}{a} \arctan \frac{b}{a} \right) \quad (w \to 0; y \to \infty) \\
G(w) &\sim \frac{2i}{\pi b} \sqrt{a^2 + b^2} \sqrt{b^2 - w} \quad (w \to b^2; x, y \to 0, 0) \\
G(w) &\sim a + ib + \frac{2ib}{3\pi a^2} \frac{(a^2 + w)^{3/2}}{\sqrt{a^2 + b^2}} \quad (w \to -a^2; x, y \to a, b)
\end{align*}
\]

Appendix B. The general initial value problem in a corner

Consider the wedge-shaped two-dimensional region

\[ 0 \leq \theta \leq \alpha \]

with an electric field with complex potential \( F(z) \sim z^{\pi/\alpha} \) so that the potential of the field is

\[
\phi(x, y) = \frac{(\alpha/\pi)}{A} r^{\pi/\alpha} \cos(\theta \pi/\alpha)
\]  \( (B.1) \)

The temperature distribution \( T \) due to the heat generated by this field is then given by

\[
\rho c \frac{\partial T}{\partial t} = k \nabla^2 T + \sigma A^2 r^{2\pi/\alpha - 2}
\]  \( (B.2) \)
with boundary conditions
\[
\frac{\partial T}{\partial \theta} = 0 \quad \text{at } \theta = 0, \theta = \alpha
\] (B.3)  

and initial conditions
\[
T(x, y, t) \equiv 0 \quad \text{at } t = 0.
\] (B.4)  

Since there are no other (point) sources in \(r = 0\) we have the additional condition of a finite field in the origin:
\[
0 \leq T(0,0,t) < \infty.
\] (B.5)  

Boundary conditions and the symmetric source imply that \(T\) is a function of \(t\) and the radial coordinate \(r\) only, so that equation (B.2) reduces to
\[
\frac{\rho c}{\sigma} \frac{\partial T}{\partial t} = k \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \sigma A^2 r^{2\pi/\alpha - 2}.
\] (B.6)  

Dimensional analysis reveals that due to the homogeneous initial and boundary conditions, the infinite geometry, and the source being a monomial in \(r\), homogeneous of the order \(2\pi/\alpha - 2\), there is no length scale in the problem other than \((kt/\rho c)^{1/2}\), so that a similarity solution is possible. It appears that
\[
T(r, t) = \frac{\sigma}{4k} A^2 \left( \frac{4kt}{\rho c} \right)^{\pi/\alpha} h(X), \quad X = \frac{\rho cr^2}{4kt}
\] (B.7)  

reduces equation (B.6) to
\[
Xh'' + (1 + X)h' - \left( \frac{\pi}{\alpha} \right)h = -X^{\pi/\alpha - 1}.
\] (B.8)  

This equation may be recognized as an inhomogeneous confluent hypergeometric equation in \(-X\), which has the general solution
\[
h(X) = C \, _1F_1(-\pi/\alpha; 1; -X) + D \, U(-\pi/\alpha, 1, -X) - (\pi/\alpha)^2 X^{\pi/\alpha}
\] (B.9)  

where \(_1F_1(a; b; z)\) and \(U(a, b, z)\) are the regular and singular confluent hypergeometric functions (ref. [11],[12],[13]), solutions of Kummer’s equation
\[
z y'' + (b - z)y' - ay = 0.
\] (B.10)  

Since \(U\) is singular in the origin:
\[
U(-\pi/\alpha, 1, -X) \simeq -\log(X)/\Gamma(-\pi/\alpha) \quad (X \to 0)
\] (B.11)  

this term has to vanish, and the integration constant \(D = 0\).
\(_1F_1(a; b; z)\) is defined by
\[
_1F_1(a; b; z) = 1 + \frac{az}{b} + \frac{a(a+1)z^2}{b(b+1)2} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots
\] (B.12)
where \((a)_n\) is Pochhammer’s symbol, defined by
\[
(a)_0 = 1,
\]
\[
(a)_n = a(a + 1)(a + 2) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}.
\]  
(B.13)

Note that for \(a\) equal to a negative integer, this series is finite and \(_1F_1(a; b; z)\) is just a polynomial of order \(-a\).

\(_1F_1(-\pi/\alpha; 1; -X)\) has the asymptotic expansion
\[
_1F_1(-\pi/\alpha; 1; -X) = \frac{1}{\Gamma(1 + \pi/\alpha)} \sum_{n=0}^{R-1} \frac{((-\pi/\alpha)_n)^2}{n!} X^{\pi/\alpha-n} + O(X^{-R}) \quad (X \to \infty)
\]  
(B.14)

so that the initial value condition implies for the other integration constant the value

\[
C = (\alpha/\pi)\Gamma(\pi/\alpha).
\]

All together we have the solution
\[
T(x, y, t) = \frac{\sigma \alpha^2 A^2}{4\pi^2 k} \left(\frac{4kt}{\rho c}\right)^{\pi/\alpha} \left\{ \Gamma\left(1 + \frac{\pi}{\alpha}\right) _1F_1\left(-\frac{\pi}{\alpha}; 1; -\frac{\rho c r^2}{4kt}\right) - \left(\frac{\rho c r^2}{4kt}\right)^{\pi/\alpha} \right\}
\]
\[
= \frac{\sigma \alpha^2 A^2}{4\pi^2 k} \left(\frac{4kt}{\rho c}\right)^{\pi/\alpha} \Gamma\left(1 + \frac{\pi}{\alpha}\right) _1F_1\left(-\frac{\pi}{\alpha}; 1; -\frac{\rho c r^2}{4kt}\right) - r^{2\pi/\alpha}. \quad (B.15)
\]

Note that, as may be expected, the behaviour for \(r \to \infty\) ceases to depend on \(\sqrt{2} T\) and is just the linear growth in time:
\[
T(x, y, t) \sim \frac{\sigma}{\rho c} A^2 r^{2\pi/\alpha - 2} t \quad (r \to \infty) \quad (B.16)
\]

Finally, a remark may be in order about the effect of a finite radius of curvature of the corner that might occur in practice. In such a situation the present solution is not valid for very small time. This is seen as follows.

The present similarity solution is available because of the absence of any length scale in the problem. Therefore, the temperature has to depend on the inherent length scale \(\ell = (4kt/\rho c)^{1/2}\). If we had a corner with a small but finite radius of (say) \(r_0\), then a length scale is introduced via the boundary condition \(\partial T/\partial r = 0\) at \(r = r_0\), which, strictly speaking, invalidates the present solution. Indeed, for very short times, when \(\ell \leq O(r_0)\) we need another solution. However, for larger times, the details of the corner become “invisible”, and the present solution is valid again.
Appendix C. Finite element solution

By means of the finite element package SEPRAN ([10]) a numerical solution has been generated of the present problem (2.8) in (2.10) with (2.9, 2.11), for a thermode of molybdenum with $V = 0.4 \text{V}$, $L_a = 20 \text{mm}$, $L_b = 10 \text{mm}$ and $a = b = 2 \text{mm}$. The finally used mesh is given in Figure 12. By halving the mesh size from (typically) $\frac{1}{2} \text{mm}$ to $\frac{1}{4} \text{mm}$ and using an automatic time step integrator with a relative accuracy of $10^{-3}$, the typical relative accuracy of the solution was estimated to be less than $O(2 \cdot 10^{-3})$.

![Figure 12: Mesh in the finite element calculation](image)
Acknowledgements:

The author gratefully acknowledges the enthusiastic support of J.K.M. Jansen and L.G.F.C. van Bree by programming and running the SEPRAN package for the present problem, and the interesting remarks by prof. J. Boersma.

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