A taxonomy of keyword pattern matching algorithms

by

Bruce W. Watson and G. Zwaan

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A taxonomy of keyword pattern matching algorithms

B.W. Watson & G. Zwaan
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Faculty of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB
Eindhoven, The Netherlands
email: watson@win.tue.nl or wsinswan@win.tue.nl

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Abstract

This paper presents a taxonomy of keyword pattern matching algorithms, including the well-known Knuth-Morris-Pratt, Aho-Corasick, Boyer-Moore, and Commentz-Walter algorithms and a number of their variants. The taxonomy is based on the idea of ordering algorithms according to their essential problem and algorithm details, and deriving all algorithms from a common starting point by adding these details in a correctness preserving way. This way of presentation not only provides a complete correctness argument of each algorithm, but also makes very clear what algorithms have in common (the details of their nearest common ancestor) and where they differ (the details added after their nearest common ancestor). Moreover, the paper provides complete derivations of the intricate precomputation algorithms, some of which either cannot be found in the literature (Commentz-Walter) or are given in several different versions (Boyer-Moore).
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1 Introduction and related work

Keyword pattern matching is one of the most extensively explored fields in computing science. Loosely stated, the problem is to find the set of all occurrences from a set of patterns in an input string.

Just as the variety of applications has grown, so has the diversity of the solutions. Many of the solutions require a simplification of the problem such as "the patterns are regular languages," or "the patterns are finite languages." The myriad of variations on the problem, along with differing program design methodology, leads to solutions that are difficult to compare to one another.

This report presents a taxonomy of keyword pattern matching algorithms. The main results are summarized in the taxonomy graph presented at the end of this section, and in the conclusions presented in Part III. The taxonomy strives for the following goals:

- to present algorithms in a common framework to permit comparison of algorithms; such a framework is to be an easy to comprehend abstract presentation.
- to emphasize the derivation of an algorithm as a series of refinements to either algorithms or to the problem.
- to factor out common portions of well-known algorithms to facilitate comparison of these algorithms.

This report systematically presents a number of variations of four well-known algorithms in a common framework. Two of the algorithms to be presented require that the set of patterns is a single keyword, while the other two require that the set of patterns is a finite set of keywords. The algorithms are

- the Knuth-Morris-Pratt (KMP) algorithm as presented in [KMP77]. This algorithm matches a single keyword against the input string. Originally, the algorithm was devised to find only the first match in the input string. We will consider a version that finds all occurrences within the input string.
- the Boyer-Moore (BM) algorithm as presented in [BM77]. This is also a single keyword matching algorithm. Several corrections and improvements to this algorithm have been published; a good starting point for these is the bibliographic section of [Aho90].
- the Aho-Corasick (AC) algorithm as presented in [AC75]. This algorithm can match a finite set of keywords in the input string.
- the Commentz-Walter (CW) algorithm as presented in [Com79a, Com79b]. This algorithm can also match a finite set of keywords in the input string. Few papers have been published on this algorithm, and its correctness, time complexity, and precomputation are ill-understood.

These four algorithms are also presented in the overview of [Aho90].

The algorithms will be derived from a common starting point. The derivation proceeds by adding either problem or algorithm details. As a problem detail is added (that is, the problem is made more specific) a change may be possible in the algorithm — in particular, an improvement of efficiency may be possible. This is because the more specific problem may permit some transformation not possible in the more general problem.

Algorithm details are of course added in a correctness-preserving way; they are usually made to improve the efficiency of the algorithm. They may be added to restrict nondeterminacy, or to make a change of representation; either of these changes to an algorithm gives a new algorithm meeting the same specification. A derivation should make clear the differences and similarities of these algorithms; the entire derivation can then be taken to be a taxonomy of the four algorithms (and other related algorithms).

This type of taxonomy development and program derivation has been used in the past. One of the most notable is Broy's sorting algorithm taxonomy [Bro83]. In this taxonomy, algorithm
and problem details are also added, starting with a naive solution; the taxonomy arrives at all of the well-known sorting algorithms. A similar taxonomy (which predates the one of Broy) is by Darlington [Dar78]; this taxonomy also considers sorting algorithms. Our particular incarnation of the method of developing a taxonomy was developed in the thesis of Jonkers [Jon82], where it was used to give a taxonomy of garbage collection algorithms. Jonkers' method was then successfully applied to attribute evaluation algorithms by Marcelis in [Mar90].

The recent taxonomy of pattern matching algorithms presented by Hume and Sunday (in [HS91]) gives variations on the Boyer-Moore algorithm; the taxonomy concentrates on many of the practical issues, and provides data on the running time of the variations, and their respective precomputation.

Two important aims of our derivations are clarity and correctness of presentation. Towards both aims, the traditional method of using indexed strings (for the input string and patterns) has been abandoned in this paper; we use a more abstract (but equivalent) presentation. In order to easily provide correctness arguments the guarded command language of [Dij76] is used, rather than a programming language such as Pascal or C.

Part I contains the derivation of the four algorithms named above, along with several intermediate algorithms that are byproducts of the derivation.

Part II details the precomputation of functions necessary for each of the four algorithms.

Part III presents the conclusions.

Part IV contains the appendices. A program skeleton that we will often instantiate is detailed in Appendix A. Definitions and properties of operators and functions are provided in Appendix B.

The taxonomy graph that we arrive at after deriving the algorithms in Part I is shown in figure 1 on page 3. Each vertex corresponds to an algorithm. If the vertex is labeled with a number that number refers to an algorithm in this report. If it is labeled with a page number that page number refers to the page where the algorithm is first mentioned. Each edge corresponds to the addition of either a problem or algorithm detail and is labeled with the name of that detail (a list of detail names follows). Each of the algorithms will either be called by their algorithm number, by their name as found in the literature (for the well known algorithms), or by the parenthesized sequence of all labels along the path from the root to the algorithm's vertex. For example, the algorithm known as the optimized Aho-Corasick algorithm can also be called (P+, E, AC, OPT) (it is also algorithm 3.3 in this report). All of the well known algorithms appear as leaves in the tree. Due to its labeling the graph can be used as an alternative table of contents to this report. Four algorithm details (P+, s+, P-, and s-) are actually composed of two separate algorithm details. For example, detail (P+) is composed of details (p) and detail (+), however the second detail must always follow either detail (p) or detail (s) and so we treat them as a single detail. The edges labeled MO and SL in figure 1 represent generic algorithm details that still have to be instantiated. Possible instantiations are given by the two small trees at the bottom of figure 1. The details and a short description of each of them are as follows:

P (§ 2) Examine prefixes of a given string in any order.

P+ Examine prefixes of a given string in order of increasing length.

P- As in (P+), but in order of decreasing length.

S (§ 2) Examine suffixes of a given string in any order.

S+ Examine suffixes of a given string in order of increasing length.

S- As in (S+), but in order of decreasing length.
Figure 1: A taxonomy of pattern matching algorithms. An explanation of the graph and its labels is given in the text of this section. Algorithm 3.3 corresponds to the optimized Aho-Corasick algorithm ([AC75], section 6). Algorithm 3.4 corresponds to the Aho-Corasick algorithm using linear search ([AC75], section 2, algorithm 1). Algorithm 3.6 corresponds to the Knuth-Morris-Pratt algorithm ([KMP77], section 2, p.326). The algorithm of the vertex labeled p.24 and with incoming edge labeled NORM corresponds to the Commentz-Walter algorithm ([Com79a], section II, and [Com79b], sections II.1 and II.2). The algorithm of the vertex labeled p.24 and with incoming edge labeled BM corresponds to the Boyer-Moore algorithm ([BM77], section 4). Algorithm 5.4 corresponds to the Boyer-Moore algorithm as well ([BM77], sections 4 and 5).
RT (§ 2.1.1) Usage of the transition function of the reverse trie corresponding to the set of keywords to check whether a string which is a suffix of some keyword, preceded by a character is again a suffix of some keyword.

FT (§ 2.2.1) Usage of the transition function of the forward trie corresponding to the set of keywords to check whether a string which is a prefix of some keyword, followed by a character is again a prefix of some keyword.

E (§ 3) Matches are registered by their endpoint.

AC (§ 3.1) A state variable is maintained while examining prefixes of the input string. The value of the variable is the longest string from the set of all suffixes of the current prefix of the input string, which are prefixes of some keyword.

OPT (§ 3.2) A single "optimized" transition function is used to update the state variable in the Aho-Corasick algorithm.

LS (§ 3.3) Use linear search to update the state variable in the Aho-Corasick algorithm.

AC-FAIL (§ 3.3.1) Implement the linear search using the transition function of the extended forward trie and the failure function.

KMP-FAIL (§ 3.3.2) Implement the linear search using the extended failure function.

OKW (§ 3.3.2) The set of keywords contains one keyword.

INDICES (§ 3.3.2) Represent substrings by indices into the complete strings, converting a string based algorithm into an indexing based algorithm.

NE (§ 4) The empty string is not a keyword.

CW (§ 4.1) Consider any shift distance that does not lead to the missing of any matches. Such shift distances are called safe.

NLA (§ 4.2) The lookahead character is not taken into account when computing a safe shift distance. The computation of a shift distance is done by using two precomputed shift functions applied to the current longest partial match.

LA (§ 4.3) The lookahead character is taken into account when computing a safe shift distance.

NEAR-OPT (§ 4.3) Compute a shift distance using a single precomputed shift function applied to the current longest partial match and the lookahead character.

NORM (§ 4.3) Compute a shift distance as in (NLA) but additionally using a third shift function applied to the lookahead character. The shift distance obtained is that of the normal Commentz-Walter algorithm.

BM (§ 4.4) Compute a shift distance using one shift function applied to the lookahead character, and another shift function applied to the current longest partial match. The shift distance obtained is that of the Boyer-Moore algorithm.

RBM (§ 5) Introduce a particular program skeleton as a starting point for the derivation of the different Boyer-Moore variants.

MO (§ 5) A match order is used to determine the order in which characters of a potential match are compared against the keyword. This is only for the one keyword case (OKW). Particular instances of match orders are

FWD (§ 5) The forward match order is used to compare the (single) keyword against a potential match in a left to right direction.
REV (§ 5) The reverse match order is used to compare the (single) keyword against a potential match in a right to left direction. This is the original Boyer-Moore match order.

OM (§ 5) The characters of the (single) keyword are compared in order of ascending probability of occurring in the input string. In this way mismatches will generally be discovered as early as possible.

SL (§ 5.1) Before an attempt at matching a candidate string and the keyword a "skip loop" is used to skip portions of the input that cannot possibly lead to a match. Particular "skips" are

NONE (§ 5.1) No "skip" loop is used.

SFC (§ 5.1) The "skip loop" compares the first character of the match candidate and the keyword; as long as they do not match, the candidate string is shifted one character to the right.

FAST (§ 5.1) As with (SFC), but the last character of the candidate and the keyword are compared, and, possibly, a larger shift distance is used.

SLFC (§ 5.1) As with (FAST), but a low frequency character of the keyword is first compared.

MI (§ 5.2) The information gathered during an attempted match is used (along with the particular match order used during the attempted match) to determine a safe shift distance.

Part I

The algorithm derivations

2 The problem and some naive solutions

The problem is to find all occurrences of any of a set of keywords in an input string. Formally, given an alphabet \( V \) (a non-empty finite set of symbols), an input string \( S \in V^* \), and a finite non-empty pattern set \( P \subseteq V^* \), establish

\[
R: \quad O = (\cup l, v, r : lvr = S : \{ l \} \times (\{ v \} \cap P) \times \{ r \}).
\]

A trivial (but unrealistic) solution to this is

Algorithm 2.1()

\[
O := (\cup l, v, r : lvr = S : \{ l \} \times (\{ v \} \cap P) \times \{ r \})
\]

\{ R \}

The sequence of details describing this algorithm is the empty sequence (sequences of details are introduced in section 1).

There are two basic directions in which to proceed while developing naive algorithms to solve this problem. Informally, a substring of \( S \) can be considered a "suffix of a prefix of \( S \)" or a "prefix of a suffix of \( S \)". These two possibilities are considered separately below.

Formally, we can consider "suffixes of prefixes of \( S \)" as follows:

---

1Throughout this paper we will adopt the convention that, unless stated otherwise, program variables and bound variables with names from the beginning of the Latin alphabet (i.e. a, b, c) will range over \( V \), while variables with names from the end of the Latin alphabet (i.e. l, q, r, u, v, w) will range over \( V^* \).
\[(\cup l, v, r : lwv = S : \{l\} \times (\{v\} \cap P) \times \{r\})\]

= \{\text{introduce } u : u = lv\}

= \{l, v \text{ only occur in the latter range conjunct, so restrict their scope}\}

\[(\cup u, r : ur = S : (\cup l, v : lv = u : \{l\} \times (\{v\} \cap P) \times \{r\}))\]

The method of implementing a computation of such a quantification is detailed in Appendix A.

A simple non-deterministic\(^2\) algorithm (the structure of which is discussed in Appendix A.1) is obtained by applying algorithm detail\(^\Box\).

**Detail (p):** Examine prefixes of a given string in any order.

to input string \(S\). It results in\(^3\)

**Algorithm 2.2(p)**

\[
\begin{align*}
W & := (\cup u, r : ur = S : \{u\} \times \{r\}); O := \emptyset; \\
\text{for } (u, r) : (u, r) \in W \text{ do} \\
O & := O \cup (\cup l, v : lv = u : \{l\} \times (\{v\} \cap P) \times \{r\}) \\
\text{rof } \{R\}
\end{align*}
\]

Again starting from algorithm 2.1() we can also consider “prefixes of suffixes of \(S\)” as follows:

\[
\begin{align*}
(\cup l, v, r : lwv = S : \{l\} \times (\{v\} \cap P) \times \{r\}) \\
= \{\text{introduce } w : w = vr\}
\end{align*}
\]

\[
\begin{align*}
(\cup l, v, r, w : lw = S \land vr = w : \{l\} \times (\{v\} \cap P) \times \{r\}) \\
= \{v, r \text{ only occur in the latter range conjunct, so restrict their scope}\}
\end{align*}
\]

\[(\cup l, w : lw = S : (\cup u, v : lw = u : \{l\} \times (\{v\} \cap P) \times \{r\}))\]

Introduction of algorithm detail\(^\Box\)

**Detail (s):** Examine suffixes of a given string in any order.

yields the simple non-deterministic algorithm (s) which is analogous to algorithm 2.2(p). Hence, it is not presented here.

The update of \(O\) (with another quantifier) in the inner repetitions of algorithms (p) and (s) can be computed with another non-deterministic repetition. In the case of (p) the inner repetition would consider suffixes of \(u\) to give algorithm (ps); similarly, in (s) the inner repetition would consider prefixes of \(u\) to give algorithm (sp).

Each of (ps) and (sp) consists of two nested non-deterministic repetitions. In each case, the repetition can be made deterministic by considering prefixes (or suffixes as the case is) in increasing (called detail (+)) or decreasing (detail (-)) order of length. For each of (ps) and (sp) this gives two binary choices. Along with the binary choice between (ps) and (sp) this gives a 3-cube representing the three binary choices; the cube is depicted in figure 2 on page 7 with vertices representing the eight possible algorithms for the two nested repetitions. The edges marked ‘\(=\)’ join algorithms which are symmetrical; for example, the order in which \((p+5+)\) considers \(S\) and \(P\) is mirrored (with respect to string reversal of \(S\) and \(P\)) by the order in which \((s_+5_+)\) considers \(S\) and \(P\). Because of this symmetry, we present only four algorithms in this section: \((p_+5_+)\), \((p_+5_-)\), \((s_-5_-)\), and \((s_+5_+)\). These algorithms were chosen because their outer repetitions examine \(S\) in left to right order. In subsection 2.1 algorithm 2.2(p) will be refined further and in subsection 2.2 algorithm (s) will be refined. In section 3 algorithm \((p_+5_+)\) will be developed into the Aho-Corakisk and Knuth-Morris-Pratt algorithms, while in sections 4 and 5 algorithm \((p_+5_+)\) will be developed into the Commentz-Walter and Boyer-Moore algorithms.

\(^2\)An algorithm is called non-deterministic if the order in which its statements are executed is not fixed.

\(^3\)The for-do-rof statement is taken from [vdE92]. Statement for \(x : P \text{ do } S \text{ od}\) amounts to executing statement list \(S\) once for each value of \(x\) that satisfies \(P\) initially. The order in which the values of \(x\) are chosen is arbitrary.
2.1 The \((p^+)\) algorithms

The \((p)\) algorithm presented in the previous section can be made deterministic by considering prefixes of \(S\) in order of increasing length. The outer union quantifier in the required value of \(O\) can be computed with a deterministic repetition. Instantiating the algorithm in Appendix A.2 with \(W = V^* \times V^*\), \(\text{RANGE}(u, r) \equiv ur = S\), \((u_0, r_0) \leq (u_1, r_1) \equiv u_0 \leq_p u_1\), \(\emptyset = \emptyset\), and \(f(u, r) = (U I, v : lv = u \times \{v\} \cap P) \times \{r\}\) results in algorithm \((p^+)\):

Algorithm 2.3\((p^+)\)

\[
\begin{align*}
  u, r &:= \varepsilon, S; O := \{\varepsilon\} \times (\{\varepsilon\} \cap P) \times \{S\}; \\
  \text{do } r \neq \varepsilon \rightarrow \\
  &\quad u, r := u(r[1]), r[1]; \\
  &\quad O := O \cup (U I, v : lv = u \times \{v\} \cap P) \times \{r\} \\
  \text{od } \{R\}
\end{align*}
\]

This algorithm will be used in section 3 as a starting point for the Aho-Corasick and Knuth-Morris-Pratt algorithms. The inner union quantification in the required value of \(O\) can be computed with a non-deterministic repetition as outlined in Appendix A.1. This algorithm is called \((p^+s)\) but will not be given here.

2.1.1 The \((p_+, s^+)\) algorithms

Starting with algorithm \((p_+s)\) we make its inner repetition deterministic by considering suffixes of \(u\) in order of increasing length. In keeping with the form in Appendix A.2, a first such algorithm is

\[\text{The operators } |, \parallel, \lor, \text{ and } | \text{ are defined in definition B.6}\]
Algorithm 2.4($P \rightarrow S_4$)

\[
\begin{align*}
    u, r &:= \varepsilon, S; O := \{\varepsilon\} \times ((\varepsilon) \cap P) \times \{S\}; \\
    \text{do } r \neq \varepsilon &\rightarrow \\
    &u, r := u(r[1]), r[1]; \\
    l, v &:= u, \varepsilon; O := O \cup \{u\} \times ((\varepsilon) \cap P) \times \{r\}; \\
    \text{do } l \neq \varepsilon &\rightarrow \\
    &l, v := l[1], (l[1])v; \\
    O &:= O \cup \{l\} \times (\{v\} \cap P) \times \{r\}
\end{align*}
\]

od {R}

This algorithm has running time $O(|S|^2)$, assuming that intersection with $P$ is a $O(1)$ operation. We will now improve the running time of this algorithm. Note that

\[
(\forall w, a : w \notin \text{suff}(P) : aw \notin \text{suff}(P)).
\]

In other words, in the inner repetition when $(l[1])v \notin \text{suff}(P)$ we need not consider any longer suffixes of $u$. The inner repetition guard can therefore be strengthened to

\[
l \neq \varepsilon \text{ cand } (l[1])v \notin \text{suff}(P).
\]

Observe that $v \in \text{suff}(P)$ is an invariant of the inner repetition. This invariant is initially established by the assignment $v := \varepsilon$. Direct evaluation of $(l[1])v \notin \text{suff}(P)$ is expensive. Therefore it is done using the transition function of the reverse trie [Fre60] corresponding to $P$, $\tau_{P,r} : \text{suff}(P) \times V \rightarrow \text{suff}(P) \cup \{\bot\}$ defined by

\[
\tau_{P,r}(w, a) = \begin{cases} 
    aw & \text{if } aw \in \text{suff}(P) \\
    \bot & \text{if } aw \notin \text{suff}(P) 
\end{cases} \quad (w \in \text{suff}(P), a \in V).
\]

Since we usually refer the trie corresponding to $P$ we will write $\tau_r$ instead of $\tau_{P,r}$. Transition function $\tau_r$ can be computed beforehand. The guard becomes $l \neq \varepsilon \text{ cand } \tau_r(v, l[1]) \neq \bot$. This amounts to the introduction of algorithm detail

Detail (RT): Usage of reverse trie function $\tau_r$ to implement expression $(l[1])v \notin \text{suff}(P)$. \square

and yields

Algorithm 2.5($P \rightarrow S_4$, RT)

\[
\begin{align*}
    u, r &:= \varepsilon, S; O := \{\varepsilon\} \times ((\varepsilon) \cap P) \times \{S\}; \\
    \text{do } r \neq \varepsilon &\rightarrow \\
    &u, r := u(r[1]), r[1]; \\
    l, v &:= u, \varepsilon; O := O \cup \{u\} \times ((\varepsilon) \cap P) \times \{r\}; \\
    \text{do } l \neq \varepsilon &\text{ cand } \tau_r(v, l[1]) \neq \bot \\
    &l, v := l[1], (l[1])v; \\
    O &:= O \cup \{l\} \times (\{v\} \cap P) \times \{r\}
\end{align*}
\]

od {R}

This algorithm has $O(|S| \cdot (\text{MAX } p : p \in P : |p|))$ running time. The precomputation of $\tau_r$ is similar to the precomputation of the transition function of the forward trie $\tau_f$ (defined in 2.2.1) which is discussed in Part II, section 6.
2.1.2 The \((p+s_-)\) algorithm

In the previous section we modified the inner repetition of algorithm \((p+s)\) to consider suffixes of \(u\) in order of increasing length. In this section, we will make use of an inner repetition which considers them in order of decreasing length. The general form of such a repetition is given in Appendix A.3. This gives us the following

**Algorithm 2.6\((p+s_-)\)**

\[
\begin{align*}
&u, r := \varepsilon, S; O := \{\varepsilon\} \times \{\varepsilon\} \cap P \times \{S\}; \\
&\text{do } r \neq \varepsilon \rightarrow \\
&\quad u, r := u(r[1]), r[1]; \\
&\quad l, v := \varepsilon, u; \\
&\quad \text{do } v \neq \varepsilon \rightarrow \\
&\quad\quad O := O \cup \{l\} \times \{\varepsilon\} \cap P \times \{r\}; \\
&\quad\quad l, v := l(v[1]), v[1] \\
&\quad\text{od}; \\
&\quad O := O \cup \{u\} \times \{\varepsilon\} \cap P \times \{r\} \\
&\text{od } \{R\}
\end{align*}
\]

This algorithm has running time that is \(O(|S|^2)\).

2.2 The \((s_-)\) algorithms

Algorithm \((s)\) can be made deterministic by considering suffixes of \(S\) in order of decreasing length. Instantiating the algorithm in Appendix A.3 with \(W = V^* \times V^*, \text{RANGE}(l, w) \equiv lw = S, (l_0, w_0) \leq (l_1, w_1) \equiv w_0 \leq w_1, \emptyset = \emptyset, \text{and } f(l, w) = (\cup v, r : vr = w : \{l\} \times \{\varepsilon\} \cap P \times \{r\})\) results in the deterministic algorithm \((s_-)\) which will not be given here. Furthermore, the assignment to \(O\) in the repetition can be written as a non-deterministic repetition (see Appendix A.1 and also section 2.1) to give the algorithm \((s_-p)\) which will not be given here.

2.2.1 The \((s_-p+)\) algorithms

Starting with algorithm \((s_-p)\) we make the inner repetition deterministic by considering prefixes of each suffix of the input string in order of increasing length, in keeping with the algorithm in Appendix A.2. The algorithm is:

**Algorithm 2.7\((s_-p+)\)**

\[
\begin{align*}
&l, w := \varepsilon, S; O := \emptyset; \\
&\text{do } w \neq \varepsilon \rightarrow \\
&\quad v, r := \varepsilon, w; O := O \cup \{l\} \times \{\varepsilon\} \cap P \times \{w\}; \\
&\quad \text{do } r \neq \varepsilon \rightarrow \\
&\quad\quad v, r := v(r[1]), r[1]; \\
&\quad\quad O := O \cup \{l\} \times \{\varepsilon\} \cap P \times \{r\} \\
&\quad\text{od}; \\
&\quad l, w := l(w[1]), w[1] \\
&\text{od}; \\
&\quad O := O \cup \{S\} \times \{\varepsilon\} \cap P \times \{\varepsilon\} \\
&\text{od } \{R\}
\end{align*}
\]

This algorithm has \(O(|S|^2)\) running time like algorithm 2.4\((p+s+)\). In a manner similar to the introduction of the reverse trie, in algorithm 2.4\((p+s+)\), we can strengthen the inner repetition guard. Note that

\[(\forall u, a : u \notin \text{pref}(P) : ua \notin \text{pref}(P)).\]
So we can strengthen the guard of the inner repetition to \( r \neq \varepsilon \text{ cand } v(r11) \in \text{pref}(P) \). Conjunct \( v \in \text{pref}(P) \) can be added to the invariant of this repetition. It is initially established by the assignment \( v := \varepsilon \). Efficient computation of this guard can be done by using the transition function of the forward trie corresponding to \( P \tau_f : \text{pref}(P) \times V \rightarrow (\text{pref}(P) \cup \{\bot\}) \), defined by

\[
\tau_f(u, a) = \begin{cases} 
u a & \text{if } ua \in \text{pref}(P) \\ \bot & \text{if } ua \notin \text{pref}(P) \end{cases} \quad (u \in \text{pref}(P), a \in V).
\]

Transition function \( \tau_f \) can be computed beforehand. The guard now becomes

\( r \neq \varepsilon \text{ cand } \tau_f(v, r11) \neq \bot. \)

**Detail (FT):** Usage of forward trie function \( \tau_f \) to implement expression \( v(r11) \in \text{pref}(P) \). □

The forward trie detail (FT) is defined and used symmetrically to the reverse trie detail (RT).

Introducing algorithm detail (FT) yields

**Algorithm 2.8(\( S+P+ \), FT)\)**

\[
\begin{align*}
& l, w := \varepsilon, S; O := \emptyset; \\
& \text{do } w \neq \varepsilon \rightarrow \\
& \quad v, r := \varepsilon, w; O := O \cup \{I\} \times (\{\varepsilon\} \cap P) \times \{w\}; \\
& \quad \text{do } r \neq \varepsilon \text{ cand } \tau_f(v, r11) \neq \bot \rightarrow \\
& \quad \quad v, r := v(r11), r11; \\
& \quad \quad O := O \cup \{I\} \times (\{v\} \cap P) \times \{r\}
& \quad \text{od}; \\
& \quad l, w := l(w1), w1; \\
& \text{od}; \\
& O := O \cup \{S\} \times (\{\varepsilon\} \cap P) \times \{\varepsilon\} \\{R\}
\end{align*}
\]

This algorithm has \( O(|S| \cdot (\text{MAX } p : p \in P : |p|)) \) running time like algorithm 2.5(\( P+S+, \text{RT} \)).

### 2.2.2 The (\( S-P- \)) algorithm

The inner repetition of algorithm (\( S-P- \)) can also be made deterministic by considering prefixes of \( w \) in order of decreasing length, as in Appendix A.3. This yields algorithm (\( S-P- \)) which is not given here. Its running time is \( O(|S|^2) \).

### 3 The Aho-Corasick algorithms

In this section, starting with algorithm 2.3(\( P_+ \)), we derive the Aho-Corasick and Knuth-Morris-Pratt algorithms. First, we make a preliminary step. The triple format of \( O \) used so far has been redundant. This redundancy can be removed by registering matches in \( S \) by their end-point; that is, the first component of the triple will be dropped. This modification is known as algorithm detail (E).

**Detail (E):** Matches are registered by their end-point. □

In the following derivation we use the symbol \( \simeq \) to indicate that the problem specification has been specialized (in this case, through projection). The postcondition of algorithm 2.3(\( P_+ \)) can be rewritten as follows:
\[(\cup u, r : ur = S : (\cup l, v : lv = u : \{l\} \times (\{v\} \cap P) \times \{r\}))\]

\approx \{\text{introduction of detail (E)}\}

\[(\cup u, r : ur = S : (\cup l, v : lv = u : \{v\} \cap P) \times \{r\})\]

\approx \{\text{definition of suff, distributivity}\}

\[(\cup u, r : ur = S : (\text{suff}(u) \cap P) \times \{r\})\]

This yields a new postcondition

\[R_e : O_e = (\cup u, r : ur = S : (\text{suff}(u) \cap P) \times \{r\})\]

which is established by a modified version of algorithm 2.3(p_+)

**Algorithm 3.1(p_+, E)**

\[
u, r := \varepsilon, S; O_e := (\{\varepsilon\} \cap P) \times \{S\};
\]

\[\text{do } r \neq \varepsilon \rightarrow \]

\[
u, r := u(r[1]), r[1];
\]

\[
O_e := O_e \cup (\text{suff}(u) \cap P) \times \{r\}
\]

\[\text{od } \{R_e\}\]

In the following sections, algorithm details unique to the Aho-Corasick and Knuth-Morris-Pratt algorithms will be introduced.

### 3.1 Algorithm detail AC

In order to facilitate the update of \(O_e\) in algorithm 3.1(p_+, E) we introduce a new variable \(U\) and attempt to maintain invariant \(U = \text{suff}(u) \cap P\). For the update of \(U\) we derive

\[
\text{suff}(ua) \cap P
\]

\[= \{\text{suff}(ua) = \text{suff}(u)a \cup \{\varepsilon\}\}\]

\[= (\text{suff}(u)a \cap P) \cup (\{\varepsilon\} \cap P)\]

\[= \{\text{suff}(u)a \cap P \subseteq \text{pref}(P)a\}\]

\[= ((\text{suff}(u) \cap \text{pref}(P))a \cap P) \cup (\{\varepsilon\} \cap P)\]

Therefore, in order to calculate the new value of \(U\) we need the set \(\text{suff}(u) \cap \text{pref}(P)\) rather than the old value of \(U (\text{suff}(u) \cap P)\). Formula \(\text{suff}(u) \cap \text{pref}(P)\) can be viewed as a generalization of formula \(\text{suff}(u) \cap P\). Hence, we try to maintain invariant

\[P_0(u, U) \equiv (U = \text{suff}(u) \cap \text{pref}(P))\]

which is initially established by assignment \(u, U := \varepsilon, \{\varepsilon\}\). Assuming \(P_0(u, U)\) we derive

\[
\text{suff}(ua) \cap \text{pref}(P)
\]

\[= \{\text{preceding derivation with } \text{pref}(P) \text{ instead of } P, \text{ pref is idempotent}^5\}\]

\[= (((\text{suff}(u) \cap \text{pref}(P))a \cap \text{pref}(P)) \cup (\{\varepsilon\} \cap \text{pref}(P)))\]

\[= \{P_0(u, U), P \neq \emptyset\}\]

\[= (Ua \cap \text{pref}(P)) \cup \{\varepsilon\} .\]

From \(P_0(u, U)\) and \(P \subseteq \text{pref}(P)\) it follows that \(\text{suff}(u) \cap P = U \cap P\). This all leads to the following modification of algorithm 3.1(p_+, E):

\(^5\text{A function } f \text{ is called idempotent if } f \circ f = f.\)
\[
u, r := \varepsilon, S; U := \{\varepsilon\}; O_e := (\{\varepsilon\} \cap P) \times \{S\};
\]

\{invariant: \(P_0(u, U)\}\n
\text{do } r \neq \varepsilon \rightarrow \n
\quad U := (U(r[1]) \cap \text{pref}(P)) \cup \{\varepsilon\}; \quad \{P_0(u(r[1]), U)\}\n
\quad u, r := u(r[1]), r[1]; \quad \{P_0(u, U)\}\n
\quad O_e := O_e \cup (U \cap P) \times \{r\}\n
\text{od } \{R_e\}\n
Since \(S\) and, therefore, \(u\) can be any string from \(V^*\) it follows from invariant \(P_0(u, U)\) that the values that \(U\) can have constitute the finite set \(\{\text{suff}(w) \cap \text{pref}(P) \mid w \in V^*\}\). Hence, the preceding algorithm can be viewed as simulating the behavior of Moore machine \([HU79]\) (or finite transducer) \(M_0 = (Q_0, \Sigma_0, \Delta_0, \delta_0, \lambda_0, s_0)\) on input string \(S\), where

- state set \(Q_0 = \{\text{suff}(w) \cap \text{pref}(P) \mid w \in V^*\}\),
- input alphabet \(\Sigma_0 = V\),
- output alphabet \(\Delta_0 = \mathcal{P}(P)\),
- transition function \(\delta_0 : Q_0 \times V \rightarrow Q_0\) is defined by
  \[
  \delta_0(q, a) = (qa \cap \text{pref}(P)) \cup \{\varepsilon\} \quad (q \in Q_0, a \in \Sigma_0),
  \]
- output function \(\lambda_0 : Q_0 \rightarrow \Delta_0\) is defined by
  \[
  \lambda_0(q) = q \cap P \quad (q \in Q_0),
  \]
  and
- start state \(s_0 = \{\varepsilon\}\).

Moore machine \(M_0\) can be viewed as a deterministic finite automaton without final states and with an additional output alphabet \(\Delta_0\) and an additional output function \(\lambda_0\). If on reading input sequence \(w\) machine \(M_0\) goes through states \(s_0, q_1, q_2, \ldots, q_{|w|}\) it will emit output sequence \(\lambda_0(s_0)\lambda_0(q_1)\lambda_0(q_2)\ldots\lambda_0(q_{|w|})\). The set \(O_e\) can be viewed as an encoding of the output sequence of Moore machine \(M_0\).

The following intermezzo shows that Moore machine \(M_0\) can be obtained in a different way.

An interesting solution to the pattern matching problem involves using an automaton for the language \(V^*P\). Usually, a nondeterministic finite automaton (NFA) is constructed. The automaton is then simulated, processing input string \(S\), and considering all paths through the automaton. Whenever a final state is entered a keyword match has been found, and the match is registered; see for example Aho, Hopcroft & Ullman in \([AHU74]\).

The state graph for the NFA is simply the forward trie for \(P\), augmented with a transition from state \(\varepsilon\) to itself on all symbols in \(V\). The NFA is defined as \((Q_N, V, \delta_N, s_N, F_N)\), where

- state set \(Q_N = \text{pref}(P)\),
- input alphabet \(V\),
- transition function \(\delta_N : Q_N \times V \rightarrow \mathcal{P}(Q_N)\) is defined by
  \[
  \delta_N(\varepsilon, a) = \begin{cases} 
  \{\varepsilon\} & \text{if } a \in \text{pref}(P) \\
  \{\varepsilon\} & \text{otherwise}
  \end{cases} \quad (a \in V),
  \]
  and
  \[
  \delta_N(q, a) = \begin{cases} 
  \{qa\} & \text{if } qa \in \text{pref}(P) \\
  \emptyset & \text{otherwise}
  \end{cases} \quad (q \in \text{pref}(P) \setminus \{\varepsilon\}, a \in V).
  \]
  and is extended to \(\delta_N^* : Q_N \times V^* \rightarrow \mathcal{P}(Q_N)\) in the obvious way,
- start state \( s_N = \varepsilon \), and
- final state set \( F_N = P \).

The simulation of this automaton can proceed as follows:

\[
\begin{align*}
u, r := \varepsilon, S; q_N := \{\varepsilon\}; \\
O_s := (q_N \cap F_N) \times \{r\}; \\
\text{(invariant: } q_N = \delta_N(\varepsilon, u)) \\
do r \neq \varepsilon \rightarrow \\
q_N := (\cup q : q \in q_N : \delta_N(q, r|1)); \\
O_s := O_s \cup (q_N \cap F_N) \times \{r\} \\
\od \{R_s\}
\end{align*}
\]

Strictly speaking, the NFA is being used as a nondeterministic Moore machine. Each path through the Moore machine is followed simultaneously; the output function is only defined for some of the states (\( F_N \) to be precise). The output alphabet \( \Delta_N \) can be written as \( \Delta_N = P \cup \{\perp_N\} \) (\( \perp_N \) is output in nonmatching states). The output function is \( \lambda_N : Q_N \rightarrow \Delta_N \) defined as

\[
\lambda_N(q) = \begin{cases} 
q & \text{if } q \in P \\
\perp_N & \text{if } q \notin P
\end{cases}
\]

The nondeterministic Moore machine is now \( M_N = (Q_N, V, \Delta_N, \delta_N, \lambda_N, s_N) \). In the algorithm, the set \( O_s \) is only updated when the output is not \( \perp_N \).

The subset construction (see [RS59]) can be applied to the nondeterministic Moore machine, to give a deterministic Moore machine \( M_D \). In the following paragraphs, we will prove that this deterministic Moore machine (with unreachable states removed) is equal to \( M_0 \) (presented above).

Under the subset construction, the state set is \( P(Q_N) = P(\text{pref}(P)) \). The set of reachable states is smaller, as will be shown below. A new output alphabet (under the subset construction) is defined as: \( \Delta_D = P(\Delta_N) \). The set of reachable states is

\[
Q_D = \{ \text{subset construction and reachability} \} \\
\delta^*_N(\varepsilon, w) \mid w \in V^* \\
\{ \text{definition of } \delta_N \} \\
\{ \{ q \mid q \in \text{pref}(P) \land w \in V^* \} \mid w \in V^* \} \\
\{ w \in V^* q \equiv q \in \text{suff}(w) \} \\
\{ \text{suff}(w) \cap \text{pref}(P) \mid w \in V^* \} \\
\{ \text{definition of } Q_0 \} \\
Q_0
\]

The deterministic output function \( \lambda_D : Q_D \rightarrow P(\Delta_N) \) is

\[
\lambda_D(q) = \{ \text{subset construction} \} \\
\{ \lambda_N(q') \mid q' \in q \land \lambda_N(q') \neq \perp_N \} \\
\{ \text{definition of } \lambda_N \} \\
\{ q' \mid q' \in q \land q' \in P \} \\
\{ \text{set calculus} \} \\
q \cap P \\
\{ \text{definition } \lambda_0 \} \\
\lambda_0(q)
\]

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Lastly, the deterministic transition function \( \delta_D : Q_D \times V \rightarrow Q_D \) is

\[
\delta_D(q, a) = \begin{cases} \text{subset construction} \\
(\cup q' : q' \in q : \delta_N(q', a)) \end{cases}
\]

\[
= (\cup q' : q' \in q \land q'a \in \text{pref}(P) : \{q'a\}) \cup \{\varepsilon\}
\]

\[
= \{\text{definition of } \delta_0\} \quad (q \cap \text{pref}(P)) \cup \{\varepsilon\}
\]

\[
= \{\text{definition of } \delta_0\}
\]

\[
\delta_0(q, a)
\]

From these derivations it follows that \( M_0 = M_D \).

Notice that the number of states of the Moore machine does not grow during the subset construction. Perrin mentions the AC and KMP Moore machines as examples of ones which do not suffer from exponential blowup (i.e. the number of states grows exponentially) during the subset construction [Per90].

In subsection 3.2 it is shown that Moore machine \( M_0 \) is minimal.

We proceed by observing that for each \( v \in V^* \) the set \( \text{suff}(v) \cap \text{pref}(P) \) is nonempty, finite, and linearly ordered with respect to the suffix ordering \( \leq_s \) (see Definition B.4) and therefore has a maximal element \((\text{MAX}_{\leq_s}, w : w \in \text{suff}(v) \cap \text{pref}(P) : w)\). Since \( \text{suff} \) is idempotent \((\text{suff}(\text{suff}(u)) = \text{suff}(u))\) we have by theorem B.5

\[
\text{suff}(v) \cap \text{pref}(P) = \text{suff}((\text{MAX}_{\leq_s}, w : w \in \text{suff}(v) \cap \text{pref}(P) : w)) \cap \text{pref}(P)
\]

so the states of machine \( M_0 \) can be represented by their maximal elements. We replace variable \( U \) in the algorithm by variable \( q \) and maintain invariant

\[
P_0(u, q) \equiv (q = (\text{MAX}_{\leq_s}, w : w \in \text{suff}(v) \cap \text{pref}(P) : w)).
\]

Introduction of \( q \) is called algorithm detail (AC).

Detail (AC): A variable \( q \) is introduced into algorithm 3.1\((P_+, E)\) such that

\[
q = (\text{MAX}_{\leq_s}, w : w \in \text{suff}(v) \cap \text{pref}(P) : w)
\]

We now have that \( \text{suff}(u) \cap P = \text{suff}(q) \cap P \). By introducing function \( \text{Output} : \text{pref}(P) \rightarrow P(P) \), defined by

\[
\text{Output}(w) = \text{suff}(w) \cap P \quad (w \in \text{pref}(P))
\]

the update of \( O_v \) can be done by assignment \( O_v := O_v \cup \text{Output}(q) \times \{r\} \). The precomputation of function \( \text{Output} \) is discussed in Part II, section 6.

We now have obtained algorithm

**Algorithm 3.2**\((P_+, E, AC)\)

\[
\begin{array}{l}
u, r := \varepsilon, \varepsilon; O_v := \text{Output}(q) \times \{S\}; \\
\{\text{invariant: } P_0(u, q)\} \\
\text{do } r \neq \varepsilon \rightarrow \\
q := (\text{MAX}_{\leq_s}, w : w \in \text{suff}(u(r|1)) \cap \text{pref}(P) : w); \quad \{P_0(u(r|1), q)\} \\
u, r := u(r|1), r|1; \quad \{P_0(u, q)\} \\
O_v := O_v \cup \text{Output}(q) \times \{r\} \\
\text{od } \{R_v\}
\end{array}
\]

The next two sections are concerned with alternative ways of implementing assignment

\[
q := (\text{MAX}_{\leq_s}, w : w \in \text{suff}(u(r|1)) \cap \text{pref}(P) : w).
\]
3.2 Method OPT

Assuming \( P(u, q) \) we derive

\[
(MAX_{\leq}, w : w \in \text{suff}(ua) \cap \text{pref}(P) : w)
\]

\[
= \{ \text{suff}(ua) = \text{suff}(u) a \cup \{ \varepsilon \}, P \neq \emptyset \}
\]

\[
(MAX_{\leq}, w : w \in \text{suff}(u)a \cap \text{pref}(P) \lor w = \varepsilon : w)
\]

\[
= \{ \text{theorem B.5} \}
\]

\[
(MAX_{\leq}, w : w \in \text{suff}((MAX_{\leq}, w' : w' \in \text{suff}(u) \cap \text{pref}(P) : w'))a \cap \text{pref}(P) \lor w = \varepsilon : w)
\]

\[
= \{ \text{suff}(qa) = \text{suff}(q)a \cup \{ \varepsilon \}, P \neq \emptyset \}
\]

\[
(MAX_{\leq}, w : w \in \text{suff}(qa) \cap \text{pref}(P) : w)
\]

By introducing function \( \gamma_f : \text{pref}(P) \times V \rightarrow \text{pref}(P) \), defined by

\[
\gamma_f(q, a) = (MAX_{\leq}, w : w \in \text{suff}(qa) \cap \text{pref}(P) : w)
\]

the assignment to \( q \) in algorithm 3.2(\( P_+, E, AC \)) can be written as \( q := \gamma_f(q, a) \). This is called algorithm

**Detail (OPT): Usage of function \( \gamma_f \) to update variable \( q \). \( \Box \)**

and leads to algorithm (cf. [AC75], section 6)

**Algorithm 3.3(\( P_+, E, AC \), OPT)**

\[
\begin{align*}
& u, r := \varepsilon, S; q := \varepsilon; O_e := \text{Output}(q) \times \{ S \}; \\
& \{ \text{invariant: } P'_0(u, q) \}
\end{align*}
\]

\[
\text{do } r \neq \varepsilon \rightarrow
\]

\[
\begin{align*}
& q := \gamma_f(q, r)1; \quad \{ P'_0(u(r)1, q) \} \\
& u, r := u(r)1, r1; \quad \{ P'_0(u, q) \} \\
& O_e := O_e \cup \text{Output}(q) \times \{ r \}
\end{align*}
\]

\[
\text{od } \{ R_e \}
\]

Note that \( \gamma_f \) is the transition function of Moore machine \( M_1 = (\text{pref}(P), V, P(P), \gamma_f, \text{Output}, \varepsilon) \). Machine \( M_1 \) is isomorphic with machine \( M_0 \) from section 3.1 since function \( enc : Q_0 \rightarrow \text{pref}(P) \) defined by

\[
enc(q) = (MAX_{\leq}, q' : q' \in q : q')
\]

is bijective. Furthermore Moore machine \( M_1 \) corresponds to the automaton in the “optimized” version of the Aho-Corasick algorithm.

Another interesting property of the Moore machine \( M_1 \) is that it is in fact the minimal Moore machine for its language. This will be shown in the following intermezzo.

For deterministic Moore machines we use the following definition of minimality:

\[
\text{Minimal}(Q, V, \Sigma, \delta, \lambda, s) \equiv
(\forall q_0, q_1 : q_0 \neq q_1 \land q_0, q_1 \in Q \land \exists q_2 \in Q : (\exists w : w \in V^* : \lambda(\delta^*(q_0, w)) \neq \lambda(\delta^*(q_1, w)))
\]

Notice that this definition can be viewed as a generalization of the definition of minimality for deterministic finite automata (replace \( \lambda(\delta^*(q, w)) \) by \( \delta^*(q, w) \in F \) in the definition where \( F \) is the set of final states of the finite automaton).

\(^6\text{Subscript } f \text{ is used to indicate that } \gamma_f \text{ corresponds to the forward trie transition function } \tau_f.\)
We now prove that the Moore machine $M_1$ is minimal by contradiction. Assume that there are $q_0, q_1 \in \text{pref}(P) \land q_1 \in \text{pref}(P) \land q_0 \neq q_1 \land |q_0| \geq |q_1|$ such that

$$(\forall w : w \in V^* : \text{Output}(\gamma_f(q_0, w)) = \text{Output}(\gamma_f(q_1, w))).$$

Choose $w_0 : q_0w_0 \in P$. Then $\gamma_f(q_0, w_0) = q_0w_0$ and $q_0w_0 \in \text{Output}(\gamma_f(q_0, w_0))$. In this case (from the assumptions)

$q_0w_0 \in \text{Output}(\gamma_f(q_1, w_0)) \Rightarrow \{ \text{definition of } \gamma_f \text{ and } \text{Output} \}$

$q_0w_0 \leq_s \gamma_f(q_1, w_0) \leq_s q_1 w_0 \Rightarrow \{ \text{property of } \leq_s \}$

$q_0 \leq_s q_1 \Rightarrow \{|q_0| \geq |q_1|\}$

$q_0 = q_1$

which is a contradiction. We conclude that Moore machine $M_1$ is minimal and end this intermezzo.

Provided evaluating $\gamma_f(q, a)$ and $\text{Output}(q)$ are $O(1)$ operations (for instance, if $\gamma_f$ and $\text{Output}$ are tabulated) algorithm 3.3$(P, E, AC, OPT)$ has $O(|S|)$ run time complexity. Precomputation of $\gamma_f$ is discussed in Part II, section 6. It involves the so-called failure function which is introduced in the next subsection. Precomputation takes $O(|\text{pref}(P)| \cdot |V|)$ time. Storage of $\gamma_f$ and $\text{Output}$ takes $O(|\text{pref}(P)| \cdot |V|)$ space.

### 3.3 Linear search

In this subsection we give an alternative way of implementing assignment

$$q := (\text{MAX}_{\leq_s} w : w \in \text{suff}(u(rl1)) \cap \text{pref}(P) : w)$$

involving linear search. We start with the following derivation, assuming $P_0(u, q)$,

(\text{MAX}_{\leq_s}, w : w \in \text{suff}(u) \cap \text{pref}(P) : w)

= \{ \text{derivation in subsection 3.2 without last step} \}

(\text{MAX}_{\leq_s}, w : w \in \text{suff}(q) \cap \text{pref}(P) \lor w = \varepsilon : w)

= \{ \text{suff}(q) \cap \text{pref}(P) \subseteq \text{pref}(P) \}

(\text{MAX}_{\leq_s}, w : w \in (\text{suff}(q) \cap \text{pref}(P)) \cap \text{pref}(P) \lor w = \varepsilon : w)

= \{ \text{domain split, introduction of } \perp_s \text{ with } \perp_s \text{max}_{\leq_s} w = w \text{max}_{\leq_s} \perp_s = w \}

\text{and } (\text{MAX}_{\leq_s}, w : w \in \emptyset : w) = \perp_s

(\text{MAX}_{\leq_s}, w : w \in (\text{suff}(q) \cap \text{pref}(P)) \cap \text{pref}(P) : w) \text{max}_{\leq_s} \varepsilon

= \{ \text{change of bound variable: } w = w'a \}

(\text{MAX}_{\leq_s}, w' : w' \in \text{suff}(q) \cap \text{pref}(P) \land w'a \in \text{pref}(P) : (w'a) \text{max}_{\leq_s} \varepsilon

= \{ \text{additional requirement on } \perp_s : \perp_s w = w \perp_s = \perp_s (\perp_s \text{ is zero of concatenation}) \}

(\text{MAX}_{\leq_s}, w' : w' \in \text{suff}(q) \cap \text{pref}(P) \land w'a \in \text{pref}(P) : (w'a) \text{max}_{\leq_s} \varepsilon

In order to compute the value of the quantified subexpression in the last expression of the derivation we use a linear search on $\text{suff}(q) \cap \text{pref}(P)$. This is called algorithm detail detail

**Detail (LS): Using linear search to update the state variable q.**

In the next two subsections we present two slightly different methods of linear search; the first leads to the standard Aho-Corasick algorithm, the second to the Knuth-Morris-Pratt algorithm.\footnote{Like q representing $\text{suff}(q) \cap \text{pref}(P)$, $\perp_s$ can be thought of as representing the empty set, provided we extend the definition of $\text{suff}$ with $\text{suff}(\perp_s) = \emptyset$.}
3.3.1 The Aho-Corasick algorithm with failure function

Given a linearly ordered, non-empty, and finite set $W$ we can define predecessor function $\text{pred} : W \setminus \{\min(W)\} \rightarrow W \setminus \{\max(W)\}$ by

$$\text{pred}(w) = (\text{MAX } w' : w' \in W \land w' < w : w') \quad (w \in W \setminus \{\min(W)\}).$$

Given a predicate $B : W \rightarrow \mathbb{B}$ linear search for the maximal element of $W$ satisfying $B$ can proceed as follows:

$$w := \max(W);$$
$$\text{do } w \neq \min(W) \land \neg B(w) \rightarrow w := \text{pred}(w) \text{ od }$$
$$(\{w = \min(W) \land \neg (\exists w' \in W : B(w')) \lor w = (\text{MAX } w' \in W : B(w') : w')\})$$

Taking

- $W = \text{suff}(q) \cap \text{pref}(P)$ (linearly ordered under $\leq_s$, $\max(W) = q (P_0'(u, q))$, $\min(W) = \varepsilon$),
- $\text{pred} = f_f|_{(\text{wump}(q) \cap \text{pref}(P))}$ where $f_f : \text{pref}(P) \setminus \{\varepsilon\} \rightarrow \text{pref}(P)$ is defined by

$$f_f(w) = (\text{MAX}_{\leq_s} w' : w' \in \text{suff}(w) \setminus \{w\} \cap \text{pref}(P) : w') \quad (w \in \text{pref}(P) \setminus \{\varepsilon\})$$

(function $f_f$ is the Aho-Corasick failure function corresponding to the forward trie [AC75]), and

- $B(w) \equiv w a \in \text{pref}(P) \quad (w \in \text{pref}(P))$

leads to the following update of variable $q$

$$\{P_0'(u, q)\}$$
$$q' := q;$$
$$\text{do } q' \neq \varepsilon \land q'a \notin \text{pref}(P) \rightarrow q' := f_f(q') \text{ od};$$
$$\{(q' = \varepsilon \land \neg (\exists w : w \in \text{suff}(q) \cap \text{pref}(P) : wa \notin \text{pref}(P)))$$
$$\lor q' = (\text{MAX}_{\leq_s} w : w \in \text{suff}(q) \cap \text{pref}(P) \land wa \in \text{pref}(P) : w)\}$$
$$\text{if } q' = \varepsilon \land a \notin \text{pref}(P) \rightarrow q := \varepsilon$$
$$\text{if } q' \neq \varepsilon \land a \in \text{pref}(P) \rightarrow q := q'a$$
$$\text{fi} \{P_0'(ua, q)\}$$

The second conjunct in the guard of the repetition can be evaluated using the forward trie $\tau_f$ ($q'a \notin \text{pref}(P) \equiv \tau_f(q', a) = \bot$). However, by introducing the extended forward trie $\tau_{ef} : \text{pref}(P) \times V \rightarrow \text{pref}(P) \cup \{\bot_s\}$ defined by

$$\tau_{ef}(w, c) = \begin{cases} wc & \text{if } wc \in \text{pref}(P) \\ \varepsilon & \text{if } w = \varepsilon \land c \notin \text{pref}(P) \\ \bot_s & \text{otherwise} \end{cases} \quad (w \in \text{pref}(P), c \in V)$$

both conjuncts can be combined:

$$q' \neq \varepsilon \land q'a \notin \text{pref}(P) \equiv \tau_{ef}(q', a) = \bot_s,$$

As a side effect of this introduction the if-fi statement can be replaced by the single assignment statement $q := \tau_{ef}(q', a)$.

---

With $f|_A$ we denote the function that is equal to $f$ with its domain restricted to set $A$. 

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By adding algorithm detail

Detail (AC-FAIL): Introduction of the extended forward trie $\tau_{e/f}$ and the failure function $f_f$ to implement the linear search updating state variable $q$. □

and eliminating variable $q'$ we obtain algorithm (cf. [AC75], section 2, algorithm 1)

Algorithm 3.4(P+, E, AC, LS, AC-FAIL)

\begin{verbatim}
    \begin{Verbatim}
    u, r := \epsilon, S; q := \epsilon; O_e := Output(q) \times \{S\};
    \{invariant: $P^0_0(u, q)$\}
    do $r \neq \epsilon$ ---
       do $\tau_f(q, r) \leftarrow \bot_s \rightarrow q := f_f(q)$ od;
    q := $\tau_f(q, r) \mid 1$; \{ $P^0_0(u(r)\mid 1), q)$ \}
    u, r := u(r)\mid 1, r\mid 1; \{ $P^0_0(u, q)$ \}
    O_e := O_e \cup Output(q) \times \{r\}
    od \{Re\}
    \end{Verbatim}
\end{verbatim}

This algorithm still has $O(|S|)$ run time complexity [Aho90] but is less efficient than the algorithm 3.3(P+, E, AC, OPT) in section 3.2. Function $\tau_f$ can be stored more efficiently than function $\gamma_f$ by use of a default value ($\bot_s$) requiring $O(|\text{pref}(P)|)$ space. Precomputation of extended forward trie $\tau_{e/f}$ and failure function $f_f$ is discussed in Part II, section 6.

3.3.2 The Knuth-Morris-Pratt algorithm

We now derive the Knuth-Morris-Pratt (KMP) algorithm, using a type of linear search different from that used for the Aho-Corasick algorithm with failure function.

As in the previous subsection we define a predecessor function on a totally ordered set $W$. In this case, however, we have a total predecessor function $\text{pred}_{ext} : W \rightarrow (W \setminus \max(W)) \cup \{\bot_W\}$ defined by

\[
\text{pred}_{ext}(w) = \begin{cases} 
\text{pred}(w) & \text{if } w \neq \min(W) \\
\bot_W & \text{if } w = \min(W)
\end{cases} \quad (w \in W)
\]

where $\bot_W$ is such that $\bot_W \max_{\leq} w = w \max_{\leq} \bot_W = w$ and $(\max_{\leq} w \in W : w \in \emptyset : w) = \bot_W$.

Assuming a selection predicate $B$ as in the previous section, linear search can proceed as follows:

\begin{verbatim}
    \begin{Verbatim}
    w := \max(W);
    do w \neq \bot_W \text{ cand } \neg B(w) \rightarrow w := \text{pred}_{ext}(w) \text{ od}
    \{w = (\max_{\leq} w' \in W : B(w') : w')\}
    \end{Verbatim}
\end{verbatim}

Taking $W = \text{suff}(q) \cap \text{pref}(P)$, $\bot_W = \bot_s$ (remember that $\bot_s$ is also defined to be a zero of concatenation), $\text{pred}_{ext} = f_{e/f}(\text{pref}(q) \cap \text{pref}(P))$, where $f_{e/f} : \text{pref}(P) \rightarrow \text{pref}(P) \cup \{\bot_s\}$ is defined by

\[
f_{e/f}(w) = \begin{cases} 
f(w) & \text{if } w \neq \epsilon \\
\bot_s & \text{if } w = \epsilon
\end{cases} \quad (w \in \text{pref}(P))
\]

$(f_{e/f}$ is called the extended failure function corresponding to the forward trie), and $B(w) \equiv w a \in \text{pref}(P)$ leads to the following instantiation of the linear search:

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\begin{align*}
\{P_0^e(u,q)\} \\
q' := q; \\
do q' \neq \perp, \text{ cand } q' \notin \text{pref}(P) \rightarrow q' := f_{ef}(q') \od; \\
\{q' = (\Max_{\leq} w : w \in \text{suff}(q) \cap \text{pref}(P) \land w \in \text{pref}(P) : w)\} \\
q := q' \text{ max}_{\leq} \varepsilon \\
\{q = (\Max_{\leq} w : w \in \text{suff}(q) \cap \text{pref}(P) \land w \in \text{pref}(P) : w) \text{ max}_{\leq} \varepsilon\} \\
\{P_0^e(ua,q)\}
\end{align*}

Adding the algorithm detail

**Detail (KMP-FAIL):** The extended failure function \( f_{ef} \) is introduced to implement the linear search for the update of \( q \).

and eliminating variable \( q' \) leads to algorithm

**Algorithm 3.5**

```
\begin{align*}
u, r := \varepsilon, S; q := \varepsilon; O_e := \text{Output}(q) \times \{S\}; \\
\{\text{invariant: } P_0^e(u,q)\} \\
do r \neq \varepsilon \rightarrow \\
do q \neq \perp, \text{ cand } q(r \upharpoonright 1) \notin \text{pref}(P) \rightarrow q := f_{ef}(q) \od; \\
q := q(r \upharpoonright 1) \text{ max}_{\leq} \varepsilon; \quad \{P_0^e(u(r \upharpoonright 1), q)\} \\
u, r := u(r \upharpoonright 1), r \upharpoonright 1; \quad \{P_0^e(u, q)\} \\
O_e := O_e \cup \text{Output}(q) \times \{r\} \\
\od \{R_e\}
\end{align*}
```

Adding indices: Historically, the KMP algorithm was designed using indexing within strings; this stems from efficiency concerns. Some of the most common uses of the KMP algorithm are in file-search programs and text editors, in which pointers to memory containing a string are a preferable method of accessing strings. In order to show the equivalence of this more abstract version of KMP, and the classically presented version we will now convert the above algorithm to make use of indexing within strings. In order to facilitate the use of indexing, we have to restrict the problem to the one keyword case, as stated in problem detail

**Detail (OKW):** \( P = \{p\} \)

We now introduce three shadow variables, and invariants that are maintained between the shadow variables and the existing program variables. Most shadow predicates and functions will be "hatted" for easy identification. Variables \( i \) and \( j \) are so named to conform to the original publication of the algorithms.

- \( i : q = p_1 \ldots p_{i-1} \) where \( i = 1 \equiv q = \varepsilon \) and \( i = 0 \equiv q = \perp \). With this convention we mirror the coding trick from the original KMP algorithm.
- \( j : u = S_1 \ldots S_{j-1} \land r = S_j \ldots S_{|S|}. \) Also \( r \upharpoonright 1 = S_j \) if \( j \leq |S| \).
- \( \hat{O}_e : \hat{O}_e = (\cup x \in \hat{O}_e : \{(p, S_x \ldots S_{|S|})\}) \).

We must of course define new predicates and a new predecessor function \( f_{ef} \) on these shadow variables.

- Define \( \hat{f}_{ef} : [1, |p| + 1] \rightarrow [0, |p|] \) as \( \hat{f}_{ef}(i) = |f_{ef}(p_1 \ldots p_{i-1})| + 1 \) and define \( |\perp| = -1 \).
- \( \hat{P}_0^e(j, i) \equiv (p_1 \ldots p_{i-1} = (\Max_{\leq} w \in V^* : w \in \text{suff}(S_1 \ldots S_{j-1}) \cap \text{pref}(p) : w)) \).
- \( \hat{R}_e \equiv (\hat{O}_e = \{ j \in \{ 1 \leq j \leq |S| + 1 \} \land p \in \text{suff}(S_1 \cdots S_{j-1}) : \{ j \}) \)

We can also note the following equivalences and correspondences:

- Since \( q \in \text{pref}(p) \) then \( q(r[1]) \notin \text{pref}(p) \equiv S_i \neq p_i \) when \( i \leq |p| \land j \leq |S| \). Similarly \( q \neq \bot \equiv 0 < i \) and \( q = p \equiv i = |p| + 1 \).
- \( q := q(r[1]) \max_{\leq \varepsilon} \varepsilon \) corresponds to \( i := i + 1 \)
- \( u, r := u(r[1]), r[1] \) corresponds to \( j := j + 1 \)
- \( r \neq \varepsilon \equiv j < |S| \)
- \( O_e := O_e \cup \text{Output}(q) \times \{ r \} \) corresponds to if \( |p| < i \rightarrow \hat{O}_e := \hat{O}_e \cup \{ j \} \land i \leq |p| \rightarrow \text{skip} \)

The complete algorithm (written without the invariants relating shadow to non-shadow variables) is now:

\[
\begin{align*}
u, r &:= \varepsilon, S; q := \varepsilon; O_e := \text{Output}(q) \times \{ S \}; \\
i &:= 1; j := 1; \\
\text{if } i = |p| + 1 &\rightarrow \hat{O}_e := \{ j \} \land i \neq |p| + 1 \rightarrow \hat{O}_e := \emptyset \text{ fi}; \\
\{ \text{invariant: } P_0^o(u, q) \land \hat{P}_0^o(j, i) \} \text{ do } \\
j &:= 0 \text{ < i cand } S_j \neq p_i \rightarrow q := P_0^o(q) ; i := \hat{f}_e i; \text{ od}; \\
q &:= q(r[1]) \max_{\leq \varepsilon} \varepsilon ; i := i + 1; \quad \{ P_0^o(u(r[1]), q) \land \hat{P}_0^o(j + 1, i) \} \\
u, r &:= u(r[1]), r[1]; j := j + 1; \quad \{ P_0^o(u, q) \land \hat{P}_0^o(j, i) \} \\
O_e &:= O_e \cup \text{Output}(q) \times \{ r \}; \\
\text{if } i = |p| + 1 &\rightarrow O_e := \hat{O}_e \cup \{ j \}; \\
\text{fi if } i \neq |p| + 1 &\rightarrow \text{skip} \\
\text{fi od } \{ R_e \land \hat{R}_e \} \\
\end{align*}
\]

We have introduced algorithm detail:

**Detail (INDICES):** Represent substrings by indices into the complete strings. □

Removing the non-shadow variables leaves us with the classic KMP algorithm (cf. [KMP77], section 2, p.326):

**Algorithm 3.6**(*P_1, E, AC, LS, KMP-FAIL, OKW, INDICES*)

\[
\begin{align*}
i &:= 1; j := 1; \\
\text{if } i = |p| + 1 &\rightarrow \hat{O}_e := \{ j \} \land i \neq |p| + 1 \rightarrow \hat{O}_e := \emptyset \text{ fi}; \\
\{ \text{invariant: } \hat{P}_0^o(j, i) \} \text{ do } \\
j &:= 0 \text{ < i cand } S_j \neq p_i \rightarrow i := \hat{f}_e i; \text{ od}; \\
i &:= i + 1; \quad \{ \hat{P}_0^o(j + 1, i) \} \\
j &:= j + 1; \quad \{ \hat{P}_0^o(j, i) \} \\
\text{if } i = |p| + 1 &\rightarrow O_e := \hat{O}_e \cup \{ j \}; \\
\text{fi if } i \neq |p| + 1 &\rightarrow \text{skip} \\
\text{fi od } \{ \hat{R}_e \} \\
\end{align*}
\]

This algorithm has \( \mathcal{O}(|S|) \) running time. Storage of \( \hat{f}_e \) requires \( \mathcal{O}(|p|) \) space. Precomputation of function \( \hat{f}_e \) can easily be derived by converting, in a similar way, the precomputation of function \( f_{e,f} \) (as discussed in Part II, section 6) into using indices.
4 The Commentz-Walter algorithms

We now consider a derivation of the Commentz-Walter algorithms starting with algorithm 2.5 \((p+s_+, \text{RT})\). We will be exploring the possibility of (safely) making shifts of more than one symbol.

To present an algorithm more closely matched to the one presented by Commentz-Walter we add the problem detail

**Detail (NE):** \(\varepsilon \not\in P\)

Consequently, assignments \(O := \{\varepsilon\} \times (\{\varepsilon\} \cap P) \times \{S\}\) and \(O := O \cup \{u\} \times (\{\varepsilon\} \cap P) \times \{\varepsilon\}\) become superfluous in algorithm 2.5\((p+s_+, \text{RT})\). Our goal is to make shifts larger than one symbol in the assignment \(u, r := u(r[1]), r[1]\). In order to do this, an attempted match should occur before this assignment. In this case, information obtained during the attempted match can be used to determine an appropriate shift. Attempted matches are performed by the inner repetition of the algorithm. A phase shift of the outer repetition will place the inner repetition immediately before the shift assignment. Such a phase shift also places an extra copy of the inner repetition after the outer repetition. Let \(m = (\text{MIN}_{p \in P} : |p|)\). Since \(|u| < m \Rightarrow \text{suff}(u) \cap P = \emptyset\) we also start with a different assignment to \(u, r\). This phase shift and assignment to \(u, r\) are not considered algorithm details. This yields algorithm

**Algorithm 4.1\((p+s_+, \text{RT, NE})\)**

\[
\begin{align*}
u, r &:= S[m], S[m]; O := \emptyset; \\
do \ r \neq \varepsilon \quad &\longrightarrow \\
l, v := u, \varepsilon; \\
do l \neq \varepsilon \quad \text{cand} \quad \tau_r(v, l[1]) \neq \perp \quad &\longrightarrow \\
l, v := l[1], (l[1]u); \\
O := O \cup \{l\} \times (\{v\} \cap P) \times \{\varepsilon\} \\
od; \\
\{v \in \text{suff}(P) \wedge (l = \varepsilon \ \text{cor} \ (l[1]v \not\in \text{suff}(P))\} \\
u, r := u(r[1]), r[1]; \\
od; \\
l, v := S, \varepsilon; \\
do l \neq \varepsilon \quad \text{cand} \quad \tau_r(v, l[1]) \neq \perp \quad &\longrightarrow \\
l, v := l[1], (l[1]u); \\
O := O \cup \{l\} \times (\{v\} \cap P) \times \{\varepsilon\} \\
od \\
\{v \in \text{suff}(P) \wedge (l = \varepsilon \ \text{cor} \ (l[1]v \not\in \text{suff}(P))\} \\
\{R\}
\end{align*}
\]

4.1 Larger shifts

We now consider larger shifts than in the assignment

\(u, r := u(r[1]), r[1]\)

in the previous algorithm.

**Detail (cw):** If \(k\) is such that

\[1 \leq k \leq (\text{MIN}_{n : 1 \leq n \leq |r| \wedge \text{suff}(u(r[n])) \cap P \neq \emptyset : n}) \min |r|\]

then the assignment to \(u, r\) may be replaced by

\(u, r := u(r[k]), r[k]\)

without missing any matches. \(\Box\)
A number $k$ satisfying the above condition is called a **safe shift distance**. Computing the upper-bound on $k$ (the maximal safe shift) is essentially the same as the problem that we are trying to solve. Therefore, we will aim at easier to compute approximations of the upper-bound. By weakening the predicate $\text{suff}(u(r|n)) \cap P \neq \emptyset$ in the range of the quantified expression approximations of the upper-bound from below are obtained.

This method of **predicate weakening** proves to be extremely important both in the derivation of the Commentz-Walter algorithm and the Boyer-Moore algorithm variants. In both cases the value of a quantified minimum must be computed. The range predicate in the quantifier is initially too strong, amounting to a problem of similar difficulty to the one which we are trying to solve. A weakening of this predicate will lead to a conservative approximation of the quantified minimum, with less computational effort.

In the following derivation we will assume the post-condition of the inner repetition in algorithm 4.1($p_4.4$, RT, NE): $Iv = u \land v \in \text{suff}(P) \land (l = \varepsilon \land \text{cor} (l[l])u \not\in \text{suff}(P))$. In fact, this post-condition can be rewritten with non-conditional disjunction in place of the conditional disjunction since $\varepsilon[l]1 = \varepsilon$ by definition.

We now proceed to weaken the predicate, assuming $1 \leq n \leq |r|$:

\[
\text{suff}(u(r|n)) \cap P \neq \emptyset \\
\equiv \quad \{ u = Iv \} \\
\text{suff}(lv(r|n)) \cap P \neq \emptyset \\
\Rightarrow \quad \{ \text{split, and discard most of } l, \text{ do not lookahead at } r, n \leq |r| \} \\
\text{suff}(V^*(l[l])uV^n) \cap P \neq \emptyset
\]

Notice that we have obtained a weaker predicate that does not depend on $r$. After substituting this predicate in the upperbound the restriction $n \leq |r|$ can be removed due to the second operand of the $\text{min}$-operator, $|r|$. We continue the derivation, assuming $n \geq 1$:

\[
\text{suff}(V^*(l[l])uV^n) \cap P \neq \emptyset \\
\equiv \quad \{ \text{property B.2} \} \\
V^*(l[l])uV^n \cap V^*P \neq \emptyset \\
\equiv \quad \{ V^*A \cap V^*B \neq \emptyset \equiv V^*A \cap B \neq \emptyset \lor V^*B \cap A \neq \emptyset \} \\
V^*(l[l])uV^n \cap P \neq \emptyset \lor V^*P \cap (l[l])uV^n \neq \emptyset \\
\Rightarrow \quad \{ l = \varepsilon: \text{trivial; } l \neq \varepsilon: \text{property B.7} \} \\
V^*(l[l])uV^n \cap P \neq \emptyset \lor V^*P \cap uV^n \neq \emptyset
\]

We now consider several further weakenings of this predicate.

### 4.2 Discarding the lookahead symbol

In the disjunct $V^*(l[l])uV^n \cap P \neq \emptyset$ we discard $(l[l])$:

\[
V^*(l[l])uV^n \cap P \neq \emptyset \lor V^*P \cap uV^n \neq \emptyset \\
\Rightarrow \quad \{ \text{monotonicity of } r: V^*(l[l]) \subseteq V^* \} \\
V^*uV^n \cap P \neq \emptyset \lor V^*P \cap uV^n \neq \emptyset
\]

We now manipulate the $\text{MIN}$ quantifier into a suitable form:

\[
(\text{MIN } n : 1 \leq n \leq |r| \land \text{suff}(u(r|n)) \cap P \neq \emptyset : n) \text{ min } |r| \\
\geq \quad \{ \text{weakening of the range predicate using the preceding derivations} \} \\
(\text{MIN } n : 1 \leq n \land (V^*uV^n \cap P \neq \emptyset \lor V^*P \cap uV^n \neq \emptyset) : n) \text{ min } |r| \\
= \quad \{ \text{property of MIN with disjunctive range} \} \\
(\text{MIN } n : 1 \leq n \land V^*uV^n \cap P \neq \emptyset : n) \\
\text{min}(\text{MIN } n : 1 \leq n \land V^*P \cap uV^n \neq \emptyset : n) \text{ min } |r|
\]
Since \( v \in \text{suff}(P) \) we can define two functions \( d_1, d_2 : \text{suff}(P) \rightarrow \mathbb{N} \) by

\[
\begin{align*}
d_1(x) &= (\text{MIN } n : 1 \leq n \leq V^*xV^n \cap P \neq \emptyset : n) \quad (x \in \text{suff}(P)) \\
d_2(x) &= (\text{MIN } n : 1 \leq n \leq V^*xP \cap xV^n \neq \emptyset : n) \quad (x \in \text{suff}(P))
\end{align*}
\]

**Detail (NLA):** The lookahead term \( l1 \) is discarded. Functions \( d_1 \) and \( d_2 \) can be precomputed and used to compute the no lookahead shift

\[
k_{\text{NLA}} = d_1(v) \min d_2(v) \min |r|
\]

\[\square\]

Using this detail gives a new algorithm \( (P, S_+, \text{RT, NE, CW, NLA}) \). Precomputation of the two functions \( d_1 \) and \( d_2 \) is discussed in Part II, subsection 7.1.

### 4.3 Using the lookahead symbol

Instead of discarding the lookahead term \( l1 \) it can also be taken into account.

**Detail (LA):** The lookahead term \( l1 \) is not discarded. \( \square \)

\[
\begin{align*}
V^*(l1)vV^n \cap P \neq \emptyset \lor V^*P \cap vV^n \neq \emptyset \\
\equiv & \quad \{ \text{monotonicity of } \cap : V^*(l1) \subseteq V^* \} \\
(V^*(l1)vV^n \cap P \neq \emptyset \land V^*vV^n \cap P \neq \emptyset) \lor V^*P \cap vV^n \neq \emptyset \\
\Rightarrow & \quad \{ \text{monotonicity of } \cap : vV^n \subseteq V[|l|+n] \} \\
(V^*(l1)\text{V}[|l|+n] \cap P \neq \emptyset \land V^*vV^n \cap P \neq \emptyset) \lor V^*P \cap vV^n \neq \emptyset
\end{align*}
\]

**Detail (NEAR-OPT):** Define function \( d_{\text{no}} : \text{suff}(P) \times V \rightarrow \mathbb{N} \) by

\[
d_{\text{no}}(x, a) = (\text{MIN } n : 1 \leq n \leq (V^*aV^n+|v| \cap P \neq \emptyset \land V^*xV^n \cap P \neq \emptyset) \lor V^*P \cap xV^n \neq \emptyset : n)
\]

for \( x \in \text{suff}(P), a \in V \), and use it to compute shift amount

\[
k_{\text{no}} = \begin{cases} 
 d_{\text{no}}(v, l1) \min |r| & l1 \neq \varepsilon \\
 d_1(v) \min d_2(v) \min |r| & l1 = \varepsilon
\end{cases}
\]

\[\square\]

Using shift amount \( k_{\text{no}} \) yields algorithm \( (P, S_+, \text{RT, NE, CW, LA, NEAR-OPT}) \). Precomputation of \( d_{\text{no}} \) is discussed in Part II, subsection 7.2.

Precomputation of \( d_{\text{no}} \) is rather expensive both in space and time. Moreover, storage of \( d_{\text{no}} \) requires \( O(|\text{suff}(P)| \cdot |V|) \) space. Therefore, we derive another approximation, resulting in a more efficient precomputation, and less storage requirements. We derive

\[
\begin{align*}
d_{\text{no}}(v, (l1)) \min |r| \\
= & \quad \{ \text{definition of } d_{\text{no}} \text{ and } d_2, \text{ disjunctive range in quantifier} \} \\
& \quad (((\text{MIN } n : 1 \leq n \leq (V^*(l1)V^n+|v| \cap P \neq \emptyset \land V^*vV^n \cap P \neq \emptyset : n)) \min d_2(v) \min |r| \\
\quad \geq & \quad \{ \text{conjunctive range in quantifier, definition of } d_1 \} \\
& \quad (((\text{MIN } n : 1 \leq n \leq (V^*(l1)V^n+|v| \cap P \neq \emptyset : n)) \max d_1(v)) \min d_2(v) \min |r| \\
\quad \geq & \quad \{ \text{calculus} \} \\
& \quad (((\text{MIN } n : 1 \leq n \leq V^*(l1)V^n \cap P \neq \emptyset : n) - |v|) \max d_1(v)) \min d_2(v) \min |r|
\end{align*}
\]
Detail (NORM): Define $d_3 : N \times V \rightarrow N$ by

$$d_3(x, a) = (\text{MIN } n : 1 \leq n \land V^* a V^n \cap P \neq \emptyset : n - z) \quad (x \in N, a \in V),$$

functions $d_1$ and $d_2$ as in subsection 4.2, and use them to compute shift amount

$$k_{\text{norm}} = \left\{ \begin{array}{ll}
(d_3(|v|, |l|) \max d_1(v) \min |r| & |l| \neq \varepsilon \\
(d_1(v) \min d_2(v) \min |r| & |l| = \varepsilon
\end{array} \right.$$
5 The Boyer-Moore family of algorithms

The Boyer-Moore algorithm derivation in the previous section only accounted for one method of traversing the variable \( u \), in increasing order of \( v \). In practice, when \( P = \{ p \} \) other methods of comparing \( v \) to keyword \( p \) can be used. We therefore introduce problem detail

**Detail (OKW):** \( P = \{ p \} \)

and starting with the original problem specification derive the Boyer-Moore algorithm and its variants.

We define a "perfect match" predicate

\[
P_M((l, v, r)) \equiv (lvr = S \land v = p)
\]

and rewrite the postcondition into

\[
R' : O = (\cup l, v, r : P_M((l, v, r)) : \{(l, v, r)\}).
\]

Define right shift function \( Sh : (V^*)^3 \times N \rightarrow (V^*)^3 \) by

\[
Sh(l, v, r, k) = (l(vr|k), (v(r|k))|k, r|k).
\]

By introduction of the "regular Boyer-Moore" algorithm detail

**Detail (RBM):** Use function \( Sh \) and maintain invariant

\[
\begin{align*}
P_1(l, v, r) &\equiv (lvr = S) \land (|v| \leq |p|) \land (|v| < |p| \Rightarrow r = \varepsilon) \\
&\land (O = (\cup l', v', r' : P_M((l', v', r')) \land (l'v' <_p lv) : \{(l', v', r')\}))
\end{align*}
\]

we obtain a first (deterministic) solution (which is a phase shifted version of the algorithm in Appendix A.2)

**Algorithm 5.1 (OKW, RBM)**

\[
\begin{align*}
l, v, r &:= \varepsilon, S|p|, S||p|; O := \emptyset; \\
\{\text{invariant: } P_1(l, v, r)\} \\
do\{R'\} &\rightarrow \\
\text{if } v = p &\rightarrow O := O \cup \{(l, v, r)\} \\
\text{if } v \neq p &\rightarrow \text{skip} \\
\text{fi; } \\
l, v, r &:= Sh(l, v, r, 1) \quad \{P_1(l, v, r)\}
\end{align*}
\]

This algorithm does not take into account how we evaluate \( v = p \). Define a "match order" to be a bijective function \( mo : [1, |p|] \rightarrow [1, |p|], \) i.e. a permutation of the integers \( j : 1 \leq j \leq |p| \). This function is used to determine the order in which the individual symbols of \( v \) and \( p \) are compared. We now have

\[
(v = p) \equiv (\forall i : 1 \leq i \leq |p| : v_{mo(i)} = p_{mo(i)}).
\]

The match order detail is:

**Detail (MO):** The characters of \( v \) and \( p \) are compared in a fixed order determined by a bijective function \( mo : [1, |p|] \rightarrow [1, |p|] \) (i.e. a permutation of \( [1, |p|] \)).

The particular match order used in an algorithm determines the third position of the algorithm name. Three of the most common match orders, which represent particular instances of detail (MO), are
**Detail (FWD):** The forward (or identity) match order given by \( mo(i) = i \). □

**Detail (REV):** The reverse match order given by \( mo(i) = |p| - i + 1 \). This is the original Boyer-Moore match order. □

**Detail (OM):** Let \( Pr: [1,|p|] \rightarrow [0,1] \) be the probability distribution of the symbols of \( p \) in input string \( S \); the domain of function \( Pr \) consists of indices into \( p \). Let an optimal mismatch match order be any permutation \( ma \) such that

\[
(\forall i,j: 1 \leq i \leq |p| \land 1 \leq j \leq i : Pr(mo(j)) \leq Pr(mo(i))).
\]

This match order is described as “optimal” because it compares characters of \( p \) in order of ascending probability of occurring in \( S \). In this way, the least probable characters of \( p \) are compared first, so on the average one can expect to find any mismatch as early as possible. □

Comparing \( v \) and \( p \) using match order \( mo \) is done by procedure \( \text{Match} \) specified by

\[
\{ |v| = |p| \}
\]

\[
\text{Match}(l.v, \downarrow p, \downarrow mo, \uparrow i)
\]

\[
\{ P_2(v,p,mo,i) : (1 \leq i \leq |p| + 1) \land (i \leq |p| \Rightarrow v_{mo(i)} \neq p_{mo(i)}) \\
\land (\forall j : 1 \leq j < i : v_{mo(j)} = p_{mo(j)}) \}
\]

From \( P_2(v,p,mo,i) \) it follows that \( (v = p) \equiv (i = |p| + 1) \), and that if \( i \leq |p| \) then \( v_{mo(i)} \) is the first (in the given order) mismatching character. An implementation of \( \text{Match} \) is

\[
i := 1;
\]

\[
do \ i \leq |p| \quad \text{cand} \quad v_{mo(i)} = p_{mo(i)} \rightarrow i := i + 1 \od
\]

Adding \( mo, i, \) and \( \text{Match} \) to the algorithm 5.1(OKW, RBM) results in

**Algorithm 5.2(OKW, RBM, MO)**

\[
l, v, r := \epsilon, S|p|, S|p|; O := \emptyset;
\]

\[
\{ \text{invariant: } P_1(l,v,r) \}
\]

\[
do \ |v| = |p| \rightarrow
\]

\[
\text{Match}(v,p,mo,i);
\]

\[
\{ P_2(v,p,mo,i) \}
\]

\[
\text{if } i = |p| + 1 \rightarrow O := O \cup \{(l,v,r)\}
\]

\[
\mid i \neq |p| + 1 \rightarrow \text{skip}
\]

\[
\text{fi};
\]

\[
(l,v,r) := \text{Sh}(l,v,r,1) \quad \{ P_1(l,v,r) \}
\]

\[
\od \{ R' \}
\]

In some versions of the Boyer-Moore algorithms \( \text{Match} \) is only executed after a successful comparison of a character of \( p \) which is least frequent in \( S \), and the corresponding character of \( v \). In the taxonomy in [HS91] this comparison is called the guard and the character of \( p \) involved the guard character. We do not consider it here since it can be viewed as additionally requiring that \( p_{mo(1)} \) is a character of \( p \) with minimal frequency in \( S \).
5.1 Larger shifts without using Match information

It may be possible to make an additional shift (immediately before Match is performed) providing no matches are missed. A shift of not greater than \((\text{MIN} k : 0 \leq k \land PM(Sh(l, v, r, k)) : k)\) will be safe. This can be done with the statement

\[
\{ |v| = |p| \} \\
(l, v, r) := Sh(l, v, r, (\text{MIN} k : 0 \leq k \land PM(Sh(l, v, r, k)) : k) \text{ min } |r|) \\
\{ |v| = |p| \land (r = \varepsilon \lor PM(l, v, r)) \}
\]

The \(\text{min } |r|\) is used to ensure that \(|v| = |p|\) is maintained. Another implementation of the shift is

\[
\{ |v| = |p| \} \\
do 1 \leq |r| \land \neg PM(l, v, r) \\
(l, v, r) := Sh(l, v, r, (\text{MIN} k : 1 \leq k \land PM(Sh(l, v, r, k)) : k) \text{ min } |r|) \\
od \{ |v| = |p| \land (r = \varepsilon \lor PM(l, v, r)) \}
\]

This could have been implemented with an if-fi construct, however, the do-od construct will prove to be more useful when the shift distance is approximated from below. The do-od version is known as a skip loop in the taxonomy of Hume and Sunday [HS91].

Calculating the \(\text{MIN}\) quantification is essentially as difficult as the problem we are trying to solve. Since any smaller shift length suffices, we consider weakenings of predicate \(PM\). Some weakenings are: \(Q_0(l, v, r) \equiv (v_1 = p_1)\), \(Q_2(l, v, r) \equiv (v_1 = p)\), and \(Q_3(l, v, r) \equiv (v_j = p_j)\) (for some \(1 \leq j \leq |p|\)); the predicates \(Q_1\) and \(Q_2\) require that \(p \neq \varepsilon\). Predicates \(Q_1\) and \(Q_2\) are special cases of \(Q_3\). We can of course take the conjunction of any of these weakenings and still have a weakening of \(PM\).

For each weakening of \(PM\), we consider the shift length as calculated with the quantified \(\text{MIN}\). In the case of \(Q_0\), the entire skip loop is equivalent to skip.

We consider the shift length for \(Q_3\) before returning to \(Q_1\) and \(Q_2\) as special cases. We need to compute

\[(\text{MIN} k : 1 \leq k \land PM(Sh(l, v, r, k)) : k)\]

In order to easily compute this we will weaken the range predicate, removing lookahead. Additionally, it is known (from the do-od guard) that \(\neg Q_2(l, v, r)\) holds. The derivation proceeds as follows (assuming \(1 \leq k \leq |r|\), \(\neg Q_3(l, v, r)\) and fixed \(j : 1 \leq j \leq |p|\)):

\[
PM(Sh(l, v, r, k)) \\
\equiv \{ \text{definition of } Sh \} \\
PM((l(vr)|k), (v(r)|k))|k,r|k) \\
\equiv \{ \text{definition of } PM \} \\
(v(r)|k) | k = p \\
\equiv \{ \text{definition of } = \text{ on strings} \} \\
(\forall i : 1 \leq i \leq |p| : ((v(r)|k)|k)_i = p_i) \\
\equiv \{ \text{rewrite } | \text{ into indexing} \} \\
(\forall i : 1 \leq i \leq |p| : (v(r)|k)_{i+k} = p_i) \\
\Rightarrow \{ \text{discard lookahead at } r, |v| = |p| \} \\
(\forall i : 1 \leq i \leq |p| - k : v_{i+k} = p_i) \\
\equiv \{ \text{change of bound variable: } i = i + k, \neg Q_3(l, v, r) \} \}
\]
\[(\forall i' : 1 + k \leq i' \leq |p| : v'_i = p_{i' - k}) \land v_j \neq p_j\]
\[\Rightarrow \{\text{one point rule}\}\]
\[1 + k \leq j \Rightarrow v_j = p_{j - k} \land v_j \neq p_j\]
\[\equiv \{\text{transitivity of = }\}\]
\[1 + k \leq j \Rightarrow v_j = p_{j - k} \land p_j \neq p_{j - k}\]

The final predicate is free of \(r\), and so the upperbound of \(|r|\) on \(k\) can be dropped.

Given \(j : 1 \leq j \leq |p|\), we can define a function and a constant

\[s_1(a) = (\text{MIN } k : 1 \leq k \land (1 + k \leq j \Rightarrow a = p_{j - k}) : k)\]
\[s_2 = (\text{MIN } k : 1 \leq k \land (1 + k \leq j \Rightarrow p_j \neq p_{j - k}) : k)\]

Functions \(s_1\) and \(s_2\) can be combined to give a shift of \((s_1(v_j) \max s_2) \min |r|\). In practice \(s_1\) and \(s_2\) can be obtained from two functions computed for a different purpose. If a conjunct of any of \(Q_0\), \(Q_1\), \(Q_2\), or \(Q_3\) is used as a weakening of \(PM\), the appropriate skip length can be approximated as the \(\max\) of the individual skip lengths. A particularly interesting skip length is that arising from predicate \(Q_1\). In this case, \(s_1(a) = 1\) and \(s_2 = 1\) and a skip length of 1 is used.

Assuming \(Q\) is a weakening of \(PM\) we introduce program detail.

**Detail (SL):** Comparison of \(v\) and \(p\) is preceded by a skip loop based upon weakening \(Q\) of \(PM\) and some appropriate skip length. □

Assuming some fixed \(j : 1 \leq j \leq |p|\) we use \(Q_3\) as an example of a weakening of \(PM\) in

**Algorithm 5.3:**

\[
\begin{align*}
& l, v, r := \epsilon, S[p], S[p]; O := \emptyset; \\
& \{\text{invariant: } P_3(l, v, r)\} \\
& \text{do } [v] = [p] \\
& \quad \{[v] = [p]\} \\
& \quad \text{do } 1 \leq |r| \land \neg Q_3((l, v, r)) \\
& \quad \text{od; } \{[v] = [p] \land (Q_3((l, v, r)) \lor r = \epsilon)\} \\
& \quad \text{Match}(v, p, mo, i); \{P_3(v, p, mo, i)\} \\
& \quad \text{if } i = |p| + 1 \\
& \quad \text{do } O := O \cup \{(l, v, r)\} \\
& \quad \text{fi; } \{P_3(l, v, r)\} \\
& \text{od} \{R'\}
\end{align*}
\]

We proceed by presenting four instances of detail (SL) (each based on a weakening of \(PM\))^10:

**Detail (NONE):** The predicate \(Q_0\) \((true)\) is used as the weakening of \(PM\) in the skip loop. Notice that in this case the skip loop is equivalent to statement skip. □

**Detail (SFC):** The predicate \(Q_1\) is used as the weakening of \(PM\) in the skip loop. □

**Detail (FAST):** The predicate \(Q_2\) is used as the weakening of \(PM\) in the skip loop. □

**Detail (SLFC):** Let \(p_j\) be a character of \(p\) with minimal frequency in \(S\). Predicate \(Q_3\), defined by

\[Q_3((l, v, r)) \equiv v_j = p_j,\]

is used as the weakening of \(PM\) in the skip loop. □

---

^10 names are taken from the taxonomy in [HS91]

^11 search first character

^12 search least frequent character
5.2 Making use of Match information

Up to now information from previous matching attempts was not used in the computation of the shift distance (in fact there was no computation and the shift distance defaulted to 1). In this subsection we will take into account the information from the immediately preceding matching attempt.

With a shift of $k$ symbols, $p$ will be compared against $(vr[k])_1[p]$. Ideally, we would like to select our shift $k$ such that it is the smallest $k$ satisfying $1 \leq k \leq |r|$ and

$$(\forall j : 1 \leq j \leq |p| : (vr[k])_j = p_j).$$

Again, we apply the technique of weakening such a predicate, thereby obtaining approximations of the optimal shift distance from below. The weakening of the predicate should, amongst others, include the removal of any reference to $r$ (no lookahead).

In the following calculations we assume $k \leq |r| \land |v| = |p|$ and the postcondition of Match, namely $P_2(v, p, m, i)$. We derive

$$(\forall j : 1 \leq j \leq |p| : \{j = |p|, k \mid 1 \leq j \leq |p|, (vr[k])_j = (vr)_{j+k}\}$$

$$(\forall j : 1 \leq j \leq |p| : (vr[k])_j = p_j)$$

$$(\forall j' : 1 + k \leq j' \leq |p| + k : (vr)_{j'} = p_{j'-k})$$

$$(\forall j' : 1 + k \leq j' \leq |p| : v_{j'} = p_{j'-k})$$

$$(\forall j : 1 \leq j \leq |p| \land 1 + k \leq mo(j) : v_{mo(j)} = p_{mo(j)-k})$$

$$(\forall j : 1 \leq j \leq |p| \land 1 + k \leq mo(j) : v_{mo(j)} = p_{mo(j)-k})$$

$$(\forall j : 1 \leq j \leq |p| \land 1 + k \leq mo(i) : v_{mo(i)} = p_{mo(i)-k})$$

$$(\forall j : 1 \leq j \leq |p| \land 1 + k \leq mo(i) : v_{mo(i)} = p_{mo(i)-k})$$

$$(\forall j : 1 \leq j \leq |p| \land 1 + k \leq mo(i) : v_{mo(i)} = p_{mo(i)-k})$$

The last predicate in the preceding derivation will be denoted by $P_3(v, i, k)$ (here we have chosen to make parameters $m$ and $p$ implicit). We now define the shift distance $k$ based on previous match information by

$$k = (\text{MIN } j : 1 \leq j \land P_3(v, i, j) : j) \text{ min}(|r| + 1).$$

Notice that this shift distance still depends on implicit parameter $m$. The predicate $P_3$ is frequently weakened further (most often the conjunct $p_{m(i)} \neq p_{m(i)-k}$ is discarded). In much of the literature, $P_3$ is broken up into

$$P_3'(i, k) \equiv (\forall j : 1 \leq j < i \land 1 + k \leq mo(j) : p_{mo(j)} = p_{mo(j)-k})$$

$$P_3''(v, i, k) \equiv (i \leq |p| \land \text{ cand } 1 + k \leq mo(i) \Rightarrow v_{mo(i)} = p_{mo(i)-k})$$

$$P_3'''(i, k) \equiv (i \leq |p| \land \text{ cand } 1 + k \leq mo(i) \Rightarrow p_{mo(i)} \neq p_{mo(i)-k})$$

This leads to three functions $s_1 : N \rightarrow N$, $char_1 : V^{|p|} \times N \rightarrow N$, and $char_2 : N \rightarrow N$ defined by

$$s_1(i) \equiv (\text{MIN } k : 1 \leq k \land P_3'(i, k) \land \text{ cand } 1 \leq k \land P_3''(v, i, k)) \quad (i \in N)$$

$$char_1(v, i) \equiv (\text{MIN } k : 1 \leq k \land P_3'(v, i, k) \land \text{ cand } 1 \leq k \land P_3''(v, i, k)) \quad (v \in V^{|p|}, i \in N)$$

$$char_2(i) \equiv (\text{MIN } k : 1 \leq k \land P_3'''(i, k)) \quad (i \in N)$$

Applying these functions yields a new, possibly smaller, shift distance

$$k = (s_1(i) \max \text{ char}_1(v, i) \max \text{ char}_2(i)) \text{ min}(|r| + 1).$$

This is known as the match information detail.
Detail (MI): Use information from the preceding match attempt by computing the shift distance using functions \( s_1, \text{char}_1, \) and \( \text{char}_2 \). □

Adding this detail results in the following Boyer-Moore algorithm skeleton (details (MO) and (SL) still have to be instantiated), for weakening \( Q_3 \) of \( PM \) (cf. [HS91], section 4, p.1224):

**Algorithm 5.4(OKW, RBM, MO, SL, MI)**

\[
\begin{align*}
    l, v, r &:= \varepsilon, S[l[p], S[l[p]]; O := \emptyset; \\
    \{\text{invariant: } P_1(l, v, r)\} \\
    \text{do } |v| &= |p| \\
    \text{do } 1 \leq |r| \land \neg Q_3((l, v, r)) \rightarrow (l, v, r) := Sh(l, v, r, (s_l(v, j) \max s_2) \min |r|) \text{ od;} \\
    \{|v| = |p| \land (Q_3((l, v, r)) \lor r = \varepsilon)\} \\
    \text{Match}(v, p, mo, i); \quad \{P_2(v, p, mo, i)\} \\
    \text{if } i = |p| + 1 \rightarrow O := O \cup \{(l, v, r)\} \\
    \text{fi} \\
    k := (s_l(i) \max \text{char}_1(v, i) \max \text{char}_2(i)) \min(|r| + 1); \\
    (l, v, r) := Sh(l, v, r, k) \quad \{P_1(l, v, r)\} \\
    \text{od } \{R'\}
\end{align*}
\]

Precomputation of functions \( s_1, \text{char}_1, \) and \( \text{char}_2 \) is discussed in Part II, subsection 7.5 for instantiations (FWD) and (REV) of algorithm detail (MO).

Given fixed \( j : 1 \leq j \leq |p| \) we can easily compute the function \( s_l \) and constant \( s_2 \) from section 5.1. This can be done for any particular \( mo \). The functions are

\[
\begin{align*}
    s_l(v, j) &= \text{char}_1(v, mo^{-1}(j)) \\
    s_2 &= \text{char}_2(mo^{-1}(j))
\end{align*}
\]
Part II

Precomputation

In this part we derive algorithms for the precomputation of the functions used in the pattern matching algorithms in Part I. The algorithms are correct due to their formal derivation. This can not always be said about the algorithms found in the literature, mostly due to the absence of any formal derivation (see for instance the single keyword Boyer-Moore precomputation algorithms given in [BM77], [KMP77], and [Ryt80], where each article shows the preceding article to give an incorrect precomputation algorithm). Moreover, we give the first formal derivation of the precomputation algorithms for the Commentz-Walter family of algorithms. They can, amongst others, be specialized to a correct precomputation algorithm for the single keyword Boyer-Moore algorithm.

6 Precomputation for the Aho-Corasick algorithms

First, we consider the transition function of the forward trie corresponding to $P \tau_{P,f} : \text{pref}(P) \times V \rightarrow (\text{pref}(P) \cup \{\bot\})$ defined by

$$
\tau_{P,f}(u,a) = \begin{cases} 
ua & \text{if } ua \in \text{pref}(P) \\
\bot & \text{if } ua \notin \text{pref}(P)
\end{cases} \quad (u \in \text{pref}(P), a \in V).
$$

Since \text{pref} is idempotent and the definition of $\tau_{P,f}$ only depends on \text{pref}(P), we have $\tau_{P,f} = \tau_{\text{pref}(P),f}$. Set $P$ being nonempty we also have $\text{pref}(P) = \{\varepsilon\} \cup \text{pref}(P)$ and, hence, $\tau_{\text{pref}(P),f} = \tau_{(\varepsilon) \cup \text{pref}(P),f}$. These observations lead to the following algorithm to compute $\tau_{P,f}$ in which variable $\tau$ is used to calculate and store $\tau_{P,f}$ (cf. [AC75], section 3, algorithm 2):

```plaintext
{\tau = \tau_{\varepsilon,f}}
for a : a \in V do \tau(\varepsilon, a) := \bot \Rightarrow;
{\tau = \tau_{\varepsilon,f}}
P_d,P_r := \emptyset,P;
{\text{invariant: } P_d \cup P_r = P \land P_d \cap P_r = \emptyset \land \tau = \tau_{(\varepsilon) \cup \text{pref}(P_d),f}}
do P_r \neq \emptyset \Rightarrow
p : p \in P_r;
\tau(\varepsilon,p) := \tau_{(\varepsilon) \cup \text{pref}(P_d),f}
do P_r \neq \emptyset \Rightarrow
v \neq \varepsilon \Rightarrow
if \tau(v,1) = 1 \Rightarrow
\tau(v,1) := \tau(\varepsilon,1)
\Rightarrow
\text{for } a : a \in V \text{ do } \tau(u(v,1), a) := \bot \Rightarrow;
\tau(u(v,1)) \neq \bot \Rightarrow \text{skip} \Rightarrow
\text{fi;}
u := u \cdot v \Rightarrow
\text{od;}
P_d,P_r := P_d \cup \{p\}, P_r \cdot \{p\}
{\tau = \tau_{P,f}}
```

Notice that the algorithm does a depth first traversal of the forward trie. Also notice that variable $P_d$ is only needed to formulate an invariant for $\tau$, so it may safely be removed from the algorithm. Furthermore, the states of the forward trie are represented by strings. In practice, one can resort to a more suitable representation, for instance a representation by natural numbers. We will not elaborate this here.
The extended forward trie corresponding to $P \tau_{P,ef} : \text{pref}(P) \times V \rightarrow (\text{pref}(P) \cup \{\bot\})$ is defined by

$$
\tau_{P,ef}(u, a) = \begin{cases} 
ua & \text{if } ua \in \text{pref}(P) \\
\varepsilon & \text{if } u = \varepsilon \land a \not\in \text{pref}(P) \\
\bot & \text{if } u \neq \varepsilon \land ua \not\in \text{pref}(P)
\end{cases}
$$

(u \in \text{pref}(P), a \in V).

It can be computed by the algorithm obtained by adding statement

```plaintext
for a : tau(\varepsilon, a) = \bot do tau(\varepsilon, a) := \varepsilon \rof
```

to the end of the algorithm computing $\tau_{P,ef}$.

Next, we focus on the computation of function $\gamma_f : \text{pref}(P) \times V \rightarrow \text{pref}(P)$, defined by

$$
\gamma_f(q, a) = (\text{MAX}_{\leq}, w : w \in \text{suff}(qa) \cap \text{pref}(P) : w) \quad (q \in \text{pref}(P), a \in V),
$$

and $f_f : \text{pref}(P) \\setminus \{\varepsilon\} \rightarrow \text{pref}(P)$, defined by

$$
f_f(q) = (\text{MAX}_{\leq}, w : w \in (\text{suff}(q) \setminus \{q\}) \cap \text{pref}(P) : w) \quad (q \in \text{pref}(P) \setminus \{\varepsilon\}).
$$

In order to arrive at an algorithm computing both $\gamma_f$ and $f_f$ we first derive (mutually) recursive definitions of $\gamma_f$ and $f_f$.

i. Let $a \in V$. We derive

\[
\gamma_f(\varepsilon, a) = (\text{MAX}_{\leq}, w : w \in \text{suff}(a) \cap \text{pref}(P) : w)
= \begin{cases} 
a & \text{if } a \in \text{pref}(P) \\
\varepsilon & \text{if } a \not\in \text{pref}(P)
\end{cases}
\]

ii. Let $u \in \text{pref}(P) \setminus \{\varepsilon\}$ and $a \in V$. We distinguish two cases.

a. Assume $ua \in \text{pref}(P)$. Then by definition of $\gamma_f$ we have $\gamma_f(u, a) = ua$.

b. Assume $ua \not\in \text{pref}(P)$. Let $u = bu_0$ where $b \in V$ and $u_0 \in V^*$. We derive

\[
\gamma_f(u, a) = (\text{MAX}_{\leq}, w : w \in \text{suff}(ua) \cap \text{pref}(P) : w)
= \begin{cases} 
ua & \text{if } ua \in \text{pref}(P) \\
\varepsilon & \text{if } u = \varepsilon \land a \not\in \text{pref}(P)
\end{cases}
\]

\[
\text{If suff is idempotent, Theorem B.5}
\]

\[
\gamma_f(u, a) = (\text{MAX}_{\leq}, w : w \in \text{suff}(\text{MAX}_{\leq}, v : v \in \text{suff}(u) \cap \text{pref}(P) : v) \cap \text{pref}(P)
\lor w = \varepsilon : w)
= \{u = bu_0, \text{suff}(u) \setminus \{u\} = \text{suff}(u_0)\}
\]

\[
\text{(MAX}_{\leq}, w : w \in \text{suff}(u) \cap \text{pref}(P) : w)
= \\text{definition of } f_f
\]

\[
(\text{MAX}_{\leq}, w : w \in \text{suff}(f_f(u)) \cap \text{pref}(P) : w)
\]

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iii. Let $a \in V$ such that $a \in \text{pref}(P)$. We derive

$$f_f(a) = \{ \text{definition of } f_f \}$$

$$(\text{MAX}_\leq, w : w \in (\text{suff}(a) \setminus \{a\}) \cap \text{pref}(P) : w)$$

$$= \{ \text{suff}(a) = \{\varepsilon, a\}, P \neq \emptyset \}$$

$\varepsilon$

iv. Let $u \in V^* \setminus \{\varepsilon\}$ and $a \in V$ such that $ua \in \text{pref}(P)$. We derive

$$f_f(ua) = \{ \text{definition of } f_f \}$$

$$(\text{MAX}_\leq, w : w \in (\text{suff}(ua) \setminus \{ua\}) \cap \text{pref}(P) : w)$$

$$= \{ \text{suff}(ua) \setminus \{ua\} = (\text{suff}(u) \setminus \{u\})a \cup \{\varepsilon\}, P \neq \emptyset \}$$

$$(\text{MAX}_\leq, w : w \in (\text{suff}(u) \setminus \{u\})a \cap \text{pref}(P) \lor w = \varepsilon : w)$$

$$= \{ \text{see derivation in ii.b.} \}$$

$$\gamma_f(f_f(u), a)$$

Summarizing, we have

$$\gamma_f(\varepsilon, a) = \begin{cases} \varepsilon & \text{if } a \not\in \text{pref}(P) \\ a & \text{if } a \in \text{pref}(P) \end{cases}$$

$$\gamma_f(u, a) = \begin{cases} ua & \text{if } ua \in \text{pref}(P) \\ \gamma_f(f_f(u), a) & \text{if } ua \not\in \text{pref}(P) \end{cases}$$

$$f_f(a) = \begin{cases} \varepsilon & \text{if } a \in \text{pref}(P), a \in \text{pref}(P) \end{cases}$$

$$f_f(ua) = \begin{cases} \gamma_f(f_f(u), a) & \text{if } u \in \text{pref}(P) \setminus \{\varepsilon\}, a \in V, ua \in \text{pref}(P) \end{cases}$$

Since $(\forall u : u \in \text{pref}(P) \setminus \{\varepsilon\} : |f_f(u)| < |u|)$ the functions $\gamma_f$ and $f_f$ can be computed by the following algorithm that is based upon the preceding recursive definitions (notice the layer wise or breadth first traversal of $\text{pref}(P)$; algorithm variables $gf$ and $ff$ are used to calculate and store $\gamma_f$ and $f_f$, respectively; cf. [AC75], a combination of algorithm 3 from section 3 and algorithm 4 from section 6):
for \( a : a \in V \) do
\[
\begin{align*}
\text{if } a \in \text{pref}(P) & \rightarrow \text{gf}(\epsilon, a) := a; fI(a) := \epsilon \\
\text{fi}
\end{align*}
\]
\[
\begin{align*}
\text{if } a \notin \text{pref}(P) & \rightarrow \text{gf}(\epsilon, a) := \epsilon \\
\text{fi}
\end{align*}
\]
\[
\begin{align*}
\text{rof} ; \\
n := 1 ;
\end{align*}
\]
\[
\text{do} \text{pref}(P) \cap V^n \neq \emptyset \rightarrow \\
\text{for } u : u \in \text{pref}(P) \cap V^n \text{ do} \\
\text{for } a : a \in V \text{ do} \\
\begin{align*}
\text{if } ua \in \text{pref}(P) & \rightarrow \text{gf}(u, a) := ua; fI(ua) := \text{gf}(fI(u), a) \\
\text{fi}
\end{align*}
\]
\[
\begin{align*}
\text{if } ua \notin \text{pref}(P) & \rightarrow \text{gf}(u, a) := \text{gf}(fI(u), a) \\
\text{fi}
\end{align*}
\]
\[
\begin{align*}
\text{rof} ; \\
n := n + 1
\end{align*}
\]
\text{od}

If the forward trie \( \tau_f \) has already been computed and represented by \( \tau_u \), then the guard \( "ua \in \text{pref}(P)" \) in the preceding algorithm can be replaced by \( "\tau_u(u, a) \neq \bot" \).

Next, we show how to compute failure function \( f_f \) without function \( \gamma_f \) using linear search. For \( u \in \text{pref}(P) \setminus \{\epsilon\}, a \in V \), and \( ua \in \text{pref}(P) \) we derive

\[
f_f(ua) = \begin{cases} 
\text{see derivation iv.} & \\
\text{(MAX}_{\leq}, w : w \in (\text{suff}(u) \setminus \{u\})a \cap \text{pref}(P) \cap w = \epsilon : w) & \\
\text{= } & \\
\text{domain split, (suff}(u) \setminus \{u\})a \cap \text{pref}(P) \subseteq \text{pref}(P)a & \\
\text{(MAX}_{\leq}, w : w \in (\text{suff}(u) \setminus \{u\})a \cap \text{pref}(P) \cap w = \epsilon : \text{max}_{\leq}, \epsilon) & \\
\text{= } & \\
\text{(MAX}_{\leq}, w' : w' \in \text{suff}(u) \setminus \{u\} \cap \text{pref}(P) \cap w/a \in \text{pref}(P) : w')\text{max}_{\leq}, \epsilon & \\
\end{cases}
\]

As in 3.3.1 this expression can be computed using a linear search

\[
\begin{align*}
\{\forall v : v \in \text{pref}(P) \land v <_s u : \text{ff}(v) = f_f(v)\} \\
u' := fI(u); \\
\text{do } \tau_f(u', \alpha) = \bot_s \rightarrow u' := \text{ff}(u') \text{ od}; \\
\text{ff}(ua) := \tau_f(u', a)
\end{align*}
\]

This leads to the following algorithm computing failure function \( f_f \) (notice the breadth first traversal of \( \text{pref}(P) \setminus \{\epsilon\} \); cf. [AC75], section 3, algorithm 3):
Finally, we consider the precomputation of function \( \text{Output} : \text{pref}(P) \longrightarrow \mathcal{P}(P) \) defined by \( \text{Output}(u) = \text{suff}(u) \cap P \). A recursive definition of \( \text{Output} \) is derived as follows:

i. By definition we have \( \text{Output}(\varepsilon) = \{\varepsilon\} \cap P \).

ii. Let \( u \in \text{pref}(P) \setminus \{\varepsilon\} \). Let \( u = bu_0 \) where \( b \in V \) and \( u_0 \in V^* \). We derive

\[
\text{Output}(u) = \{ \text{definition of Output} \} \cr \text{suff}(u) \cap P = \{ \text{suff}(u) = (\text{suff}(u) \setminus \{u\}) \cup \{u\} \} \cr = ((\text{suff}(u) \setminus \{u\}) \cap P) \cup ((\{u\} \cap P) \cr = \{ u = bu_0, \text{suff}(u) \setminus \{u\} = \text{suff}(u_0), P \subseteq \text{pref}(P) \} \cr = (\text{suff}(u_0) \cap \text{pref}(P) \cap P) \cup (\{u\} \cap P) \cr = \{ \text{suff is idempotent, Theorem B.5} \} \cr (\text{suff}((\text{MAX}_{\leq}, w : w \in \text{suff}(u_0) \cap \text{pref}(P) : w)) \cap \text{pref}(P) \cap P) \cup (\{u\} \cap P) \cr = \{ P \subseteq \text{pref}(P), u = bu_0, \text{suff}(u_0) = \text{suff}(u) \setminus \{u\} \} \cr (\text{suff}(\text{MAX}_{\leq}, w : w \in (\text{suff}(u) \setminus \{u\}) \cap \text{pref}(P) : w)) \cap P \cup (\{u\} \cap P) \cr = \{ \text{definition of } f_f, u \in \text{pref}(P) \setminus \{\varepsilon\}, \text{definition of Output} \} \cr \text{Output}(f_f(u)) \cup (\{u\} \cap P) \]

By preceding the algorithm on page 34 with assignment “\( \text{out}(\varepsilon) := \{\varepsilon\} \cap P \)”, and by adding assignment “\( \text{out}(a) := \text{out}(\varepsilon) \cup (\{a\} \cap P) \)” to the end of the first alternative of its first \( \text{if} \)-\( \text{fi} \) statement and assignment “\( \text{out}(ua) := \text{out}(ff(ua)) \cup (\{ua\} \cap P) \)” to the end of the first alternative of its second \( \text{if} \)-\( \text{fi} \) statement one obtains an algorithm computing function \( \text{Output} \) as well.
7 Precomputation for the Commentz-Walter algorithms

In this section we will be using the reverse trie corresponding to \( \mathcal{P} \in \mathcal{V} \),

\[
\text{suff}(\mathcal{P}) \xrightarrow{V} \text{suff}(\mathcal{P}) \cup \{\perp\}
\]

defined by

\[
\tau_r(u, a) = \begin{cases} 
au & \text{if } au \in \text{suff}(\mathcal{P}) \\
\perp & \text{if } au \notin \text{suff}(\mathcal{P})
\end{cases} \quad (u \in \text{suff}(\mathcal{P}), a \in V),
\]

its optimal transition function \( \gamma_r : \text{suff}(\mathcal{P}) \times V \rightarrow \text{suff}(\mathcal{P}) \) defined by

\[
\gamma_r(q, a) = (\text{MAX}_{w} w : w \in \text{pref}(qa) \cap \text{suff}(\mathcal{P}) : w) \quad (q \in \text{suff}(\mathcal{P}), a \in V),
\]

and its failure function \( f_r : \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \rightarrow \text{suff}(\mathcal{P}) \) defined by

\[
f_r(q) = (\text{MAX}_{w} w : w \in (\text{pref}(q) \setminus \{q\}) \cap \text{suff}(\mathcal{P}) : w) \quad (q \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\}).
\]

These functions are the mirror image of the functions corresponding to the forward trie and can
be computed by algorithms that are the mirror images of the algorithms in the previous section.

7.1 Computation of \( d_1 \) and \( d_2 \)

Next, we consider the computation of function \( d_1 : \text{suff}(\mathcal{P}) \rightarrow \mathbb{N} \) defined by

\[
d_1(x) = (\text{MIN } n : n \geq 1 \land V^*xV^n \cap \mathcal{P} \neq \emptyset : n) \quad (x \in \text{suff}(\mathcal{P})).
\]

Let \( x \in \text{suff}(\mathcal{P}) \). We derive

\[
(\text{MIN } n : n \geq 1 \land V^*xV^n \cap \mathcal{P} \neq \emptyset : n)
\]

\[
= \{ \text{property B.2} \}
\]

\[
(\text{MIN } n : n \geq 1 \land (xV^n) \cap \text{suff}(\mathcal{P}) \neq \emptyset : n)
\]

\[
= \{ \text{change of bound variable: } n = |x| \}
\]

\[
(\text{MIN } n s : s \in V^+ \land xs \in \text{suff}(\mathcal{P}) : |x|)
\]

\[
= \{ \text{change of bound variable: } t = xs \}
\]

\[
(\text{MIN } t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \land x <_p t : |t| - |x|)
\]

\[
= \{ \text{domain split} \}
\]

\[
(\text{MIN } t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \land x = f_r(t) : |t| - |x|)
\]

\[
\min(\text{MIN } t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \land x <_p f_r(t) : |t| - |x|)
\]

\[
= \{ \text{see following note} \}
\]

\[
(\text{MIN } t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \land x = f_r(t) : |t| - |x|)
\]

Note

In order to show that the second operand of the min-operator can be omitted we distinguish two cases:

i. Assume \( \exists t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} : x <_p f_r(t) \). The second operand now equals the unity of the min-operator.

ii. Assume \( \exists t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} : x <_p f_r(t) \). We derive

\[
(\text{MIN } t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \land x <_p f_r(t) : |t| - |x|)
\]

\[
> \{ (\exists t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} : x <_p f_r(t), t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \Rightarrow |f_r(t)| < |t| \}
\]

\[
(\text{MIN } t : t \in \text{suff}(\mathcal{P}) \setminus \{\epsilon\} \land x <_p f_r(t) : |f_r(t)| - |x|)
\]
\[
\{ f_r(t) \in \text{suff}(P), x < p, f_r(t) \Rightarrow f_r(t) \neq \epsilon \}
\]
\[
(\text{MIN } t : t \in \text{suff}(P) \setminus \{\epsilon\} \land x = f_r(t) : |f_r(t) - |x|) \geq \{ \text{omitting first predicate in domain, change of bound variable: } t' = f_r(t) \}
\]
\[
(\text{MIN } t' : t' \in \text{suff}(P) \setminus \{\epsilon\} \land x < p, t' : |t'| - |x|)
\]
\[
= \{ \text{see first part of the previous derivation} \}
\]
\[
d_1(x)
\]

Since \( a = b \min c \land c > a \Rightarrow a = b \) the second operand of the \text{min}-operator can be omitted in this case as well.

(End of Note)

Summarizing, we have
\[
d_1(x) = (\text{MIN } t : t \in \text{suff}(P) \setminus \{\epsilon\} \land x = f_r(t) : |t| - |x|) \quad (x \in \text{suff}(P)).
\]

(cf. [Com79a], sections I and III, and [Com79b], sections II.1 and III, functions \text{shift1}, \text{set1}, and \text{set1}'). Function \( d_1 \) can be computed during the computation of \( \gamma_r \) and \( f_r \) without having to compute the (generalized) inverse of \( f_r \) explicitly.

Before giving an algorithm demonstrating this we will first deal with the computation of function \( d_2 : \text{suff}(P) \rightarrow \mathbb{N} \) defined by
\[
d_2(x) = (\text{MIN } n : n \geq 1 \land V^* P \cap x V^n \neq \emptyset : n) \quad (x \in \text{suff}(P)).
\]

We will show that \( d_2 \) can also be expressed in terms of \( f_r \). We distinguish two cases:

i. Let \( x = \epsilon \). We derive
\[
(\text{MIN } n : n \geq 1 \land V^* P \cap x V^n \neq \emptyset : n) = \{ \epsilon = \epsilon, \text{property B.2} \}
\]
\[
= \{ \epsilon \notin P \land \text{suff}(V^n) \neq \emptyset : n \}
\]
\[
= \{ \epsilon \notin P \}
\]
\[
= \{ \text{MIN } p : p \in P : |p| \}
\]

ii. Let \( x \in \text{suff}(P) \setminus \{\epsilon\} \). We derive
\[
(\text{MIN } n : n \geq 1 \land V^* P \cap x V^n \neq \emptyset : n) = \{ \text{property B.2} \}
\]
\[
= \{ \epsilon \notin P \land \text{suff}(x V^n) \neq \emptyset : n \}
\]
\[
= \{ \epsilon \notin P \land \text{suff}(x V^n) \neq \emptyset : n \}
\]
\[
= \{ \text{change of bound variable: } n = |s|, \text{definition of } d_2 \}
\]
\[
= \{ \text{MIN } s : s \in V^+ \land xs \in P : |s| \} \min d_2(x|1)
\]
\[
= \{ \text{change of bound variable: } p = xs \}
\]
\[
= \{ \text{MIN } p : p \in P \land x < p : |p| - |x| \} \min d_2(x|1)
\]
\[
= \{ \epsilon \notin P, \text{hence } p \in \text{suff}(P) \setminus \{\epsilon\}, x \in \text{suff}(P), \text{definition B.9, corollary B.12} \}
\]
\[
= \{ \text{MIN } p : p \in P \land (\exists i : 0 < i \leq \nu(p) : x = f_r(p) : |p| - |x|) \min d_2(x|1) \}
\]

The result in case i. can be made to look more like the result in case ii.:
\[(\text{MIN } p : p \in P : |p|) = \{ f^\nu(p) = \epsilon, \epsilon \not\in P, \text{ hence } \nu(p) > 0, |\epsilon| = 0 \}\]

\[(\text{MIN } p : p \in P \land (\exists i : 0 < i \leq \nu(p) : \epsilon = f_i^\nu(p)) : |p| - |\epsilon|)\]

Summarizing, we have, for \(x \in \text{suff}(P) \setminus \{\epsilon\},\)

\[
d_2(\epsilon) = (\text{MIN } p : p \in P \land (\exists i : 0 < i \leq \nu(p) : \epsilon = f_i^\nu(p)) : |p| - |\epsilon|)\]

\[
d_2(x) = (\text{MIN } p : p \in P \land (\exists i : 0 < i \leq \nu(p) : x = f_i^\nu(p)) : |p| - |x|) \min d_2(x, 1).
\]

(cf. [Com79a], sections I and III, and [Com79b], sections II.1 and III, functions shift2, set2, and set2', the restriction from set2 to set2' for the computation of shift2 is not explained there and seems, in view of our results for \(d_2\), to be incorrect).

The expressions derived for \(d_1\) and \(d_2\) lead to the following algorithm computing \(\gamma_r, f_r, d_1,\) and \(d_2\) in program variables \(gr, fr, dl,\) and \(d2,\) respectively:

```plaintext
for u : u \in \text{suff}(P) do d1(u), d2(u) := +\infty, +\infty rof;
for a : a \in V do
    if a \in \text{suff}(P) ->
        gr(\epsilon, a) := a;
        fr(a) := \epsilon;
        d1(\epsilon) := d1(\epsilon) \min 1;
        if a \in P ->
            d2(\epsilon) := d2(\epsilon) \min 1
        | a \not\in P ->
            skip
    fi
    if a \not\in \text{suff}(P) ->
        gr(\epsilon, a) := \epsilon
    fi
fi
rof;
for u : u \in \text{suff}(P) \cap V^n do
    for a : a \in V do
        if au \in \text{suff}(P) ->
            gr(au, a) := au;
            fr(au) := gr(fr(au), a);
            d1(fr(au)) := d1(fr(au)) \min |au| - |fr(au)|;
            if au \in P ->
                v := fr(au); i := 1;
                { invariant: v = fr(au) \land 0 < i \leq \nu(au) }
                do v \not\in \epsilon ->
                    d2(v) := d2(v) \min |au| - |v|;
                    v := fr(v); i := i + 1
                od;
                d2(\epsilon) := d2(\epsilon) \min |au|
            | au \not\in P ->
                skip
            fi
        fi
    fi
for u : u \in \text{suff}(P) \setminus V^n do
    for a : a \in V do
        if au \in \text{suff}(P) ->
            gr(au, a) := gr(fr(au), a)
        fi
    fi
```

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\[\text{rof}\]
\[\text{rof} \]
\[n := n + 1\]
\[
\text{od};
\begin{align*}
\{ (\forall u, a : u \in \text{suff}(P) \wedge a \in V : g(u, a) = \gamma_r(u, a) ) \\
\wedge (\forall u : u \in \text{suff}(P) \setminus \{e\} : f_r(u) = f_r(u)) \\
\wedge (\forall u : u \in \text{suff}(P) : d1(u) = d1(u)) \\
\wedge (\forall u : d2(u) = (\text{MIN } p : p \in P \wedge (\exists i : 0 < i \leq \nu(p) : u = f^*_r(p)) : |p| - |u|) )
\}
\]
\[n := 1;\]
\[
\text{for } u : u \in \text{suff}(P) \cap V^n \neq \emptyset \rightarrow \text{ do } d2(u) := d2(u) \min d2(u|1) \text{ rof;}
\]
\[n := n + 1\]
\[
\text{od}
\]

7.2 Computation of \(d_{no}\)

Let \(x \in \text{suff}(P)\) and \(a \in V\). We derive

\[d_{no}(x, a) = \{ \text{definition of } d_{no} \}
\]
\[\text{(MIN } n : n \geq 1 \wedge ((V^*aV^n+|x|) \cap P \neq \emptyset \wedge V^*xV^n \cap P \neq \emptyset ) \vee V^*P \cap (xV^n) \neq \emptyset : n) \}
\]
\[\text{def. of } d_2\]
\[\text{(MIN } n : n \geq 1 \wedge V^*aV^n+|x| \cap P \neq \emptyset \wedge V^*xV^n \cap P \neq \emptyset : n) \min d_2(x)\]
\[\text{def. of } d_2\]
\[\text{MIN } s : s \in V^+aV^{|x|} \cap \text{suff}(P) \neq \emptyset \wedge xs \in \text{suff}(P) : |s| \min d_2(x)\]
\[\text{change of bound variable: } |s| = n\]
\[\text{MIN } t : t \in \text{suff}(P), t \neq x, \text{ corollary B.12, def. of } \text{oxx}_r (\text{after derivation}) \}
\[\text{(MIN } t : t \in \text{suff}(P) \setminus \{e\} : aV^{|x|} \cap \text{suff}(P) \neq \emptyset \wedge x < p : t |t| - |x| \min d_2(x)\]
\[\text{change of bound variable: } t = xs\]
\[\text{MIN } \text{oxx}_r(a) = \{ n | n \in N \wedge aV^n \cap \text{suff}(P) \neq \emptyset \} \quad (a \in V).\]

Observe that \(\text{oxx}_r\) can easily be computed beforehand, e.g., during the computation of \(\tau_r\). Therefore, the computation of the first operand of the \text{min}-operator is similar to the first part of the computation of \(d_2\). Finally, function \(d_{no}\) can be computed during the second and final part of the computation of \(d_2\). We do not give an algorithm here since with these observations the reader may easily adapt the preceding algorithm to also compute \(d_{no}\).

7.3 Computation of \(d_3\)

Function \(d_3 : N \times V \rightarrow N\) can be expressed in terms of function \(\overline{d}_3 : V \rightarrow N\), defined by

\[\overline{d}_3(a) = (\text{MIN } n : n \geq 1 \wedge V^*aV^n \cap P \neq \emptyset : n) \quad (a \in V),\]
as follows
\[
d_3(z, a) = \begin{cases} 
+\infty & \text{if } \overline{d}_3(a) = +\infty \\
\overline{d}_3(a) - z & \text{if } \overline{d}_3(a) \neq +\infty
\end{cases}
(z \in \mathbb{N}, a \in V).
\]

Let \(a \in V\). We derive
\[
d_3(a) = [\text{definition of } \overline{d}_3]
\]
\[= (\text{MIN } n : n \geq 1 \land V^* a V^n \cap P \neq \emptyset : n)\]
\[= [\text{property B.2}]
\]
\[= (\text{MIN } n : n \geq 1 \land a V^n \cap \text{suff}(P) \neq \emptyset : n)\]
\[= [\text{definition of } \text{occ}_r]\]
\[= (\text{MIN } n : n \geq 1 \land n \in \text{occ}_r(a) : n).\]

This derivation shows that \(\overline{d}_3\) can be computed at the same time as \(\text{occ}_r\).

### 7.4 Computation of \(d_{bm}\) and \(\text{char}\)

Let \(x \in \text{suff}(P)\). We derive
\[
d_{bm}(x) = [\text{definition of } d_{bm}]\]
\[= (\text{MIN } n : n \geq 1 \land V^* x V^n \cap V^* P \neq \emptyset : n)\]
\[= [V^* A \cap V^* B \neq \emptyset \equiv V^* A \cap B \neq \emptyset \lor A \cap V^* B \neq \emptyset, \text{domain split}]\]
\[= (\text{MIN } n : n \geq 1 \land V^* x V^n \cap P \neq \emptyset : n) \min (\text{MIN } n : n \geq 1 \land x V^n \cap V^* P \neq \emptyset : n)\]
\[= [\text{definition of } d_1 \text{ and } d_2]\]
\[d_1(x) \min d_2(x).\]

Hence, we have
\[
d_{bm}(x) = d_1(x) \min d_2(x) \quad (x \in \text{suff}(P)),
\]
showing that \(d_{bm}\) can be computed from \(d_1\) and \(d_2\).

Let \(a \in V\). We derive
\[
\text{char}(a) = [\text{definition of } \text{char}]\]
\[= (\text{MIN } n : n \geq 1 \land V^* a V^n \cap V^* P \neq \emptyset : n)\]
\[= [V^* A \cap V^* B \neq \emptyset \equiv V^* A \cap B \neq \emptyset \lor A \cap V^* B \neq \emptyset, \text{domain split}]\]
\[= (\text{MIN } n : n \geq 1 \land V^* a V^n \cap P \neq \emptyset : n) \min (\text{MIN } n : n \geq 1 \land a V^n \cap V^+ P \neq \emptyset : n)\]
\[= [\text{definition of } \overline{d}_3, P \neq \emptyset, e \notin P, a V^n \cap V^+ P \neq \emptyset \equiv n \geq (\text{MIN } p : p \in P : |p|)\]
\[\overline{d}_3(a) \min (\text{MIN } p : p \in P : |p|)\]

Defining
\[
m_p = (\text{MIN } p : p \in P : |p|)
\]
we have
\[
\text{char}(a) = \overline{d}_3(a) \min m_p \quad (a \in V),
\]
showing that \(\text{char}\) can be computed from \(\overline{d}_3\).

Having derived expressions for \(d_{bm}\) and \(\text{char}\) in terms of \(d_1, d_2,\) and \(\overline{d}_3\) we are able to compare the amount of shift for the normal Commentz-Walter algorithm, \(k_{\text{norm}}\), to the amount of shift for the Boyer-Moore algorithm, \(k_{bm}\). First, we derive
Next, we derive

\[ k_{bm} = \]

\[ ((\text{char}(l|l) - |v|) \max d_{bm}(v)) \min |r| \]

\[ = \]

\[ \{ \text{preceding derivation, } d_{bm} \text{ expressed in } d_3 \text{ and } d_2 \} \]

\[ ((d_3(v), l|l) \min(m_P - |v|)) \max(d_1(v) \min d_2(v)) \min |r| \]

\[ = \]

\[ \{ \text{distributivity} \} \]

\[ ((d_3(v), l|l) \min(m_P - |v|)) \max d_1(v) \]

\[ \min((d_2(|v|, l|l) \min(m_P - |v|)) \max d_2(v)) \min |r| \]

\[ = \]

\[ \{ (\forall n: 1 \leq n < m_P - |v|: V^*P \cap vV^n = \emptyset), \text{definition of } d_2, \text{hence } m_P - |v| \leq d_2(v) \} \]

\[ ((d_3(v), l|1) \min(m_P - |v|)) \max d_1(v) \min d_2(v) \min |r| \]

\[ \leq \]

\[ \{ \text{calculus} \} \]

\[ (d_3(|v|, l|1) \max d_1(v)) \min d_2(v) \min |r| \]

\[ = \]

\[ \{ \text{definition of } k_{norm} \} \]

\[ k_{norm}, \]

showing that the amount of shift in the normal Commentz-Walter algorithm is at least the amount of shift in the Boyer-Moore algorithm.

### 7.5 Precomputation of \( s_1 \), \( \text{char}_1 \), and \( \text{char}_2 \)

Here we discuss the precomputation of functions \( s_1 \), \( \text{char}_1 \), and \( \text{char}_2 \) for the variants of the one keyword Boyer-Moore algorithm obtained by instantiating detail (MO) by (FWD) and (REV), respectively.

#### 7.5.1 Forward matching

In the forward matching scheme (algorithm detail (FWD)) we have \( m_0(i) = i \). In this case \( P_3 \) can be manipulated further:

\[ P_3(v, i, k) \]

\[ \equiv \]

\[ \{ \text{definition of } P_3 \text{ and } m_0 \} \]

\[ (\forall j: 1 \leq j < i \land 1 + k \leq j : p_j = p_{j-k}) \]

\[ \land (i \leq |p| \land 1 + k \leq i \Rightarrow v_i = p_i-k \land p_i \neq p_i-k) \]

\[ \equiv \]

\[ \{ \text{simplifying ranges, } 1 \leq i \leq |p| + 1 \} \]

\[ (\forall j: 1 + k \leq j < i : p_j = p_{j-k}) \]

\[ \land (1 + k \leq i \leq |p| \Rightarrow v_i = p_i-k \land p_i \neq p_i-k) \]

We continue with only the first conjunct, assuming \( 1 + k \leq i \):

---

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\((\forall j : 1 + k \leq j < i : p_j = p_{j-k})\)
\[\equiv \quad \{\text{rewrite using} \ j \text{ and} \ k\}\]
\[(p|(i-1)|k = p|(i-1-k))\]
\[\equiv \quad \{\text{set calculus}\}\]
\[\{(p|(i-1)|k) \cap \{p|(i-1-k)\} \neq \emptyset\]\n\[\equiv \quad \{(x|j) \cap Y \neq \emptyset \equiv \{x\} \cap V^j Y \neq \emptyset \ (0 \leq j \leq |x|), \ k \leq i - 1\}\]
\[\{p|(i-1)\} \cap V^k(p|(i-1-k)) \neq \emptyset\]
\[\equiv \quad \{X \cap Y(x|j) \neq \emptyset \equiv XV^{|x|-j} \cap Y \neq \emptyset \ (0 \leq j \leq |x|), \ k \leq i - 1, \ i \leq |p| + 1\}\]
\[\{p|(i-1)\}V^{|p|+k-i+1} \cap V^kp \neq \emptyset\]
\[\equiv \quad \{(Vx, y : x \in X \land y \in Y : |x| = |y|) \Rightarrow (X \cap Y \neq \emptyset \equiv V^*X \cap V^*Y \neq \emptyset)\}\]
\[V^*(p|(i-1))V^{|p|+k-i+1} \cap V^*p \neq \emptyset\]

Notice that this predicate is similar to the predicate in the definition of function \(d_{\text{perm}}\) (see subsection 4.4). Precomputation of functions \(s_1\), \(char_1\), and \(char_2\) is similar to the precomputation for the Boyer-Moore variant derived from the Commentz-Walter algorithm (Part II, subsection 7.4). For this reason we do not elaborate the precomputation any further.

### 7.5.2 Backward matching

With backward matching (algorithm detail \((REV)\)), \(p\) is compared to \(v\) from right to left, i.e. we have \(mo(i) = |p| - i + 1\), the reverse permutation of the integers from 1 to \(|p|\). Predicate \(P_3\) can be manipulated further. We have

\[P_3(v, i, k)\]
\[\equiv \quad \{\text{definition of} \ P_3 \text{ and} \ mo\}\]
\[(\forall j : 1 \leq j < i \land j \leq |p| - k : p_{|p|-j-k+1} = p_{|p|-j-k+1})\]
\[\land (i \leq |p| - k \Rightarrow v_{|p|-i+1} = p_{|p|-i-k+1} \land p_{|p|-i-k+1} \neq p_{|p|-i-k+1})\]

We concentrate on the first conjunct and distinguish three cases. If \(i \leq |p| - k\) the first conjunct becomes

\[(\forall j : 1 \leq j < i : p_{|p|-j-1} = p_{|p|-j-k+1})\]
\[\equiv \quad \{\text{rewrite using} \ j \text{ and} \ k\}\]
\[(p|(i-1)) \equiv (p|(k+i-1)|k)\]
\[\equiv \quad \{\text{set calculus}\}\]
\[\{p|(i-1)\} \cap \{p|(k+i-1)|k\} \neq \emptyset\]
\[\equiv \quad \{X \cap \{y|j\} \neq \emptyset \equiv XV^j \cap \{y\} \neq \emptyset \ (0 \leq j \leq |y|), \ i + k \leq |p|\}\]
\[\{p|(i-1)\}V^k \cap \{p|(k+i-1)|k\} \neq \emptyset\]
\[\equiv \quad \{X \cap \{y|j\} \neq \emptyset \equiv V^{|y|-j}X \cap \{y\} \neq \emptyset \ (0 \leq j \leq |y|), \ k + i - 1 < |p|\}\]
\[V^{|p|+k-i+1}(p|(i-1))V^k \cap \{p\} \neq \emptyset\]
\[\equiv \quad \{(Vx, y : x \in X \land y \in Y : |x| = |y|) \Rightarrow (X \cap Y \neq \emptyset \equiv V^*X \cap V^*Y \neq \emptyset)\}\]
\[V^*(p|(i-1))V^{|p|+k-i+1} \cap V^*p \neq \emptyset\]

If \(i > |p| - k\) and \(k \leq |p|\) the first conjunct becomes

\[(\forall j : 1 \leq j \leq |p| - k : p_{|p|-j-1} = p_{|p|-j-k+1})\]
\[\equiv \quad \{\text{rewrite using} \ j \text{ and} \ k\}\]
\[p|k = p|k\]
\[\equiv \quad \{\text{set calculus}\}\]
\[\{p|k\} \cap \{p|k\} \neq \emptyset\]
\[
\begin{align*}
\equiv & \quad \{ \{ x | j \} \cap Y \neq \emptyset \equiv \{ x \} \cap V^j Y \neq \emptyset \ (0 \leq j \leq |x|), \ k \leq |p| \} \\
\{ p \} \cap V^k(p|k) \neq \emptyset & \\
\equiv & \quad \{ X \cap Y | (y | j) \neq \emptyset \equiv X V^j \cap Y y \neq \emptyset \ (0 \leq j \leq |y|), \ k \leq |p| \} \\
\equiv & \quad \{ (\forall x, y : x \in X \land y \in Y : |x| = |y|) \Rightarrow (X \cap Y \neq \emptyset \equiv V^*X \cap V^*Y \neq \emptyset) \} \\
p^V \cap V^k p \neq \emptyset & \\
\equiv & \quad \{ (\forall x, y : x \in X \land y \in Y : |x| = |y|) \Rightarrow (X \cap Y \neq \emptyset \equiv V^*X \cap V^*Y \neq \emptyset) \}
\end{align*}
\]

If \( i > |p| - k \) and \( k > |p| \) the first conjunct holds by definition. Notice that in this case \( V^*p V^k \cap V^*p \neq \emptyset \) holds as well, so the last two cases can combined. From these derivations and the definition of \( d_{bm} \) (see subsection 4.4, \( P = \{ p \} \)) it follows that \( s_1(i) = d_{bm}(p|i-1) \) for \( i \geq 1 \). Notice that \( p|i-1) \in \text{suff}(P) \). Precomputation of function \( s_1 \) is therefore equal to the precomputation of \( d_{bm} \) (see Part II, subsection 7.4).

In a similar way one can derive

\[ char_1(v, i) = (\text{MIN } k : i \leq k \land V^*p|_{|i|-1} V^k \cap V^*p \neq \emptyset : k) - (i - 1) \]

in which the quantified expression can approximated from below by \( char(p|_{|i|-1}) \) (see subsection 4.4, \( P = \{ p \} \)) by enlarging the range to \( 1 \leq k \). Precomputation of \( char_1 \) is similar to the precomputation of \( char \) (see Part II, subsection 7.4).

The expression for \( char_2(i) \) becomes

\[ (\text{MIN } k : i \leq k \leq |p| - 1 \land V^*p|_{|i|-1} V^k \cap V^*p = \emptyset : k - i + 1) \min(|p| - i + 1) \]

Equivalence

\[ V^*p|_{|i|-1} V^k \cap V^*p = \emptyset \equiv V^*(V \setminus \{ p|_{|i|-1} \}) V^k \cap V^*p \neq \emptyset \]

indicates that the precomputation of \( char_2 \) is analogous to the precomputation of \( char_1 \) and \( char \).
Part III
Conclusions

The taxonomy presented in Parts I and II has achieved the goals set out in the introduction. The highlights of this taxonomy fall into two categories: general results of the derivation method and specific results of the taxonomy. The general results can be summarized as:

- The method of refinement used in each of the derivations presented the algorithms in an abstract, easily digested format. This presentation allows a correctness proof of an algorithm to be developed simultaneously with the algorithm itself.

- The presentation method proves to be more than just a method of deriving algorithms: the derivations themselves serve in the classification (in the taxonomy) of the algorithms. This is accomplished by dividing the derivation at points which involve the introduction of either problem or algorithm details. A sequence of such details serves to identify an algorithm. By prefix-factoring these sequences, common parts of two algorithm derivations can be presented simultaneously.

- The taxonomy of all algorithms considered can be depicted as a graph (in our particular case a tree); the root represents the original solution $O := (U, v, r : lor = S : (l) \times (v) \cap P) \times (r)$, edges represent the addition of a detail, and the internal vertices and leaves represent derived algorithms. This graph is shown in Figure 1. The utility of this graph is that it can be used as an "alternative table of contents" to the taxonomy. Being interested in only a subset of the algorithms, for example the Aho-Corasick (AC) algorithms, does not necessitate reading all of the derivations; only the root-leaf paths that lead to the AC algorithms need to be read for a complete view of these algorithms.

- The presentation was also more than just a taxonomy. Instead of using completed derivations of known algorithms, which are possibly in different styles of derivation, all of the algorithms were derived in a common framework. This made it easy to see what the algorithms have in common (or where they differ) for the purposes of classifying them.

- The pattern matching overview presented in [Aho90] is an excellent introduction to many of the algorithms presented in this paper. Unfortunately, it does not present all variants of the algorithms, or present them in a fashion that allows one to contrast the algorithms with one another. Our taxonomy accomplished precisely this goal, of presenting algorithms in one framework for comparison. In deriving the algorithms for this taxonomy every attempt was made to thoroughly explore all of the possible variants. Our taxonomy should be a thorough introduction to all variants of the four principal pattern matching algorithms presented in [Aho90].

Results concerning particular algorithms can be summarized as follows:

- As stated in [AC75], the AC algorithm is intended to be a generalization of the original Knuth-Morris-Pratt (KMP) algorithm — making use of automata theory. The classical derivations of the two (using automata and indices, respectively) do not serve to highlight their similarities, or differences.

When derived in the same framework, it becomes apparent that the AC algorithm cannot be specialized to arrive at KMP; this can be seen from the derivation of the AC algorithm subtree of the taxonomy tree. The linear search (introduced in subsection 3.3) used in the failure function AC algorithm (algorithm 3.4) is quite different from the linear search used in the abstract KMP algorithm (algorithm 3.5). Indices could have been introduced in algorithm 3.4, although this does not yield the classically presented KMP algorithm. The AC-KMP relationship is in fact that they have a common ancestor algorithm ($P_+, E, AC, LS$).
• The abstract intermediate KMP algorithm (algorithm 3.5) is in fact a new algorithm, albeit a variant of the AC algorithm. The running time of this new algorithm does not appear to be any better than algorithm 3.4. The transformation (by adding indices) of algorithm 3.5 into the classically presented KMP algorithm (algorithm 3.6) was demonstrated to be straightforward.

• The original Aho-Corasick article [AC75] presented the “optimal” version of the algorithm after the failure function version of the algorithm. The optimal algorithm was explained as using a transition function \( \gamma_f \) which is a composition of the extended forward trie \( \tau_{ef} \) and failure function \( f_f \). While this is indeed the case, our derivation proceeded much more smoothly by deriving an algorithm which is a common ancestor of both the optimal and the failure function algorithms.

• “Predicate weakening” (of sections 4 and 5) was instrumental in deriving various algorithms (and their correctness proofs) from the Commentz-Walter (CW) algorithm, in particular the Boyer-Moore (BM) algorithm. The CW algorithm has not emerged as a popular string pattern matching algorithm partly due to the difficulty in understanding it. The derivation presented in Part I arrives at the CW algorithm through a series of small transformations, starting with a naive (quadratic running time) algorithm. This derivation makes the CW algorithm considerably easier to understand. Predicate weakening was also heavily used in deriving the “match-order” variant of the BM algorithm.

• Commentz-Walter’s intention was to combine the BM algorithm with automata theory, to produce an algorithm dealing with multiple keywords. The relationship between the two algorithms has previously remained obscured by the styles of presentation of the two algorithms (indices in BM, and automata in CW). As seen from section 4 the BM algorithm can indeed be arrived at in the same framework (as the CW algorithm) as a special case. The publication of the Hume-Sunday taxonomy [HS91] motivated us to also derive the BM algorithm in an entirely different manner — making use of the concept of “match-orders”.

• In both papers by Commentz-Walter describing her algorithm (in particular [Com79a]), the differences between methods of determining a safe shift amount were not made explicit. Indeed, that some of these shift functions were distinct was not mentioned in all cases. Our derivation of the CW algorithm clearly defines the differences between the shift functions. The (NEAR-OPT) shift function was only mentioned in passing in the original paper; this derivation provides a definition of this function; Part II provides the only full derivation of a precomputation algorithm for this function.

• In the BM algorithm the functions contributing to a shift have been presented in several separate papers since the introduction of the original algorithm. Until the publication of the taxonomy by [HS91] it was difficult to examine the contribution of each shift function. Both section 5 and [HS91] present a shift as consisting of components that can be readily replaced by an equivalent component, for example: the “skip” loops, or the “match-orders”. [HS91] emphasized effects on running-time of each component. Our taxonomy has emphasized the derivation of each of these components from a common specification.

• The precomputation of the BM shift functions has been troublesome; many solutions were published, corrected, and re-published (for a good bibliography of these see [Aho90]). The precomputation presented in Part II provides an understandable derivation of a correct precomputation algorithm.
A Calculating the value of a quantification

The problem is, given an associative, commutative operator $\oplus$ on set $U$ with unit $e_0$, a set $W$, a range predicate $RANGE : W \rightarrow \mathbb{B}$, and a function $f : W \rightarrow U$ on $W$, calculate:

$$w = (\forall x \in W : RANGE(x) : f(x))$$

We now present three solutions.

A.1 A nondeterministic solution

This can be done with the following nondeterministic repetition:

```plaintext
RW := \{x \mid x \in W \land RANGE(x)\}; w := e_0;
for x : x \in RW do w := w \oplus f(x) endfor
\{w = (\forall x \in W : RANGE(x) : f(x))\}
```

A.2 A deterministic solution in the ascending direction

Given the set $RW = \{x \mid x \in W \land RANGE(x)\}$ and a linear order $\leq$ on $RW$ we can define a function $next : RW \rightarrow (RW \cup \{\top\})$ as:

$$next(v) = (\min \{x \in RW : v < x : x\})$$

Function $next$ is extended to map the maximum element of $RW$ to fictitious element $\top$ (to make $next$ total). Assume $RW \neq \emptyset$.

This allows us to implement a deterministic algorithm which processes $RW$ in $\leq$-ascending order:

```plaintext
v := (\min \{x \in RW : x\}); w := f(v);
\{invariant: w \oplus (\forall x \in RW : v < x : f(x)) = (\forall x \in W : RANGE(x) : f(x)) \}
do next(v) \neq \top 
    v := next(v);
    w := w \oplus f(v)
od 
\{w = (\forall x \in W : RANGE(x) : f(x))\}
```

A.3 A deterministic solution in the descending direction

Given the set $RW$ defined above in Appendix A.2, we define a function $prev : RW \rightarrow (RW \cup \{\bot\})$ as:

$$prev(v) = (\max \{x \in RW : x < v : x\})$$

extended to map the minimum element in $RW$ to $\bot$. We can now implement a deterministic algorithm which processes $RW$ in $\leq$-descending order. Assume $RW \neq \emptyset$. The following algorithm is symmetrical to that presented above in Appendix A.2, with the exception that the repetition is phase shifted, leaving an additional assignment after the repetition:
\(v := (\text{MAX}_{x \in RW} x); w := e_{\emptyset};\)  
\{\text{invariant: } w \oplus (\exists x \in RW : x \leq v : f(x)) = (\exists x \in W : \text{RANGE}(x) : f(x))\}  
\text{do } \text{preu}(v) \neq \bot \rightarrow  
\quad w := w \oplus f(v);  
\quad v := \text{preu}(v)  
\text{od};  
\quad w := w \oplus f(v)  
\{w = (\exists x \in W : \text{RANGE}(x) : f(x))\}

\section*{A.4 Nested quantifications}

Nested quantifications can similarly be dealt with using nested repetitions. When two operators of nested quantifications are in fact the same, the accumulation variable (in the above programs \(w\)) of the two corresponding nested repetitions can be identified. This is useful in our case, where most of the quantifications will consist of two nested union quantifications.

For example, given the requirement to compute:
\[w = (\exists x \in W : \text{RANGE}(x) : (\exists y \in W' : \text{RANGE}'(x,y) : f(y)))\]

we can make the following first nondeterministic solution\(^\text{13}\):

\[
\begin{align*}
\text{RW} & := \{x | x \in W \land \text{RANGE}(x)\}; w := e_{\emptyset}; \\
\text{for } x : x \in \text{RW} \text{ do } & (\text{RW}' := \{y | y \in W' \land \text{RANGE}'(x,y)\}; w' := e_{\emptyset};  \\
\text{for } y : y \in \text{RW}' \text{ do } & w' := w' \oplus f(y) \text{ rof};  \\
\{w' = (\exists y \in W' : \text{RANGE}'(x,y) : f(y))\}  \\
\text{rof } & \{w = (\exists x \in W : \text{RANGE}(x) : (\exists y \in W' : \text{RANGE}'(x,y) : f(y)))\}
\end{align*}
\]

The program variable \(w'\) in the inner repetition is not needed, and \(w\) can instead be updated directly. The (slightly) shortened version is now:

\[
\begin{align*}
\text{RW} & := \{x | x \in W \land \text{RANGE}(x)\}; w := e_{\emptyset}; \\
\text{for } x : x \in \text{RW} \text{ do } & (\text{RW}' := \{y | y \in W' \land \text{RANGE}'(x,y)\};  \\
\text{for } y : y \in \text{RW}' \text{ do } & w := w \oplus f(y) \text{ rof}  \\
\{w = (\exists x \in W : \text{RANGE}(x) : (\exists y \in W' : \text{RANGE}'(x,y) : f(y)))\}
\end{align*}
\]

\section*{B Definitions and properties}

This section provides a series of definitions and properties which are used throughout this paper.

For any language \(L\), we take \(L^R\) to denote the reversal of the language. For a string \(w \in V^*\), we take \(w^R\) to denote the reversal of \(w\).

\textbf{Definition B.1} Let \(V\) be an alphabet. Define \(\text{pref} : \mathcal{P}(V^*) \rightarrow \mathcal{P}(V^*)\) and \(\text{suff} : \mathcal{P}(V^*) \rightarrow \mathcal{P}(V^*)\) as \(\text{pref}(L) = \{w | w \in V^* \land (\exists x : x \in V^* : wx \in L)\}\) and \(\text{suff}(L) = (\text{pref}(L^R))^R\). \(\Box\)

\(^{13}\)The deterministic solution follows similarly.
For \( w \in V^* \) we will write \( \text{pref}(w) \) (\( \text{suff}(w) \)) instead of \( \text{pref}(\{w\}) \) (\( \text{suff}(\{w\}) \)).

**Property B.2** Let \( A, B \subseteq V^* \). Then \( \text{pref}(A) \cap B \neq \emptyset \iff A \cap BV^* \neq \emptyset \) and \( \text{suff}(A) \cap B \neq \emptyset \iff A \cap V^*B \neq \emptyset. \)

The following two theorems are used in the derivation of the Aho-Corasick precomputation algorithm.

**Theorem B.3** Let \( V \) be an alphabet, \( A, B, C \subseteq V^* \), and \( V^*C \cap B = BC \cap B \). Then

\[
\text{suff}(A)C \cap B = \text{suff}(\text{suff}(A) \cap B)C \cap B.
\]

**Proof**

\[
\begin{align*}
\text{suff}(A)C \cap B &= \{ \text{suff}(A) \subseteq V^*, \text{distributivity} \} \\
\text{suff}(A)C \cap V^*C \cap B &= \{ V^*C \cap B = BC \cap B \} \\
\text{suff}(A)C \cap BC \cap B &= \{ \text{distributivity} \} \\
(\text{suff}(A) \cap B)C \cap B &\subseteq \{ X \subseteq \text{suff}(X) \text{ for all } X \subseteq V^*, \text{monotonicity} \} \\
\text{suff}(\text{suff}(A) \cap B)C \cap B &\subseteq \{ \text{suff}(A) \cap B \subseteq \text{suff}(A), \text{monotonicity} \} \\
\text{suff}(\text{suff}(A))C \cap B &= \{ \text{suff} \text{ is idempotent, since } \leq_s \text{ is transitive} \} \\
\text{suff}(A)C \cap B &
\end{align*}
\]

If \( C = \{e\} \) or \( B = \text{pref}(B) \) then condition \( V^*C \cap B = BC \cap B \) is satisfied.

**Definition B.4** Define the relations \( \leq_p \) and \( \leq_s \) over \( V^* \times V^* \) as \( u \leq_p v \equiv u \in \text{pref}(v) \) and \( u \leq_s v \equiv u \in \text{suff}(v) \).

**Theorem B.5** Let \( V \) be an alphabet, \( A, B, C \subseteq V^* \), \( V^*C \cap B = BC \cap B \), and \( A \) is nonempty, finite, and linearly ordered with respect to \( \leq_s \). Then

\[
\text{suff}(A)C \cap B = \text{suff}(\text{MAX}_{\leq}, w : w \in \text{suff}(A) \cap B : w))C \cap B.
\]

**Proof**

\[
\begin{align*}
\text{suff}(A)C \cap B &\supseteq \{ (\text{MAX}_{\leq}, w : w \in \text{suff}(A) \cap B : w) \in \text{suff}(A), \text{monotonicity, } A \neq \emptyset \} \\
\text{suff}(\text{MAX}_{\leq}, w : w \in \text{suff}(A) \cap B : w))C \cap B &\supseteq \{ \text{suff}(A) \cap B \leq_s (\text{MAX}_{\leq}, w : w \in \text{suff}(A) \cap B : w), \text{monotonicity} \} \\
\text{suff}(\text{MAX}_{\leq}, w : w \in \text{suff}(A) \cap B : w))C \cap B &= \{ \text{Theorem B.3} \} \\
\text{suff}(A)C \cap B &
\end{align*}
\]
Definition B.6 The infix operators $\mid, [ , ] : V^* \times N \to V^*$ are defined by

\[\begin{align*}
    v|0 &= \varepsilon \quad (v \in V^*) \\
    \varepsilon|\{k + 1\} &= \varepsilon \quad (k \geq 0) \\
    (aw)|\{k + 1\} &= a(w|k) \quad (k \geq 0, a \in V, w \in V^*) \\
    v|0 &= v \quad (v \in V^*) \\
    \varepsilon|\{k + 1\} &= \varepsilon \quad (k \geq 0) \\
    (aw)|\{k + 1\} &= w|k \quad (k \geq 0, a \in W, w \in V^*)
\end{align*}\]

Define $\uparrow$ as $v|k = (v^R|k)^R$ and $\uparrow$ as $v|k = (v^R|k)^R$. The operators $\mid, [ , ]$, and $\uparrow$ are called “left take,” “left drop,” “right take,” and “right drop” respectively. □

For $A \subseteq V^*$ and $k \geq 0$ we define $A|k = (\cup w : w \in A : w|k)$ and $A\uparrow k = (\cup w : w \in A : w|k)$, and likewise for $\mid$ and $\downarrow$.

Property B.7 Let $V$ be an alphabet, $A, B \subseteq V^*$, $A \neq \emptyset$, and $\varepsilon \notin A$. Then

\[V^* A \uparrow B \neq \emptyset \vee V^* B \uparrow A \neq \emptyset \implies V^* A \cap B \neq \emptyset \vee V^* B \cap (A\uparrow 1) \neq \emptyset\]

Proof

\[\begin{align*}
    V^* A \cup B \neq \emptyset \vee V^* B \cup A \neq \emptyset \\
    \quad \{\text{split second disjunct: } V^* = V^+ \cup \{\varepsilon\}\} \\
    \quad V^* A \cup B \neq \emptyset \vee B \cup A \neq \emptyset \vee V^+ B \cap A \neq \emptyset \\
    \quad \{A \subseteq (A\uparrow 1)(A\uparrow 1); B \cap A \neq \emptyset \implies V^* A \cap B \neq \emptyset\} \\
    \quad V^* A \cap B \neq \emptyset \vee V^* B \cap (A\uparrow 1) \neq \emptyset \\
    \quad \{\text{left factoring of } V\}\} \\
    \quad V^* A \cap B \neq \emptyset \vee V^* B \cap (A\uparrow 1) \neq \emptyset
\end{align*}\]

□

We continue with some properties of the failure function that are used in the derivation of the Commentz-Walter precomputation algorithm.

Lemma B.8 For $x, y \in \text{suff}(P)$ and $y \neq \varepsilon$ we have

\[x <_p y \equiv x \leq_p f_r(y).\]

Proof

Let $x, y \in \text{suff}(P)$ and $y \neq \varepsilon$. We derive

\[\begin{align*}
    x &<_p y \\
    \equiv &\{\text{definition of } <_p \text{ and pref}\} \\
    x &\in \text{pref}(y) \setminus \{y\} \\
    \equiv &\{x \in \text{suff}(P)\} \\
    x &\in \text{pref}(y) \setminus \{y\} \cap \text{suff}(P) \\
    \Rightarrow &\{\text{pref}(y) \setminus \{y\} \cap \text{suff}(P) \text{ is finite and linearly ordered w.r.t. } \leq_p\} \\
    x &\leq_p (\text{MAX}_{<_p} w : w \in \text{pref}(y) \setminus \{y\} \cap \text{suff}(P) : w) \\
    \equiv &\{y \neq \varepsilon, \text{definition of } f_r\} \\
    x &\leq_p f_r(y) \\
    \Rightarrow &\{y \neq \varepsilon, f_r(y) <_p y \text{ (by definition of } f_r\text{), transitivity of } <_p\} \\
    x &<_p y
\end{align*}\]

□
Definition B.9 We define $\nu : \text{suff}(P) \longrightarrow \mathbb{N}$ by

$$
\nu(\varepsilon) = 0
$$

and

$$
\nu(y) = \nu(f_\nu(y)) + 1 \quad (y \in \text{suff}(P) \setminus \{\varepsilon\}).
$$

Property B.10 We have for all $y \in \text{suff}(P) \setminus \{\varepsilon\}$

$$
f_{\nu(y)}(y) = \varepsilon \land (\forall n : 0 \leq n < \nu(y) : f_\nu^n(y) \neq \varepsilon).
$$

Lemma B.11 For $x, y \in \text{suff}(P)$ and $y \neq \varepsilon$ we have

$$(\forall n : 0 \leq n \leq \nu(y) : x <_p y \equiv (\exists i : 0 < i \leq n : x = f_\nu^i(y)) \lor x <_p f_\nu^n(y))$$

Proof

Let $x, y \in \text{suff}(P)$ and $y \neq \varepsilon$. We proceed by induction on $n$.

base Let $n = 0$. Observe that $\nu(y) > 0 = n$. The equivalence is satisfied trivially.

step Let $n = k + 1$ for some $k : 0 \leq k < \nu(y)$. Assume

$$
x <_p y \equiv (\exists i : 0 < i \leq k : x = f_\nu^i(y)) \lor x <_p f_\nu^k(y)
$$

We derive

$$
x <_p y
\equiv \quad \{\text{induction hypothesis}\}
\equiv (\exists i : 0 < i \leq k : x = f_\nu^i(y)) \lor x <_p f_\nu^k(y)
\equiv \quad \{0 \leq k < \nu(y), \text{ hence by property B.10 } f_\nu^k(y) \neq \varepsilon, \text{ lemma B.8} \}
\equiv (\exists i : 0 < i \leq k : x = f_\nu^i(y)) \lor x <_p f_\nu^{k+1}(y)
\equiv \quad \{x <_p f_\nu^{k+1}(y) \equiv x = f_\nu^{k+1}(y) \lor x <_p f_\nu^{k+1}(y)\}
\equiv (\exists i : 0 < i \leq k + 1 : x = f_\nu^i(y)) \lor x <_p f_\nu^{k+1}(y)
$$

By instantiating $n$ with $\nu(y)$ in this lemma we obtain

Corollary B.12 For $x, y \in \text{suff}(P)$ and $y \neq \varepsilon$ we have

$$
x <_p y \equiv (\exists i : 0 < i \leq \nu(y) : x = f_\nu^i(y))
$$
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